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Abstract

A new p-version finite element formulation for steady, incompressible fluid flow and convective heat transfer problems is presented. The steady-state residual equations are obtained by considering a limiting case of the least-squares formulation for the transient problem. The method circumvents the Babuška-Brezzi condition, permitting the use of equal-order interpolation for velocity and pressure, without requiring the use of arbitrary parameters. Numerical results are presented to demonstrate the accuracy and generality of the method.

1. Introduction

Methods used to improve the quality of finite element solutions include h-methods, p-methods and h-p methods[1]. Finite element programs generally employ the h-version of the method, where the accuracy of the solution is improved by refining the mesh while using fixed low order element interpolation. This approach can be contrasted with spectral methods, which use high order global approximation functions. Spectral methods have been used to obtain highly accurate solutions to a variety of fluid dynamics problems[2].

During the past two decades p- and h-p versions of the finite element method have been developed which combine the geometric flexibility of standard low-order finite element techniques with the rapid convergence properties of spectral methods. In the p-version of the method, accuracy is achieved by increasing the element interpolation rather than refining the mesh. The h-p version involves a combination of mesh and polynomial refinement. These finite element developments have been primarily applied to structural mechanics analysis.

In recent years, 'spectral element' methods which are similar to the p- and h-p version finite element methods have been developed specifically for incompressible fluid flow[3]. Adaptive mesh strategies for the spectral element method have recently been investigated[4]. Compressible flow problems have already been solved using h-p adaptive finite element methods[5].

Finite element methods are typically based on either variational or weighted residual formulations. Variational formulations are generally viewed as producing the 'best' approximation to the exact solution of a variational problem. Unfortunately, variational principles cannot be constructed for the Navier-Stokes equations. In the absence of a variational principle, a weighted residual method is usually employed, the Galerkin formulation being the most popular[6].

A standard Galerkin formulation for the steady, incompressible Navier-Stokes equations in primitive variable form unfortunately requires the use of velocity and pressure interpolation functions which satisfy the Babuška-Brezzi condition. Several methods of circumventing this problem have been investigated and reported in the literature[7-9]. The approach derived here is based on the formulation presented by DeSampaio[10], although the present method does not require an 'upwinding' parameter. In fact, the numerical studies to be presented indicate that the use of 'upwinding' (Petrov-Galerkin weighting) may actually degrade the solution when high order approximation functions are used.

The paper is organized as follows. The hierarchical p-version approximation functions are described in the next section. Using these p-version approximation functions, the finite element method is applied to the one-dimensional Burgers' equation in section 3 and the two-dimensional convection-conduction equation in section 4. The steady-state residual equations are obtained by considering the steady-state limit of the least-squares formulation for the transient problem. Numerical examples with known solutions are used to evaluate the accuracy of the method. The method is extended to the incompressible Navier-Stokes equations in section 5. Accurate numerical results are obtained in section 6 using equal-order interpolation for velocity and pressure without requiring the use of arbitrary parameters. Conclusions are provided in the last section of the paper.

2. p-Version Finite Element Approximation

The p-version approximation functions and nodal variables for the two-dimensional element shown in figure 1 can be obtained by taking products of one-dimensional approximation functions and nodal variable operators in the ξ and η natural coordinate directions.

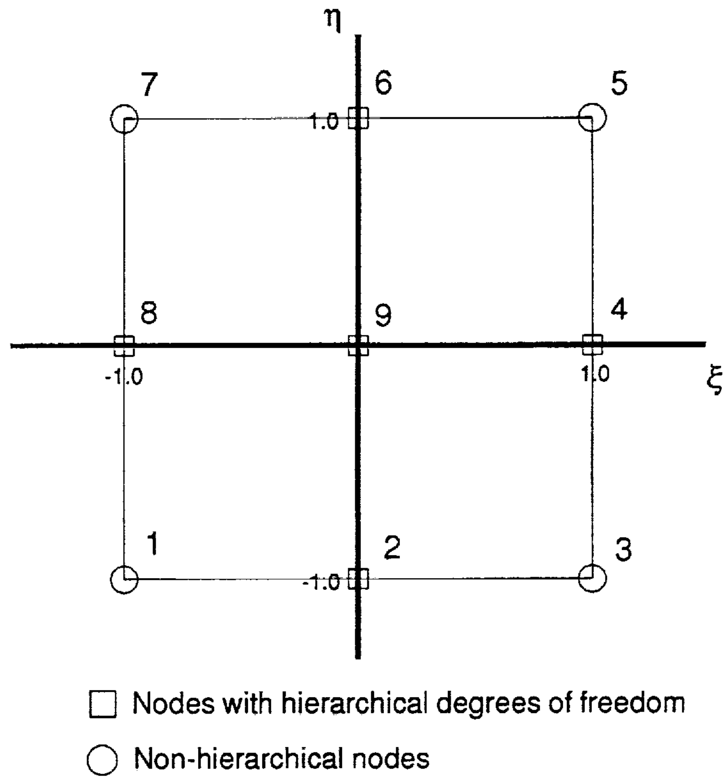

Figure 1: p-Version Element in natural coordinate space (ξ, η)

The one-dimensional approximation can be obtained from a Taylor series expansion, as shown in Appendix 1. In the ξ direction the approximation is given by:

$$\Theta^h(\xi) = N_1^{0\xi}(\Theta)_1 + N_2^{0\xi}(\Theta)_2 + N_3^{0\xi}(\Theta)_3 + \sum_{i=3}^{p} N_2^{i\xi}(\Theta_{\xi^i})_2 \tag{1}$$

where

$$N_1^{0\xi} = -\frac{\xi}{2}(1-\xi), \;\; N_2^{0\xi} = (1-\xi^2), \;\; N_3^{0\xi} = \frac{\xi}{2}(1+\xi)$$

$$N_2^{i\xi} = \xi^i - \xi^k, \;\; k = \begin{cases} 2 \text{ for even i} \\ 1 \text{ for odd i} \end{cases} \tag{2}$$

and

$$(\Theta)_1 = \Theta|_{\xi=-1},\ \ (\Theta)_2 = \Theta|_{\xi=0},\ \ (\Theta)_3 = \Theta|_{\xi=1}$$

$$(\Theta_{\xi^i})_2 = \frac{1}{i!}\frac{\partial^i\Theta}{\partial\xi^i}\Big|_{\xi=0} \tag{3}$$

Similarly, in the η direction:

$$\Theta^h(\eta) = N_1^{0\eta}(\Theta)_1 + N_2^{0\eta}(\Theta)_2 + N_3^{0\eta}(\Theta)_3 + \sum_{j=3}^{p} N_2^{j\eta}(\Theta_{\eta^j})_2 \tag{4}$$

where

$$N_1^{0\eta} = -\frac{\eta}{2}(1-\eta),\ \ N_2^{0\eta} = (1-\eta^2),\ \ N_3^{0\eta} = \frac{\eta}{2}(1+\eta)$$

$$N_2^{j\eta} = \eta^j - \eta^l,\ \ l = \begin{cases} 2 \text{ for even j} \\ 1 \text{ for odd j} \end{cases} \tag{5}$$

and

$$(\Theta)_1 = \Theta|_{\eta=-1},\ \ (\Theta)_2 = \Theta|_{\eta=0},\ \ (\Theta)_3 = \Theta|_{\eta=1}$$

$$(\Theta_{\eta^i})_2 = \frac{1}{j!}\frac{\partial^j\Theta}{\partial\eta^j}\Big|_{\eta=0} \tag{6}$$

Note that the first three terms on the r.h.s. of eqns (1) and (4) are the standard Lagrange interpolation functions for a parabolic element. Higher order terms are included by simply adding hierarchical degrees of freedom at the center node.

The two-dimensional approximation functions and nodal variables can be obtained by simply taking products of expressions (1) and (4), which yields the element approximation:

$$\Theta^h(\xi,\eta) = [N]\{\Theta\} \tag{7}$$

The contents of $[N]$ and $\{\theta\}$ are given in table 1.

Adaptive p-level refinement is extremely easy when hierarchical approximation functions are used. The element nodal configuration and geometry do not change. The p-level is increased (or decreased) by simply adding (or subtracting) degrees of freedom at nodes 2,4,6,8 and 9 shown in figure 1. Inter-element continuity conditions are satisfied by constraining appropriate degrees of freedom at the mid-side nodes 2,4,6 and 8.

