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METAL-TRUSS WING SPARS

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INTRODUCTION

Since metal-truss wing spars are coming into general use in the airplane industry, it is necessary that rational methods for their design be developed.

The purpose of the study recorded in this thesis was to develop improvements in the current methods for the calculation of the loads in the members of metal-truss wing spars which are subjected to combined bending and compression.

If there were no axial load in the metal-truss spar, its design would be very simple, because ordinary truss analysis methods could be used to determine the loads in its members. However, when axial compression is acting together with a side load, the loads in the members of the truss spar are functions of the deflections of the spar, since the combination of these deflections with axial load produces additional bending moments and shears. These additional bending moments and shears may be referred to as the secondary bending moments and secondary shears. It is necessary, then, to calculate the effect of the deflections of the panel points of a truss spar to determine the true loads in its members.

The present design rule of the Department of Commerce specifies that equations** for the calculating of bending moments and shears on uniform beams subjected to combined bending and compression shall be used for calculating the bending moments and shears on metal-truss wing spars. In order to use these equations, which will be referred to

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below as the Precise Formulas,* a value of effective moment of inertia is needed.

Once these bending moments and shears have been determined, the loads in the various truss members can be calculated by the use of ordinary truss analysis methods.

The effective moment of inertia of a metal-truss wing spar is something which is not as easily determined as the moment of inertia of an ordinary wooden spar. At first, one might erroneously believe that the moment of inertia at any section of a truss spar is the moment of inertia of the areas of the two chord members about the centroid of these areas. In the following discussion, this moment of inertia will be called the "chord moment of inertia." The chord moment of inertia at any section would be the true moment of inertia if the web members were of infinite area and did not deform under load. The fundamental beam equation, $M = EI\,\frac{d^2y}{dx^2}$, upon which all beam equations are based, was derived under the assumption that the shear deformation was so small that it could be neglected. This assumption of negligible deformation, resulting from shear, does not really fit even the case of ordinary wooden beams; consequently the value of "E" is arbitrarily reduced a certain small percentage when the beam equations are being used for that material. With metal trusses, the shear is carried by the web members instead of by a continuous web; consequently the web deformation is so great that the deflection resulting from this deformation cannot be neglected. This deflection will be referred to below as the "web deflection." As a result, the value of a chord moment of inertia must be decreased to allow for the decreased stiffness which is caused by the deformation of the web members. The portion of the truss spar deflection which results from the deformation of the chord members will be referred to as the "chord deflection."

The Department of Commerce rule specifies that the moment of inertia to be used in the Precise Formulas shall be determined by backfiguring from deflections which result when the truss spar is subjected to side load. The truss spar deflections may be calculated by any convenient

*See Chapter XI "Airplane Structures," by Niles and Newell.

deflection method,* or may be experimentally determined from full-scale tests. Assuming the truss spar to be an ordinary beam, a value of effective moment of inertia is backfigured from the appropriate beam deflection equation by a substitution of the previously determined deflection values.

One can readily see that this backfigured effective moment of inertia cannot be greatly in error. However, the entire backfiguring process is so mechanical that the designer does not see the theoretical considerations which are automatically included in that process. In this thesis, a direct method of calculating the effective moment of inertia of a metal-truss wing spar is developed. This direct method is built up from consideration of the actions of the individual truss members; consequently the designer acquires a much better understanding of the quantities which affect the effective moment of inertia than he would by using the backfiguring method.

It was originally thought that the effective moment of inertia of a metal-truss wing spar might be determined directly from the geometrical properties of the chord and web members. Further study, however, proved that the ratios of the strains of the web members to the strains of the chord members must be known in addition to the geometrical properties of the truss spar to determine the correct value for the effective moment of inertia. Since these ratios are functions of the external loading, it is necessary to know the type of load to which the truss spar is to be subjected, before the effective moment of inertia can be calculated.

The material of this thesis is divided into three parts. The derivations of the theoretical concepts are given first. The practical applications of the theory follow. Finally, in the form of an appendix, the effective moment of inertia of an actual metal-truss wing spar is calculated. This wing spar was built and tested for deflection under combined bending and compression by the Boeing Airplane Company. The calculated value of the effective moment of inertia is checked against the test data, and conclusions are drawn regarding the accuracy of the calculated value of effective moment of inertia.

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*See page 311 "Airplane Structures" by Niles and Newell.

THEORETICAL DERIVATIONS

The purpose of the following derivations is the rational determination of the effective moment of inertia of a metal-truss wing spar.

If there is no shear deformation, the beam equation $M = EI \frac{d^2y}{dx^2}$, or $I = \frac{M}{E \frac{d^2y}{dx^2}}$ is accurate. With metal-truss wing spars, the shear deflection resulting from the strain of the web members is so large that the ordinary beam equation does not hold. The beam equation may be made to apply if the value of I, the chord moment of inertia, is properly reduced to allow for shear deformation. In other words, the deflection of the spar produced by web member deformation increases $\frac{d^2y}{dx^2}$, and consequently reduces the effective value of I. Consequently, the first step in the solution of the problem is to derive an equation which gives the increments of web deflection between adjacent panel points of the truss spar in terms of the deformation of the web members. Next, the relation between the web deflection increments between panel points and the resulting change in $\frac{d^2y}{dx^2}$ must be determined. Then an accurate method of calculating the decrease of effective moment of inertia due to the changes in $\frac{d^2y}{dx^2}$ must be developed. Since the moment of inertia of an ordinary truss spar varies from one panel to the next, it is necessary to work out a means of weighting the effect of the moments of inertia in the various panels upon the stiffness of the spar as a whole. Finally a method of computing the effective moment of inertia of the entire truss spar must be developed from the reduction of chord moment of inertia in each panel, and from the relative importance of the moments of inertia in the various panels.

The theoretical derivations below include three sections which have only an indirect bearing on the main developments of the thesis. The first of these sections covers the derivation of an equation for the increment of deflection between adjacent panel points produced by the deformation, or strain, of the chord members. The second is the derivation of rules for calculating the total deflections of the panel points of a truss directly from the web and chord increments of deflection between panel points. This method of calculating deflections is very simple and direct, consequently it can often be used instead of the standard deflection methods. Its main value, however, is that it is developed from simple geometrical relations in the truss and thus gives one a very concrete concept of the action which takes place when a truss deflects under load. The third of the three sections is one which gives an exact method of calculating the total bending moments and shears to which a metal truss wing spar is subjected when acted upon by axial and side loads. It is an extremely lengthy method, and is only of value in checking the approximate method of calculating the effective moment of inertia. This exact method is not a development of the thesis; it has been known to structural engineers for some time.

(I) INCREMENT OF TRUSS DEFLECTION

BETWEEN ADJACENT PANEL POINTS

PRODUCED BY THE STRAIN OF THE WEB MEMBERS OF THE PANEL

Parallel chord trusses.- Refer to Figure 1 of the diagram sheets. ABCD represents the center lines of the members of one panel of a truss. When the truss is subjected to load, web member BD is strained; consequently panel point D deflects an amount D-D' above panel point A. The object of the following derivation is to determine the relation between the deflection D-D', and the strain in web member BD.

There are three assumptions on which the following derivation is based:

1) The deflections of the panel points of the truss are so small that the arc traced by one end of a truss member when the member is considered to rotate about the pin at its other end approximates a straight line.

2) The members of the truss are assumed to be connected by pins.

3) Since the principle of superposition is implied when one speaks of the total deflections as being the sum of chord deflections and web deflections, it is logical to calculate the deflections produced by the strain of the web members using the assumption that the chord members are unstrained.

Web member BD is strained an amount represented by eD'. This strain allows member AD to rotate about A, and take up a new position AD'.

$\angle$DeD', $\angle$eDB, and $\angle$DD'A can be considered to be right angles, since the radius of an arc is perpendicular to the tangent at the point of intersection of the radius and the arc.

