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No. 529

ANALYSIS OF A STRUT WITH A SINGLE ELASTIC SUPPORT IN THE SPAN, WITH APPLICATIONS TO THE DESIGN OF AIRPLANE JURY-STRUT SYSTEMS

PART I - DERIVATION OF FORMULAS

By A. Murray Schwartz

PART II - EXPERIMENTAL INVESTIGATION OF FORMULAS

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PREFACE

The problem of a precise method of analysis for airplane jury-strut systems was selected by Mr. A. Murray Schwartz as the subject of his Engineer's thesis at Stanford University. Mr. Schwartz's study resulted in the derivation of suitable theoretical equations and the development of a system of using them in practical design. He did not have time, however, to carry out any experimental work to prove the validity of his formulas. In the winter of 1933-34 another graduate student at Stanford, Mr. Reid Bogert, made the experimental investigation of Mr. Schwartz's formulas the subject of his Engineer's thesis, and obtained data to prove their validity.

Owing to the length of these theses, the N.A.C.A. did not consider it advisable to publish them in full, but accepted the offer of the writer, under whose direction the two theses were prepared, to combine them into a single report of length suitable for publication. The work of the writer has been primarily editorial, the theoretical derivations of the first part of the report being that of Schwartz, and the experimental work of the second part that of Bogert. While the theses on which the present report is based were written under the direction of the writer, his supervision was not very close and by far the greater part of the credit belongs to the two students.

The title of Schwartz's thesis was "Structural Analysis of Airplane Jury Strut Systems". Study of the problem showed that its essential feature was the analysis of a strut with a single elastic support at any specified point between its ends. This is a general problem of which the airplane jury-strut system is only a special case. Bogert's thesis was accordingly entitled "Tests on Struts with a Lateral Elastic Support in the Span". The title of the present paper was chosen to indicate both the essential problem attacked and its most important application in aeronautical design.

Alfred S. Niles.

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PART I

DERIVATION OF FORMULAS

By A. M. Schwartz

I. INTRODUCTION

The need of a precise analysis of airplane jury-strut systems was suggested by Mr. Richard C. Gazley in an article entitled "Late Developments in Airplane Stress Analysis Methods and Their Effect on Airplane Structures."* In the paragraphs on the jury strut, he says, "Among designers of strut-braced monoplanes, there is an increasing tendency to reduce the weight of the external wing bracing by providing the main struts with lateral support at about one third the distance in from the outer ends. This support is furnished by a small auxiliary strut, commonly called a jury strut, which is attached to the wing spar at its upper end. This type of design is quite effective for the purpose intended, but it has introduced some difficult analysis problems.

"The case in which the auxiliary strut and the upper end of the main strut are both pinned at their intersection is fairly simple, and has been successfully analyzed by a number of designers. The more common case, however, where the main lift strut is continuous, is greatly complicated by a number of factors affecting the force distribution. A precise solution of this problem probably would result in unwieldy formulas but would enable the importance of the various factors to be determined."

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*S.A.E. Journal, September 1932.

The problem assumes a difficult aspect because of the simultaneous deflection of both strut and spar, one depending upon the other, and because of the secondary stresses and deflections present due to axial loads in the members. This can easily be illustrated by reference to figure 1, in which the broken line represents the unstressed structure, and the solid lines an exaggerated view of the members when under load. The elastic curve of the spar is, of course, quite dependent upon the side or air load upon it and the amount of overhang. Since axial compression is the critical load on the lift strut, the airplane is assumed to be in inverted flight. This condition imposes a "down" side load upon the spar, and consequently the spar and strut are under axial tension, and compression, respectively. If a jury strut is connected to the spar at point "B" where the spar is deflected due to the external air loads, the strut will also be deflected. Then, due to axial load in the strut, secondary stresses will be produced which will tend to increase the deflection of B. This increased deflection will cause increased secondary stresses which will multiply until a state of equilibrium is reached; that is, the supporting forces developed in the spar will equilibrate the buckling of the strut. If it were possible that a point of zero deflection such as point "A" could be used as the upper jury-strut connection, there would be no bending of the strut and, as a consequence, no secondary stresses to cause a further deflection of point A. This situation can only be present for one actual loading, however, because a change of the axial load or the side loading on the spar moves the point of zero deflection. Moreover, even if it were possible that such a state of affairs could exist, it would be necessary to investigate the structure so that the elastic stability of the strut could be checked. It will be shown later that both of the above cases are almost alike, and that the presence of initial deflection due to the air loads on the spar does not alter or complicate the determination of the critical load on the strut or that of the size of the load on the jury strut to a very great extent. When the size of the load in the jury strut has been calculated, it is quite easy to determine the maximum unit stress in the lift strut by means of the Newell extended equation for a beam with supports deflected. (Precise three-moment equation found in "Airplane Structures" by Niles and Newell, p. 192.)

It was believed at the beginning of the work on the precise solution of jury strut systems that rather complicated formulas would be obtained, but it was found that the solution was little more unwieldy than the calculation for the moments on a continuous beam by means of the extended three-moment equation.

A review of the obstacles encountered in the precise solution of jury strut systems, shows that the problem may be resolved into four distinct phases. These are:

1. Determination of the "spring constant" of a point in the span of an axially loaded beam, and the relations between this spring constant and the stability of the member.

2. Determination of the stability of a strut supported at some point along its span by means of a jury or auxiliary strut connected to another strut or beam, the supporting beam being under either axial compression or tension, a) when the supporting member has no initial deflection caused by external side load, and b) when external side load causing deflection is present.

3. Determination of the load in the jury strut and the deflection present for a given side load on the supporting member.

4. An investigation of the critical conditions through which the system passes as the axial load is increased from zero to the final critical load, and the formulation of a method of determining the maximum critical load in a supported strut.

In order to give a complete explanation of the formulas and methods derived, several numerical examples are presented, three consisting of very simple structures, and the fourth being a representative jury-strut system.

II. REFERENCES

1. "Late Developments in Airplane Stress Analysis Methods and Their Effect on Airplane Structures", by Richard C. Gazley, S.A.E. Journal, September 1932.

This article outlined the problem and presented the difficulties to be encountered as follows: "The design of the main lift strut then resolves itself into the problem of finding the critical load for a pin-ended, long column initially straight but deflected laterally a constant amount at some point along its length. To be more accurate, the deflection could be taken as a linear function of the axial load in the column. The solution of this problem would be a valuable addition to our knowledge of strength of materials. Pending such a solution, we must rely on empirical formulas and meager test data for our allowable loads and therefore need to provide ample margins of safety."

2. "Airplane Structures", by Niles and Newell, Wiley, New York.

This volume supplied a basic theory and equations for the formulas derived in this paper.

3. "On the Buckling Strength of Beams Under Axial Compression, Bridging Elastic Intermediate Supports", by W. B. Klemperer and H. B. Gibbons, contributed by the Applied Mechanics Division of the A.S.M.E. for presentation at the National Applied Mechanics Meeting, New Haven, June 1932.

Although this paper did not consider the case of struts having unsymmetrical bays with supports having deflections due to external loads, and was of no direct use, it did supply valuable information as to methods of attack for which the writer is very grateful. This paper was also used as a check for the special case which it covers in common with this paper.

III. DEFINITIONS

For convenience of reference, definitions are given here for a group of terms that will be used frequently in the subsequent text.

Beam - the supporting member of a pair (the wing spar in the airplane jury-strut system).

Strut - the supported member of a pair (the lift strut in the airplane jury-strut system).

Load - (P), axial tension (+) or compression (-) in the beam or the strut as indicated by subscripts or the context.

Total load - (P_T), the algebraic sum of the loads in the beam and strut when they are parallel.

Supporting force - (-W), the lateral force required to hold the strut in equilibrium.

Supporting load - (W), the lateral force imposed on the beam in supporting the strut. Supporting load and supporting force are necessarily equal in magnitude but opposite in sign. They are also the external forces acting on the jury strut.

Jury strut - member joining the beam and the strut which causes these two members to interact.

Side load - (S), any lateral force other than the supporting load which acts on the beam.

Spring constant - (k), the rate of change in the lateral load required to maintain equilibrium at a point along the span of the beam (or strut) with respect to the lateral deflection of that point.

IV. THE SPRING CONSTANT FOR AN AXIALLY LOADED BEAM

The first step in the study of the stability of an elastically supported strut is to explain the derivation and significance of what will be termed the "spring constant" of a point on the span of an axially loaded beam. Mathematically this spring constant may be defined as the partial derivative of the transverse load at the point with respect to the deflection of that point. This can also be expressed in simpler though less concise language. If the beam is assumed to remain in equilibrium, although the deflection of some point on the span is changed, there must be some corresponding change in the external loading. If it be assumed that the only change in the external loading is the addition of a transverse force at the point in question and suitable reactions at the supports, there will be some definite relationship between the changes in the deflection of and the external force at the point. Inspection of the formulas for the deflection of an axially loaded beam will show that there is a linear relation between the magnitude of the transverse load on an axially loaded beam and the deflection of its point of application. This is shown by the fact that they can all be written in the form

$$\delta = \frac{W}{k} + C \qquad (1)$$

where δ is the deflection of the point in question

W is the lateral load at the same point

k and C are constants depending on the location of the point, the dimensions and material of the beam, the magnitude and character of the axial load, end moments, transverse loads at other points, etc., but independent of the transverse load W.

From equation (1) it is apparent that if the load W is the only one to vary as the deflection of its point of application changes, it must change by k pounds for each inch of change in that deflection. The quantity k is therefore the spring constant as defined mathematically above. It may also be defined as the transverse load required to cause a unit (one inch) deflection of its point of application.

The formula to be used for the computation of the spring constant in any given case will depend on the character of the axial load and whether or not the cross section of the beam is constant. In this report only two groups of cases will be considered.

1. Beams of constant section with the axial load the same on both sides of the point for which the spring constant is to be computed.

2. Beams in which the cross section and axial load are constants on each side of the point for which the spring constant is to be computed, but in which one or both of those quantities change at that point. This class of cases is of interest as the axial load in, and cross section of, the lift strut may change at the point of connection of the jury strut.

In the main body of the report attention will be directed, in general, to those cases of the first group in which the axial load is compression. It is to be understood, however, that the conclusions arrived at in respect to the significance of the spring constant, the criteria for stability, and the general equations for the deflection of the supported strut system in terms of the spring constants involved, apply equally to all cases (even those in which the axial load and the cross section vary continuously along the span) unless otherwise noted. Formulas will also be derived for the computation of the spring constant in cases of group 2 in which the axial load is compression. No attempt will be made to derive more general formulas for the spring constant, but the derivations given should be a sufficient guide to permit the engineer to handle any other case in which he is able to compute the deflection due to a unit transverse load.

Before attempting to discuss the relationships between the spring constant and the stability of a strut, it is desirable to develop the formulas for the spring constant in a representative case. For simplicity, the case studied will be that of a strut of constant section and constant axial load. Three conditions must be considered, depending on whether the axial load is compression, tension, or zero.

The formula for the deflection of a point on the span of a constant section beam subjected to axial compression and a single concentrated transverse load is*

$$\delta = \frac{1}{P}\left(M_1 + \frac{M_2 - M_1}{L}x - \frac{Wbx}{L} - C_1 \sin\frac{x}{j} - C_2 \cos\frac{x}{j}\right) \qquad (2)$$

where

$$C_1 = \frac{M_2}{\sin(L/j)} - \frac{M_1 - W\,j\,\sin(a/j)}{\tan\,(L/j)} - W\,j\,\cos\frac{a}{j}$$

$$C_2 = M_1$$

If the axial load P, and the end moments M_1 and M_2 be assumed not to vary, equation (2) is obviously a special case of equation (1). From the discussion on page 199 of reference 2 it can be seen that the only effect of adding other transverse loads to the system of forces acting on the beam would be to make necessary the addition of some more constants to equation (2), which would modify the value of the constant C of equation (1). The constant k of equation (1) would not be affected.

Combining equations (1) and (2) we have as the formula for the spring constant of a beam subjected to axial compression

$$\frac{1}{k_c} = -\frac{1}{P}\left(\frac{a\,b}{L} + \frac{j\,\sin^2(a/j)}{(\tan\,(L/j))} - j\,\sin\frac{a}{j}\cos\frac{a}{j}\right) \qquad (3)$$

In this formula the subscript c is used to indicate that the axial load is compression.

k_c, spring constant for the point and axial compression in question

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*Page 205 of reference 2. This formula applies only to the section of the beam between the left end and the point of application of the side load W. By substituting a for x it gives the deflection of the point of application of W. The same result could be obtained by placing a for x in the expression also given on page 205 of reference 2, for the deflection between the load and the right-hand end of the span.

a, distance from left end of span to the point in question

L, length of span

$b = L - a$

$j = \sqrt{EI/P}$

Similarly we can obtain as the formula for the spring constant of a beam subjected to axial tension*

$$\frac{1}{k_t} = \frac{1}{P}\left(\frac{a\ b}{L} + \frac{j\ \sinh^2 (a/j)}{\tanh (L/j)} - j \sinh \frac{a}{j} \cosh \frac{a}{j}\right) \quad (4)$$

The spring constant for the case of zero axial load can be obtained by setting $P = 0$ in equation (3) or (4), but the resulting indeterminate form 0/0, is awkward to evaluate. A simpler method of obtaining this spring constant k_0, is to differentiate the formula for the deflection of a simple beam subjected to a single concentrated side load as given on page 264 of reference 2. This gives

$$k_0 = \frac{6\ E\ I\ L}{a\ b\ (L^2 - a^2 - b^2)} = \frac{3\ E\ I\ L}{a^2\ b^2} \quad (5)$$

The spring constants as given by equations (3), (4), and (5) apply to all cases of beams, struts, or ties of constant section and constant axial load regardless of the presence or absence of end moments and other side loads on the member. In practical computations they may be used to determine either the spring constant of the member that requires support or that of the member which furnishes support.