3. Burgers' Equation

To illustrate the basic ideas, let's first consider the one-dimensional Burgers' equation:

Table 1: Hierarchical approximation functions and nodal variables

Node Number	Approximation Functions	Hierarchical Nodal Variables	Order of Derivatives
1	$N_1^{0\eta} N_1^{0\xi}$	$(\Theta)_1 = \Theta\|_{\xi=-1,\eta=-1}$	0
3	$N_1^{0\eta} N_3^{0\xi}$	$(\Theta)_3 = \Theta\|_{\xi=+1,\eta=-1}$	0
5	$N_3^{0\eta} N_3^{0\xi}$	$(\Theta)_5 = \Theta\|_{\xi=+1,\eta=+1}$	0
7	$N_3^{0\eta} N_1^{0\xi}$	$(\Theta)_7 = \Theta\|_{\xi=-1,\eta=+1}$	0
2	$N_1^{0\eta} N_2^{0\xi}$	$(\Theta)_2 = \Theta\|_{\xi=0,\eta=-1}$	0
	$N_1^{0\eta} N_2^{i\xi}$	$(\Theta_{\xi^i})_2 = \frac{1}{i!}\left(\frac{\partial^i\Theta}{\partial\xi^i}\right)_{\xi=0,\eta=-1}$	i=3,4,..., p
6	$N_3^{0\eta} N_2^{0\xi}$	$(\Theta)_6 = \Theta\|_{\xi=0,\eta=+1}$	0
	$N_3^{0\eta} N_2^{i\xi}$	$(\Theta_{\xi^i})_6 = \frac{1}{i!}\left(\frac{\partial^i\Theta}{\partial\xi^i}\right)_{\xi=0,\eta=+1}$	i=3,4,..., p
4	$N_2^{0\eta} N_2^{0\xi}$	$(\Theta)_4 = \Theta\|_{\xi=+1,\eta=0}$	0
	$N_2^{j\eta} N_3^{0\xi}$	$(\Theta_{\eta^j})_4 = \frac{1}{j!}\left(\frac{\partial^j\Theta}{\partial\eta^j}\right)_{\xi=+1,\eta=0}$	j=3,4,..., p
8	$N_2^{0\eta} N_1^{0\xi}$	$(\Theta)_8 = \Theta\|_{\xi=-1,\eta=0}$	0
	$N_2^{j\eta} N_1^{0\xi}$	$(\Theta_{\eta^j})_8 = \frac{1}{j!}\left(\frac{\partial^j\Theta}{\partial\eta^j}\right)_{\xi=-1,\eta=0}$	j=3,4,..., p
9	$N_2^{0\eta} N_2^{0\xi}$	$(\Theta)_9 = \Theta\|_{\xi=\eta=0}$	0
	$N_2^{0\eta} N_2^{i\xi}$	$(\Theta_{\xi^i})_9 = \frac{1}{i!}\left(\frac{\partial^i\Theta}{\partial\xi^i}\right)_{\xi=\eta=0}$	i=3,4,..., p
	$N_2^{j\eta} N_2^{0\xi}$	$(\Theta)_{\eta^j})_9 = \frac{1}{j!}\left(\frac{\partial^j\Theta}{\partial\eta^j}\right)_{\xi=\eta=0}$	j=3,4,..., p
	$N_2^{j\eta} N_2^{i\xi}$	$(\Theta_{\xi^i\eta^j})_9 = \frac{1}{i!j!}\left[\frac{\partial^j}{\partial\eta^j}\left(\frac{\partial^i\Theta}{\partial\xi^i}\right)\right]_{\xi=\eta=0}$	$i=3,4,\ldots,p$ $j=3,4,\ldots,p$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \tag{8}$$

Integrating with respect to time:

$$u^{n+1} - u^n + \int_t^{t+\Delta t}\Big(u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}\Big)dt = 0 \tag{9}$$

Using the trapezoidal rule to approximate the integral in eqn (9):

$$u^{n+1} - u^n + \frac{\Delta t}{2}\Big(u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}\Big)\Big|^{n} + \frac{\Delta t}{2}\Big(u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}\Big)\Big|^{n+1} = 0 \tag{10}$$

The least-squares criterion is now applied by minimizing the integral of the squared residual with respect to the degrees of freedom at time level 'n+1'. The resulting equations can be interpreted as a weighted residual formulation with weight functions as:

$$\{W\} = [N]^T + \frac{\Delta t}{2}\{\Phi\} \tag{11}$$

where

$$\{\Phi\} = u\Big[\frac{dN}{dx}\Big]^T + \frac{\partial u}{\partial x}\Big[N\Big]^T - \nu\Big[\frac{d^2N}{dx^2}\Big]^T$$

Note that the product rule of differentiation has been used to obtain the above expression. In the steady-state limit, the weighted residual formulation becomes

$$\int_\Omega \{W\}\Big(u\frac{du}{dx} - \nu\frac{d^2u}{dx^2}\Big)d\Omega = \{0\} \tag{12}$$

The above expression is clearly a Petrov-Galerkin formulation where velocity dependent 'least-squares' type terms have been included in the weighting functions given by eqn (11). The time step Δt serves as the upwinding parameter.

The Galerkin weighting $[N]^T$ can be used in the standard fashion to integrate the diffusive term by parts:

$$\int_\Omega \Big[N\Big]^T u\frac{du}{dx}d\Omega + \nu\int_\Omega \Big[\frac{dN}{dx}\Big]^T\frac{du}{dx}d\Omega + \frac{\Delta t}{2}\int_\Omega \Big\{\Phi\Big\}\Big(u\frac{du}{dx} - \nu\frac{d^2u}{dx^2}\Big)d\Omega = \nu\frac{du}{dx}\Big|_\Gamma \tag{13}$$

Note that the boundary term only applies to portions of the boundary where the primary variable 'u' has not been prescribed.

For a one-dimensional convection-diffusion problem with a constant convective velocity, the parameter Δt may be selected such that nodally exact steady-state solutions are obtained. However, for general problems in fluid dynamics and heat transfer the optimal value of the upwinding parameter is not known.

Numerical Example

Consider the solution of the steady-state Burgers' equation with the boundary conditions u(0)=1, u(1)=0 and let $\nu = 0.01$. Solutions to equation (13) were obtained using Newton-Raphson iteration. The non-symmetric matrix equation resulting from (13) was solved using a frontal solver[11] modified for use with p-version elements. No difficulties were encountered in obtaining converged solutions.

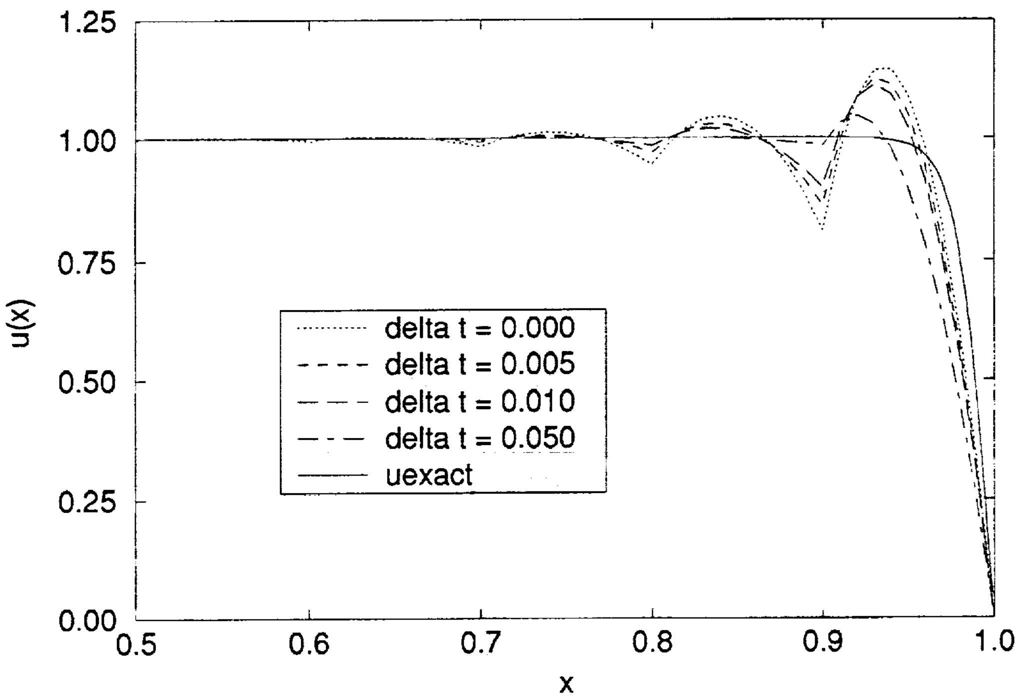

Figure 2: Solutions to Burgers' equation at p=2

A uniform mesh of ten elements at a p-level of p=2 was used to produce the oscillatory results shown in figure 2. The standard Galerkin formulation corresponds to the $\Delta t = 0.000$ solution, which is quite oscillatory. Petrov-Galerkin weighting is commonly prescribed as a cure for this oscillation problem[8], and it is clear in this case that the upwinding parameter Δt is in fact suppressing the oscillations. Unfortunately, it is also clear that none of the 'solutions' shown in figure 2 are very accurate. As has been previously pointed out by Gresho and Lee[12], the appropriate solution to this 'wiggle' problem is adequate mesh refinement. Here the use of p-refinement rather than h-refinement will be explored.

The exact solution to this problem[13] is given by:

$$u = \overline{u}\left\{\frac{1 - \exp[\overline{u}(x-1)/\nu]}{1 + \exp[\overline{u}(x-1)/\nu]}\right\} \tag{14}$$

where $\bar{u}$ satisfies:

$$\frac{\overline{u} - 1}{\overline{u} + 1} = \exp(-\overline{u}/\nu)$$

Using eqn (14) the L_2 norm:

$$||e||_0 = \left(\int_\Omega \left(u - u^h \right)^2 d\Omega \right)^{\frac{1}{2}} \tag{15}$$

can be used to measure the solution error.