$\angle eDD' = \angle(90^\circ - \eta)$, since the sides of the angles are mutually perpendicular.

$$\sin\,(\angle eDD') = \frac{eD'}{DD'}.$$

Substituting the value of $\angle$eDD':

$$\sin\,(90^\circ - \eta) = \frac{eD'}{DD'}; \quad \text{or} \quad \cos\,\eta = \frac{eD'}{DD'}$$

$$DD' = \frac{eD'}{\cos\,\eta}.$$

Since eD' = the strain of the web member, and DD' is the increment of web deflection,

$$\text{deflection increment} = \frac{\frac{PL}{AE}\,(\text{web})}{\cos\,\eta}. \qquad (1)$$

<u>Nonparallel chord trusses</u>.- Refer to Figure 2 of the diagram sheet. ABCD represents the members of one panel of a nonparallel chord truss. The line CDV represents the direction in which the deflection of the panel point D is to be calculated.

When the diagonal web member BD is strained, chord member AD is allowed to rotate about A, and takes up the position AD'. The point D traces the arc DD' when AD rotates about A. Member BD rotates about B and occupies the position BD'. De is an arc drawn with a radius BD about B as a center; consequently eD' represents the amount BD was strained. The same assumptions underlying the parallel chord analysis apply to the following derivation; consequently

$\angle eDD' = \angle\alpha$, $\angle D'DV = \angle\gamma$, and $\angle DeD'$ is a right angle. Therefore

$$DD' = \frac{eD'}{\sin \angle eDD'}.$$

D'V is drawn from D' perpendicular to VD.

$$\cos \angle VDD' = \frac{VD}{D'D}.$$

Thus,

$$VD = \frac{\cos\gamma \times eD'}{\sin\alpha}.$$

But VD is the deflection increment, and ED' is the strain of the diagonal web member. Therefore

$$\text{deflection increment} = \frac{PL}{AE}\frac{\cos\gamma}{\sin\alpha}. \qquad (2)$$

In the foregoing discussion the web member considered was diagonal. For a vertical web member, equation (2) applies if the angles γ and α are taken properly. α is the angle between the vertical web member and a chord member at the intersection of the vertical member with a chord member, which intersection is separated from the left support by the greater number of primary web members. γ is the angle between this same chord member and the perpendicular to the direction of deflection. Thus for member CD in Figure 2; α is angle BCD and γ is 0°.

In the case of truss which has no vertical web members, equation (2) applies directly since no vertical web member was considered in its derivation.

Where the vertical web member is secondary, its deformation affects only the deflection of the panel point where the member is attached. It should be noted that the type of vertical web member shown in Figures 1 and 2 increases the deflection increment of only one of the two intersections of the web member with the chord members.

Thus in Figure 2, it can be readily seen that the strain of member DC increases the deflection increment between panel points B and C, but does not affect the deflection increment between panel points A and D.

(II) CORRECTION OF CHORD MOMENT OF INERTIA FOR WEB DEFLECTION

If the web members of a truss spar were not strained, the truss spar could be considered to be an ordinary beam which had values of moments of inertia equal to the corresponding values of chord moments of inertia.*

The equation which is the basis of beam theory is

$$M = EI \frac{d^2y}{dx^2}.$$

$$I = \frac{M}{E \frac{d^2y}{dx^2}}. \qquad (3)$$

By examining equation (3), it is apparent that any modification of $\frac{d^2y}{dx^2}$ represents a change of moment of inertia. At any section of the truss spar, the strain of the web members changes $\frac{d^2y}{dx^2}$ and consequently, the effective moment of inertia. The following derivation is a calculation of the increment of $\frac{d^2y}{dx^2}$ produced by the strain of the web members. The change of moment of inertia represented by this increment of $\frac{d^2y}{dx^2}$ is then calculated from equation (3).

There are several concepts upon which the following derivation is based, and they will be stated before the derivation is given.

*See page 311 "Airplane Structures" by Niles and Newell.

The first concept is a demonstration of the effect of web deflection upon the slope of the elastic curve of a truss. In Figure 3 of the diagram sheet, AC is the line joining the two supports of the truss. b" represents the position of panel point B when both chord and web member deflections are considered.

$$\text{Now} \quad \tan j = \frac{Bb''}{AB} \quad \text{and} \quad \tan k = \frac{Bb'}{AB}.$$

The change of slope produced by the deflection of the web members is

$$\tan j - \tan k = \frac{Bb''}{AB} - \frac{Bb'}{AB} = \frac{b'b''}{AB}.$$

Therefore, the change of slope produced by the web members is the web deflection increment divided by the proper panel length.

The average rate of change of slope from one panel to the next is then the difference in web deflection slopes of the two panels divided by the distance between the points where the slopes are taken.

By similar reasoning, the average rate of change of slope produced by chord deflection is the difference between the chord deflection slopes in adjacent panels divided by the distance between the points where the slopes are taken.

The total average rate of change of slope between two adjacent panels is the sum of the average web and chord rates of change of slope.

The second concept is concerned with the relation between the slopes of the chord members of a truss spar and the slopes of the elastic curve of the spar. If straight lines are drawn so that they connect the deflected panel points of the upper or lower chords of a truss, a polygon will result. For most truss spars, the deflections of the panel points of the two chord members are slightly different; consequently, if the polygon representing the deflected neutral axis of the truss spar is to be constructed, it should be the "average" of the polygons for the two chord panel point deflections. However, the difference between the deflections of the two chord member panel points is so small that the neutral axis polygon can be considered to be the same as the polygon of either chord. If a smooth curve is drawn through the points of either one of these deflection polygons, a deflection curve will result. Since the slopes of the chords of the polygons are very small, this smooth curve may be considered the elastic curve of the truss spar.

In Figure 4 AD represents the undeflected position of the lower chord members of a truss. ABCD represents the deflected position of these members of the truss. The smooth curve ABCD is the elastic curve of the truss, and is a flat "parabola" for all ordinary deflections. Naturally, the slope of the elastic curve varies at different points along the span. For flat parabolas the slope of the chord is approximately the slope of the tangent to the curve at the mid-point of the arc subtended by the chord. Therefore, the slope of a chord member can be considered to be the slope of the elastic curve at the middle of the panel where the chord member occurs.

The web rate of change of slope will now be determined from the web deflection increments of adjacent panels of a truss. Refer to Figure 5 of the diagram sheet. ABC represents the undeflected position of the upper or lower chord members of a truss. Abc represents the position of the chord members when only the strain of the web members has produced deflection. Bb is the deflection increment produced by the strain of the web members in panel AB. c'c is the deflection increment produced by the strain of the web members in panel BC. As was previously demonstrated, the slope of the chord member of a panel approximates the slope of the elastic curve at the mid-point of the panel. Therefore, the slope of Ab (referred to ABC) is the slope of the elastic curve at the mid-point of panel AB. Similarly, the slope of chord member bc is the slope of the elastic curve at the mid-point of panel BC.

$$\text{The slope of member Ab} = \frac{Bb}{AB}.$$

$$\text{The slope of member bc} = \frac{cc'}{BC}.$$

The difference between the slope at the mid-point of panel BC and the slope at the mid-point of panel AB is therefore

$$\frac{cc'}{BC} - \frac{Bb}{AB}.$$

Then the average rate of change of slope between the mid-points of the two panels is

$$\frac{\frac{cc'}{BC} - \frac{Bb}{AB}}{\frac{1}{2}\,(AB + BC)} = \frac{d^2y}{dx^2} \quad \text{(average).} \qquad (4)$$

Assuming that the adjacent panels are of equal length, "X," equation (4) becomes $\frac{cc' - Bb}{x^2}$.

Since the deflection curve is a flat parabola, this average rate of change of slope is the "exact" rate of change of slope at a point half way between the mid-points of the two panels. Thus, since the panels are of equal length $\frac{cc' - Bb}{x^2}$ is the exact rate of change of slope at the panel point B. Therefore, the rate of change of slope at a panel point is approximately the difference between the deflection increments of the two adjacent panels divided by the square of the panel length.