V. VARIATION OF SPRING CONSTANT WITH AXIAL LOAD

It would be interesting to make a general study of the effect of varying the axial load upon the sign and magnitude of the spring constant, but the trigonometric expressions involved are too complex for this to be done conveniently. Before going into the relation between the

*For derivation, see Section III of the Appendix.

sign of the spring constant and the stability of a member, it is desirable, however, to see how the spring constant varies with axial load in a typical case. The example chosen is that of a strut 180 inches long with $EI =$ 9,000,000. The values of the spring constant for the third point of this member, as obtained from equations (3), (4), and (5) are plotted as ordinates in figure 2. For convenience, the abscissas in figure 2 are the values of the ratio L/j instead of the corresponding values of the axial load P.

The curve of figure 2 is representative of the curves of spring constant for all cases. Whenever the axial load is tension, the spring constant will be positive and will increase with the magnitude of the axial load. When the cross section is constant and the axial load is compression and constant along the entire span, as the axial load increases the spring constant will be a decreasing positive quantity until $L/j = \pi$, when the spring constant becomes zero. As L/j continues to increase, the spring constant is a negative quantity of increasing magnitude until a critical load is reached at which k_c = negative infinity. At this critical load $\pi < \frac{L}{j} \lessgtr 2\pi$. If L/j for the critical load is less than 2π, the spring constant varies from positive infinity at the critical load to zero $L/j = 2\pi$. When L/j exceeds 2π the spring constant is negative and increases with L/j, at least until $L/j =$ 10.0 in the case under consideration. The values of spring constant for higher values of L/j have not been investigated in this study but there is probably a critical load at which the spring constant passes through infinity for every increase of 2π in L/j.

The values of the spring constant for L/j values in excess of that for the critical load between $L/j = \pi$ and $L/j = 2\pi$ apply to elastic curves of the beam which are unstable unless the member is provided with more than one transverse supporting force. For this reason they are not of direct interest in this study which is limited to cases in which there is but one supporting force in the span.

The curves of spring constant vs. axial load for beams of nonuniform section and axial load would be similar to that shown in figure 2. In such cases, however, P/EI is not constant along the span and the expressions for the loading at which the spring constant becomes zero and infinite become much more complex than in the case represented in figure 2. The only case of that character that will be considered in this report will be that in which the cross section and the axial load are constant on each side of the point for which the spring constant is determined, but in which there is a sudden change in these quantities at the point in question. This case will be treated later.

VI. RELATION BETWEEN SPRING CONSTANT AND STABILITY

The simplest method of determining the relations between the spring constant of a beam or strut and its stability is to make a parallel study of the sign and magnitude of the spring constant and the mechanical action of a strut as the axial load varies.

The Unsupported Strut

The simplest case to discuss is that of an ideal pin-ended strut having no lateral support. When such a strut is loaded in tension or with a compression below the critical Euler load, $\left(\frac{L}{j} < \pi^2,\ P < \pi^2 \frac{EI}{L^2}\right)$, it will be found that for equilibrium a lateral deflection must be accompanied by a force in the same direction as the displacement of the strut. This merely means that the strut resists a side force. Formulas (3), (4), and (5) give positive values for the spring constant in this range of L/j, which is a mathematical way of expressing the same fact. The strut is then elastically stable, and if it is deflected from its normal straight position by an external force, it will immediately snap back into place when the force is removed.

Now suppose the strut is loaded with the critical Euler load $\left(\frac{L}{j} = \pi,\ P = \pi \frac{EI}{L}\right)$. The spring constant as determined from formula (3) is zero. This means that changes in the lateral deflection do not have to be accompanied by changes in the side load in order to maintain equilibrium. From this it can be deduced that the strut will be in equilibrium in any deflected position, and has no tendency to spring back into place due to its own stiffness. The strut is therefore elastically indifferent.

If now we load the strut over the critical Euler load $\left(\frac{L}{j} > \pi,\ P > \pi^2 \frac{EI}{L^2}\right)$, the spring constant is found to be negative, that is, for equilibrium a lateral deflection must be accompanied by a force in the opposite direction. Thus the strut within itself is elastically unstable, for the slightest lateral deflection will cause buckling if no external supporting force is available.

The practical strut differs from the ideal strut primarily in that it may be subjected to end moments and for side loads as well as axial loads. As noted above, the absence or presence of such forces has no influence on the magnitude of the spring constant. They do, however, make a little difference in the physical action of the member under load. In case the axial load is tension, or a compression such that $L/j < \pi$ (the range of positive spring constants), they cause the strut to deflect until a position of equilibrium is reached. The imposition of an additional side load will cause additional deflection. On its removal, the strut, instead of becoming straight, as an ideal strut would, returns to the equilibrium position it had assumed under the remaining loads when they acted alone. If L/j is equal to or greater than π, the side loads and end moments on the practical strut cause it to deflect, and the axial load produces secondary bending moments causing increased deflection as fast as, or faster than, the deflection itself. The strut therefore never reaches a condition of equilibrium unless a sufficient supporting force acting in the direction opposite to the deflection is added to the system.

One method of obtaining such a supporting force is to connect the strut, which is unstable by itself, through a more or less rigid link (or jury strut) to a member of sufficient stiffness that the resistance to deflection of the latter will provide the force required. One of the chief objects of this report is to determine the stiffness required in the supporting member so that the combination will be in stable equilibrium. For simplicity the member which requires support will be called "the strut" and the one which provides such support "the beam". In the normal airplane jury-strut system, the lift strut is "the strut" and the wing spar is "the beam". Furthermore, the force acting on the strut required to maintain stability will be called the "supporting force". The equal and opposite force acting on the beam will be termed the "supporting load".

The above discussion indicates that, if the strut is to be adequately supported, the beam must be loaded below the critical Euler load, so that it is elastically stable in itself (spring constant is positive). Or, to state the requirement less formally, the supporting beam must have "excess stiffness" to make up for the tendency of the supported strut to buckle. The above discussion also indicates that a strut loaded to or above the critical Euler load (i.e., spring constant is negative) must be supported if it is to be stable. Accordingly, we must next turn our attention to the elastic stability of a strut which is loaded above the Euler load but supported elastically.

The Supported Strut

The mechanics of the elastic stability of the supported strut is very similar to that of the unsupported strut. There are three conditions to be studied: the elastically stable, indifferent, and unstable. In this discussion it will first be assumed that there is no initial deflection of the beam due to transverse forces other than the supporting load. This will be followed by a consideration of the effect of the presence of such an initial deflection of the supporting beam.

We have an elastically stable supported strut when the spring constant of the beam is greater in magnitude than the negative spring constant of the strut. In this case we see that to produce any deflection of the support point on the beam, a greater force is needed than that required to prevent the strut from buckling, and if any deflection is produced by a momentary force, the strut is forced back to its original position as soon as the momentary force is removed. This condition corresponds to the case in which the unsupported strut was loaded below the critical Euler load.

Now suppose that the spring constants of the strut and beam are of equal magnitude but of opposite sign. If the support should be deflected a distance δ, and the force causing the deflection should be removed, the beam would be capable of exerting a supporting force equal to $k\,\delta$, where k is the spring constant of the beam, i.e., the load required to produce unit deflection of the support point. This supporting force, however, is of just the right magnitude to provide the necessary support for the strut and there would be no tendency either to spring back to the original positions or to deflect further. A state of equilibrium is thus found for any deflection of the members; the system is elastically indifferent as in the case of the unsupported strut loaded with the critical Euler load.

The last case follows when the spring constant of the supporting beam is smaller in magnitude than that of the strut. At the smallest lateral deflection the force required to hold the strut in equilibrium will be larger than the available supporting force, and the system will buckle. This is an elastically unstable condition, corresponding to the failure of the unsupported strut which is loaded above the critical Euler load.

Effect of Initial Deflection

The action of the strut and beam combination when there is initial deflection of the beam can be studied most conveniently with the aid of figure 3, in which are plotted the curves of variation of supporting force (or load) with deflection. As the partial derivatives of W with respect to δ are constants, these curves are straight lines with slopes numerically equal to the respective spring constants, k_s for the strut and k_b for the beam. In the figure OD represents the variation with deflection of the supporting force required by the strut. The actual values plotted, however, are those of the equal and opposite supporting loads that would be imposed on the beam. As it is assumed that there is no initial deflection of the strut, the equation of OD is

$$W_s = -k_s \delta \tag{6}$$

As the strut would need no support when k_s is positive, it is assumed that k_s is negative and therefore $-k_s$ is positive. The variation in the available supporting force is shown by the line AC, the equation of which is

$$W_b = k_b (\delta - \delta_o) \tag{7}$$

where δ_o is the deflection of the beam due to all forces other than the supporting load. It will be called the "initial deflection" of the beam.

In the case shown in figure 3, the beam and strut would deflect to the equilibrium position indicated by B, the intersection of OD and AC. At this deflection, δ_e, both the required supporting force for the strut and the available supporting force are equal to W_e. When the deflection is less than δ_e, the supporting force is insufficient to prevent further deflection, but if the deflection is greater than δ_e. the supporting force is the greater and the beam would force the strut back until the deflection was reduced to δ_e.

It should be clear from this figure that as long as k_b is numerically greater than k_s the two curves will intersect at a positive value of δ and there will thus be an equilibrium position. If k_b is equal to or smaller than k_s, however, there will be no such intersection and the combination will be unstable. The magnitude of the initial deflection δ_o, has no bearing on the question of whether or not there will be an equilibrium position indicated by the intersection of the two curves. It will, however, have considerable influence on the location of that intersection in any given case. The larger the value of δ_o, the greater will be the deflection before the equilibrium position is reached. In practice this may be important, as the result of a large initial deflection may be that plastic failure of the strut may take place before the equilibrium position is reached, whereas this might not have been the case if the initial deflection had been small.

If it should happen that the strut as well as the beam had an initial deflection, figure 3 and equation (6) could easily be modified to take the situation into account, but it should be obvious that this would affect only the magnitudes of the deflection and supporting force when the equilibrium position was reached but not the question of whether there was such a position, i.e., whether the strut is stable or unstable.

VII. DETERMINATION OF EQUILIBRIUM POSITION AND SUPPORTING FORCE

The magnitude of the deflection at the equilibrium position and the corresponding supporting force can easily be found by solving equations (6) and (7) simultaneously.

By this means we obtain

$$\delta_e = \frac{k_b \delta_o}{k_b + k_s} \tag{8}$$

$$W_e = - \frac{k_b k_s \delta_o}{k_b + k_s} \tag{9}$$

If there is initial deflection of the strut equation (9) can be used to determine the supporting load if δ_o be taken as the initial deflection of the beam minus that of the strut when the initial deflections are in the same direction, or as the sum of the initial deflections when they are in opposite directions. In such cases (8) will give the additional deflection of the strut which should be added to or subtracted from the initial deflection to obtain the total net deflection. In any specific case it should be obvious whether the deflections should be added or subtracted.

In the derivation of equations (8) and (9) it was tacitly assumed that there was no change in length of the jury strut connecting the strut and beam. This assumption is reasonable in nearly all practical cases. If it is not made, the necessary modifications in equations (8) and (9) can be developed without special difficulty.

From figure 3 it can be seen that the curves of W_b and W_s will always intersect except in the special case where $k_b = - k_s$. The intersection represents a condition of stable equilibrium of the system, however, only when the algebraic sum of k_b and k_s is positive.

In brief, then, the criteria for the stability of the system of a strut and beam with a single tie are:

1. If the algebraic sum of the spring constants of the two members is positive the system is stable.

2. If the algebraic sum of the spring constants of the two members is negative the system is unstable.

3. If the algebraic sum of the spring constants of the two members is zero the system is elastically indifferent.

It should be remembered, however, that these criteria apply only when the load in the strut is less than the critical load, i.e., the load at which the spring constant is infinite.

In this discussion it should be noted that it has been assumed that the two members are parallel. If they are not parallel, as in the case of an airplane jury-strut system, proper corrections must be made to formulas (8) and (9) to allow for the angle between the two members. How this should be done will be illustrated later in the numerical example of an airplane jury-strut system.