Using (15), the error was computed from the solutions obtained using selective p-refinement. The p-level was changed in the last element ($0.9 \leq x \leq 1.0$) where the solution is changing rapidly and the p-level was fixed at p=2 for all the other elements. The solution error was computed using 12 point Gauss quadrature to evaluate the integral in eqn (15). The computed error values are plotted as a function of the Δt parameter in figure 3 for several p-levels. The results at p=2 confirm that the use of Petrov-Galerkin weighting can in fact improve the solution accuracy, provided the correct value of the upwinding parameter is selected. At p=3, the best possible Δt parameter improves the accuracy only slightly, and at p=4 the best parameter choice is $\Delta t = 0$.

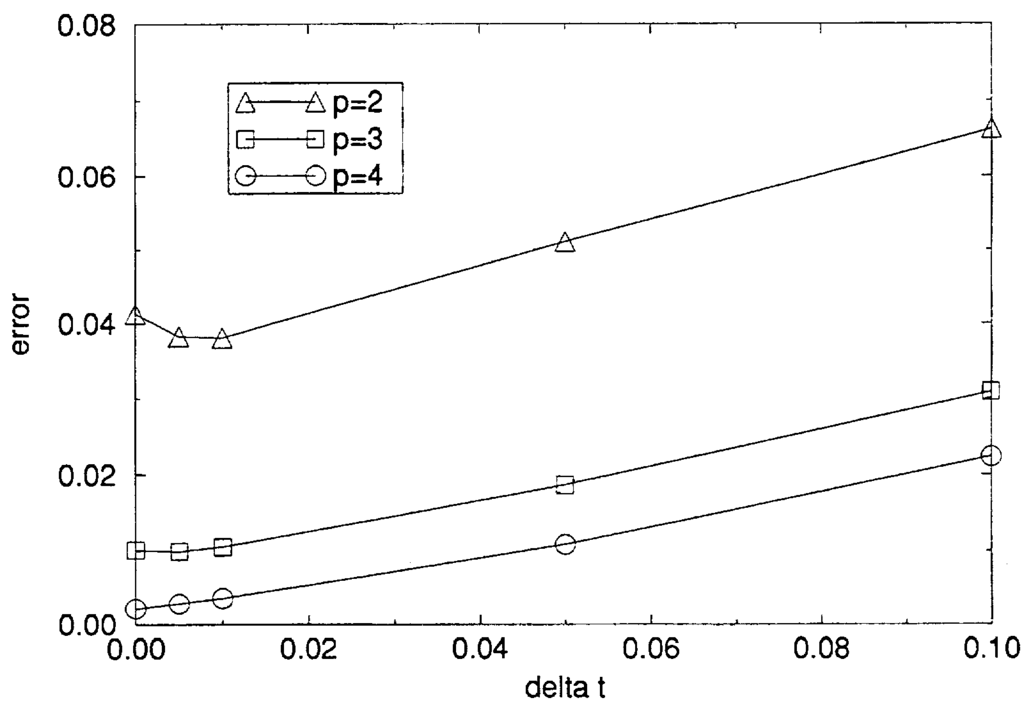

Figure 3: Burgers' equation solution error

For this problem it appears that the standard Galerkin formulation is best when higher order approximations are used. In figure 4, Galerkin solutions in the oscillatory region are compared with the exact solution. The oscillations are quite small at p=4, and at p=8 the solution is essentially exact. Highly accurate solutions have been obtained without the use of upwinding by simply employing local p-refinement.

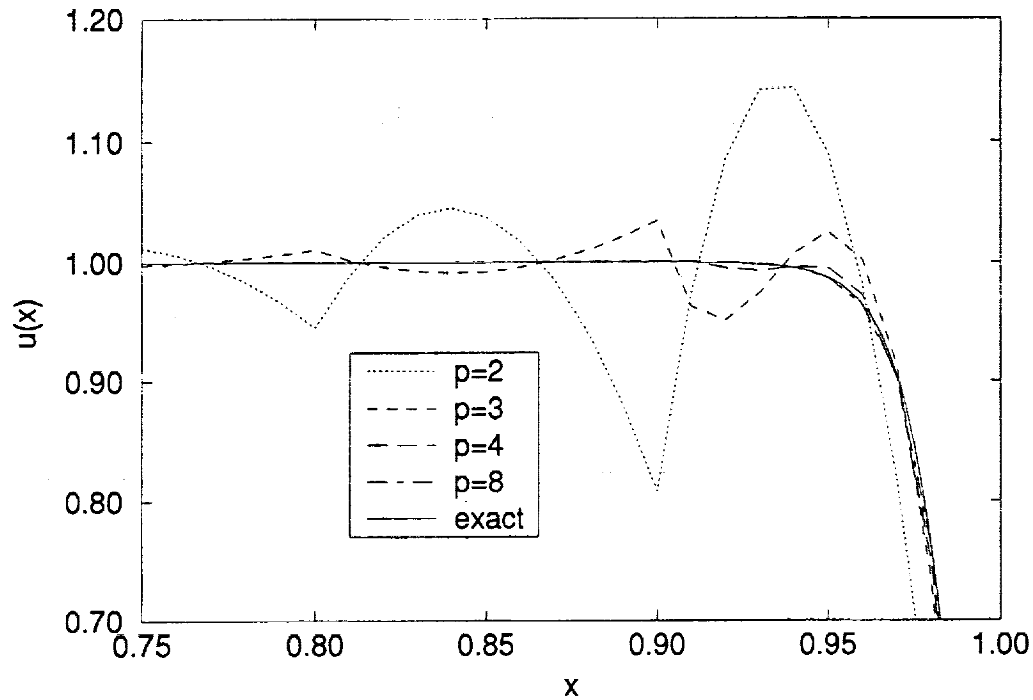

Figure 4: Solutions to Burgers' equation at higher p-levels

4. Convection-Conduction Equation

Before proceeding to the Navier-Stokes equations, let's first examine the simpler convection-conduction equation in two dimensions:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{1}{Pe}\Big(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\Big) = 0 \tag{16}$$

where u and v are known dimensionless velocity components and T is the dimensionless temperature. Applying the same approach to eqn (16) as was applied to Burgers' equation (8) yields the following steady-state result:

$$\int_\Omega \Big[N\Big]^T\Big(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\Big)d\Omega + \frac{1}{Pe}\int_\Omega\Big(\Big[\frac{\partial N}{\partial x}\Big]^T\frac{\partial T}{\partial x} + \Big[\frac{\partial N}{\partial y}\Big]^T\frac{\partial T}{\partial y}\Big)d\Omega$$

$$+\frac{\Delta t}{2}\int_\Omega\Big\{\Phi\Big\}\, r\, d\Omega = \frac{1}{Pe}\int_\Gamma\Big[N\Big]^T\nabla T\cdot n\, d\Gamma \tag{17}$$

here n is the outward normal vector at the boundary, r is the residual:

$$r = u\frac{\partial T^h}{\partial x} + v\frac{\partial T^h}{\partial y} - \frac{1}{Pe}\Big(\frac{\partial^2 T^h}{\partial x^2} + \frac{\partial^2 T^h}{\partial y^2}\Big)$$

and the vector $\{\Phi\}$ is given by:

$$\Big\{\Phi\Big\} = \frac{\partial r}{\partial\{T\}} = u\Big[\frac{\partial N}{\partial x}\Big]^T + v\Big[\frac{\partial N}{\partial y}\Big]^T - \frac{1}{Pe}\Big[\frac{\partial^2 N}{\partial x^2}\Big]^T - \frac{1}{Pe}\Big[\frac{\partial^2 N}{\partial y^2}\Big]^T$$

Once again, the boundary term in (17) only needs to be computed on portions of the boundary where the primary variable 'T' is not specified.