Let $\frac{\delta_2 - \delta_1}{x^2}$ be the rate of change of slope at any panel point, where $\delta_2 - \delta_1$ is the difference between the web deflection increments of the two adjacent panels. Equation (3) is

$$I = \frac{M}{E\,\frac{d^2y}{dx^2}}; \qquad \frac{1}{I} = \frac{E\,\frac{d^2y}{dx^2}}{M}.$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dx^2}\ \text{(chord)} + \frac{d^2y}{dx^2}\ \text{(web)}.$$

$$\frac{1}{I} = \frac{E\,\frac{d^2y}{dx^2}\,\text{(chord)}}{M} + \frac{E\,\frac{d^2y}{dx^2}\,\text{(web)}}{M} = \frac{1}{I_{\text{chord}}} + \frac{E\,\frac{d^2y}{dx^2}\,\text{(web)}}{M}.$$

Since $\frac{d^2y}{dx^2}$ (web) $= \frac{\delta_2 - \delta_1}{x^2}$;

$$\frac{1}{I} = \frac{1}{I_c} + \frac{E(\delta_2 - \delta_1)}{MX^2} . \qquad (5)$$

Equation (5) applies to a section of a truss spar occurring at a panel point; so M is the moment to which the spar is subjected at the panel point, and I_c is the chord moment of inertia at the panel point.

The sign of the deflection increments must be taken correctly, or the quantity $\delta_2 - \delta_1$ will be in error. Moving along the truss from one support to the other, if the strain of the web member tends to increase the deflection of the truss, its sign is positive; if the strain tends to decrease the deflection of the truss, its sign is negative. Thus refer to Figure 6.

Considering support A as a datum, the tension in members AH and BG produces strains which tend to allow the truss to deflect upward (with reference to support A); the strains in members GD and FE tend to decrease this upward deflection when one passes from support A to support E.

Consider equation (5) which is

$$\frac{1}{I} = \frac{1}{I_c} + \frac{E(\delta_2 - \delta_1)}{MX^2} .$$

δ_2 is the web deflection increment of the panel the farther from the "datum" support, and δ_1 is the web deflection increment of the panel the nearer to the "datum" support.

For <u>parallel chords</u>, δ is the sum of $\frac{PL}{AE \cos \eta}$ values for all the web members within the panel.

For nonparallel chords, δ is the sum of the $\frac{PL \cos \gamma}{AE \sin \alpha}$ values of all of the web members within the panel.

The web deflection increment produced by a vertical web member should be divided equally between the two adjacent panels. However, it is more conservative to place the deflection increment entirely in the panel which is the nearer to the datum support.

III. THE EFFECT OF THE MOMENT OF INERTIA OF A PANEL OF A TRUSS SPAR UPON THE DEFLECTION OF ANY PANEL POINT OF THE TRUSS

It is often desirable to know the effect of the moment of inertia of any one panel of a truss spar upon the deflections of all of the panel points. In the following derivation, the truss spar will be treated as a beam with varying moment of inertia; consequently, the ordinary beam theory methods of calculating deflections can be employed. For simplicity, it will be assumed that $\frac{M}{EI}$ is zero for all panels except the one which contains the $\frac{M}{EI}$ for which the deflection effect is being calculated.

In Figure 7 of the diagram sheet, the deflection of the point b will be calculated by the method of elastic weights for beams.*

The elastic reaction at f is $Q\alpha \frac{C}{L}$.

The elastic bending moment at b which is numerically equal to the actual deflection, is

$$\frac{Q\alpha CD}{L} = \frac{M}{EI} \frac{\alpha}{L} CD = \text{deflection increment.} \tag{6}$$

Since panel ps is any panel of the truss spar, and b is any panel point of the truss, equation (6) gives the effect of the moment of inertia of any panel of a truss spar upon the deflection of any panel point.

*Page 303 "Airplane Structures" by Niles and Newell.

Examine equation (6). For any truss spar, E and L are constants. If the deflection of any particular panel point is being investigated, D is a constant. As a result, the effect of the moment of inertia, I, of a panel upon the deflection of the panel point in question is determined by:

1) the length of panel, α, where I occurs;

2) the magnitude of the bending moment, M, at the middle of the panel where I occurs; and

3) the relative location, C, of the panel containing the moment of inertia.

In equation (6): D is always the distance from b to a support such that $Q\alpha$ is not included in the distance. C is the distance from $Q\alpha$ to a support such that b is not included in the distance.

IV. METHOD OF CALCULATING THE EFFECTIVE MOMENT OF INERTIA OF A METAL-TRUSS WING SPAR

The following method of calculating the effective moment of inertia of a truss spar is based upon direct analytical considerations of the loads in, and the sizes of the members of the truss spar. This direct analytical method has the advantage over any backfiguring method in that it gives one a very much clearer idea of the various factors which enter into the effective moment of inertia determination.

The following derivation is concerned with determining the proper "average" of the corrected moments of inertia of the various panels of a truss spar.

Since there is a different value of moment of inertia in each panel of a truss spar, it is obviously not possible to determine a single value of moment of inertia which, when substituted in the proper deflection formula, will result in absolutely correct deflections for all panel points. However, it is possible to determine a value of moment of inertia which will give a correct value of deflection for any one panel point. The question then arises as to which panel point should have its deflection correct.

Since all deflection curves are more or less flat and parabolic in shape, the maximum ordinate and the two zero ordinates of the curve are the most important in locating the curve. In other words, if the maximum ordinate and the zero ordinates at the supports are located, a smooth curve can be passed through the three ordinates, which curve will result in fairly accurate deflection values for all panel points. Practically all truss spars over two supports have their point of maximum deflection fairly close to their midspan points. Consequently, the midspan point will be chosen as the point which is to have the correct deflection.

Consider equation (6):

$$\text{deflection increment} = \frac{M}{EI}\,\frac{\alpha\,(C)\,(D)}{L} \quad \text{for one panel.}$$

Equation (6) gives the deflection at any panel point, when all of the panels except one are considered to have zero $\frac{M}{EI}$ values. Now, if all the panels of a truss spar are considered to have finite values of $\frac{M}{EI}$, the deflection at any panel is the sum of the values obtained from equation (6) for all of the panels. Thus equation (6) becomes:

$$\begin{array}{c}\text{total deflection}\\ \text{at any point}\end{array} = \sum_{i=1}^{i=\eta}\left(\frac{M}{EI}\,\frac{\alpha\,(C)\,(D)}{L}\right)_i \qquad (7)$$

where η is the number of panels.

Since the midspan point has been selected as the point which is to have the correct deflection, the deflection given by equation (7) will be made that of the midspan point.

Then D of equation (7) = 1/2 span = 1/2 L. (See fig. 7 of diagram sheet.) Equation (7) becomes:

$$\text{total deflection} = \sum_{i=1}^{i=\eta}\left[\frac{M\alpha}{I}\,(C)\,K\right]_i. \qquad (8)$$

Where $K = \frac{D}{L \times E} = \frac{1/2\ L}{L \times E} = \frac{1}{2E}$.

Now the I in equation (8) varies from panel to panel, and it is desired to determine a constant value which will give the same deflection at the midspan point as do the various different values. Let I_k be this constant value. Then, considering equation (8):

$$\text{total deflection at midspan point} = \Sigma_{i=1}^{i=\eta}\left[\frac{M\alpha}{I}\ C\ K\right]_i = \frac{1}{I_k}\,\Sigma_{i=1}^{i=\eta}\left[M\alpha\ C\ K\right]_i .$$

Solving for $\frac{1}{I_k}$:

$$\frac{1}{I_k} = \frac{K\ \Sigma_{i=1}^{i=\eta}\left[\frac{M\alpha}{I}\ C\right]_i}{K\ \Sigma_{i=1}^{i=\eta}\left[M\alpha\ C\right]_i} = \frac{\Sigma_{i=1}^{i=\eta}\left[M\alpha\ C\right]_i}{\Sigma_{i=1}^{i=\eta}\left[M\alpha\ C\right]_i} . \qquad (9)$$

In equation (9), C is the distance from a support to the panel containing the moment of inertia, such that the midpoint of the span is not included in the distance. M is the average bending moment in the panel, and α is the panel length.