VIII. SPRING CONSTANT OF A STRUT WITH A CHANGE OF SECTION

In many cases of practical importance, the cross section of the member and/or the axial load changes at the intermediate elastic support. Thus in the usual airplane jury-strut system, one component of the load in the jury strut causes the compression in the lower portion of the lift strut to be larger than that in the upper section though the difference is usually so small as to be negligible. A more important practical situation is that it may be found desirable to reduce the section of the lift strut between the jury strut and the wing spar. The spring constant for such members can be derived from the extended three-moment equation. Thus a strut having two bays of span a and b and moments of inertia I_1 and I_2, etc., with the center support deflecting δ inches, may be considered as a continuous beam and is subject to calculation by the three-moment equation. The derivation of the special three-moment equation for a beam with deflection of supports may be found in the Appendix.

The general three-moment equation for a beam having deflection of the supports, but no side load, may be written as follows:*

$$\frac{M_1\ a\ \alpha_1}{I_1} + 2M_2\left(\frac{a\ \beta_1}{I_1} + \frac{b\ \beta_2}{I_2}\right) + \frac{M_3\ b\ \alpha_2}{I_2} =$$

$$-\frac{6E}{a}(\delta_2 - \delta_1) + \frac{6E}{b}(\delta_3 - \delta_2) \qquad (10)$$

*For derivation, see Section II of the Appendix. Note that a and b are used here in place of the more usual L_1 and L_2.

The strut to be considered here will be assumed to have pin ends. Let P_1 and I_1 be the axial load and moment of inertia of bay 1-2, and P_2 and I_2 be the axial load and moment of inertia of bay 2-3. Also let $\Phi I_1 = I_2$ $j_1 = \sqrt{EI_1/P_1}$ $j_2 = \sqrt{EI_2/P_2}$ $\gamma P_1 = P_2$ $\delta_1 = \delta_3 = 0$. Since the member is pin-ended, $M_1 = M_3 = 0$. Equation (10) then reduces to

$$\delta_2 = - \frac{1}{3} \frac{M_2}{EI_1} \frac{a\ b}{L} (a\ \beta_1 + b\ \beta_2/\Phi)$$

Since $1/j^2 = P_1/EI$

$$P_1\ \delta_2 = - \frac{1}{3} \frac{M_2}{j_1^{\ 2}} \frac{a\ b}{L} (a\ \beta_1 + b\ \beta_2/\Phi) \qquad (11)$$

but

$$R_1 = \frac{M_1 - M_3}{L} + \frac{b\ W}{L} + \frac{P_1\ \delta_2\ (\gamma - 1)}{L}^{*} \qquad (12)$$

For equilibrium about the center support

$$M_2 = M_1 - P_1\ \delta_2 - a\ R_1$$

where M_1 is the applied end moment, assumed zero in this case.

$P_1\ \delta_2$ is the moment due to the eccentricity of the axial load P_1 about the center support, i.e., the center support is deflected from the line of the two end supports an amount $\delta_2 - \delta_1$. In this case $\delta_1 = 0$.

$a\ R_1$ is the moment due to the end reaction, assumed positive when the reaction acts down, times the moment arm a. Substituting from equation (12)

$$M_2 = - (P_1\ \delta_2 + \frac{a\ b}{L} W + P_1\ \delta_2\ (\gamma - 1) \frac{a}{L} \qquad (13)$$

If we let

$$\theta' = \frac{a\ b}{3\ L\ j_1^{\ 2}} (a\ \beta_1 + b\ \beta_2/\Phi) \qquad (14)$$

equation (11) becomes $P_1\ \delta_2 = - M_2\ \theta'$ (15)

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*See figure 4.

From equations (13) and (15)

$$P_1\ \delta_2 = \theta' \left(P_1\ \delta_2 + a\ b\ \frac{W}{L} + P_1\ \delta_2\ (\gamma - 1)\ \frac{a}{L}\right) \qquad (16)$$

whence

$$k = \frac{\partial W}{\partial \delta_2} = \frac{P_1\ L}{a\ b} = \frac{\left[1 - \theta' \left\{1 + (\gamma - 1)\ \frac{a}{L}\right\}\right]}{\theta'} \qquad (17)$$

Equation (17) thus gives the spring constant for the case under consideration.

IX. CHECK OF SPRING-CONSTANT FORMULAS

Formulas (17) and (3) for the spring constant of a beam or strut subjected to axial compression should become identical when the axial load and cross section are assumed constants. In this case we would have $\Phi = \gamma = 1$ since $P_1 = P_2$ and $I_1 = I_2$. Equation (17) then reduces to

$$k = \frac{P\ L}{a\ b}\ \frac{(1 - \theta)}{\theta} \qquad (18)$$

where θ is the value of θ' given by equation (14) when $\Phi = 1$. The calculations needed to check equations (18) and (3) are somewhat complicated and are omitted to conserve space, but if they are followed through the two equations are found to be identical.*

As tables of β are given in "Airplane Structures" (reference 2), it will usually be found that equation (18) is more convenient for practical use than equation (3).

A further check of equations (3) and (18) for the spring constant can be obtained by assuming $a = 0.5L$ and comparing with the results of Klemperer and Gibbons in reference 3. In this case again the resulting equations are identical.* These checks of equation (18) do not prove the validity of the more general equation (17), but the writer has been unable to devise any alternative method of proving the general case.

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*Schwartz's detailed proof of this statement is omitted from this report to conserve space. Ed.

X. INVESTIGATION OF CRITICAL LOAD CONDITIONS

The theory of the elastically supported strut is not quite complete without the development of a method for determining the magnitude of the critical load. The critical load for a strut may be defined as the smallest value of axial compression which, according to the theory of elastic members, could produce infinite bending moments in the strut. For the unsupported strut the critical load is the Euler load $\left(P = \pi^2 \frac{EI}{L^2} \quad \frac{L}{j} = \pi\right)$. The critical load of the continuous or supported strut, however, remains to be determined. This will be the maximum load under which the strut can be stable, regardless of the stiffness of the support. As the previously developed criteria of stability apply only when the axial load is less than the critical load the importance of being able to determine the latter is obvious.

For purposes of determining the critical load the supported strut can be considered as a continuous beam with deflection of the supports. As no attempt is made in this report to study struts with more than one supporting force within the span, the supported strut may be considered more specifically as a continuous beam of two spans with deflection of the intermediate support.

In Art. 11:7 of reference 2, Niles and Newell discuss the determination of the critical load of a two-span continuous beam with a uniformly distributed side load and no deflection of the supports. Their conclusions are as follows:

1. If L/j for both spans is less than π, the critical load has not been reached.

2. If L/j for both spans is greater than π, the critical load has been exceeded.

3. If L/j for one span is less than π and L/j for the other span is greater than π, the question of whether or not the critical load has been reached depends on the sign of the quantity

$$\frac{L_1}{I_1} \beta_1 + \frac{L_2}{I_2} \beta_2$$

where the subscripts 1 and 2 refer to the two spans and β is the coefficient for the extended three-moment equation as defined on page 191 of reference 2. If this quantity is negative, the critical load has not been reached, but if it is positive, the critical load has been exceeded.

These criteria apply equally to the supported strut.* From these criteria it is seen that the critical load is that at which

$$\frac{L_1 \beta_1}{I} + \frac{L_2 \beta_2}{I} = 0 \quad \text{or} \quad a \frac{\beta_1}{I_1} + b \frac{\beta_2}{I_2} = 0 \qquad (19)$$

When equation (19) is satisfied, θ' of equation (14) will also be equal to zero since $I_2 = \phi I_1$. Under these conditions equation (17) will evidently indicate that the spring constant k, is infinite. Thus the critical load is smallest axial load at which the spring constant becomes infinite.

XI. FORMULAS FOR USE IN ANALYSIS

In order to use equations (8) and (9) to determine the equilibrium position and the corresponding load in the jury strut, it is necessary to be able to compute the spring constants k, and the initial deflections δ_0, of the two members. The formulas for these quantities are the same regardless of whether the member to which they are applied is the strut or the beam. For convenience, all the formulas likely to be needed in practice are either listed in this section or references are given to places where they can be found. The nomenclature and sign conventions are those of reference 2. The most important items are as follows:

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*In his original thesis Schwartz proved this statement in detail for the case of a supported strut with a single concentrated load on each side of the deflected intermediate support. In his proof he followed the line of argument used in Art. 11:7 of reference 2, making the changes required by the difference in the type of side load and the presence of deflection of the intermediate support. In a recent article in Michigan Technic these criteria have been proved to apply regardless of the type of side load. For this reason, and to conserve space, Schwartz's detailed proof has been omitted from this report. Ed.

Upward forces and deflections are positive

L, length of member

a, distance from left end of member to jury strut

$b = L - a$

P, axial load (tension or compression as indicated)

E, modulus of elasticity

I, moment of inertia

$j = \sqrt{EI/P}$

β, function of L/j found in tables of reference 2 (page 212)

k, spring constant

δ_o, deflection of point of connection of jury strut due to all loads other than the supporting force or load

Formulas for Spring Constant

Case 1.- Member of constant axial tension load and cross section,

$$\frac{1}{k_t} = \frac{1}{P}\left(\frac{a\ b}{L} + \frac{j\ \sinh^2 (a/j)}{\tanh (L/j)} - j \sinh \frac{a}{j} \cosh \frac{a}{j}\right)^* \qquad (4)$$

Case 2.- Member of constant section with no axial load,

$$k_o = \frac{3EI}{a^2}\frac{L}{b^2} \qquad (5)$$

Case 3.- Member of constant axial compression load and cross section,

$$\frac{1}{k_c} = -\frac{1}{P}\left(\frac{a\ b}{L} + \frac{j\ \sin^2 (a/j)}{\tan (L/j)} - j \sin \frac{a}{j} \cos \frac{a}{j}\right) \qquad (3)$$

An alternative formula somewhat simpler for practical

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*For derivation, see Section III of the Appendix.

use is

$$k_c = \frac{P\ L}{a\ b} \frac{(1 - \theta)}{\theta} \tag{18}$$

where

$$\theta = \frac{a\ b}{3L\ j^2} (a\ \beta_1 + b\ \beta_2) \tag{14a}$$

β_1 and β_2 are the values of β for a/j and b/j, respectively.

Case 4.- Member with axial compressive load changing at the connection of the jury strut and of constant cross section on each side of the jury-strut connection,

$$k_c = \frac{P_1\ a}{a\ b} \frac{\left\{1 - \theta' \left[1 + (\gamma - 1) \frac{a}{L}\right]\right\}}{\theta'} \tag{17}$$

where

$$\theta' = \frac{a\ b}{3L\ {j_1}^2} \left(a\ \beta_1 + b\ \frac{\beta_2}{\Phi}\right) \tag{14}$$

$$\gamma = P_2/P_1 \qquad \Phi = I_2/I_1 \qquad j = \sqrt{EI_1/P_1}$$

Formulas for δ_0

Group 1.- Beams of constant section subjected to axial tension.

Unless great precision is desired the effect of the axial tension can be neglected and a conservative figure obtained from the formulas for the deflection of a beam without axial load. Formulas for such cases are given in Art. 15:1 of reference 2. For more precision the method of computing deflections derived in Section IV of the Appendix of this report may be used.

Group 2.- Beams of constant section without axial load.

Formulas for these cases are given in many texts and handbooks. To avoid difficulties with sign conventions, those in reference 2 are recommended.

Group 3.- Beams of constant section subjected to axial compression.

Formulas for these cases are given in Art. 11:5 of reference 2.

XII. NUMERICAL EXAMPLES

In order to render a clearer explanation of the use of the supported strut formulas, four numerical examples are worked out below. The first three examples illustrate very simple structures, but give the more important steps that should be taken in designing the members. The last example is a specific airplane jury strut system.

Example I.- Two pin-ended struts (fig. 5) are connected by a crosstie C and loaded, as shown. Member A has a large enough cross section that it is loaded below the Euler critical load. How large must the right-hand member be in order that the system will still be statically stable?

Given:	Item	Member A	Member B
	Name	Beam	Strut
	P	20 pounds	80 pounds
	L	15 inches	15 inches
	EI	1125	?

As the size of the beam is given, its spring factor may first be calculated according to equation (3) or (18). The required size for the strut is then found from the knowledge that its spring factor must be numerically equal to or less than that of the beam, otherwise the denominator of equation (8) will not be positive as is required for stability.