Numerical Example

The well known Smith-Hutton problem[14] will be solved using eight uniform p-version elements, shown in figure 5. The condition at the inlet is prescribed as:

$$T = 1 + tanh[10(2x+1)]; \quad y = 0,\ -1 \leq x \leq 0\ (inlet) \qquad (18)$$

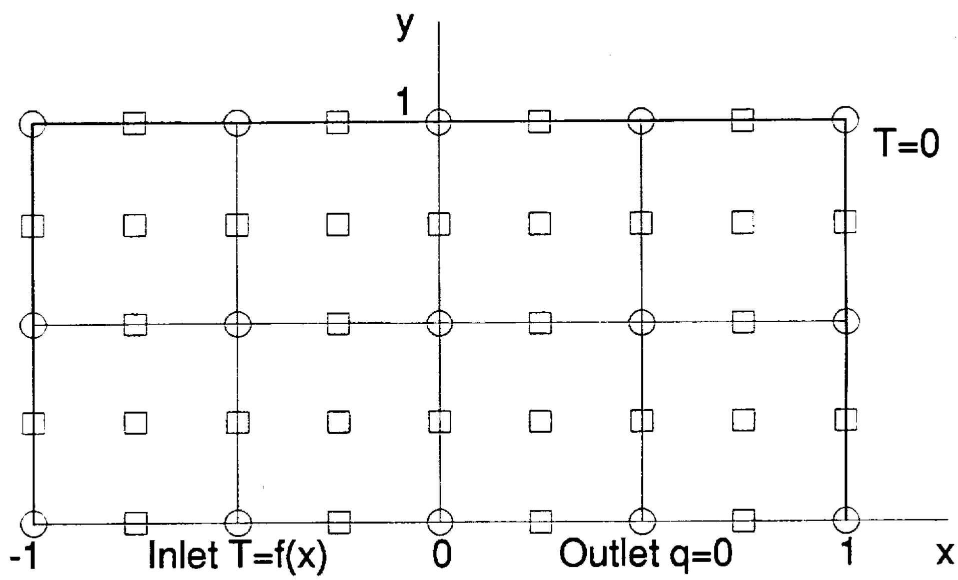

Figure 5: Smith-Hutton Problem

The dependent variable 'T' is essentially zero on $x = \pm 1$ and $y = 1$ and very nearly 2 at the origin of the coordinates. The climb from 0 to 2 occurs very sharply halfway along the inlet. A zero flux boundary condition ($q = \nabla T \cdot n = 0$) is assumed at the outlet. For this problem the velocity field is specified analytically as:

$$u = 2y(1-x^2), \quad v = -2x(1-y^2) \qquad (19)$$

corresponding to a streamfunction

$$\Psi = -(1-x^2)(1-y^2) \tag{20}$$

In the case of pure convection ($Pe \to \infty$), the exact solution to this problem is easily obtained in terms of the streamfunction Ψ:

$$T = T(\Psi) = 1 + tanh[10(1 - 2\sqrt{1+\Psi})] \tag{21}$$

For very large Peclet numbers ($Pe \geq 10^6$), eqn (21) can be used as an exact solution and the L_2 norm

$$||e||_0 = \left(\int_\Omega \left(T - T^h \right)^2 d\Omega \right)^{\frac{1}{2}} \tag{22}$$

can be used to measure the solution error.

It is well known that the standard Galerkin formulation is optimal for pure conduction ($Pe \to 0$) problems. It is commonly held that the Galerkin formulation is acceptable in low Pe cases, but inadequate for convection dominated problems. The availability of an exact solution for the convection dominated case permits an objective evaluation of the optimal upwinding parameter to be used with p-version finite elements.

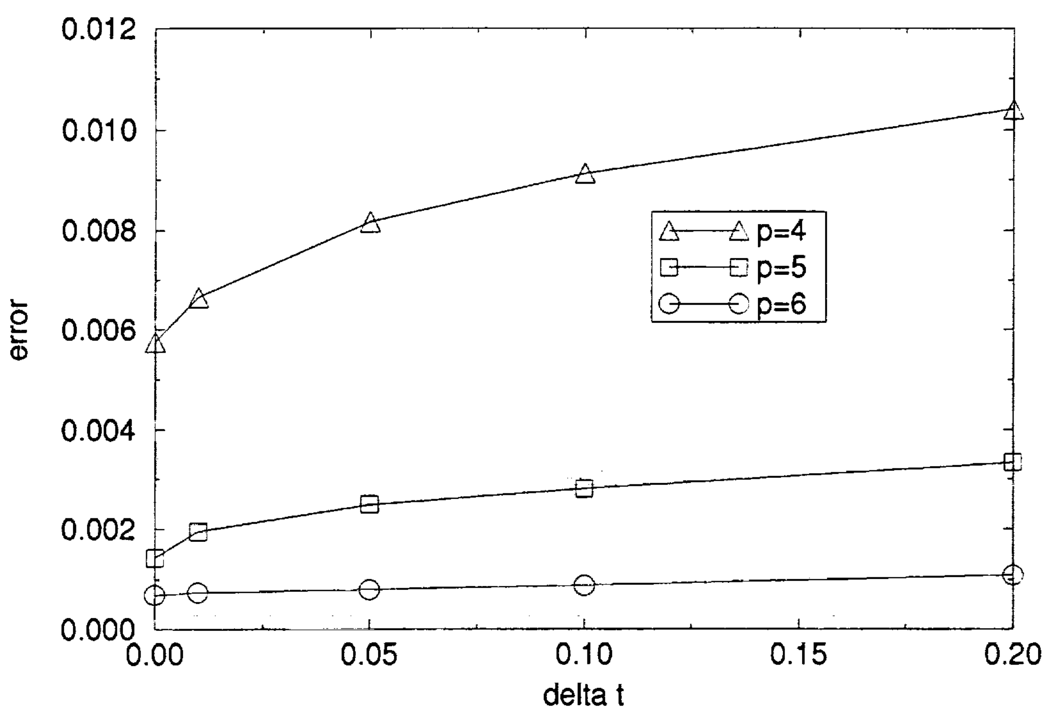

Figure 6: Smith-Hutton problem error

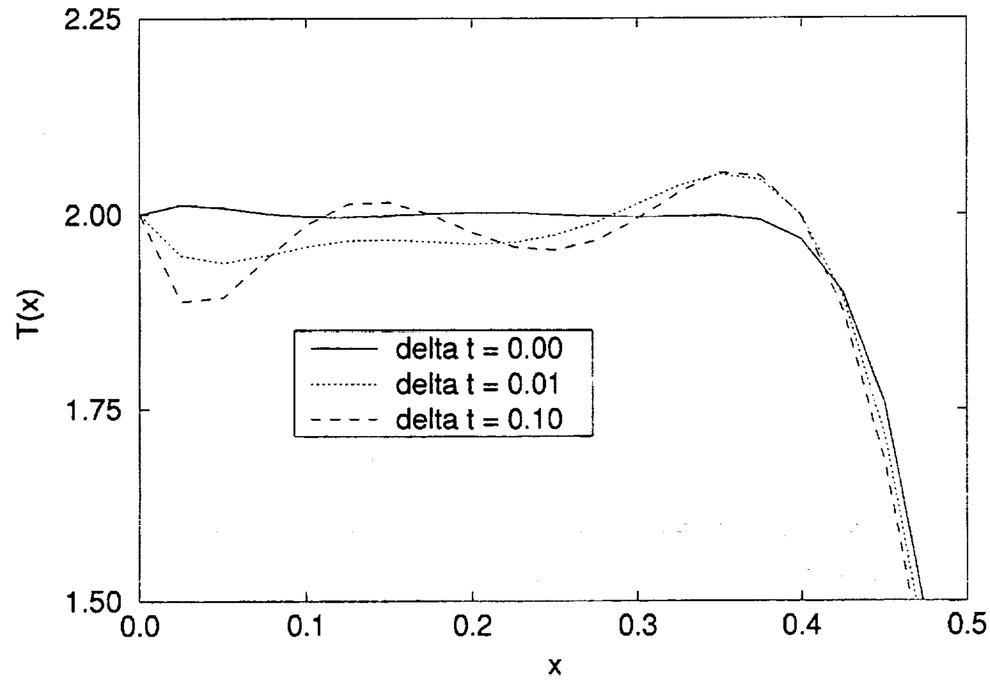

Figure 7: Solutions at $Pe = 10^6$

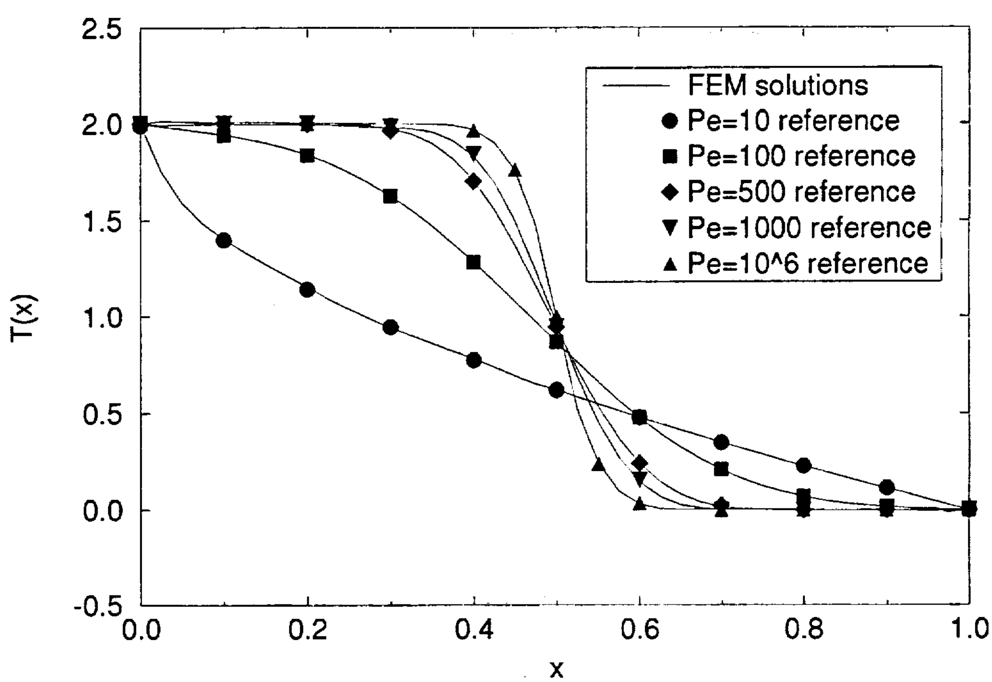

Figure 8: Smith-Hutton problem solutions

Eqn (17) was placed in matrix form and solved using the same p-version non-symmetric matrix solver used in section 3. Solutions obtained with $Pe = 10^6$ were used in eqn (22) to compute the error for several values of the upwinding parameter Δt. The results for several p-levels are presented in figure 6. Results for $p < 4$ are not presented because the inlet boundary conditions cannot be properly represented at low p-levels with the present mesh. From figure 6 it is clear that the use of Petrov-Galerkin weighting increases the solution error. The standard Galerkin formulation ($\Delta t = 0.0$) produces the best results at these p-levels.