Divide the numerator and the denominator of equation (9) by $M_m \times \alpha_m \times L/2$, where M_m is the maximum average bending moment in any panel, α_m is the greatest panel length, and $L/2$ is one half of the span.

The "effective" moment of inertia given by equation (9) is not changed by this division, because the M_m, α_m, and $L/2$ terms factor out of both the numerator and denominator, and then cancel each other.

Consequently

$$\frac{1}{I_k} = \frac{\Sigma_{i=1}^{i=\eta}\left[\frac{M}{M_m}\ \frac{\alpha}{I}\ \frac{C}{\alpha_m L/2}\right]_i}{\Sigma_{i=1}^{i=\eta}\left[\frac{M}{M_m}\ \frac{\alpha}{\alpha_m}\ \frac{C}{L/2}\right]_i} \qquad (10)$$

V. INCREMENT OF DEFLECTION BETWEEN ADJACENT PANEL POINTS PRODUCED BY THE STRAIN OF CHORD MEMBERS

Since the deflection produced by the strain of the chord members is the only deflection to be considered, the web members will be assumed to be without strain in the following derivation. In this discussion, the same fundamental assumptions will be made about the truss deflections and construction as were made in the derivations of the formulas for increments of web deflection.

In Figure 8, ABCD is a panel of an undeflected truss. When the truss deflects, the strain of the lower chord members between D and the support causes D to move to D'. The strain of the upper chord members between B and the support causes A to move to A' and B to move to g. Consider the pin to have been removed from the joint at B; then member AB will have to rotate about A', and member DB will have to rotate about D' until their free ends meet at B' before the pin can again be inserted. The problem of calculating the increment of deflection between panel point A and panel point B is then to calculate the length of gB'.

According to the previous assumptions, the deflections are so small that:

a) B'g and fB' can be considered to be straight lines which are perpendicular to AB and DB respectively.

b) The angle included between DB and D'B' is so small when compared to θ that D'B' can be considered to coincide with DB as far as the truss as a whole is concerned.

Now, $\angle fB'g = \angle\theta$, because the sides of the angles are mutually perpendicular.

$$\tan \angle\theta = \tan \angle fB'g = \frac{fg}{gB'} = \frac{fg}{\text{deflection inc.}} \qquad (11)$$

$$fg = fB + Bg. \qquad (11a)$$

Also, $fB = DD'$ since the chords are parallel, and web member DB is unstrained. Bg is the sum of the values of $\frac{PL}{AE}$ of upper chord members between B and the support, and D'D is the sum of $\frac{PL}{AE}$ of lower chord members between panel point D and the support. Therefore, substituting in equation (11a):

$$fg = \Sigma \frac{PL}{AE} \binom{\text{upper}}{\text{chord}} + \Sigma \frac{PL}{AE} \binom{\text{lower}}{\text{chord}}. \qquad (12)$$

Substituting from equation (12) in equation (11):

$$\text{deflection inc.} = \frac{\Sigma \frac{PL}{AE} \binom{\text{upper}}{\text{chord}} + \Sigma \frac{PL}{AE} \binom{\text{lower}}{\text{chord}}}{\tan \theta} .$$

VI. DERIVATION OF CHORD AND WEB DEFLECTION RULES

In two of the foregoing sections, formulas for calculating the increments of web and chord deflection between adjacent panel points were derived. From a knowledge of these deflection increments, it is possible to determine the shape of the resulting deflection curve, and then by locating this deflection curve so that the deflection of the supports are zero, the actual deflection of the various panel points can be determined.

Thus, consider Figure 9 of the diagram sheet. a, b, c, d, etc., are panel points of a truss. The increment of deflection between panel points a and b is x, between panel points b and c is y, and so on. Since, in the general case, the true slope at a is not known, any slope can be assumed with the result that the deflection of the panel point g at the other support will not be zero. Now, if the entire truss is considered to be rotated about a until g falls on the support B, the true position of the various panel points will be located.

Z, the angle through which the truss is rotated, is very small; consequently, ag is very nearly the same length as ap, and lines corresponding to de are the same length practically as lines corresponding to dh.

Therefore, considering panel point d, the correction to be applied to the deflection is ef. Now, ef = m tan Z. Consequently, the correction to the deflection of any panel point is the product of tan Z and the distance from the panel point to the support a.

There are several special cases of deflection calculation which do not require the use of the above correction procedure. a) In calculating the chord or web deflections of cantilever trusses no correction is required, because the slope of the center line of the truss is known to be zero at the support. Also, in calculating the web deflections for a truss which is symmetrical about its midspan point, no deflection correction is necessary, because the deflection of the right hand support will be zero without any rotation of the truss. The fact that the web deflection of the right hand support is zero can readily be seen by considering such a symmetrical truss. Refer to Figure 6. Starting from support A, the strain of web members AH, BH, and BG produce upward web deflection of the panel points. Members GD, DF, and FE produce corresponding downward deflections; consequently, the deflection of the support at E is calculated to be zero directly from the deflection increments, and no correction is required.

VII. EXACT METHOD OF CALCULATING THE TOTAL BENDING MOMENTS AND SHEARS TO WHICH A METAL-TRUSS WING SPAR IS SUBJECTED

This exact method, as applied to a beam, is given in books on airplane stress analysis*, and only a condensed treatment of its application to a truss will be given here.

If the total deflections of all of the panel points of a metal-truss wing spar which is subjected to combined bending and compression were known, the total bending moments and shears could be easily calculated. The total bending moment at any section would be the primary bending moment plus the product of the deflection at the section and the axial compressive load. The total shear at a section would be the primary shear plus the product of the

*Page 67, "Structural Analysis and Design of Airplanes," by B. C. Boulton.

slope of the elastic curve at the section and the axial compressive load.

The total deflections can be obtained by a repeated deflection calculation process. The deflections produced by the side load can be calculated by any standard deflection method. The first secondary bending moments can be obtained by multiplying the primary deflections by the axial load. The secondary shears can be obtained by multiplying the slopes of the elastic curve of the primary deflections by the axial load. The increase in the loads of the various members of the truss spar produced by the secondary bending moments and shears can be calculated by any standard truss analysis method. Then, a new set of panel point deflections can be calculated from the new loads by ordinary deflection methods. This process can be continued until the increase of deflection becomes negligible; consequently, the total deflections of the truss spar can be determined.

This exact method offers a means of checking the value of effective moment of inertia calculated by the methods developed in the previous parts of this thesis. In general the approximate method will give results which are 3-5% more conservative than will the exact method. Consequently the approximate method is entirely satisfactory for practical design work, and should be used instead of the exact method because it is so much shorter.

PRACTICAL RESULTS

The theory which has been developed above has two important practical applications. One is the calculation of the effective moment of inertia of a truss spar from the geometry of the spar and the loads to which the spar is to be subjected. The second is the determination of the most economical location of metal for stiffening a truss spar which has too much deflection.

Calculation of effective moment of inertia.- The effective moment of inertia is calculated from equation (10) of the theoretical derivations.

Equation (10) is:

$$\frac{1}{I_k} = \frac{\Sigma_{i=1}^{i=\eta}\left[\frac{M}{M_m}\frac{\alpha}{I\ \alpha_m}\frac{C}{L/2}\right]_i}{\Sigma_{i=1}^{i=\eta}\left[\frac{M}{M_m}\frac{\alpha}{\alpha_m}\frac{C}{L/2}\right]_i}$$

where I_k is the effective moment of inertia of the metal truss spar.

$\frac{M}{M_m}$ is the <u>bending moment weight</u> of a panel, and is determined by dividing the average bending moment in a panel by the maximum of the average bending moments in the panels of the truss.

$\frac{\alpha}{\alpha_m}$ is the <u>panel length weight</u> of a panel and is determined by dividing the length of a panel by the maximum panel length.

$\frac{C}{L/2}$ is the <u>distance from support weight</u> of a panel, and is determined by dividing the distance from the middle of a panel to the nearest support by one half the length of the span of the truss spar.