From (3) the spring constant of the beam can be found from

$$\frac{P}{k_c} = \frac{a\ b}{L} + \left(\frac{\sin\frac{a}{j}}{\tan\frac{L}{j}} - \cos\frac{a}{j}\right) j \sin\frac{a}{j} \qquad (3a)$$

$$j = \sqrt{\frac{EI}{P}} = \sqrt{\frac{1125}{20}} = 7.5$$

$$a = 5 \text{ inches}$$
$$b = 10 \text{ inches}$$

$$\frac{a}{j} = \frac{5}{7.5} = 0.667$$

$$\frac{b}{j} = \frac{10}{7.5} = 1.333$$

$$\frac{L}{j} = 2.000$$

Substituting for $\frac{P}{k_c}$ in (3a)

$$\frac{P}{k_c} = +1.63$$

$$k_c = + \frac{20}{1.63} = 12.3, \quad \text{spring constant of the beam}$$

Trial values of EI for member B are next substituted in (3) or (18) until a spring constant is found that is equal to or less than 12.3. This may be accomplished most advantageously by plotting several values of k_c vs. j, $(j = \sqrt{EI/P})$, and taking from the resulting curve a value of L/j which will give a spring constant in the range of the above limitations. In the above example, equation (18) was used to obtain the points of figure 6 for k_c vs. j. Table I in Section V of the Appendix shows the calculations that were made. In actual practice it is probable that only a few points would have to be calculated to obtain a suitable strut size.

Figure 6 shows that for a spring constant algebraically greater than -12.3, j must be 3.90 or more. Taking j = 3.90 as representing the smallest possible value of EI:

$$j = \sqrt{EI/P} = 3.90$$

$$P = 80 \quad L = 15$$

$$\frac{L}{j} = \frac{15}{3.90} = 3.85$$

$$EI = Pj^2 = 1220$$

L/j is well above π, the critical load for an <u>unsupported</u> strut.

It is interesting to note that efficiency of the above strut system is very high. The value of EI required to support 100 pounds by a single strut is:

$L/j = \pi$ (critical load for an unsupported strut)

$L = 15 \quad P = 100$

$j = 15/\pi = 4.77$

$EI = Pj^2 - 2280$ required for a single support

But the sum of EI of members A and B is:

EI (total) = 1220 + 1125 = <u>2345</u>

Efficiency = 2280/2345 = 97 percent

Of course, for equal values of moments of inertia, a single strut would be lighter than two struts. It must be remembered that in the above example, stability has been considered in one plane only. Member B must have either a support or a large enough section in a perpendicular plane to be truly stable in all directions. In streamline struts this would probably be the case.

<u>Example II</u> (see fig. 5).- Using the same structure as in problem I, but with a concentrated side load of 10 pounds on member A, at the connection of member C, find the smallest possible value of EI for member B. The maximum allowable side load applied at point a on member A is assumed to be 15 pounds. (In an actual structure, the maximum side load on member A would depend upon the yield point of the material or its modulus of rupture.) Also determine the lateral deflection of point a or b when the system is in static equilibrium.

Since the axial extension or compression of member C is very small as compared to the lateral deflection of the struts, it will be neglected. Having an allowable force of 15 pounds at point a and an external side load of 10 pounds, we see that the maximum supporting load W_e in equation (9) is 5 pounds. The spring constant of a member A has already been calculated in example I: $k_c = 12.3$. Only two unknowns remain: the required spring factor of the strut and the "initial deflection" of member A due to external loads. The initial deflection δ_o, may be found from the given data and the formulas for the deflection of axially loaded beams.

As the only external load in this case happens to be the one applied at the supporting point a, δ_o can be computed from the spring constant found in example I. $k_c = 12.3$, i.e., it requires 12.3 pounds of load at a to deflect 1.00 inch. The deflection of point a due to 10 pounds will therefore be $\delta_o = \frac{10}{12.3} = 0.814$ inch. Substituting the spring constant and initial deflection of A and the maximum allowable supporting force W in equation (9):

$$5.00 = - \frac{12.3 \times 0.814 \times k_s}{12.3 - k_s}$$

whence $k_s = - 4.10$

This indicates that the beam A can supply the lateral support needed to prevent the strut B from buckling provided the latter is of such size that its spring constant is algebraically greater than - 4.10. (If the value of k_s for the strut taken from curve I is positive, it indicates that the strut requires no supporting force, but if it is negative and larger in magnitude than - 4.10, it will require a larger supporting force than the beam A can provide without failure of its material.)

Figure 6 shows that the value of j for member B is 4.50 when $k_s = - 4.10$.

Since $j = \sqrt{EI/P}$ and $P = 80$

$$EI = Pj^2 = 80 \times 4.50^2 = 1620$$

The deflection of points a and b may be found from the known values of k_s for member B and the supporting force W. (See equation (6) and fig. 3.)

Thus $W = 5$ pounds

$$k_s = - 4.10$$

$$\delta_e = \frac{W}{k_s} = \frac{5}{4.10} = + 1.22 \text{ inches}$$

This deflection might also be found by applying the proper deflection formula to member A, as all the loads on it are known (both external and supporting load). In the above example, both external and supporting loads are applied at the same point and equation (1) may be used.

Total side load = 10 pounds + 5 pounds = 15 pounds

$$k_c = 12.3$$

$$\delta = \frac{15}{12.3} + 0 = 1.22 \text{ inches}$$

which checks with the above calculation for δ.

It should be noticed that the supporting load will always be in the same direction as the external side load on the beam. This should be fairly obvious if it is remembered that the system deflects in the direction of the external side load with a consequent tendency for the strut to buckle in the same direction. Due to this fact the supporting load W, and the initial deflection δ_o, are always the same sign.

Example III.- The 100-pound weight of problem I is moved to the right of member B (fig. 7). Member A has the same moment of inertia as before. Find the smallest size (value of EI) of member B which will allow the system to be elastically stable.*

Given: Item	Member A	Member C
P	20-pound tension	120-pound compression
L	15 inches	15 inches
EI	1125	?

This example is, of course, the same type as number I except that the supporting member is in tension. The stiffness of member A will accordingly be found from equation (4). The value of j for the strut which will give an equal required stiffness will then be found from figure 6. Thus, from (4):

$$\frac{1}{k_t} = \frac{1}{P}\left\{\frac{a\ b}{L} + \frac{j \sinh^2 (a/j)}{\tanh (L/j)} - j \sinh\left(\frac{a}{j}\right) \cosh \frac{a}{j}\right\}$$

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*It should be remembered that a strut having a required stiffness just equal to the available support stiffness is really not elastically stable but elastically indifferent. Such a condition is exactly the same as the case of an unsupported strut loaded with the critical Euler load. In actual design, struts are usually designed for the critical Euler load when a suitable load factor or safety factor has been used to obtain the design loading.

$a = 5$, $b = 10$, $j = \sqrt{EI/P} = \sqrt{1125/20} = 7.5$ $L/j = 2.00$

$a/j = 0.667$ $b/j = 1.333$ $\sinh(a/j) = 0.7172$ $\cosh(a/j) =$

1.2306 $\tanh(L/j) = 0.9640$

whence $k_t = 27.9$, the spring constant for point a of member A when that member is subjected to 20 pounds tension.

From figure 6, the value of j for member B is 3.04 when $k_c = -27.9$.

$$EI = j^2 P = 3.04 \times 120 = 1110$$

It is interesting to note that the strut system of problem III requires a smaller value of EI for member B than in example I, even though the axial compression on B is 50 percent greater. This is due, of course, to the greater support given by A when it is in tension.

An Airplane Jury Strut System

Example IV.- The structure consisting of the spar, lift strut, and jury strut is shown in figure 8 loaded for the inverted flight condition.

Given data:

Item	Spar	Lift strut
M_1	-62,200 (due to overhang)	0
M_3	0 (pin joint)	0
w	-12.92	0
L	168 inches	177 inches
a	56 inches	59 inches
b = (L-a)	112 inches	118 inches
I	152	?
P	+7830 pounds	-8250 pounds
E	1.3×10^6 (spruce)	30×10^6 (steel)
$j = \sqrt{EI/P}$	159	?

Given data (continued):

Item	Spar	Lift strut
L/j	1.057	?
a/j	.3521	
sinh a/j	.3593	
cosh a/j	1.0626	
sinh L/j	1.2652	
cosh L/j	1.6126	
tanh L/j	.7845	

M_1 is the end moment due to the cantilever overhang. The axial loads are computed in the ordinary manner, i.e., using the given data for side loading.

A strut size must be selected which, first, will allow the system to remain elastically stable, and second, which will not cause a supporting load on the wing spar that is large enough to stress it beyond the modulus of rupture.

The procedure is much the same as in the previous examples, but a slight approximation and a correction are necessary because of the angularity between the jury strut and the lift strut.

It is quite obvious that as the force in the jury strut of figure 8 is not normal to the lift strut, there is an axial component imposed on the lower bay of the strut. As the supporting force will be quite small in proportion to the total axial load in the lift strut, the above axial component may be neglected with no very great error.

As noted above, a correction must also be made to compensate for the angularity of deflection and supporting force of the wing spar to the lift strut; this correction will be made on the spring constant of the strut. An exaggerated view of the system when deflected is given in figure 9. In this figure ab and a'b' represent the jury strut in the undeflected and deflected positions, respectively. Points a and b are assumed to deflect in directions normal to the members when they are in the undeflected positions. If aa' is assumed to equal bb'', the deflection of the lift strut (bb') equals the deflection of the spar divided by $\cos\alpha$ ($aa'/\cos\alpha$). The above assumption is made on the basis that a'b' equals a'b'', which is obviously untrue. However, the error involved is very small for the small deflections allowable in ordinary structures. Thus, if (ab) = 18 inches, bb' = 2 inches, and $\sin\alpha = 1/3$, $b'b'' = 2\sin\alpha = 0.67$ inches. Thus, $a'b' - a'b'' = 0.67/(2\times18) = 0.0125$ inch which, in comparison with 2 inches, may be neglected. As the deflection bb' would be much less than 2 inches in practice, the assumption that a'b' = a'b'' is justified.

It is also apparent (fig. 9) that the supporting force normal to the lift strut (W_s) opposes a force equal to the load on the jury strut (W_b) times $\cos\alpha$, that is, W_s (strut) = $-W_b$ (spar) $\times \cos\alpha$. Thus, from the above,

$$\frac{\delta_b}{\cos\alpha}\,(\text{spar}) = \delta_s\,(\text{strut}) \tag{20}$$

and

$$W_b\,(\text{spar}) = -\frac{W_s}{\cos\alpha}\,(\text{strut}) \tag{21}$$

and dividing (21) by (20)

$$\left(\frac{W}{\delta}\right)_b (\text{spar}) = \frac{1}{(\cos\alpha)^2} \times \left(\frac{W}{\delta}\right)_s (\text{strut})$$

or

$$k_b\,(\text{spar}) = k_s\,(\text{strut})/\cos^2\alpha \tag{22}$$

Having the necessary corrections for the special case in which the strut and supporting beam are not parallel, we may now proceed with the calculations in the conventional manner, using equation (9).

The initial deflection of the spar δ_o, can be computed by the method outlined in Section IV of the Appendix.

$$\delta = -\frac{1}{P}\,(M_o - M) \tag{23}$$

Since we have uniformly distributed side load, when $x = a$

$$M_o = M_1 + (M_2 - M_1)\frac{a}{L} - \frac{w\,a\,L}{2} + \frac{w\,a^2}{2}$$

$$M = \frac{M_2 - (M_1 + wj^2)\cosh(L/j) + wj^2}{\sinh(L/j)}\sinh\frac{a}{j}$$

$$+ (M_1 + wj^2)\cosh\frac{a}{j} - wj^2$$

Substituting the given data in the above expressions, we obtain:

$$M = -1199$$

$$M_o = -949$$

$$M_o - M = +250$$

$$\delta = \frac{-250}{P} = \frac{-250}{7830} = -0.0320 \text{ inch}$$

The spring constant of the spar is obtained from equation (4)

$$\frac{1}{k_t} = \frac{1}{P}\left(\frac{a\ b}{L} + \frac{j\sinh^2(a/j)}{\tanh(L/j)} - j\sinh\frac{a}{j}\cosh\frac{a}{j}\right)$$

Substituting the given data gives $k_t = +2800$

In order to facilitate the selection of the best size of lift strut for this structure, a curve (fig. 10) of k_s vs. j was drawn as in example I. The calculations for figure 10 are shown in table II of Section V of the Appendix. This curve shows that the spring constant of the strut changes very rapidly from the critical point to values of j in the neighborhood of 33. This indicates that the best strut size is one having a value of j near the sharp break in the curve (about j equals 33). It is quite apparent that reductions of strut size below this region ($j = 33$) are accompanied by a very high rate of change of k_s which, of course, causes the supporting force W of equation (9) to increase very rapidly. On the other hand, if a larger size strut is chosen (with a larger value of j), very little is gained in reduction of supporting force W because of the small rate of change of spring constant above $j = 33$.

Accordingly, the value of k_s in figure 10 at $j = 33$ is found to be -380. Applying the correction derived above in equation (2): Since

$$\cos \alpha = \frac{56}{59} = 0.95$$

corrected $$k_s = \frac{-380}{(0.95)^2} = -421$$

Calculation of W_e from equation (9)

$$W_e = -0.0320 \times \frac{2800 \times 421}{2800 - 421}$$

where
$$-0.0320 = \delta_0$$
$$2800 = k_b$$
$$-421 = k_s$$

whence

$W = -15.9$ pounds supporting load on spar (down).