The effect of the use of Petrov-Galerkin weighting at p=6 is illustrated in figure 7, where the upwinding parameter Δt is seen to introduce rather than eliminate oscillations in the outlet solution. Petrov-Galerkin weighting was also tested at a low Peclet number (Pe=10) and found to be of no benefit.

Galerkin solutions at the outlet for a range of Peclet numbers are presented in figure 8 along with reference solutions[14] considered to be reasonably accurate. The finite element solutions presented were obtained with p=6 for all cases except Pe=1000, where it was necessary to use p=7. It should be noted that these results are quite superior to previously published p-version least-squares results[15].

5. Incompressible Fluid Flow

The p-version finite element method used in sections 3 and 4 will now be extended to the incompressible Navier-Stokes equations. Two-dimensional, incompressible fluid flow with constant properties can be described by the following set of equations expressed in dimensionless form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{23a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \tag{23b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{24}$$

The approach used in sections 3 and 4 can be applied to the momentum equations (23) to give:

$$u^{n+1} - u^n + \frac{\Delta t}{2}\left(f_1^n + f_1^{n+1}\right) = 0 \tag{25a}$$

$$v^{n+1} - v^n + \frac{\Delta t}{2}\left(f_2^n + f_2^{n+1}\right) = 0 \tag{25b}$$

where

$$f_1 = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{26a}$$

$$f_2 = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \frac{1}{Re}\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) \tag{26b}$$

Let r_1,r_2 and r_3 represent the residuals in equations (25a),(25b) and (24), respectively. Applying the least squares criterion at time level 'n+1' to the integrated sum of the squared residuals yields:

$$\int_\Omega \Big[\frac{\partial r_1}{\partial\{u\}}r_1 + \frac{\partial r_2}{\partial\{u\}}r_2 + \frac{\partial r_3}{\partial\{u\}}r_3\Big]d\Omega = \{0\} \tag{27a}$$

$$\int_\Omega \Big[\frac{\partial r_1}{\partial\{v\}}r_1 + \frac{\partial r_2}{\partial\{v\}}r_2 + \frac{\partial r_3}{\partial\{v\}}r_3\Big]d\Omega = \{0\} \tag{27b}$$

$$\int_\Omega \Big[\frac{\partial r_1}{\partial\{P\}}r_1 + \frac{\partial r_2}{\partial\{P\}}r_2 + \frac{\partial r_3}{\partial\{P\}}r_3\Big]d\Omega = \{0\} \tag{27c}$$

Expressions for the derivatives of the residuals are obtained assuming equal order interpolation for field variables u,v and P.
From eqn (25a):

$$\frac{\partial r_1}{\partial\{u\}} = [N]^T + \frac{\Delta t}{2}\frac{\partial f_1^{n+1}}{\partial\{u\}}; \quad \frac{\partial r_1}{\partial\{v\}} = \frac{\Delta t}{2}\frac{\partial f_1^{n+1}}{\partial\{v\}}; \quad \frac{\partial r_1}{\partial\{P\}} = \frac{\Delta t}{2}\frac{\partial f_1^{n+1}}{\partial\{P\}} \tag{28a}$$

Similarly, from eqn (25b):

$$\frac{\partial r_2}{\partial\{u\}} = \frac{\Delta t}{2}\frac{\partial f_2^{n+1}}{\partial\{u\}}; \quad \frac{\partial r_2}{\partial\{v\}} = [N]^T + \frac{\Delta t}{2}\frac{\partial f_2^{n+1}}{\partial\{v\}}; \quad \frac{\partial r_2}{\partial\{P\}} = \frac{\Delta t}{2}\frac{\partial f_2^{n+1}}{\partial\{P\}} \tag{28b}$$

And from eqn (24):

$$\frac{\partial r_3}{\partial\{u\}} = \Big[\frac{\partial N}{\partial x}\Big]^T; \quad \frac{\partial r_3}{\partial\{v\}} = \Big[\frac{\partial N}{\partial y}\Big]^T; \quad \frac{\partial r_3}{\partial\{P\}} = \Big\{0\Big\} \tag{28c}$$

Substituting eqns (28) into (27) and considering the steady-state limit yields the following system of equations:

$$\int_\Omega \Big[\Big([N]^T + \frac{\Delta t}{2}\frac{\partial f_1}{\partial\{u\}}\Big)f_1 + \frac{\Delta t}{2}\frac{\partial f_2}{\partial\{u\}}f_2 + \Big[\frac{\partial N}{\partial x}\Big]^T\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big)\Big]d\Omega = \Big\{0\Big\} \tag{29a}$$

$$\int_\Omega \Big[\frac{\Delta t}{2}\frac{\partial f_1}{\partial\{u\}}f_1 + \Big([N]^T + \frac{\Delta t}{2}\frac{\partial f_2}{\partial\{u\}}\Big)f_2 + \Big[\frac{\partial N}{\partial y}\Big]^T\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big)\Big]d\Omega = \Big\{0\Big\} \tag{29b}$$

$$\int_\Omega \left[\frac{\partial N}{\partial x}\right]^T \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x} - \frac{1}{Re}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big)\right) d\Omega \ +$$

$$\int_\Omega \left[\frac{\partial N}{\partial y}\right]^T \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y} - \frac{1}{Re}\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big)\right) d\Omega = \Big\{0\Big\} \tag{30}$$

where the factor $\frac{\Delta t}{2}$ has been cancelled out of eqn (30).

The above derivation is similar to the formulation presented by DeSampaio[10]. However, DeSampaio did not include the continuity equation (24) directly in the least squares minimization. As a result, DeSampaio's formulation requires the selection of a nonzero value of Δt in order to circumvent the Babuška-Brezzi condition. Because the Δt parameter was not helpful in the p-version finite element solutions of Burgers' equation (section 3) and the convection-conduction equation (section 4) it is unlikely to be of benefit here. With Δt set to zero, equations (29) represent a Galerkin treatment of the momentum equations modified by a least-squares term to enforce the continuity equation. Equation (30) represents a least-squares minimization with respect to the pressure degrees of freedom. Setting Δt to zero and integrating the viscous terms in equations (29) by parts yields:

$$\int_\Omega \Big[N\Big]^T \Big(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial P}{\partial x}\Big) d\Omega + \frac{1}{Re}\int_\Omega \Big(\Big[\frac{\partial N}{\partial x}\Big]^T \frac{\partial u}{\partial x} + \Big[\frac{\partial N}{\partial y}\Big]^T \frac{\partial u}{\partial y}\Big) d\Omega$$

$$+ \int_\Omega \Big[\frac{\partial N}{\partial x}\Big]^T \Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) d\Omega = \frac{1}{Re}\int_\Gamma \Big[N\Big]^T \nabla u \cdot n \, d\Gamma \tag{31a}$$

$$\int_\Omega \Big[N\Big]^T \Big(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial P}{\partial y}\Big) d\Omega + \frac{1}{Re}\int_\Omega \Big(\Big[\frac{\partial N}{\partial x}\Big]^T \frac{\partial v}{\partial x} + \Big[\frac{\partial N}{\partial y}\Big]^T \frac{\partial v}{\partial y}\Big) d\Omega$$

$$+ \int_\Omega \Big[\frac{\partial N}{\partial y}\Big]^T \Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) d\Omega = \frac{1}{Re}\int_\Gamma \Big[N\Big]^T \nabla v \cdot n \, d\Gamma \tag{31b}$$

where n is the outward normal vector at the boundary. Equations 30, 31a and 31b constitute the required set of equations.