η is the number of panels.

I is the corrected chord moment of inertia of a panel. This corrected chord moment of inertia is calculated from equation (5) of the derivations. Equation (5) is:

$$\frac{1}{I} = \frac{1}{I_c} + \frac{E\ (\delta_2 - \delta_1)}{MX^2}.$$

I_c is the chord moment of inertia and equals $\frac{d^2 A_u A_L}{A_u + A_L}$ (approximately), where d is the distance between chord center lines, A_u is the cross-sectional area of the upper chord, and A_L is the cross-sectional area of the lower chord.

$\delta_2 - \delta_1$ is the difference in the web deflection increments of adjacent panels. For parallel chord trusses,

δ is the sum of the $\frac{PL}{AE \cos \eta}$ values of all of the web members within the panel. (See eq. (1).) For nonparallel chord trusses, δ is the sum of the $\frac{PL}{AE}\frac{\cos \gamma}{\sin \alpha}$ values for all web members within the panel. (See eq. (2).) Angles η, γ, and α are illustrated in Figures 1 and 2 and are explained in Section I of the theory.

M is the bending moment at the panel point in question.

X is the panel length.

The value of I obtained from equation (5) is the corrected moment of inertia at a panel point. Since a corrected moment of inertia for a panel is required, the values of I at the two panel points of a panel must be averaged.

The following procedure will be found expedient in calculating the effective moment of inertia:

1) Calculate the loads in all of the members of the truss spar when only the side load is acting.

2) Obtain the corrected values of moments of inertia for all of the panels.

3) Calculate the bending moment weight, the panel length weight, and the distance from the support weight for each panel.

4) Obtain the product of the three types of weights for each panel.

5) Multiply the inverse of the corrected moment of inertia in each panel by the products of the weights of the panel.

6) Divide the sum of the products obtained in 5) for all panels, by the sum for all panels of the products obtained in 4). The reciprocal of this quotient is the effective moment of inertia of the metal truss spar.

Some of the terms of equation (10) depend upon the loads in the members of the truss spar. If a metal truss spar is subjected to combined bending and compression, the loads in the various members cannot be calculated until the effective moment of inertia is known. Consequently, the primary loads must be used in solving equation (10) for the effective moment of inertia. This approximation is good because the ratio of the chord member load to web member load in a panel does not differ greatly between the condition where side load is acting alone and the condition where side load is acting with axial load.

However, if greater accuracy is desired, equation (10) can first be solved assuming that only side loads are acting. The calculated value of effective moment of inertia can then be substituted in the Precise Formulas and the total bending moments and shears determined. From these values of moments and shears, the loads in the various members can be calculated. Equation (10) can be solved again with these new loads, and a more accurate value of effective moment of inertia is obtained. This process can be repeated until the effective moment of inertia is as accurate as the designer desires.

Economical location of metal for stiffening.- If a metal-truss spar is found to have too much deflection, it is desirable to know the panel in which a given increase in the size of chord members will produce the greatest stiffening effect.

Equation (10) of the theoretical derivations gives the designer the necessary information for economical location of metal for stiffening.

It is apparent from that equation that the quantities which are important in selecting the panel for economical location of metal for stiffening are the bending moment weight, $\frac{M}{M_m}$, and the distance from support weight, $\frac{C}{L/2}$. Since an addition of metal to the chord members of a panel increases I, this increase in metal is going to have the greatest effect in the panel where an increase of I will have the greatest effect. It is obvious that I will have the greatest effect upon I_k, the effective moment of inertia, in the panel where $\frac{M}{M_m} \times \frac{C}{L/2}$ is a maximum. Thus, the panel in which the chord members are to be increased in size should be the one which has the largest product of bending moment weight and distance from the support weight.

APPENDIX

In the body of this thesis, several purely theoretical concepts have been developed. In this appendix, the numerical application of these concepts to a practical metal-truss wing spar will be made. This practical truss has previously been subjected to lateral and axial load, and the deflection of its panel points measured. By using the theoretical concepts, the deflections of the panel points under the same loading are calculated. This resulting set of measured and calculated deflection values will enable one to check the accuracy of the theory used in determining the calculated values. The metal-truss spar used in the following calculations was built and tested by the Boeing Airplane Company.*

Figure 1 of the appendix is a line diagram of the test truss spar and shows the type and intensity of the load to which the spar was subjected. End moment was placed upon the truss spar at the left support by means of a 2000 lb. weight on the end of the steel plate, AB. The truss spar was loaded at its panel points by means of metal straps which carried weights at their lower ends. The connection of strut BC to the spar at B is accomplished by a pin, and the other end of BC is connected to a foundation which is sufficiently distant from B to allow the axis of member BC to represent the direction of the load carried by BC. The spar is supported at D by another pin connection.

The angularity of member BC places axial compression in the truss spar, which compression is a function of this angularity, and of the spar reaction at B.

The deflections at several panel points of the spar were measured when the spar was deflected under the loading shown in Figure 1.

The object of the following set of computations is to calculate the deflections of the truss spar at the panel points where the deflections were actually measured in the test, and under the same loading as that used in the test.

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*See Test No. 10096, Boeing Airplane Company, Seattle, Washington.

Outline of calculations.-

1) The chord moments of inertia of the various panels are corrected for web deflection.

2) The effective moment of inertia of the truss spar is calculated from the corrected moment of inertia values.

3) This effective value of moment of inertia is substituted in the proper Precise Formula for deflection; consequently, the deflections of the spar under the loading shown in Figure 1 are determined. Since the deflections have been measured under the same loading, a comparison between the measured deflection values and the calculated deflection values is had.

Explanations and Assumptions.- Panels (1-2) and (13-14) are not considered separately because it is impossible to determine the cross-sectional areas of the members of these panels. These areas are indeterminate because gusset plates are included between the chord and web members.

Since the cross-sectional areas are indeterminate, some sort of approximation is necessary if the web deflection effect upon chord moment of inertia is to be calculated. It seems reasonable to assume that the effect of web deflection in panels (1-2) and (13-14) is the same as that of the corresponding adjacent panels (2-3) and (12-13). It would probably be more accurate to assume the web deflection effect of panels (1-2) and (13-14) to be zero, since the area of the gusset plates is quite large. However, it is more conservative to assume the deflection effect to be greater than zero, so the first mentioned deflection assumption is used in the calculations.

TABLE I

WEB DEFLECTION INCREMENTS

(A) Web member (diagonal)	(B) Load (lb.)	(C) $\frac{PL}{A}$	(D) $\frac{PL}{A \cos \eta}$	(E) Web member (vertical)	(F) Load (lb.)	(G) $\frac{PL}{A \cos \eta}$
2-23	+ 2385.	277,000	588,000	3-23	-1122	61,300
3-24	2002.	232,400	494,000	4-24	942	51,400
4-25	1620.	188,000	399,000	5-25	762	41,600
5-26	1238.	143,700	305,000	6-26	582	31,800
6-27	855.	99,200	210,500	7-27	402	21,950
7-28	472.	54,800	116,500	8-28	222	12,130
8-29	89.3	10,370	22,000			
29-10	293.5	34,050	72,300	10-30	318	17,380
30-11	676.5	78,500	166,800	11-31	498	27,200
31-12	1060.	123,100	261,800	12-32	678	37,050
32-13	1443.	167,700	356,000			
Explanation	Fig. 1A	L = 17.00 in. A = .1464 sq.in.	$\cos \eta = \frac{8}{17}$	Fig. 1A	Fig. 1A	$\cos \eta = 1.00$ A = 0.1464 sq.in.