Since the value of j for the strut is 33

$$j = \sqrt{EI/P} \qquad I = j^2 \frac{P}{E}$$

$$I = \frac{(33)^2 \times 8250}{30 \times 10^6} = 0.2995$$

For comparison, a strut having a value of $j = 32$ will be tried. From figure 10, $k_s = -520$ at $j = 32$. Corrected

$$k_s = \frac{-520}{(0.95)^2} = -576$$

$$W = -0.0320 \times \frac{2800 \times 576}{2800 - 576} = -23.2 \text{ pounds}$$

$$I = j^2P/E = \frac{(32)^2 \times 8250}{30 \times 10^6} = 0.282$$

Thus we see that a 6-percent reduction of I causes a 34-percent increase in W.

It is interesting to make a comparison of the size of tube needed when the strut is supported and unsupported. The size of an unsupported tube is:

$$L/j = \pi \quad \text{for unsupported strut}$$

and $L = 177$ inches

or $J = 177/\pi = 56.3$

$$I = \frac{56.3^2 \times 8250}{30 \times 10^6} = 0.872$$

The nearest commercial size of the above supported tube having a value of $I = 0.282$ would be 2-1/4 by 0.083 inches and for a length of 177 inches would weigh 28.3 pounds. The size of the unsupported tube having an I of 0.872 would be 3-1/4 by 0.120 inches and would weigh 59 pounds, or twice as much as the other. It is quite evident that the reduction in weight due to the jury-strut system is quite worth while. Of course, the weight of the jury strut should be added, in the case of the supported strut, but as the above calculations show, this member carries such a small load that it will be a relatively small tube.

In order to select the best point to connect the jury strut, some idea must be had of the deflection curve of the spar due to external loads, for it is obvious that if the jury strut happens to be connected to the point of zero deflection, the supporting force is zero. It has been mentioned before, however, that this could only be true for one particular loading as the point of zero deflection moves as a function of the axial load in the spar. Nevertheless, this movement is relatively small and a connection in the vicinity of the zero deflection point is the most logical place to make a joint. Accordingly, a deflection curve of the spar due to the external loads in example IV has been prepared (fig. 11). This curve shows that the jury strut in the above example was placed close to the point of zero deflection. Due to the lack of knowledge at the present time of the actual wing loading which occurs, it is possible that the external deflection at the jury-strut connection is much greater than the assumed loading predicts. Some calculation should accordingly be made to determine just how serious and how large the supporting force might be under some extreme condition. Formula (9) shows, however, that no extremely rapid change may take place as W changes only in direct proportion to the initial deflection δ_0. For an actual numerical calculation, suppose the deflection at the jury-strut connection due to some very unusual loading should equal the maximum deflection shown by the curve. This appears to be as unusual a condition as might be encountered because, under ordinary loads, the deflection of the above point is very small.

From the curve, we see that the maximum external deflection is about 0.15 inch. Taking the same values used above in the calculation of W for a strut having a value of j = 32:

$$W = -\ 0.15 \times \frac{2800 \times 576}{2800 - 576} = -\ 109 \text{ pounds}$$

Thus, for an extreme case, the supporting load is only 109 pounds, which does not appear to be excessive. A check should be made, of course, of the bending moments occurring due to the 109 pounds concentrated load added to the external loading.

The above calculations deal with support in one plane only. In most designs of today, the lift strut has a streamline section so that no support is necessary in the wind direction. However, if a round tube is to be used, some means of side support must be made in another plane besides that one calculated in example IV. This is probably most easily accomplished as shown in figure 12. As a and b would both have approximately the same deflection under a given loading condition, point c will have very little horizontal deflection. The above assumptions indicate that (ca) will carry very little load. If the designer feels that a more precise calculation should be made, the method used in example IV may be used if some corrections are added to take care of the angularity of the members of figure 12.

PART II

EXPERIMENTAL INVESTIGATION OF FORMULAS

By Reid Bogert

I. APPARATUS AND TESTS

The tests conducted to check the validity of the formulas derived in the first part of this paper were carried out on an Olsen 20,000-pound, hand-operated, testing machine located in the Materials Laboratory at Stanford University. The apparatus required for the tests is shown in the photographs of figures 13, 14, and 15, and in the detailed drawings of figures 16a to 16e and, except where otherwise noted, was constructed from cold-rolled steel-bar stock. The principal parts of this apparatus are (I) an upper loading bar, (II) a lower loading bar, (III) a tie rod between the midspan points of the beam and strut, (IV) a pulley system for applying side load, and (V) a screw micrometer for measuring deflections.

The upper loading bar had five 120-degree notches milled in the top surface to take a 90-degree hardened steel knife-edge mounted on the head of the testing machine. On the bottom surface of the upper loading bar and the top surface of the lower loading bar there were corresponding pairs of 90-degree notches, five inches apart, to take test members in compression. At corresponding ends of the upper and lower loading bars a slot was milled to take fittings for the tension test members (figs. 16a,b,e).

The tie rod is shown in detail in figure 16c. This rod was required to prevent relative lateral movement of the midspan points of the beam and strut and not to interfere with the bending in the beam and strut. It was constructed of two side pieces separated by four blocks, longitudinally adjustable to take different sized test members, and held in position by four machine bolts passing through slots in the side pieces. The whole assembly, supported by rollers resting on a standard fitted into the lower loading bar, was free to move laterally.

Horizontal side load was applied to the midspan point of the test beam through a length of piano wire, attached to the tie rod and passing over a ball-bearing pulley, to which was fastened a weight pan-loaded with shot bags. The pulley was supported on an arm extending from the lower loading bar.

The screw micrometer was a 1/4-inch steel screw threaded through a micarta block. It was mounted on a bar pivoted at the lower loading bar and slotted at the upper so that vertical movement of the upper loading bar was unimpeded (fig. 16d). The scale of the micrometer was calibrated to 0.025 inch and a dial to 0.001 inch. Contact of the micrometer screw with a bolthead mounted on the tie rod closed an electrical circuit containing a small flashlight bulb and battery.

Test members were made up from solid, cold-rolled steel-bar stock.

Compression members were 1/2 inch in width and 1/4, 3/8, or 1/2 inch in thickness. All were 20 inches long and were ground to 60-degree knife-edge ends.

The tension member was 1 inch in width and 1/4 inch in thickness. Special end fittings were required (fig. 16e) and the length between supports was 22 inches.

Values of EI in lb. in.^{2} were determined for all test members from bending tests. The members were simply supported on knife-edges and deflection measurements made for side load applied at midspan. The experimentally determined values of EI were as follows.*

Test Members

No.	Size	Type	EI lb.in.^{2}
1	1/2 by 1/4 inch	compression	19,000
2	1/2 by 1/4 inch	"	19,000
3	1/2 by 3/8 inch	"	67,000
4	1/2 by 3/8 inch	"	67,000
5	1/2 by 1/2 inch	"	156.000
6	1 by 1/4 inch	tension	39,500

*The bending test data and calculations for EI were given in Bogert's thesis, but are omitted from this report to conserve space. Ed.

The set-up for tests with the beam in compression is shown in the photograph of figure 15. The test members were placed at A and B (figs. 16a, 16b) and the external load at various positions between A and B.

The set-up for tests with the beam in tension is shown in the photograph of figure 14. The strut was placed at B and the beam at C (figs. 16a, 16b). The external load was placed at various positions outside of B.

Deflection measurements were made at the midspan points on each of two sizes of compression members tested as struts when supported at the midspan point by each of three sizes of compression members and the tension member as beams. For each combination of test members, the position of the external load on the loading bar was varied, thereby varying the proportions of loads in the beam and strut, and for each combination of members and position of external load, the value of the side load was varied.

The test program was as follows: (Positions of members and load as shown in fig. 16a.)

Schedule of Tests

Test	Member at A	Member at B	Member at C	Total load acting at	Side loads in pounds
A - 1	#5	#3		4	0 - 15 - 30
- 2	5	3		3	0 - 15 - 30
- 3	5	3		2	0 - 15 - 30
B - 1	5	1		4	0 - 10 - 30
- 2	5	1		3	0 - 10 - 30
- 3	5	1		2	0 - 10 - 30
C - 1	4	3		4	0 - 10 - 20
- 2	4	3		3	0 - 10 - 20

Schedule of Tests (Cont.)

Test	Member at A	Member at B	Member at C	Total load acting at	Side loads in pounds
D - 1	3	1		4	0 - 5 - 15
- 2	3	1		3	0 - 5 - 15
- 3	3	1		2	0 - 5 - 15
E - 1	2	1		4	0 - 5 - 10
- 2	2	1		3	0 - 5 - 10
F - 1		4	6	5	0 - 15 - 30
- 2		4	6	3	0 - 15 - 30
- 3		4	6	1	0 - 15 - 30
G - 1		1	6	5	0 - 10 - 15
- 2		1	6	3	0 - 10 - 15
- 3		1	6	1	0 - 10 - 15

In making the tests, the test members were first set up in the apparatus, in a vertical position, and just sufficient total load applied to hold them in place. The tie-rod blocks were adjusted so that the loading edges were in contact with the test members, and the clamping nuts tightened. Total load was then increased and the deflection of the strut with no side load measured. Such deflections were due to initial eccentricity in the test members and, as it was desirable to eliminate the effect of this eccentricity as much as possible, readjustments of the tie-rod blocks were made until the deflections obtained with no side load were the minimum possible with the test apparatus. Data for deflection and total load were then recorded for the condition of no side load and increments of the total load from the minimum to a maximum just under the failure load. Similar runs were made for the beam subjected to constant side loads. In every case initial and final deflection readings were taken for the minimum total load with and without the side load.

II. DISCUSSION

Scope of Tests

The formulas of Part I were developed for the analysis of a structure consisting of a pin-ended strut supported at any point, through a tie rod perpendicular to the axis of the strut, by a parallel beam axially loaded and subjected to side loads. Application of the formulas to an airplane jury-strut system, in which the strut and beam are not parallel, required the use of correction factors for the relative angularity of the members. In these tests, however, a set-up was used similar to the conditions for which the basic formulas were derived. Due to limitations in time, it was possible to investigate but one position for the tie rod and one type of side load on the beam. The position chosen for the tests was the midspan point on the strut and beam, for at this point it is obvious that the deflections obtained would be greatest, and the relative effect on the deflections of inaccuracies in the set-up would be least. The spring constants of the beam and strut, however, depend upon their geometrical dimensions and the type and value of the axial loads. The spring constant, then, for a given strut, varies with a change in position of the tie rod when the axial load is held constant, and varies with the axial load when the position of the tie rod is held constant. In the tests, the effect of a change in the value of the spring constant was investigated by varying the axial load. The agreement between the theoretical and experimental results, however, indicated that the change in spring constant due to axial load was correctly accounted for by the formulas. It is reasonable to conclude, therefore, that the formulas would also be correct for a variation in spring constant due to change in the position of the tie rod.

The lateral loading applied to the beam in the tests consisted of concentrated loads applied at the midspan, or supporting point, on the beam. The formulas, however, show that stability of the strut and beam system is unaffected by the side load, although the position of equilibrium of the system is dependent upon the deflection of the beam at the supporting point due to the side loads. In each of the test runs the side load was kept constant. Since the deflection of the beam for a constant side load is a function of the axial load, the variation in the initial deflection of the beam covered a wide range. If the stability of the system were dependent upon some function of the initial deflection, the effect would then be apparent in a comparison of the theoretical and experimental results. As no such general effect is evident from the curves of figures 17 to 26, it can be concluded that the stability of the system is independent of the initial deflection and therefore of the type of side load. It seems, therefore, that the present tests are sufficient proof of the validity of the formulas for any conditions.

Apparatus

As the apparatus required for the making of these tests was, of necessity, somewhat complicated, some comment on the difficulties involved in its development and operation seems advisable.

Axial load was applied simultaneously to the two test members through an upper and lower loading bar in which pin-end conditions for the test members were obtained by using knife-edge loading points. The apparatus and test members were constructed from cold-rolled steel and no attempt was made to harden the knife-edges. Only on test member 4, which was subjected to the greatest loads, however, was any mutilation of the knife-edges evident. In preliminary tests with beams in tension, the tension member was loaded through circular pins. The deflections obtained, however, were considerably smaller than the results of theoretical calculations indicated. It was thought that this might be due to friction in the loading pins, so the pins were ground down to provide a simple knife-edge support (fig. 16e). Although this did not completely eliminate the presence of friction, the agreement between the theoretical and experimental deflections was greatly improved. The effect of the alteration is shown in figure 25.

Some difficulty was encountered in satisfactorily adjusting the tie rod to reduce the effect of initial bow and knife-edge eccentricities, especially in the larger test members. This difficulty was due primarily to the design of the tie rod, the adjustment of the loading blocks of which was made by sliding clamping bolts in slots. A suggested improvement, but one which limitations of time made impossible to take advantage of for these tests, would be an arrangement for adjusting the blocks' longitudinally by means of screws.