6. Numerical Examples

Two numerical examples are presented in this section to demonstrate the accuracy and convergence characteristics of this p-version formulation.

a) Example 1: Driven Cavity, Re=1000

Consider the well known 'driven cavity' problem shown in figure 9. The velocity components are constrained along the entire boundary and the pressure is constrained to

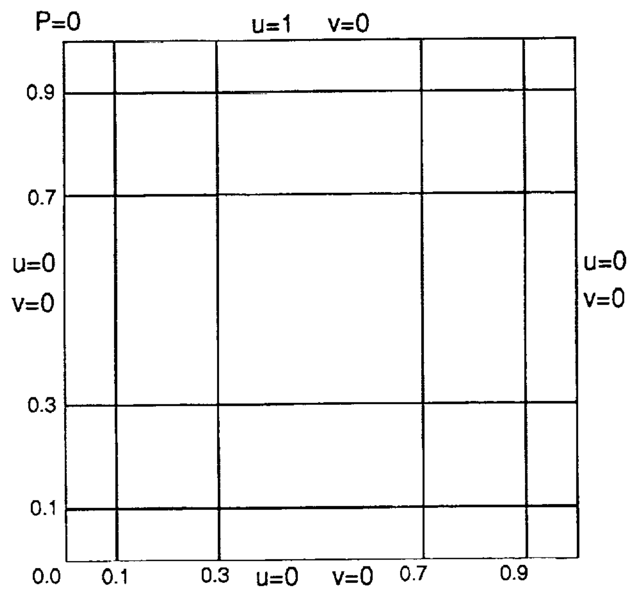

Figure 9: Driven Cavity Problem

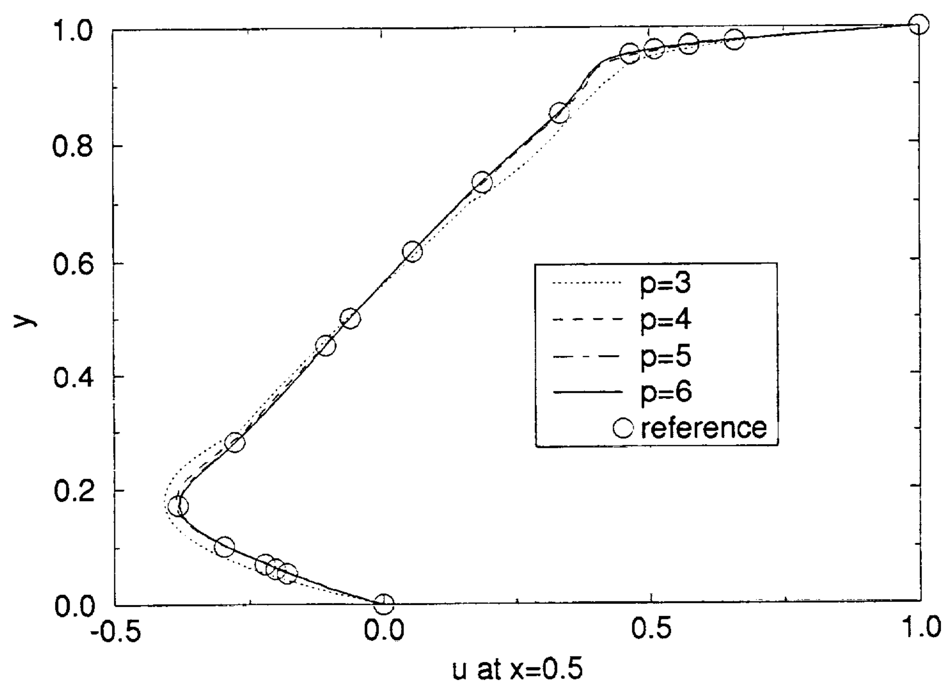

Figure 10: Plots of u(x=0.5,y) for driven cavity

a reference value of $P = 0$ at a single point. Computations were performed up to a p-level of p=6 using uniform p-refinement with the 25 element mesh shown in figure 9.

The coupled, nonlinear system of equations (30, 31a and 31b) was solved by succcessive substitution using the p-version finite element non-symmetric matrix frontal solver used in the previous sections. The p-levels were sequentially increased starting with p=2. A starting vector of $\{\delta\} = \{0\}$ was assumed at $p = 2$. For $p \geq 3$, the solution obtained at the previous p-level was used to generate an initial solution vector. This procedure is quite easy to implement because of the hierarchical nature of the degrees of freedom. The solution was considered to be sufficiently converged (in the iterative sense) when the maximum change in the solution vector was less than 10^{-4}. Generally around 10 iterations were required at each p-level.

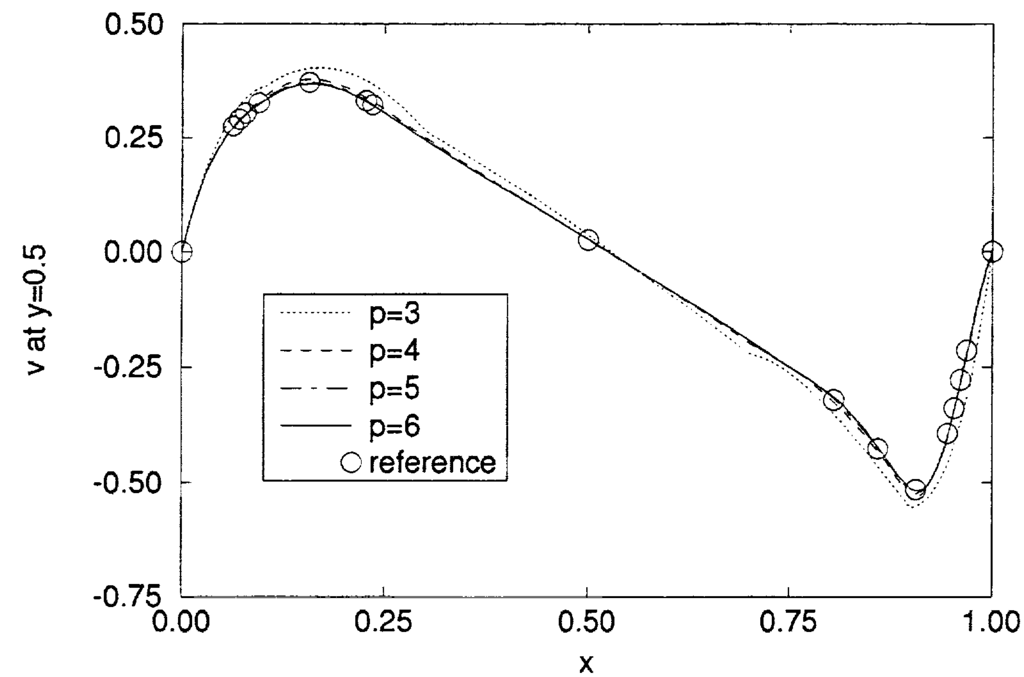

Figure 11: Plots of v(x,y=0.5) for driven cavity

The solutions for 'u' along the vertical centerline (x=0.5) are presented in figure 10 and the solutions for 'v' along the horizontal centerline are shown in figure 11. The results reported by Ghia et.al.[16] are also presented as a reference. Note that the solutions converge rapidly and the agreement with the reference solution is excellent.

The vector and streamline plot is shown in figure 12 and the pressure contours are presented in figure 13. Both the velocity and pressure solutions are quite smooth, even though equal order interpolation has been used without upwinding or other special techniques.

b) Example 2: Asymmetric Sudden Expansion, Re=229

In the second example a 3:2 asymmetric sudden expansion problem is examined and the numerical results are compared with the experimental results of Denham and Patrick[17]. The highest Reynolds number from the experiment, Re=229, is considered,

