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DIFFERENTIAL EQUATIONS IN AIRPLANE MECHANICS.

By

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For determining the motion of an airplane, we have adopted
the hypothesis that the reactions of the air depend entirely on
the relative speed of the airplane. Even if we adopt the sim-
plest laws of resistance, we obtain differential equations which
we can not integrate explicitly. If we confine ourselves to the
motion in a vertical plane and, at the same time, assume a con-
stant angle of attack, we still obtain differential equations
which can not be integrated by elementary methods.

In the following paragraphs, we will first draw some con-
clusions of purely theoretical interest, from the general equa-
tions of motion. At the end, we will consider the motion of an
airplane, with the engine dead and with the assumption that the
angle of attack remains constant. Thus we arrive at a simple
result, which can be rendered practically utilizable for deter-
mining the trajectory of an airplane descending at a constant
steering angle.

Let us assume that the airplane moves in its plane of symme-
try, considered vertical. Let x and y represent a system of
coordinates in this plane, the axis of the x lines being hori-
zontal and the axis of the y lines being vertical and upward.
Let us designate by u and v the projections of the velocity v

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of the center of gravity of the airplane on the axes of x and y
and by the angle ψ the angle of the velocity with the axis of
the x lines. Let us designate by M the mass of the airplane
and by MI its moment of inertia with relation to an axis pass-
ing through the center of gravity at right angles to the plane
xy. The forces acting on the airplane are:

Its weight Mg directed toward -y:

Propeller thrust T. of which we will designate by MA and
MB the components in the direction of w and at right angles
to w;

Drag, M a w², in the direction -w;

Lift, M b w², at right angles to w.

Under these conditions the theory of the motion of the cen-
ter of gravity gives us

$$M\frac{du}{dt} = - M \, a \, w^2 \cos \, \psi \; + M \, A \, \cos \, \psi \; - M \, b \, w^2 \sin \, \psi \; - M \, B \, \sin \, \psi .$$

$$M\frac{dv}{dt} = - M \, a \, w^2 \sin \, \psi \; + M \, A \, \sin \, \psi \; + M \, b \, w^2 \cos \, \psi \; + M \, B \, \cos \, \psi -$$
$$-Mg \, dt \, .$$

By substituting

$$\cos \, \psi \; = \frac{u}{w} \qquad \sin \, \psi \; = \frac{v}{w}$$

we have

(1) $\frac{du}{dt} = - w \, (a \, u + b \, v) + \frac{Au \, - Bv}{w}$

(2) $\frac{dv}{dt} = - w \, (a \, v - b \, u) + \frac{Av + Bu}{w} - g$

$$w = \sqrt{u^2 + v^2} \, .$$

If we project these forces on the tangent and on the perpendicular to the trajectory, we obtain

(3) $\frac{dw}{dt} = - aw^2 - g \sin \psi + A.$

(4) $w\frac{d\psi}{dt} = bw^2 - g \cos \psi + B.$

By letting

$$a = \rho \sin \alpha \qquad A = K \cos \beta$$
$$b = \rho \cos \alpha \qquad B = K \sin \beta$$

We can, by introducing imaginary quantities, combine equations (1) and (2) into a single equation

(5) $\frac{d(u + iv)}{dt} = \left(i w \rho e^{i\alpha} + \frac{Ke^{i\beta}}{w} \right)(u + iv) - gi.$

The moment of the air resistances about the center of gravity can be expressed in the form

$$M w^2 G(\varphi) - M w \frac{d(\varphi + \psi)}{dt} H(\varphi).$$

in which G and H are the periodic functions of the angle of attack φ, which varies with the steering angle. We then have, according to the theory of motion about the center of gravity,

(6) $I \frac{d^2 (\varphi + \psi)}{dt^2} = w^2 G(\varphi) - w \frac{d(\varphi + \psi)}{dt} H(\varphi).$

Let us assume that the influence of altitude may be neglected. By multiplying equation (1) by u and equation (2) by v and adding the results, we obtain

(7) $\dfrac{1}{2}\dfrac{dw^2}{dt} = -aw^3 + Aw - gv.$

The quantity a, necessarily implying a lower positive limit a', is obtained from equation (7).

(8) $\dfrac{1}{2}\dfrac{dw^2}{dt} < -w\left(a'w^3 - (P+g)\right).$

MP representing the maximum propeller thrust, from which we reason that the speed can not go on increasing indefinitely. To be more exact, it must remain below

$$\sqrt{\dfrac{P+g}{a'}}$$

in which a" is any quantity smaller than a'.

By means of equation (1) we can easily demonstrate the following theorem, which is practically self-evident. <u>If the engine is dead, it is not possible by any maneuver of the elevator to keep the airplane above any fixed horizontal line for any arbitrary period of time.</u> We find, in fact, by integrating equation (7) between t_0 and t(A and B being zero)

$$\dfrac{1}{2}(w_t{}^2 - w_0{}^2) = -\int_{t_0}^{t} aw^3\,dt - g(y_t - y_0).$$

Taking into account the inequality a > a' > 0, the hypothesis $y_t > 0$ gives us

$$a'\int_{t_0}^{t} w^3\,dt < \dfrac{1}{2}w_0{}^2 + gy_0.$$

It follows that the integral

$$\int_{t_0}^{\infty} w^3\,dt$$

is convergent. Remarking that by reason of equations (7) and (8), $\frac{dw^2}{dt}$ is limited, we reason from this that w tends towards zero, when t tends toward infinity. Hence we have also

(9) $\quad \lim\limits_{t\,=\,\infty} v = 0, \qquad \lim\limits_{t\,=\,\infty} u = 0$

Consequently, there is an infinity of values of t, t_1, t_2 ... t_n such that we have

(10) $\quad \lim\limits_{n\,=\,\infty}\left(\dfrac{dv}{dt}\right)_{t_n} = 0, \qquad \lim\limits_{n\,=\,\infty}\left(\dfrac{du}{dt}\right)_{t_n} = 0, \qquad \lim\limits_{t\,=\,\infty} t_n = \infty$

By substituting $t = t_n$ in the equation

$$\frac{dv}{dt} = -w\,(av - bu) - g$$

and by making n tend toward infinity, we arrive at the absurdity $g = 0$. The hypothesis $y_t > 0$ is therefore inadmissible.