In a preliminary study of the apparatus required for the tests, it was thought that the deflections could be measured on a calibrated dial scale, the pointer of which was fastened to a small pin rotated by the movement of a wire attached to the tie rod, wrapped once around the pin, and loaded with the side load weights. For the desired magnification of the deflections, however, it was necessary to use a pin of such small radius that wire was not sufficiently flexible to wrap around it. A strong cord was used, but due to friction in the pin bearings and elasticity of the cord, the method was entirely unsatisfactory. The apparatus was then altered and side load applied directly through a wire, passing over a large-diameter pulley carried on ball bearings, as shown in figures 14 and 15. Deflection was measured by means of the screw micrometer, used in the final tests, but mounted on a bar fastened only to the lower loading bar. It was found, however, that due to uneven action of the loading screws in the testing machine used, one side of the head of the machine was pulled down before the other, thereby giving it a slight lateral movement. As this motion was transmitted to the upper ends of the test members the deflection readings were affected. Several other testing machines were tried but the same motion of the head was present in a greater or less degree in all. The deleterious effect of this movement on the deflection readings was finally eliminated by mounting the micrometer on a bar fastened by single bolts to both loading bars. In order that the supporting bar should take no vertical load from the upper loading bar, the slot, shown in figure 16d was provided in the upper end of the supporting bar. This method provided a parallelogram motion and maintained a constant distance between the micrometer support and the normal unloaded position of the strut.

Precision of Dimensional Quantities and Measurements

The dimensional quantities and measurements for the apparatus and tests were made as accurate as was practicable with the set-up.

Dimensions of the test apparatus and test members were made to an allowable variation of ±0.025 inch. The maximum possible error in the value of the axial load in the beam or strut would then be about 2 percent and a maximum error in the initial deflection of the beam due to the length of the strut would be about 1/3 of 1 percent. A valid quantitative estimate of the probable error due to inconsistencies of the material, initial bend, or slight eccentricity of the knife-edge ends of the test members is impossible.

The value of EI in $lb.in.^2$ used for each of the test members in the theoretical calculations, is the average of a number of experimentally determined values as found from bending tests. The maximum variation of EI determined from these tests was about 3 percent. This would indicate a possible error in the initial deflection of the beam of about 5 percent.

Side load in the tests was applied by weights, in the form of canvas bags filled with shot, placed in a light sling. The sling weighed approximately one quarter pound but this weight was neglected in the computations under the assumption that it would be balanced by friction in the pulley assembly and in the rollers supporting the tie rod. An error in this assumption would affect the deflections directly in proportion to the ratio of the error to the side load. The canvas bags were 5, 10, and 25 pounds in weight and their values were checked on a balance scale to within an ounce.

Values of the total load were read from the balance arm of the testing machine to the nearer five pounds, and corrected for a tare weight of 80 pounds. Small errors in this quantity would have a negligible effect when plotted to the scales used for the curves. Readings of deflection were made to 0.001 inch and estimated to 0.0001 inch. It was assumed that movement of the head of the testing machine was small enough to neglect the effect of rotation of the axis of the screw micrometer and that play in the pins holding the micrometer support was negligible.

The deflection values used in plotting the experimental results in figures 17 to 26 are the differences between the deflection readings for the corresponding total load and side load, and the deflection reading for the minimum total load and no side load. The validity of these deflection measurements therefore depends upon the assumption of a negligible deflection of the system for the condition of the minimum total load and no side load. An examination of the curves of deflection for no side load in figures 17 to 26 will show that the rate of increase of deflection is small for low values of the total load. The assumption is therefore justified.

Experimental Curves

The experimental and theoretical data are shown in figures 17 to 26 in the form of curves of deflection as a function of total load. The agreement, in general, between the curves is sufficiently within the possible experimental error to justify the validity of the formulas.

In the theoretical formulas it is assumed that ideal conditions of loading and material of the strut and beam are obtained. In the actual case this is impossible of realization, as there is always some slight heterogeneity of the material or small eccentricity in loading which will affect the action of the strut or beam. In the present tests it was found impossible to eliminate the deflections of the strut under the condition of no side load. These deflections, however, were reduced as much as possible and were in the direction in which the side loads were applied for all the tests. From a consideration of the curves it can be seen that, in general, initial differences in the experimental and theoretical curves are increased as the load increases until a value of the load near the maximum is reached. At loads approaching the maximum the curves tend to more nearly agree. The major exception to this is in test G, in which the beam is in tension. Friction in the loading pin is undoubtedly the cause of the high maximum loads, in comparison to the ideal condition, obtained.

Attention is called to the fact that the scales of deflection for all the tests are similar although the load scales vary for different combinations of test members.

Practical Application of Results

A study of the experimental curves of figures 17 to 26 inclusive shows that, in general, the effect of initial eccentricities becomes less important as the L/ρ ratio of the strut and the initial deflection due to side load on the beam increase. Bending moments on the strut due to small eccentricities, however, are relatively unimportant when compared with the moments induced by the deflection of the system. For minimum bending moments the deflection of the supported point due to side loads on the beam should be as small as possible. The jury strut, in an airplane jury-strut system, therefore, should be connected to the wing spar at or near the point of zero deflection for the design load. This point will usually be near the point of inflection. The point of zero deflection, however, will move as the load on the spar is reduced and the initial deflection at the supporting point will then be increased. This increase in initial deflection, in spite of a reduction of the external loads may cause the critical bending moments on the strut to be those for a loading less than the design load. The designer should take this possibility into account unless further study of the general problem should prove this to be unnecessary.

A curve has been included in figure 25 to show the effect of friction in pin bearings on the deflection characteristics of a jury-strut system. Friction is evidently desirable from a consideration of structure rigidity, since it materially reduces the deflections by inducing restraining end moments in the strut and spar. Such restraining moments in pin bearings, however, are small, and should be neglected in practical design.

In test B3, in which were tested the smaller strut and the largest beam, bending occurred in the unsupported semispans of the strut before the midspan point of the strut had reached a maximum deflection. The strut assumed an S shape rather than the usual simple bow. This bending in the individual spans was first noticeable at an axial load in the strut of about 1,750 pounds, which gives, for the whole strut, a value of $L/j = 6.0$ and, for the semispans a value of $L/j = 3.0$. In no other test was there any apparent S bending of this type in the unsupported spans. The maximum load, however, on the smaller strut (No. 1) was obtained in test G1 for a total load of 910 pounds. In this test the load in the strut was 1,910 pounds and L/j of the semispan was 3.17. The maximum load on the larger strut (No. 4) was 3,820 pounds in test F1 corresponding to a value of L/j of 2.45 for the semispan. It will be noticed that, while bending in the unsupported spans occurred in test B3 at a value of $L/j = 3.0$, there was no noticeable bending in the unsupported spans in test G1 for a value of $L/j = 3.17$, or slightly more than π. The curves show, however, that, in test G1, the maximum load was obtained at a deflection approximately five times that of test B3. Although the results of these tests are not, in any way, conclusive evidence on this point, they would indicate that the restraint coefficient c, for the unsupported spans, increases as the deflection at the point of support of the strut increases. It may be remarked also that the midspan point of support, used in these tests, would be expected to be least effective in increasing the restraint coefficients for the individual spans. If the strut were supported at the midspan point by a rigid support, the individual spans of the strut could act as simple pin-ended struts, the whole strut bending in an S shape. If the point of support were shifted to either side of the midspan point, however, the shorter unsupported span would provide an end restraint to the longer span. As the airplane jury strut is usually located between the third and mid points of the strut, an investigation of the effective restraint coefficients when an elastic support is used at locations other than midspan would be highly desirable.

An interesting observation, from the results of these tests, is that for the type of set-up used, and if the beam and strut are the same length, the maximum load carried by the combination is approximately the same regardless of the proportion of load in the beam snd strut. The maximum total load is reduced only slightly as the proportion of load in the strut is increased.

III. RESULTS

The results of the tests described in this paper are shown graphically in the curves of figures 17 to 26, inclusive.* Theoretical curves, calculated from the formulas for the corresponding test conditions, are included as a basis for comparison. The values of total load for the experimental curves have been corrected for a tare weight of 80 pounds, and the deflections given are equal to the difference between the deflection reading for the corresponding total load and side load and the deflection reading for the minimum total load and no side load.

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*These figures cover only about half of the tests made, but they include those in which the divergences between the experimental and theoretical curves are a maximum as well as those in which they are a minimum. As these ten sets of curves are a fair sample, including both the best and the poorest experimental results, the other nine sets have been omitted to conserve space. Ed.

A P P E N D I X

SECTION I

Nomenclature

The derivations of formulas in the following sections are carried out parallel to the derivations of the Newell equations in Chapter XI of reference 2. Formula numbers preceded by the letter N denote references to the equations in that book. The nomenclature used is also, so far as possible, the same as that employed by Newell. The more common symbols and their meanings are as follows:

w, intensity of distributed lateral load in lb. per in., positive when acting upward.

W, magnitude of concentrated lateral load in lb., positive when acting upward.

M, bending moment in in. lb., positive when it tends to cause compression in the upper fibers of the beam.

i, slope of elastic curve of the beam in radians, positive when the tangent rises from left to right.

δ, deflection in in., positive when the deflected position of a point is above the original position.

P, axial load, lb.

E, modulus of elasticity of the material, lb. per sq. in.

I, moment of inertia of the section in $in.^4$

L, length of span between supports in inches.

x, distance to a section from the left end of the span in which it is located, in inches.

a, distance from the left end of the span to the point of application of a concentrated side load.

j, $\sqrt{EI/P}$.

SECTION II

Derivation of the Extended Three-Moment Equation with Deflection of Supports

The first step is the derivation of formulas for a single span beam having a uniformly distributed side load, axial compression, and deflected supports. It must be noted that the axial load P is not in line with the support points but is always parallel to the base line from which δ is measured. (See fig. 27.)

Taking moments about support (2), we have for R_1

$$R_1 = \frac{wL_1^2}{2L_1} - \frac{M_2 - M_1}{L_1} + \frac{P\,(\delta_1 - \delta_2)}{L_1} \qquad \text{(A.1)}$$

and the moment at any point is

$$M = M_1 + \frac{wx^2}{2} - \left(\frac{wL_1}{2} - \frac{M_2 - M_1}{L_1} + \frac{P(\delta_1 - \delta_2)}{L_1}\right) x - P\,(y - \delta_1) \qquad \text{(A.2)}$$

This expression is the same as that obtained in reference 2 (p. 188), equation (N 11:1) except for the addition of the deflection terms.

On differentiating twice with respect to x, the deflection terms vanish and the differential equation obtained is the same as that given on page 188 of reference 2 for the case without deflection of supports. As the boundary conditions are the same for the two cases, the expressions for M, equation (N 11:2), including even the constants of integration, are also identical for the two cases. The same would be true for any other condition of side load, so we find the interesting fact that the expression for moment in a span of continuous beam in terms of the end moments and side loads on the span is independent of any deflection of the supports. This does not mean that the moment in the span is unaffected by such deflections, since the end moments are definitely influenced by it as will be seen below, but only that the formula for intermediate moments in terms of end moments is unchanged.

The effect of support deflection reappears when we obtain an expression for deflection at any point in the span by substituting the value of M from equation (N 11:2) in equation (A.2) and solving for y. This gives an expression identical to (N 11:6) except for the addition of the deflection terms

$$\frac{1}{P} + P\,\delta_1 - P\,(\delta_1 - \delta_2)\,\frac{x}{L}$$

The slope of the tangent to the elastic curve at any point is obtained by differentiating the deflection equation. This gives an equation identical to (N 11:7) except for the addition of the deflection term

$$-\frac{1}{P}\left(\frac{P\,(\delta_1 - \delta_2)}{L}\right)$$

Three-Moment Equation

The three-moment equation is obtained in exactly the same way as described in article 11:3 of reference 2 on page 190. The final equation is exactly the same as equation (N 11:11) except for the addition of the deflection terms.

$$+\frac{6\,(\delta_1 - \delta_2)\,E}{L_1} + \frac{6\,(\delta_3 - \delta_2)\,E}{L_2}$$

It should be noted that after M_2 has been obtained for a specific case by means of the three-moment equation, a deflection term must be added in the calculation of the reactions of the beam as indicated in equation (A.1). Thus, the deflection term $P\,(\delta_1 - \delta_2)/L_1$ must be included in the expression for R_1 and $-P\,(\delta_1 - \delta_2)\,L_1$ in the expression for S_{-2}.

SECTION III

Derivation of Spring Constant for a Beam

Subjected to Axial Tension

In order to determine the spring constant for a point on a beam subjected to axial tension, it is necessary to develop the formulas for the deflection of a beam of this type due to a single concentrated side load W. Such a beam and the forces acting on it are shown in figure 28.