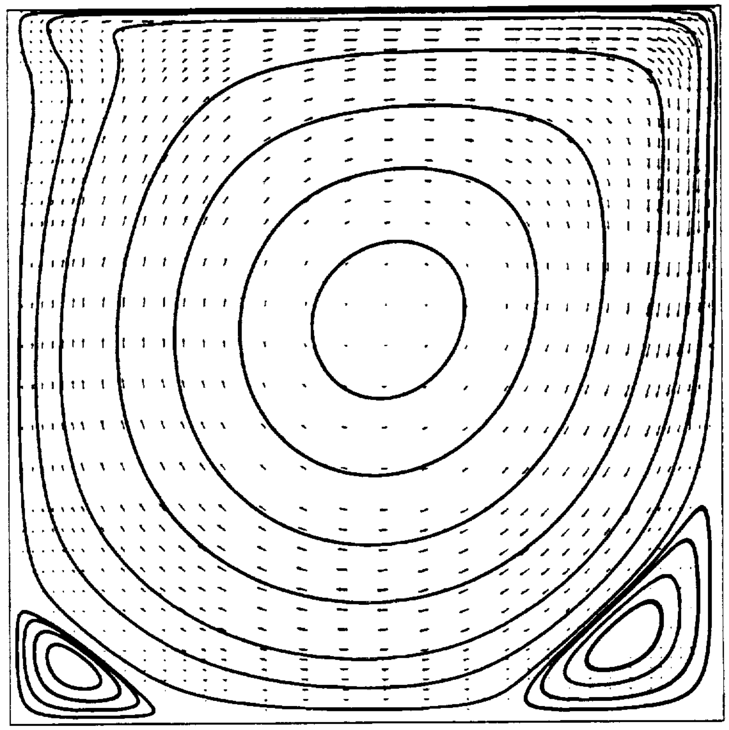

Figure 12: Vector and streamline plot for driven cavity

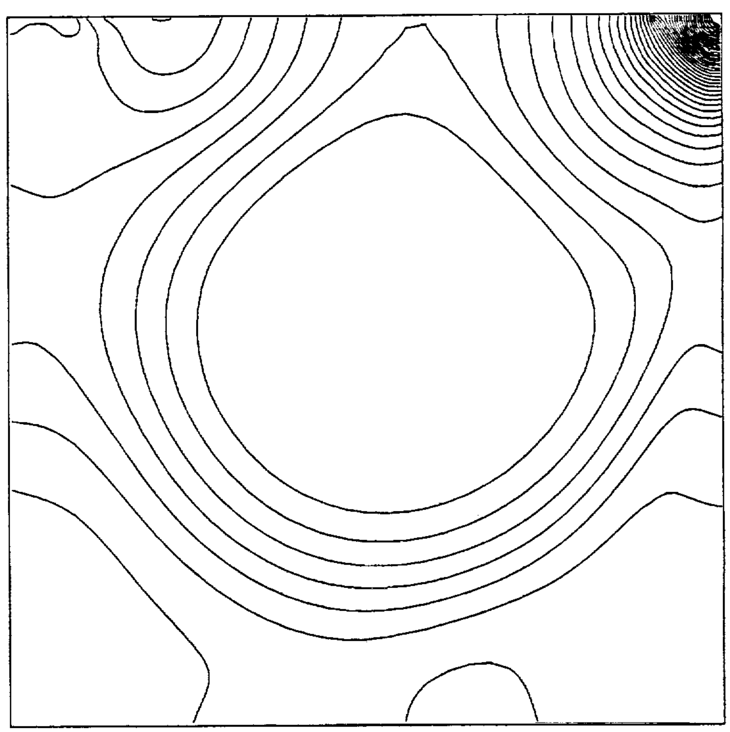

Figure 13: Pressure contour plot for driven cavity

where the Reynolds number is based on the mean velocity and channel half-width at the inlet. This problem was selected because the experimental data is very nearly two-dimensional and steady in the expansion region. At higher Reynolds numbers, the flow pattern is generally three-dimensional and often not truly laminar.

The problem geometry (expressed in step height units), boundary conditions and finite element mesh are shown in figure 14. The inlet boundary conditions for 'u' were determined using a least-squares fit to the inlet velocity measurements, and the vertical velocity 'v' was set to zero. Fully developed conditions were imposed 40 step heights downstream. In addition to the natural boundary conditions ($\nabla u \cdot n = \nabla v \cdot n = 0$), the pressure was constrained to a reference value of $P = 0$ at the outlet.

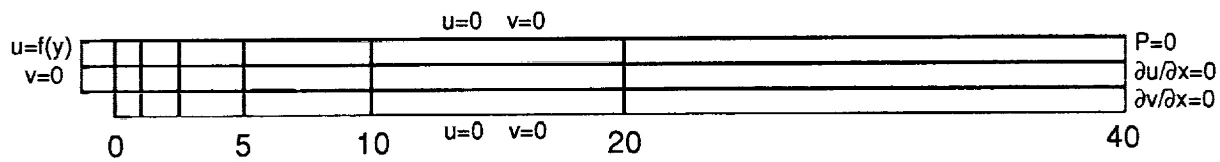

Figure 14: Asymmetric Sudden Expansion Problem

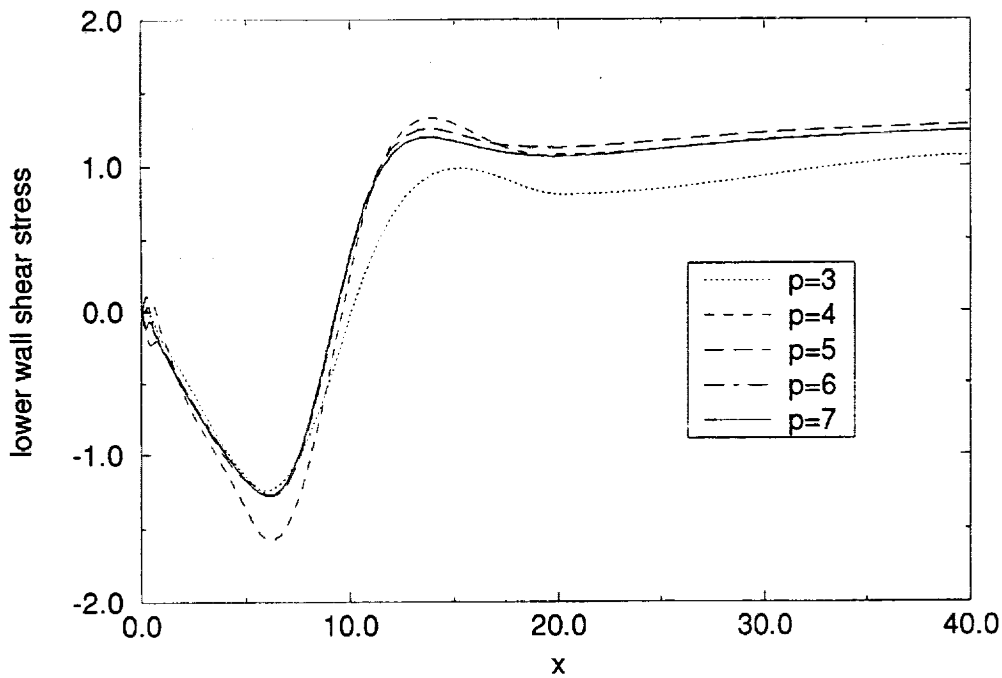

Figure 15: Plot of viscous shear stress along lower wall

Computations were performed up to a p-level of p=7 using the same iterative procedure described in the first example. In figure 15 the dimensionless viscous shear stress ($\tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$) along the lower wall (y=-1) is presented for several p-levels. The converged

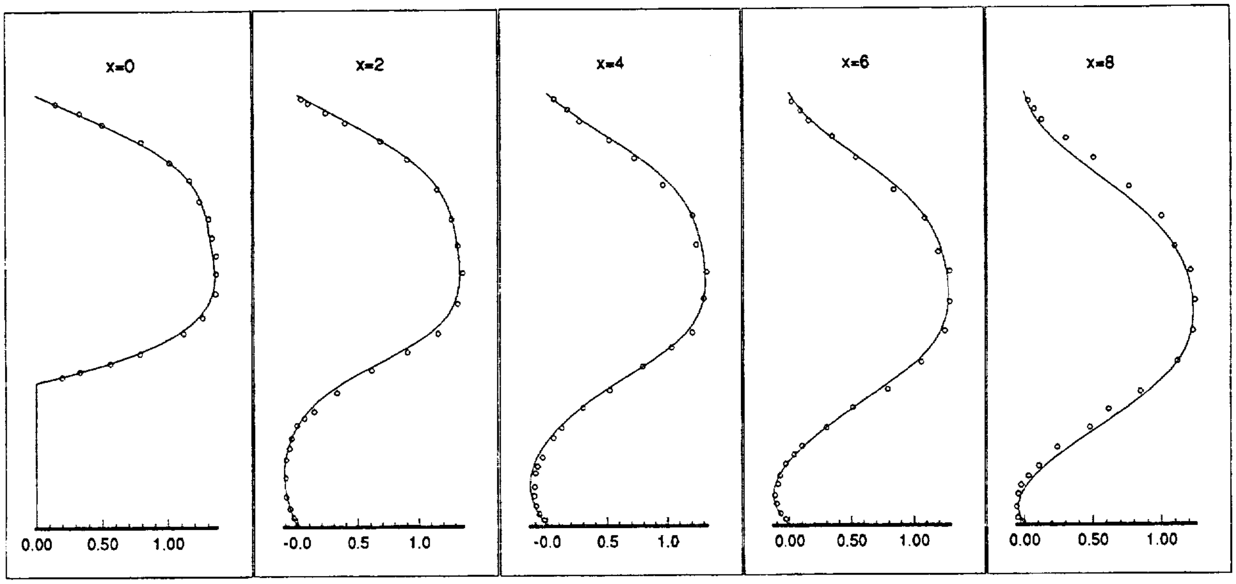

Figure 16: Velocity Profiles

solution indicates a recirculation zone length L_r prediction of $L_r = 9.3$, which agrees fairly well with the experimental value ($L_r \simeq 9.8$).

The converged velocity profiles at x=0,2,4,6 and 8 are compared with the experimental results in figure 16. The agreement with the experimental values is excellent for x=0 through x=6 and good for the x=8 profile. Similar numerical results were obtained using a p-version least-squares formulation [18]. Some error ($\approx 2\%$) may have been introduced in the process of extracting the velocity values from the graphical results of Denham and Patrick.

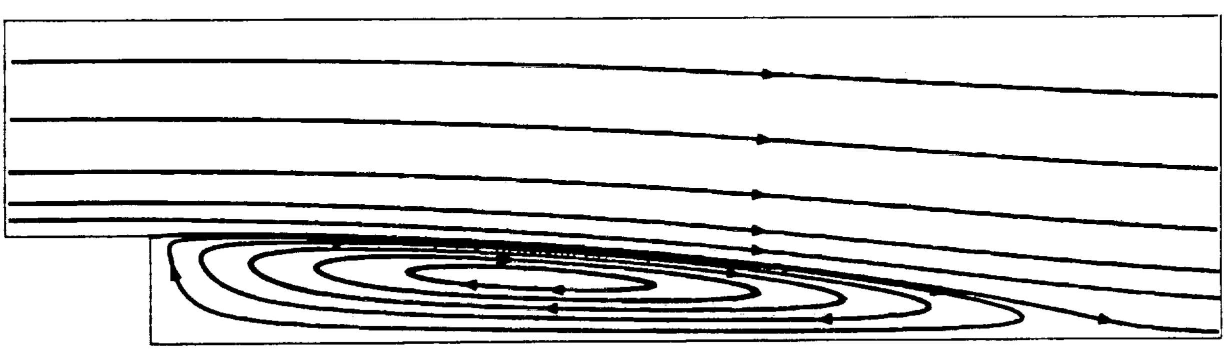

Figure 17: Streamlines for the sudden expansion problem

The recirculation zone streamline plot is shown in figure 17 and the pressure contours are presented in figure 18. Again, smooth solutions have been obtained using equal order interpolation without requiring the use of arbitrary parameters.