TABLE II

CORRECTION OF CHORD MOMENTS OF INERTIA

(A) Panel	(B) Total deflection increment	(C) Δ Increment	(D) M	(E) Mx^2	(F) $\frac{\Delta}{Mx^2}$	(G) Av. $\frac{1}{I_w}$	(H) $\frac{1}{I_c}$	(I) $\frac{1}{I_t}$
* 1-3	+649,300	103,900	-13,400	3,013,000	.0345	.0345	.153	.188
3-4	+545,400	104,800	+ 744	167,000	.626	.3302	.153	.483
4-5	+440,600	103,800	+12,200	2,743,000	.0378	.3299	.153	.483
5-6	+336,800	104,350	+20,900	4,700,000	.0222	.0300	.147	.177
6-7	+232,450	103,820	+26,950	6,060,000	.0171	.0196	.1414	.1610
7-8	+128,630	106,630	+30,300	6,820,000	.01561	.0163	.1414	.1577
8-9	+ 22,000	111,680	+30,900	6,930,000	.01611	.01586	.1414	.1573
9-10	- 89,680	104,320	+28,800	6,480,000	.0161	.01610	.1414	.1575
10-11	-194,000	104,850	+24,020	5,410,000	.01935	.01772	.1414	.1591
11-12	-298,850	57,150	+16,580	3,730,000	.01532	.01733	.1414	.1587
** 12-14	-356,000					.01733	.1414	.1587
Explanation	Col. (D) + (G) of Table I	Difference in values of Col. (B)	From Fig. 1A by method of moments	x = 15 in. M is from Col. (D)	$\frac{\text{Col. (C)}}{\text{Col. (E)}}$	Average of values in Col. (F)	See *	G + H

Calculation of I_c:

$$^{*}I_c = \frac{d^2 A_u A_L}{A_u + A_L} = \frac{64 \times 0.2041 \times 0.2041}{\underset{.4082}{0.2041 + 0.2041}} = \underline{\underline{6.54 \text{ in.}^4}}$$

$$^{**}I_c = \frac{64 \times 0.2041 \times 0.2400}{\underset{.4441}{0.2400 + 0.2041}} = \underline{\underline{7.075 \text{ in.}^4}}$$

TABLE III

CALCULATION OF EFFECTIVE MOMENT OF INERTIA

Panel	(A) Av. M	(B) Moment weight	(C) Panel length	(D) Panel weight	(E) Dist. from support	(F) Dist. weight	(G) Total weight	(H) $\frac{1}{I_t}$	(I) Products
1-3	-26,700	.8725	22.5	1.000	11.25	.125	.1091	.188	.0205
3-4	- 6,328	.2066	15.0	.667	30.00	.333	.0458	.483	.0221
4-5	+ 6,470	.2112	15.0	.667	45.00	.500	.0705	.483	.03405
5-6	+16,550	.541	15.0	.667	60.00	.667	.2410	.177	.04265
6-7	+23,925	.782	15.0	.667	75.00	.833	.4350	.1610	.0700
7-8	+28,625	.936	15.0	.667	90.00	1.00	.6240	.1577	.0984
8-9	+30,600	1.000	15.0	.667	75.00	.833	.555	.1573	.0873
9-10	+29,850	.976	15.0	.667	60.00	.667	.435	.1575	.0685
10-11	+26,410	.8635	15.0	.667	45.00	.500	.288	.1591	.0458
11-12	+20,300	.663	15.0	.667	30.00	.333	.1472	.1587	.0234
12-14	+ 8,290	.271	22.5	1.000	11.25	.125	.0339	.1587	.00538
	Average of D, Table II	$\frac{(A)}{30,600}$	Fig. 1A	$\frac{(C)}{22.5}$	Fig. 1A	$\frac{(E)}{90}$	(BxDxF)	Col. (I) Table II	(G) x (H)
							2.9845		.51808

$$\frac{1}{I_t} = \frac{0.51808}{2.9845}; \qquad I_t = \underline{5.76 \text{ in.}^4}$$

CALCULATION OF DEFLECTIONS PRODUCED BY AXIAL AND SIDE LOADS

$$\delta = \frac{1}{P}\left[\overset{(a)}{M_1} + \overset{(b)}{\frac{M_2-M_1}{L}x} - \overset{(c)}{\frac{wLx}{2}} + \overset{(d)}{\frac{wx^2}{2}} - \overset{(e)}{\frac{D_2-D_1\cos L/j}{\sin L/j}\sin x/j} - \overset{(f)}{D_1\cos x/j} - \overset{(g)}{wj^2}\right]^{*}$$

$M_1 = -40{,}000$ in.lb.; $\dfrac{M_2-M_1}{L} = \dfrac{0+40{,}000}{180} = +222$; $w = -12$ lb./in.; $wL = -12 \times 180 = -2160$; $\dfrac{wL}{2} = -1080$;

$j = \sqrt{\dfrac{EI}{P}} = \sqrt{\dfrac{10{,}000{,}000 \times 5.76}{6{,}000}} = 97.98$; $j^2 = 9600$; $L/j = \dfrac{180}{97.98} = 1.838$; $\sin L/j = +0.96450$

115,200

$\cos L/j = -.26403$; $D_1 = M_1 - wj^2 = -40{,}000-(-12) \times 9600 = +75{,}200$; $D_2 = M_2 - wj^2 = 0 + 12 \times 9600 = +115{,}200$

+19,890

$wj^2 = -12 \times 9600 = -115{,}200$; $\dfrac{D_2-D_1\cos L/j}{\sin L/j} = \dfrac{+115{,}200-(+75{,}200)\ (-.26403)}{+.96450} = +140{,}100$

TABLE IV

Panel point	x	x^2	x/j	(c)	(b)	(d)	sin x/j	cos x/j	(e)	(f)	Σ	(H) Deflections (in.)
3	22.5	506.2	.2298	+ 24300	+ 5000	-3040	+.2278	+.9737	-31900	-73200	-3640	-.606
5	52.5	2756.	.5360	+ 56650	+11670	-16520	+.5107	+.8597	-71600	-64600	-9200	-1.534
7	82.5	6806.	.8425	+ 89100	+18320	-40850	+.7466	+.6656	-104800	-50000	-13030	-2.170
9	112.5	12650.	1.149	+121600	+25000	-75900	+.9124	+.4094	-128000	-30800	-12900	-2.150
11	142.5	20300.	1.455	+154000	+31650	-121900	+.9933	+.1155	-139200	- 8690	-8840	-1.473
	Fig. 1A	See above equations and data -										

*Page 202 - "Airplane Structures," by Niles and Newell.

TABLE V

CALCULATED AND MEASURED DEFLECTIONS

Panel point	Calculated deflection	Measured deflection	Error	% Error
	(in.)	(in.)	(in.)	
3	- .606	- .55	+.056	+10.18
5	-1.534	-1.51	+.024	+ 1.59
7	-2.170	-2.10	+.070	+ 3.33
9	-2.150	-2.10	+.050	+ 2.38
11	-1.473	-1.43	+.043	+ 3.00
	H Table IV	Page 4 Test No. 10096, Boeing Air- plane Co.		

Conclusions.- An examination of Table V shows that there is good agreement between the measured and calculated deflection values. Thus, although the method used in calculating the effective moment of inertia contains several approximations, the good agreement between the two sets of deflections shows that the effective moment of inertia has been calculated fairly accurately.

The greatest percentage of error occurs at a point which is very close to the support where end moment is applied. Since the method used in calculating the effective moment of inertia was based upon having the maximum ordinate accurate and the other ordinates only approximately so, the greatest percentage of error would naturally be expected to be at points near the supports. The error in inches does not vary as much from one end of the span to the other as does the percentage error, because at the points of small deflection, any error at all produces a large percentage of error. Also, small errors in the measurement of deflections produce large percentage errors in the deflection values if these values are very small. Consequently, a sizable portion of the percentage "error" at points of small deflection can be attributed to measurement errors in the deflection test.

Since the calculated deflection values are greater at all panel points than the measured deflections, the effective moment of inertia must have been calculated in too conservative a manner. There were two assumptions made in the calculation of the effective moment of inertia which were obviously conservative. One was the assumption that the panels containing the gusset plates had as much shear deformation as the adjacent panels which did not contain gusset plates. The second assumption was that the elastic curve was represented by the deflected position of the panel points of the lower chord member, which panel points were deflected more under load than the panel points of the upper chord. A more accurate procedure would have been to consider the elastic curve of the truss spar to be an average of the upper and lower chord deflection polygons.