The bending moment at any point in the span will be

$$M = M_0 + P\ y \tag{A.3}$$

where M_0 is the bending moment due to the effect of the end moments, M_1 and M_2, and the transverse load W, acting alone, while $P\ y$ is the secondary moment due to the axial load and the deflection. In this case, with the only side load a concentrated one, the variable x does not appear in M_0 to any power but the first. On differentiating twice with respect to x, therefore, the term M_0 disappears and the expression for moment becomes

$$\frac{d^2M}{dx^2} - \frac{M}{j^2} = 0$$

The solutions of this differential equation are

$$M_1 = C_1 \sinh \frac{x}{j} + C_2 \cosh \frac{x}{j} \quad \text{when } x = a \tag{A.4}$$

$$M_2 = C_3 \sinh \frac{x}{j} + C_4 \cosh \frac{x}{j} \quad \text{when } x = a \tag{A.5}$$

Although the general form of the equation for moment is the same for the two sections into which the beam is divided by the supporting load, the presence of that load makes it necessary to use two equations with separate constants of integration for the two sections of the beam.

Three of the constants of integration can be evaluated from the boundary conditions that

when $x = 0,\ M = M_1$

$x = L,\ M = M_2$

$x = a,\ M_a$ is the same, regardless of which equation is used.

For the fourth constant of integration, the simplest method of evaluation is to differentiate equations (A.4) and (A.5) with respect to x, thus obtaining expressions for the shear on sections normal to the elastic curve of the beam. When $x = a$, these shear expressions should give values which differ by the amount of the concentrated load W.

Proceeding along these lines, we obtain

$$C_1 = \frac{M_2}{\sinh \frac{L}{j}} - \frac{M_1 - W\, j \sinh \frac{a}{j}}{\tanh \frac{L}{j}} - W\, j \cosh \frac{a}{j}$$

$$C_2 = M_1$$

$$C_3 = \frac{M_2 - (M_1 - W\, j \sinh \frac{a}{j} \quad \cosh \frac{a}{j}}{\sinh \frac{L}{j}}$$

$$C_4 = M_1 - W\, j \sinh \frac{a}{j}$$

For purposes of computing deflections, equation (A.3) may be written

$$y = - \frac{1}{P} (M_0 - M) \tag{23}$$

Substituting the values of M from equations (A.4) and (A.5)

$$y = - \frac{1}{P} \left(M_0 - C_1 \sinh \frac{x}{j} - C_2 \cosh \frac{x}{j}\right) \text{ when } x = a \tag{A.6}$$

$$y = - \frac{1}{P} \left(M_0 - C_3 \sinh \frac{x}{j} - C_4 \cosh \frac{x}{j}\right) \text{ when } x = a \tag{A.7}$$

Since, in the case in which we are interested, the beam is assumed pin-ended, $M_1 = M_2 = 0$.

$M_o = -W\, b\, x/L$ when $x = a$ and $M_0 = -W\, a(L-x)/L$ when $x = a$

When $x = a$ at the supporting point, the deflection $y = \delta$ can be obtained by substituting these qualities in either equation (A.6) or (A.7), whence

$$\frac{P}{k_t} = \frac{ba}{L} + j \sinh \frac{a}{j} \left(\frac{\sinh \frac{a}{j}}{\tanh \frac{L}{j}} - \cosh \frac{a}{j}\right) \tag{A.8}$$

or

$$\frac{1}{k_t} = \frac{1}{P}\left(\frac{ab}{L} + \frac{j \sinh^2 (a/j)}{\tanh (L/j)} - j \sinh \frac{a}{j} \cosh \frac{a}{j}\right) \qquad (4)$$

SECTION IV

Deflection of a Beam Subjected to Axial Tension

The deflections of a beam subjected to axial tension and any side load may be obtained from formula (23) above if proper changes be made in M_0 and M to allow for the difference in the type of side loading.

When the side load is uniformly distributed over the span the deflection can be obtained from the formula given in reference 2 for this case. (See p. 208.)

Thus

$$M_0 = + \left(M_1 + \frac{M_2 - M_1}{L} x - \frac{wLx}{2} + \frac{wx^2}{2}\right)$$

$$M = \frac{D_2 - D_1 \cosh \frac{L}{j}}{\sinh \frac{L}{j}} \sinh \frac{x}{j} + D_1 \cosh \frac{x}{j} - w j^2$$

Where $D_1 = M_1 + w j^2$

$D_2 = M_2 + w j^2$

The formulas for this and other types of side loading can also be obtained from formulas for the same type of side load and axial compression by the following procedure.

1. Substitute for each trigonometric function the corresponding hyperbolic function.

2. Reverse the sign of $w j^2$ (but not that of $W j$) where it appears.

3. Substitute $-P$ for $+P$ to indicate the effect of changing the character of the axial load in the formula for deflection. Applying these rules to the case of a side load varying uniformly from w lb./in. at the left support

to (w + kw) lb./in. at the right support, we have*

$$M = C_1 \sinh \frac{x}{j} + C_2 \cosh \frac{x}{j} - \left(1 + \frac{kx}{L}\right) w\ j^2$$

Where

$$C_1 = \frac{M_2 + (1 + k)\ w\ j^2 - (M_1 + w\ j^2) \cosh \frac{L}{j}}{\sinh \frac{L}{j}}$$

$$C_2 = (M_1 + w\ j^2)$$

and

$$\delta = -\frac{1}{P} \Big[M_1 + (M_2 - M_1) \frac{x}{L} - \frac{wLx}{2} - \frac{kwLx}{6}$$

$$+ \frac{wx^2}{2} + \frac{kwx^3}{6L} - C_1 \sinh \frac{x}{j} - C_2 \cosh \frac{x}{j} + (1 + \frac{kx}{L}\ w\ j^2)\Big]$$

---

*Note that k in these formulas is not a spring constant but the ratio between the side loads at the ends of the span.

SECTION V

Computation of Spring Constants for Numerical Examples

TABLE I: Calculations for k_c of Figure 6*

j	$\frac{a}{j}$	$\frac{b}{j}$	β_1	β_2	$a_1\beta_1 + b_2\beta_2$	θ	k_c
4.545	1.10	2.20	1.0912	1.6124	21.580	1.1590	-3.293
4.348	1.15	2.30	1.1009	1.7325	22.829	1.3407	-6.100
4.167	1.20	2.40	1.1114	1.8854	24.402	1.5607	-8.625
4.000	1.25	2.50	1.1225	2.0864	26.476	1.8385	-10.950
3.846	1.30	2.60	1.1345	2.3618	29.290	2.2000	-13.090
3.571	1.40	2.80	1.1610	3.3963	39.768	3.4653	-17.070
3.333	1.50	3.00	1.1915	7.3486	79.443	7.9443	-20.980
2.941	1.70	3.40	1.2673	-3.0787	-24.451	-3.1404	-31.640

*k_c is computed from equation (18).

TABLE II: Calculations for k_c of Figure 10*

j	$\frac{a}{j}$	$\frac{b}{j}$	β_1	β_2	$a_1\beta_1+b_2\beta_2$	θ	k_c
31	1.093	3.808	+1.3723	-0.7970	-13.080	-0.1784	-1386
32	1.844	3.688	1.3382	-1.1242	-53.702	-.6870	-516
34	1.735	3.470	1.2833	-2.3537	-202.022	-2.290	-301
36	1.640	3.278	1.2423	-6.3943	-681.231	-8.550	-235
40	1.476	2.950	1.1839	5.5875	729.175	5.970	-175
45	1.312	2.622	1.1401	2.4367	354.797	2.295	-119
50	1.180	2.360	1.1072	1.8195	280.026	1.469	-67
56.34	1.048	2.094	1.0819	1.5106	242.083	1.000	0

*k_c is computed from equation (18).

SECTION VI

Samples of Experimental Data and Deflection Computations

Although it has seemed advisable to omit most of the detailed experimental data and deflection curves, it appears advisable to include samples of this part of the work in order to show a little more clearly how the curves of figures 17 to 26 were obtained. For this purpose the experimental data for test E-1 reductions of the general formulas for the special cases studied, and values computed from these formulas for the theoretical curves for tests E-1 and G-1 are given in this section of the Appendix.

Experimental Data - Test E-1

No. 2 at A			No. 1 at B			Load at 4		
	P_T	δ		P_T	δ		P_T	δ
N.S.L.	20	0	N.S.L.	20	0	N.S.L.	20	0
	340	-0.0003	5 lb.	20	0.0252	10 lb.	20	0.0460
	550	+ .0001		195	.0293		135	.0539
	735	.0044		295	.0332		300	.0660
	855	.0097		415	.0402		420	.0809
	20	-.0002		540	.0521		535	.0998
				705	.0781		660	.1465
				805	.1205		735	.1718
				870	.1849		20	.0476
				20	.0261	N.S.L.	20	-.0002
			N.S.L.	20	.0013			

For locations of loads and specimens, see figure 16.

P_T, total load indicated by testing machine corrected for tare weight of 80 pounds.

δ, deflection of mid-point of specimens measured from position with minimum total load and no side load.

N.S.L. indicates the condition of minimum total load and no inside load.

Reduction of General Formulas of Part I for the

Special Cases Investigated Experimentally

A. Reduction of equation (3) for the spring constant of a strut in compression for the case where $a = 0.5\ L = b$.

Equation (3) is

$$\frac{1}{k_c} = -\frac{1}{P}\left(\frac{a\ b}{L} + \frac{j\ \sin^2\ (a/j)}{\tan\ (L/j)} - j\ \sin\frac{a}{j}\ \cos\frac{a}{j}\right)$$

Substituting a for b, 2a for L, and A for a/j

$$-\frac{P}{k_c} = \frac{a}{2} + j\ \left(\frac{\sin^2\ A}{\tan\ 2A} - \sin\ A\ \cos\ A\right)$$

$$= \frac{a}{2} + j\ \left(\frac{\sin^2\ A\ \cos\ 2A}{\sin\ 2A} - \sin\ A\ \cos\ A\right)$$

$$= \frac{a}{2} + j\ \left(\frac{\sin^2\ A\ (\cos^2\ A - \sin^2\ A)}{2\ \sin\ A\ \cos\ A} - \sin\ A\ \cos\ A\right)$$

$$= \frac{a}{j} + j\ \left(\frac{\cos^2\ A - \sin^2\ A - 2\ \cos^2\ A}{2\ \cot\ A}\right)$$

$$= \frac{a}{2} - \frac{j}{2}\tan\ A$$

$$\frac{1}{k_c} = -\frac{j}{2\ P}\left(\frac{a}{j} - \tan\frac{a}{j}\right) \qquad (A.6)$$

B. Reduction of equation (4) for the spring constant of a strut in tension for the case where $a = b = 0.5\ L$.

Equation (4) is

$$\frac{1}{k_t} = \frac{1}{P}\left(\frac{a\ b}{L} + \frac{j\ \sinh^2\ (a/j)}{\tanh\ (L/j)} - j\ \sinh\frac{a}{j}\ \cosh\frac{a}{j}\right)$$

Substituting a for b, 2a for L, and A for a/j,

$$\frac{P}{k_t} = \frac{a}{2} + j\ \left(\frac{\sinh^2\ A\ \cosh^3\ A}{\sinh\ A} - \sinh\ A\ \cosh\ A\right)$$

$$\frac{P}{k_t} = \frac{a}{2} + j\left(\frac{\sinh^2 A\ (\cosh 2A - 2\cosh^2 A)}{2\sinh A\cosh A}\right)$$

$$= \frac{a}{2} + j\left(\frac{\cosh^2 A + \sinh^2 A - 2\cosh^2 A}{2\coth A}\right)$$

$$= \frac{a}{2} - \frac{j}{2}\,\frac{1}{\coth A}$$

$$\frac{1}{k_t} = \frac{j}{2P}\left(\frac{a}{j} - \tanh\frac{a}{j}\right) \qquad (A.7)$$

C. Derivation of formula for deflection of support point when the only external load on the beam is a concentrated force at the support point.

In the case represented by the tests, the initial deflection of the beam δ_o can be obtained from the spring constant of that member. Thus

$$\delta_o = S/k_b$$

where S is the side load in pounds

k_b, spring constant of the beam in pounds per inch.

Substituting this relation in equation (8) to determine the deflection δ_e of the combination of beam and strut, we have

$$\delta = \frac{k_b\ \delta_o}{k_b + k_s} = \frac{S}{k_b + k_s} \qquad (A.8)$$

Numerical Values of Spring Constants

The values of the spring constants for specimens 1 and 2 in compression, and specimen 6 in tension, are given below. The constants for specimens 1 and 2 were computed from equation (A.6), and those for specimen 6 from equation (A.7).