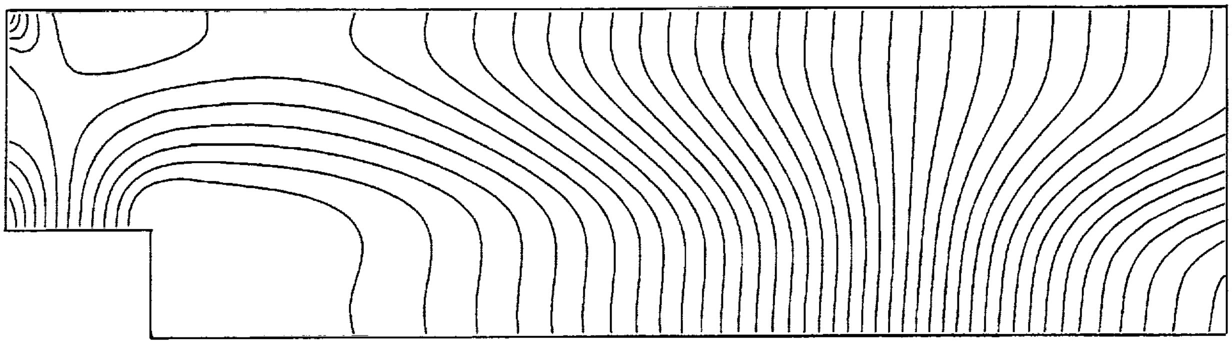

Figure 18: Pressure contours for the sudden expansion problem

7. Conclusions

Petrov-Galerkin formulations for the steady solution of Burgers' equation, convective heat transfer, and incompressible fluid dynamics problems have been presented. The weighted residual formulations were obtained by applying the least- squares approach to the transient problems, with the temporal derivatives approximated by finite differences. In the steady-state limit, this approach leads to a Petrov-Galerkin statement with the time step Δt serving as the upwinding parameter.

Numerical results using p-version finite elements for the one-dimensional Burgers' equation and a two-dimensional convection-conduction problem revealed that the standard Galerkin method ($\Delta t = 0$) performed better than the Petrov-Galerkin formulation when higher order approximations ($p \geq 4$) were used. This evaluation was made by comparing the L_2 norm of the solution errors.

Based on these numerical findings, a value of $\Delta t = 0$ was selected for the incompressible Navier-Stokes formulation, resulting in a method of circumventing the Babuška-Brezzi condition without requiring the selection of parameter values. The numerical solutions obtained using this formulation agree well with reference solutions and experimental data.

The following specific conclusions can be made concerning these results:

1. The Babuška-Brezzi condition can be circumvented in a very simple manner without the use of arbitrary parameters.

2. 'Upwinding'(Petrov-Galerkin weighting) is only helpful if the element approximation is not capable of accurately representing the solution.

3. The p-version of the finite element method is well-suited to problems involving incompressible fluid flow and convective heat transfer, producing non-oscillatory and highly accurate solutions.

4. Although h-p methods may be necessary for problems with extremely sharp gradients, the p-version of the finite element method can be very effective at increasing the local order of approximation to produce accurate solutions to a range of incompressible flow problems.

The primary goal of this work was to demonstrate the accuracy and generality of this new method. The computational efficiency of the method can be improved by using the technique of 'static condensation', where the degrees of freedom on element interiors are computed after determining the solution on the element boundaries. Also, for large scale problems, indirect methods are more efficient than direct methods such as the frontal solver. Future work will concentrate on these efficiency improvements and the development of reliable error indicators for use with adaptive methods.

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Appendix A

Consider a three node, one dimensional element: node 1 at $\xi = -1$, node2 at $\xi = 0$ and node3 at $\xi = +1$, where ξ is a local coordinate. A dependent variable $\Theta(\xi)$ can be expanded in a Taylor series about node 2 ($\xi = 0$):

$$\Theta(\xi) = \Theta_2 + \xi \frac{\partial \Theta_2}{\partial \xi} + \frac{\xi^2}{2} \frac{\partial^2 \Theta_2}{\partial \xi^2} + \sum_{i=3}^{\infty} \frac{\xi^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \tag{A.1}$$

It is desireable to replace $\frac{\partial \Theta_2}{\partial \xi}$ and $\frac{\partial^2 \Theta_2}{\partial \xi^2}$ in eqn (A.1) by $\Theta(\xi = -1) = \Theta_1$ and $\Theta(\xi = 1) = \Theta_3$. From eqn (A.1):

$$\Theta_1 = \Theta_2 - \frac{\partial \Theta_2}{\partial \xi} + \frac{1}{2} \frac{\partial^2 \Theta_2}{\partial \xi^2} + \sum_{i=3}^{\infty} \frac{(-1)^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \tag{A.2}$$

$$\Theta_3 = \Theta_2 + \frac{\partial \Theta_2}{\partial \xi} + \frac{1}{2} \frac{\partial^2 \Theta_2}{\partial \xi^2} + \sum_{i=3}^{\infty} \frac{(+1)^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \tag{A.3}$$

Subtracting eqn (A.2) from (A.3) and rearranging gives

$$\frac{\partial \Theta_2}{\partial \xi} = \frac{1}{2}\Theta_3 - \frac{1}{2}\Theta_1 - \frac{1}{2} \sum_{i=3}^{\infty} \frac{1-(-1)^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \tag{A.4}$$

Adding eqn (A.2) to (A.3) and rearranging gives

$$\frac{\partial^2 \Theta_2}{\partial \xi^2} = \Theta_1 - 2\Theta_2 + \Theta_3 - \sum_{i=3}^{\infty} \frac{1+(-1)^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \tag{A.5}$$

Substituting eqns (A.4) and (A.5) into (A.1) yields

$$\begin{aligned}\Theta(\xi) = \Theta_2 + \frac{\xi}{2}\Theta_3 - \frac{\xi}{2}\Theta_1 - \frac{\xi}{2} \sum_{i=3}^{\infty} \frac{1-(-1)^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \\ -\xi^2\Theta_2 + \frac{\xi^2}{2}\Theta_3 + \frac{\xi^2}{2}\Theta_1 - \frac{\xi^2}{2} \sum_{i=3}^{\infty} \frac{1+(-1)^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \\ + \sum_{i=3}^{\infty} \frac{\xi^i}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i}\end{aligned}$$

which can be rewritten as

$$\Theta(\xi) = -\frac{\xi}{2}(1-\xi)\Theta_1 + (1-\xi^2)\Theta_2 + \frac{\xi}{2}(1+\xi)\Theta_3 + \sum_{i=3}^{\infty} \frac{(\xi^i - \xi^k)}{i!} \frac{\partial^i \Theta_2}{\partial \xi^i} \tag{A.6}$$

where

$$k = \begin{cases} 2 \text{ for even i} \\ 1 \text{ for odd i} \end{cases}$$

If terms of up to order 'p' are retained, then $\Theta(\xi)$ can be approximated as:

$$\Theta(\xi) = N_1^{0\xi}(\Theta)_1 + N_2^{0\xi}(\Theta)_2 + N_3^{0\xi}(\Theta)_3 + \sum_{i=3}^{p} N_2^{i\xi}(\Theta_{\xi^i})_2 \tag{A.7}$$

where

$$N_1^{0\xi} = -\frac{\xi}{2}(1-\xi), \ \ N_2^{0\xi} = (1-\xi^2), \ \ N_3^{0\xi} = \frac{\xi}{2}(1+\xi)$$

$$N_2^{i\xi} = \xi^i - \xi^k, \ k = \begin{cases} 2 \text{ for even i} \\ 1 \text{ for odd i} \end{cases} \tag{A.8}$$

and

$$(\Theta)_1 = \Theta|_{\xi=-1}, \ \ (\Theta)_2 = \Theta|_{\xi=0}, \ \ (\Theta)_3 = \Theta|_{\xi=1}$$

$$(\Theta_{\xi^i})_2 = \frac{1}{i!}\frac{\partial^i \Theta}{\partial \xi^i}\bigg|_{\xi=0} \tag{A.9}$$

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13. ABSTRACT ***(Maximum 200 words)***

A new p-version finite element formulation for steady, incompressible fluid flow and convective heat transfer problems is presented. The steady-state residual equations are obtained by considering a limiting case of the least-squares formulation for the transient problem. The method circumvents the Babuska-Brezzi condition, permitting the use of equal-order interpolation for velocity and pressure, without requiring the use of arbitrary parameters. Numerical results are presented to demonstrate the accuracy and generality of the method.

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