Diagram Sheet

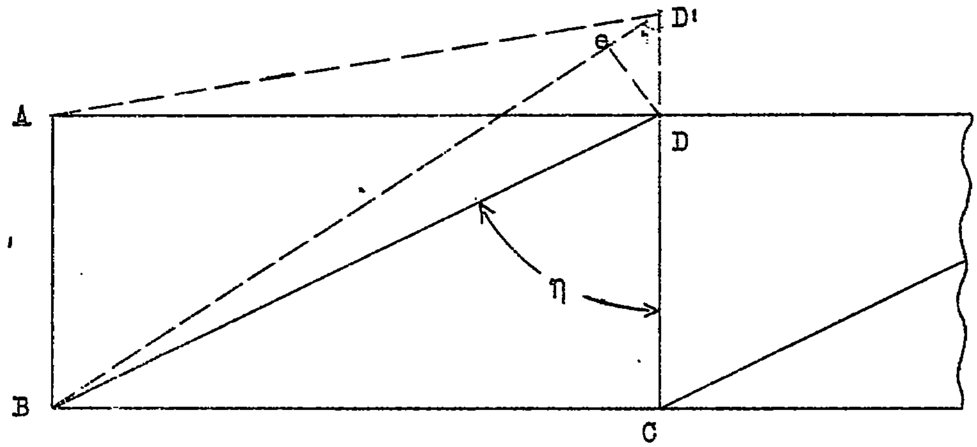


Fig.1

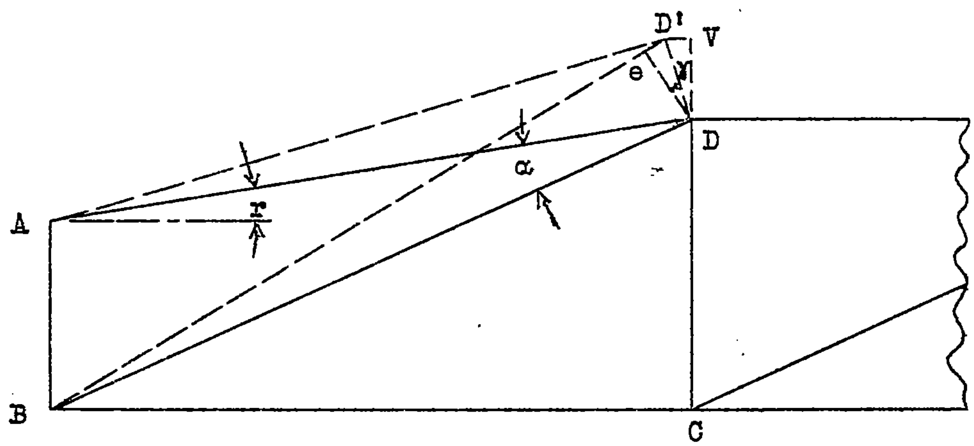


Fig.2

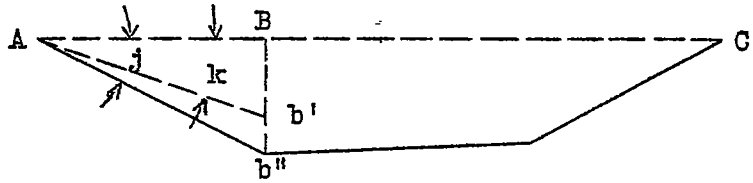
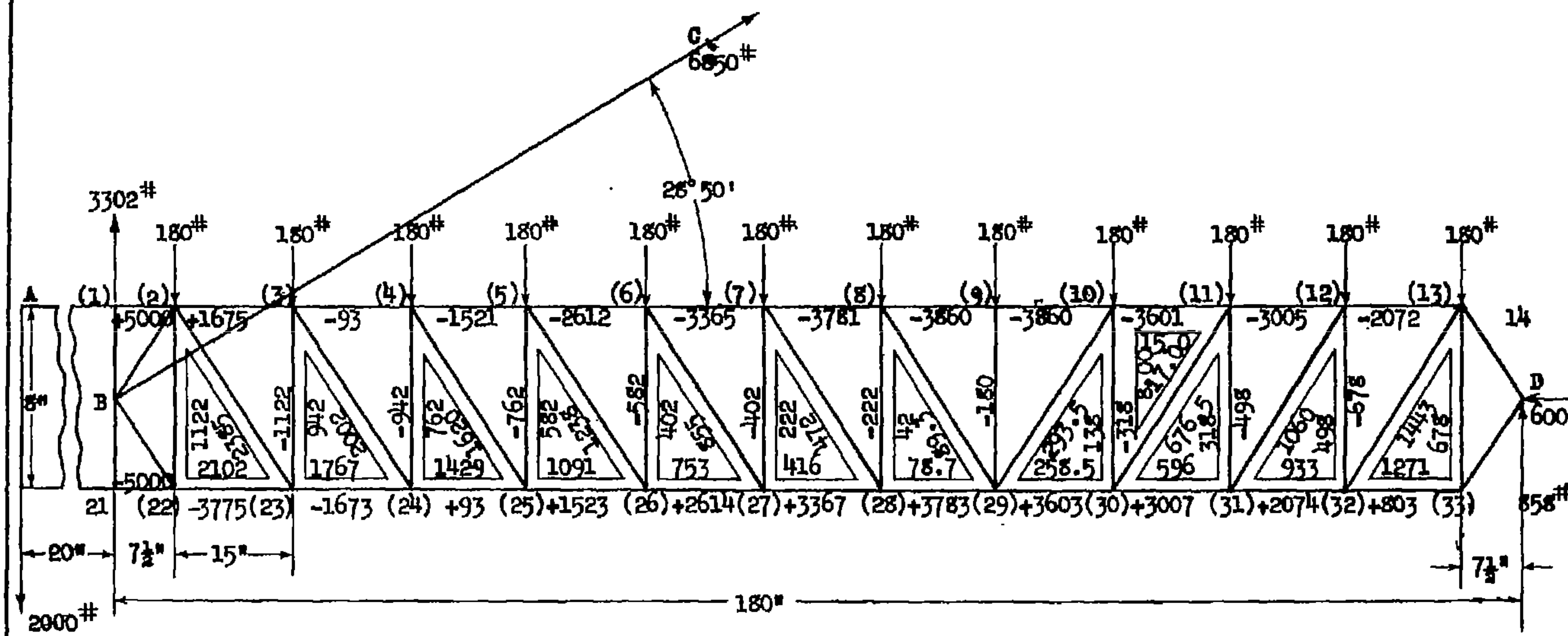


Fig.3



Calculation of Reactions:

(1) ΣM about B:

$$\overset{-40{,}000}{-2000 \times 20} + \overset{180 \times 1080 = +194{,}400}{180\,(7.50 + 22.5 + 37.5 + 52.5 + 67.5 + 82.5 + 97.5 + 112.5}$$

$$+ 127.5 + 142.5 + 157.5 + 172.5) - V_D \cdot 180 = 0 \quad ; \quad \therefore V_D = \frac{154{,}400}{180} = \underline{\underline{858}}^{\#}$$

(2) $\Sigma V = 0$: $\quad +V_B + 858 - 2160 - 2000 = 0 \quad ; \quad V_B = \underline{\underline{+3302}}^{\#}$