Let us now assume the thrust and angle of attack to remain constant. On dividing equation (6) by w^2, we see that for a constant steering angle of the elevator, this hypothesis concerning the angle of attack is judtified in proportion as I. and H are smaller and w is larger. From equations (1) and (3) we deduce the differential equation

(11) $\quad \dfrac{du}{(u^2 + v^2)\,(au + bv) - Au + Bv} =$

$$\dfrac{dv}{(u^2 + v^2)\,(av - bu) - Av - Bu + g\,\sqrt{u^2 + v^2}}$$

in which a, b, A and B are constants. Once this equation is
integrated, t, x and y are obtained by quadratures. We can
accordingly confine ourselves to the study of the integral curves
of equation (11), that is, to determining the hodograph of the
motion. We can apply to equation (11) the methods given by Mr.
Poincaré in his memoirs: "On curves defined by a differential
equation" (Journal de Mathématiques pures et appliquées, 1881-
1886). We see that equations (1) and (2) define u and v for
every value of t, when the initial velocity $u_0 v_0$ is given.
On considering u and v as cartesian coordinates of a point
in the plane, the point (u,v) describes a certain characteris-
tic curve, when t varies from 0 to infinity. We call "sin-
gular points" those where the second members of equations (1)
and (2) disappear at the same time or become discontinuous. In
the case under consideration, there is, aside from the origin,
only one other singular point of finite distance. We have al-
ready seen (inequality 8) that the characteristics remain at a
finite distance. Under this condition, the theory of Poincaré
shows us that a knowledge of the singular points and closed char-
acteristics (limited cycles) at a finite distance suffices for
finding the course of the integral curves, when t tends toward
infinity.

Instead of going deeply into this study here, we shall take
up by a simpler method, the case where the propeller thrust is
zero. Equation (11) is reduced to

(13)
$$\frac{du}{w\,(au + bv)} = \frac{dv}{w\,(av - bu) + g}$$

For $\alpha = C$,* that is to say, $a = 0$, $b = \rho$, this equation is readily integrated by means of the elementary functions. We have, in fact,

$$\frac{du}{\rho\,vw} = \frac{dv}{-\,\rho\,uw + g} = \frac{w\,d\,w}{gv}$$

Hence

$$gdu = \rho\,w^2\,dw$$

$$\rho\,\frac{w^3}{3} - gu = C \qquad (C = \text{arbitrary constant})$$

By varying C we obtain a system S of closed, non-intersecting curves, which embrace the whole plane.

In the case $\alpha = \pi/3$, corresponding to the motion of a sphere in a resisting medium, equation (12) can be integrated by quadratures, as demonstrated by Liouville.

Although we do not know for α any general integral of equation (12), we can construct its integral curves by a simple method. We have, in fact, the following theorem. The integral curves of equation (12) are the trajectories of the angle α of the system of S curves.

$$\rho\,\frac{(u^2 + v^2)^{3/2}}{3} - gu = \text{constant}$$

provided we consider these trajectories with reference to the rec-

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* In his article "Le vol Aérien" (Aerial Flight), Mr. Lanchester discussed the fall of an airplane under this hypothesis, which returns to the assumption that the airplane does not lose energy.

tangular axes which make the angle α (in the positive direction)
with the original trajectories (see figure).

For the demonstration, we write the differential equations in
the following condensed form

(13) $$\frac{d\,(u + iv)}{dt} = iw\,\rho e^{i\alpha}\,(u + iv) - gi$$

which is obtained from equation (5) by letting $K = 0$. By making
$\alpha = 0$ in equation (13) we find, for the S system, the differen-
tial equation

(14) $$\frac{d\,(u + iv)}{dt} = iw\,\rho\,(u + iv) - gi.$$

We obtain the differential equation of the trajectories of the an-
gle α by multiplying the second member of equation (14) by $e^{i\alpha}$.
Hence

(15) $$\frac{d\,(u + iv)}{dt} = iw\,\rho\,e^{i\alpha}\,(u + iv) - gie^{i\alpha}$$

The change of axes indicated in the above statement is obtained by
multiplying $u + iv$ in equation (15) by $e^{i\alpha}$. By dividing the
ratio thus obtained by $e^{i\alpha}$, we return to equation (13), which dem-
onstrates the theorem.

All the characteristics tend toward the point

$$u = \sqrt{\frac{g}{\rho}}\,\cos\alpha, \qquad v = -\sqrt{\frac{g}{\rho}}\,\sin\alpha$$

which corresponds to a rectilinear and uniform descent.* If the in-

* If an integral curve passes near the point $u = v = 0$, we can
hardly expect that it will correspond to the real motion of the air-
plane, because the hypothesis that the angle of attack remains con-
stant is not justified for low speeds.

tegral curve cuts the axis of the negative u's, n times, the airplane executes n loops. We come, therefore, to the following conclusion. Whatever be the initial velocity, the airplane, after executing, if necessary, a finite number of loops, acquires a motion which approaches indefinitely a state of rectilinear and uniform descent.

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