(3) $\Sigma H = 0$: $\quad +H_{BC} - H_{DC} = 0 \quad ; \quad V_{DC} = 3302 \quad ; \quad BC = \dfrac{3302}{\underset{0.4823}{\sin 28°50'}} = \underline{\underline{6{,}850}} \quad ; \quad H_{BC} = 6850 \times \cos 28°50'$

$$H_{BC} = \underline{\underline{6000}}^{\#}$$

Fig. 1A Calculation of primary loads

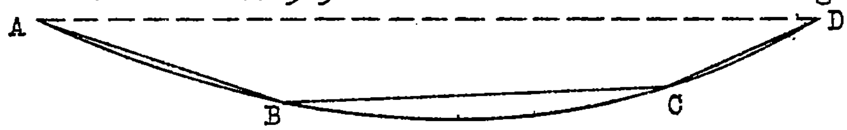


Fig.4

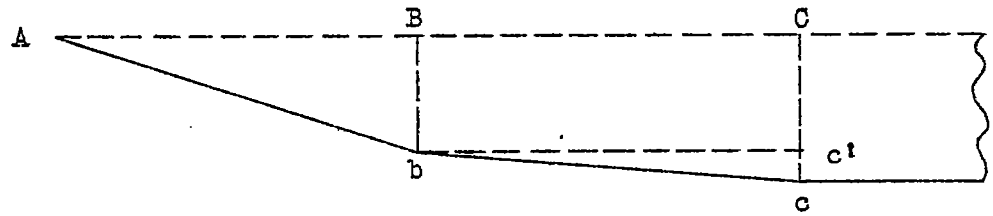


Fig.5

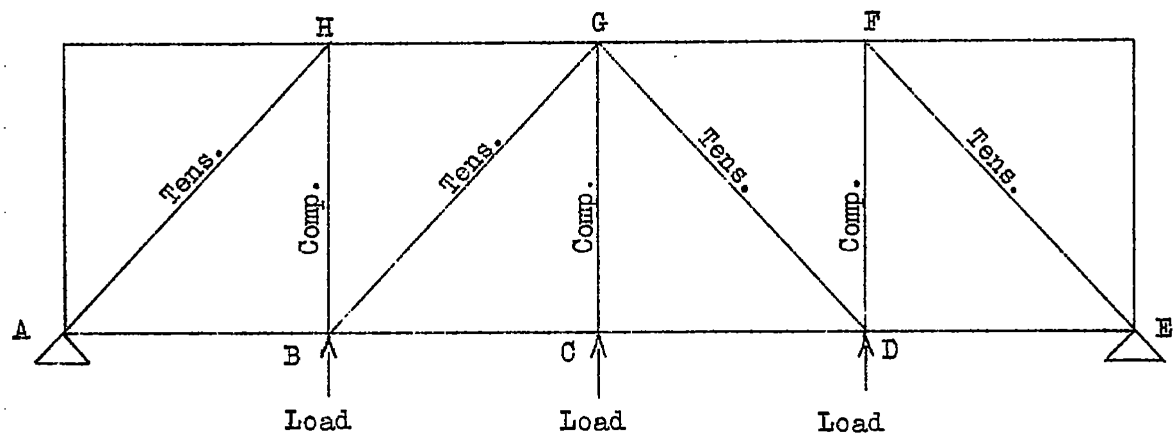


Fig.6

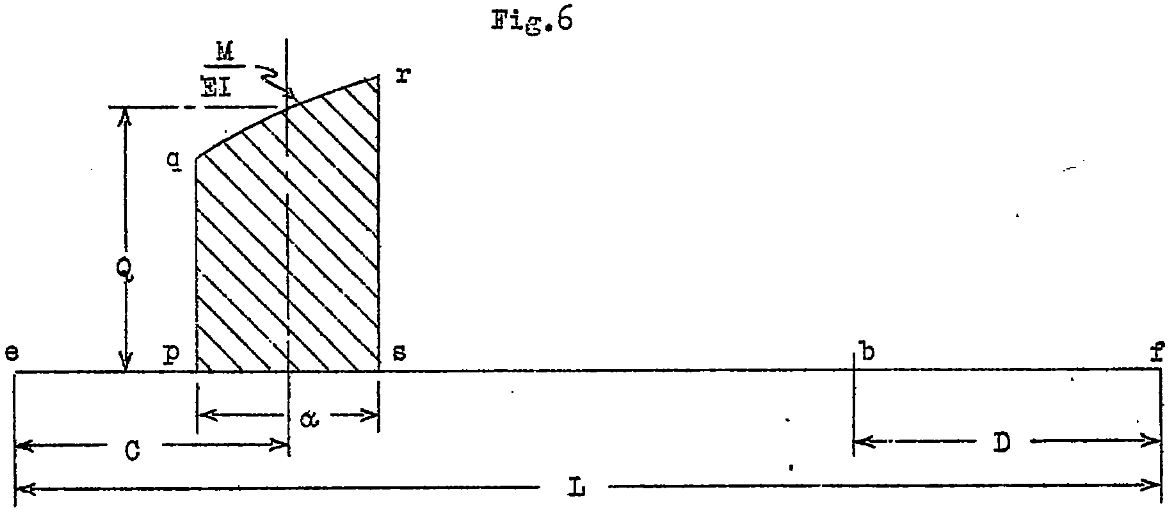


Fig.7

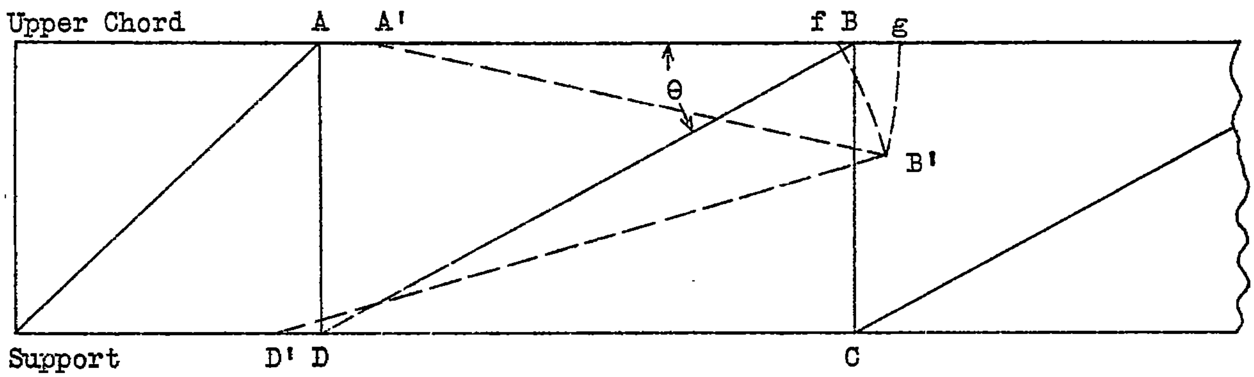

Upper Chord
A
A'
f B
g
θ
B'
Support
D' D
C

Fig. 8

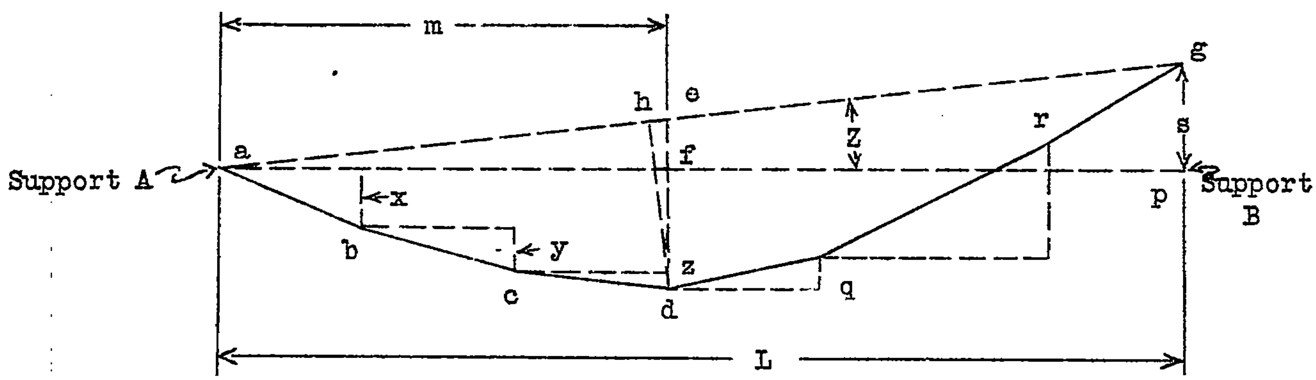

m
g
h
e
r
a
f
Z
s
Support A
x
Support B
p
b
y
z
c
d
q
L

Fig. 9