Spring Constants for Specimens 1 and 2

Size 1/2 by 1/4 inch EI = 19,000 lb.in.^{2} a = 10 inches

Axial load compression							
P	k_c	P	k_c	P	k_c	P	k_c
0	+114.0	280	44.9	600	-32.3	1200	-186.0
60	99.1	300	41.5	625	-38.7	1225	-193.2
100	90.3	350	29.4	660	-47.3	1250	-199.5
120	85.4	375	22.8	700	-58.1	1260	-202.2
140	80.3	400	17.1	750	-70.3	1400	-240.0
150	77.9	420	12.2	770	-75.0	1470	-258.8
180	70.2	440	7.1	840	-92.6	1500	-266.5
200	65.8	450	4.3	875	-101.1	1680	-317.5
210	63.7	490	-5.7	880	-102.9	1750	-337.5
220	60.5	500	-7.7	900	-108.8	1875	-377.5
240	56.6	525	-13.6	1000	-133.5	2100	-445.0
250	53.4	560	-22.1	1050	-147.0		

Spring Constants for Specimen 6

Size 1 by 1/4 inch EI = 39,500 lb.in.^{2} a = 11 inches

Axial load tension							
P	k_t	P	k_t	P	k_t	P	k_t
0	+177.6	120	203.5	350	255.2	800	352.0
20	180.0	140	209.0	400	270.0	880	368.2
40	186.8	160	212.0	440	274.0	900	372.2
60	191.5	180	218.5	600	307.5	1320	461.0
70	193.2	200	222.1	660	320.0	1540	509.5
80	294.5	220	226.2	700	329.8	1760	551.0
100	197.5	300	243.5	770	344.5	1980	600.0

Theoretical Deflections for Test E-1

Beam No. 2 $P_b = 0.3\ P_T$
Strut No. 1 $P_s = 0.7\ P_T$

P_T	P_b	P_s	k_b	k_s	k_b+k_s	δ_e S = 5	δ_e S = 10
0	0	0	+114.0	+114.0	+228.0	0.0219	0.0438
200	60	140	99.1	80.3	179.4	.0278	.0556
400	120	280	85.4	44.9	130.3	.0383	.0766
500	150	350	77.9	29.4	107.3	.0465	.0930
600	180	420	70.2	12.2	82.4	.0607	.1214
700	210	490	63.7	-5.7	58.0	.0863	.1726
800	240	560	56.6	-22.1	34.5	.1450	.2900

Theoretical Deflections for Test G-1

Beam No. 6 P_b=1.10 P_T(tens.)
Strut No. 1 P_s=2.10 P_T(comp.)

P_T	P_b	P_s	k_b	k_s	k_b+k_s	δ_e S = 10	δ_e S = 15
0	0	0	+177.6	+114.0	+291.6	0.0343	0.0515
200	220	420	226.2	12.2	238.4	.0419	.0628
400	440	840	274.0	-92.6	181.4	.0551	.0827
600	660	1260	321.0	-202.2	118.8	.0841	.1263
700	770	1470	344.5	-258.8	85.7	.1168	.1750
800	880	1680	368.2	-317.5	50.7	.1970	.2958

Stanford University,
California, February 1935.

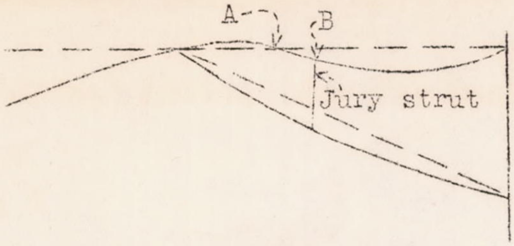


Figure 1.

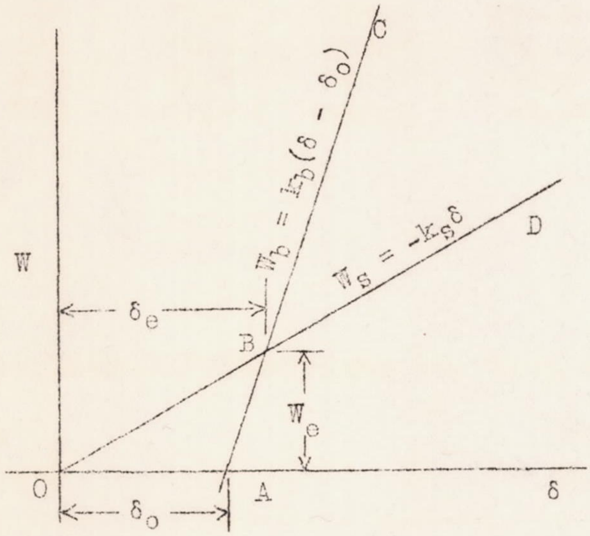


Figure 3.-Supporting load and force with initial deflection of beam.

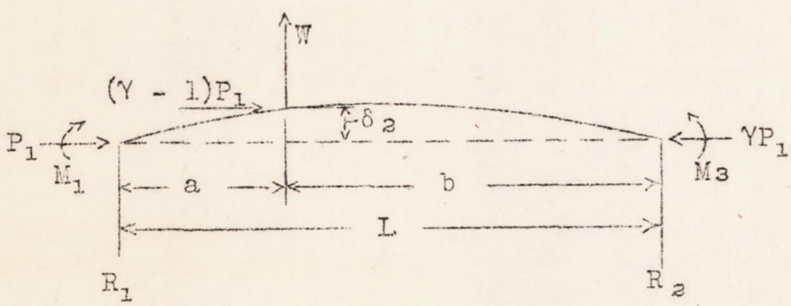


Figure 4.

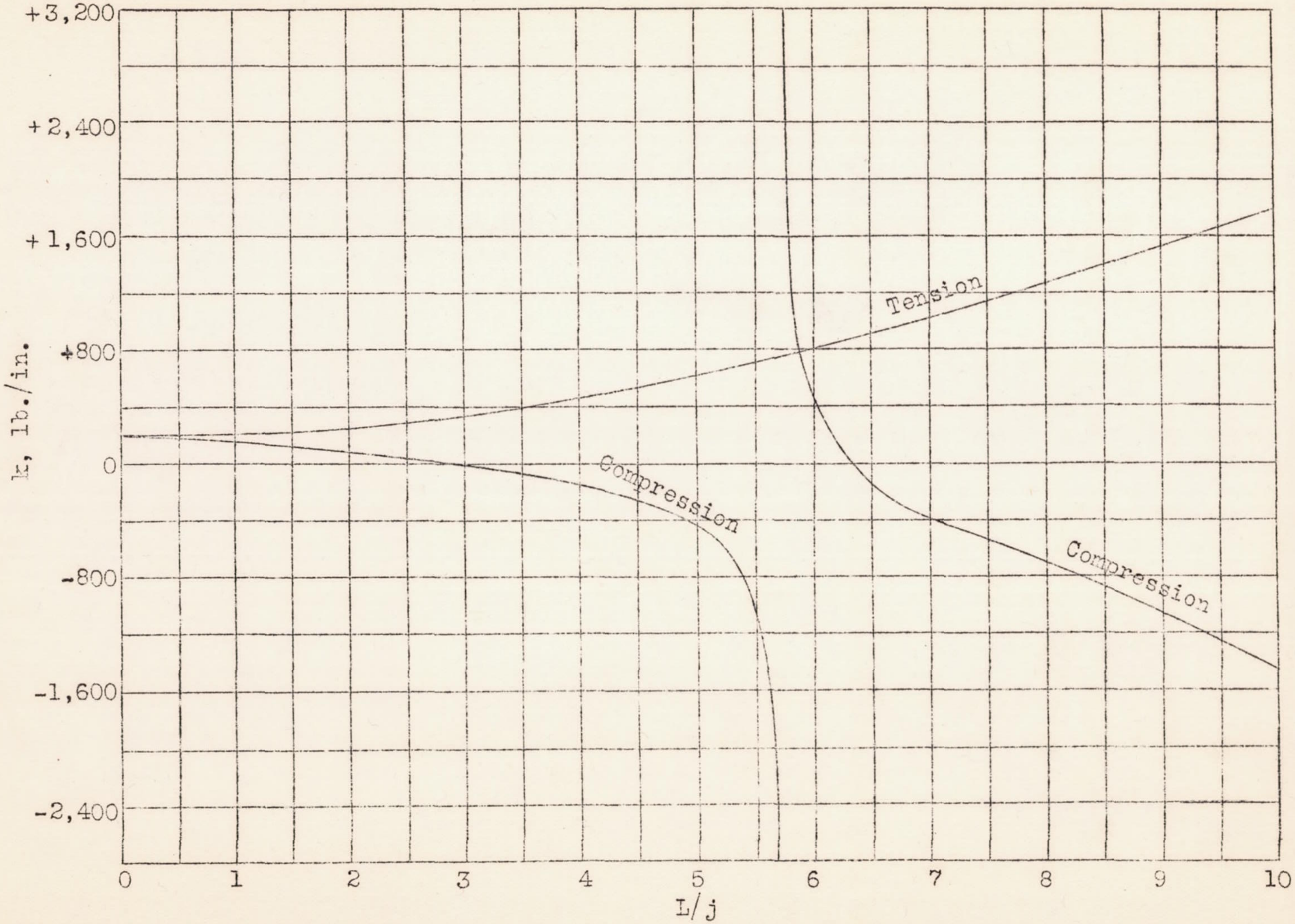


Figure 2.-Typical variation of spring constant.

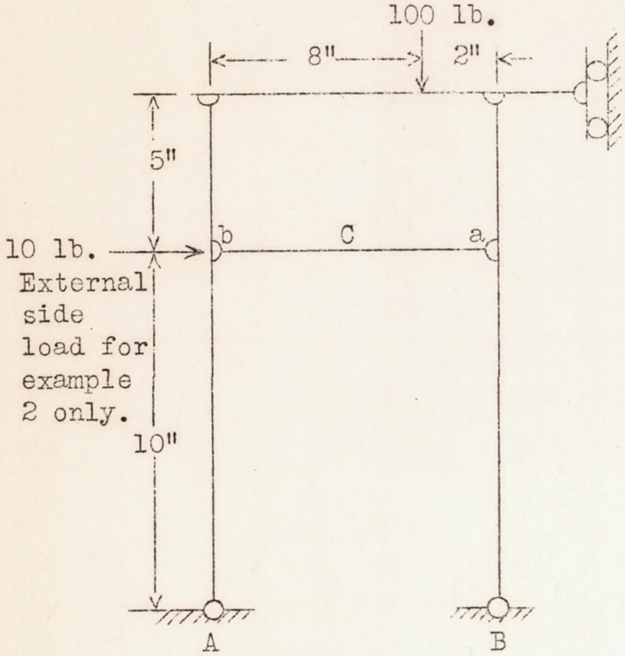


Figure 5.

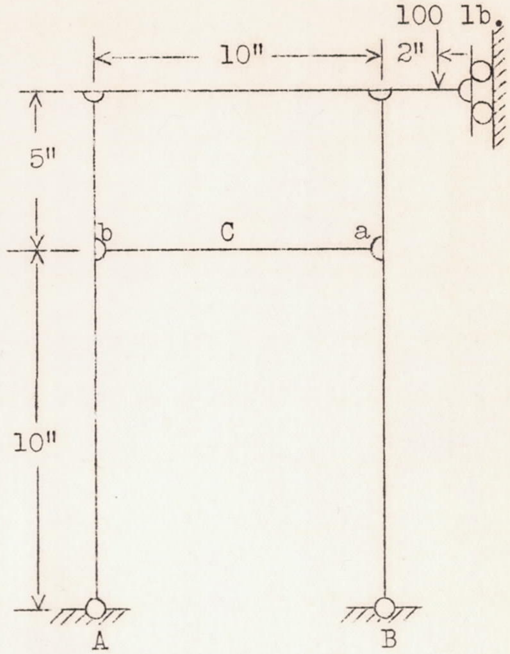


Figure 7.

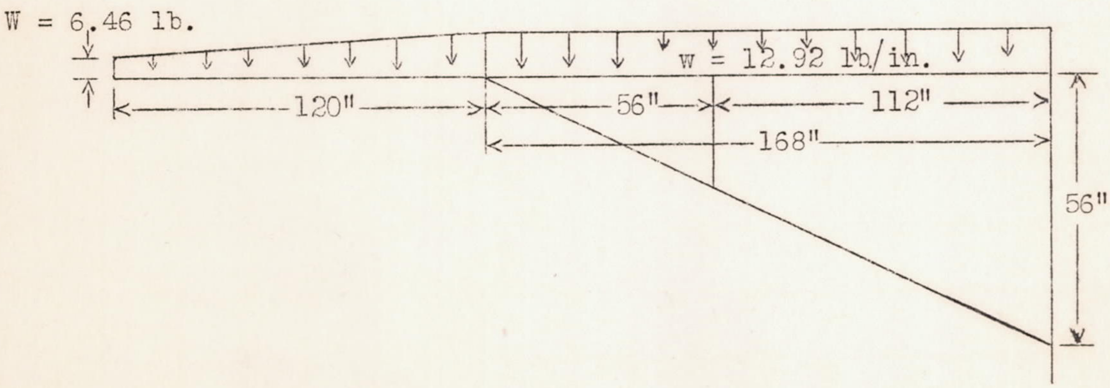


Figure 8.

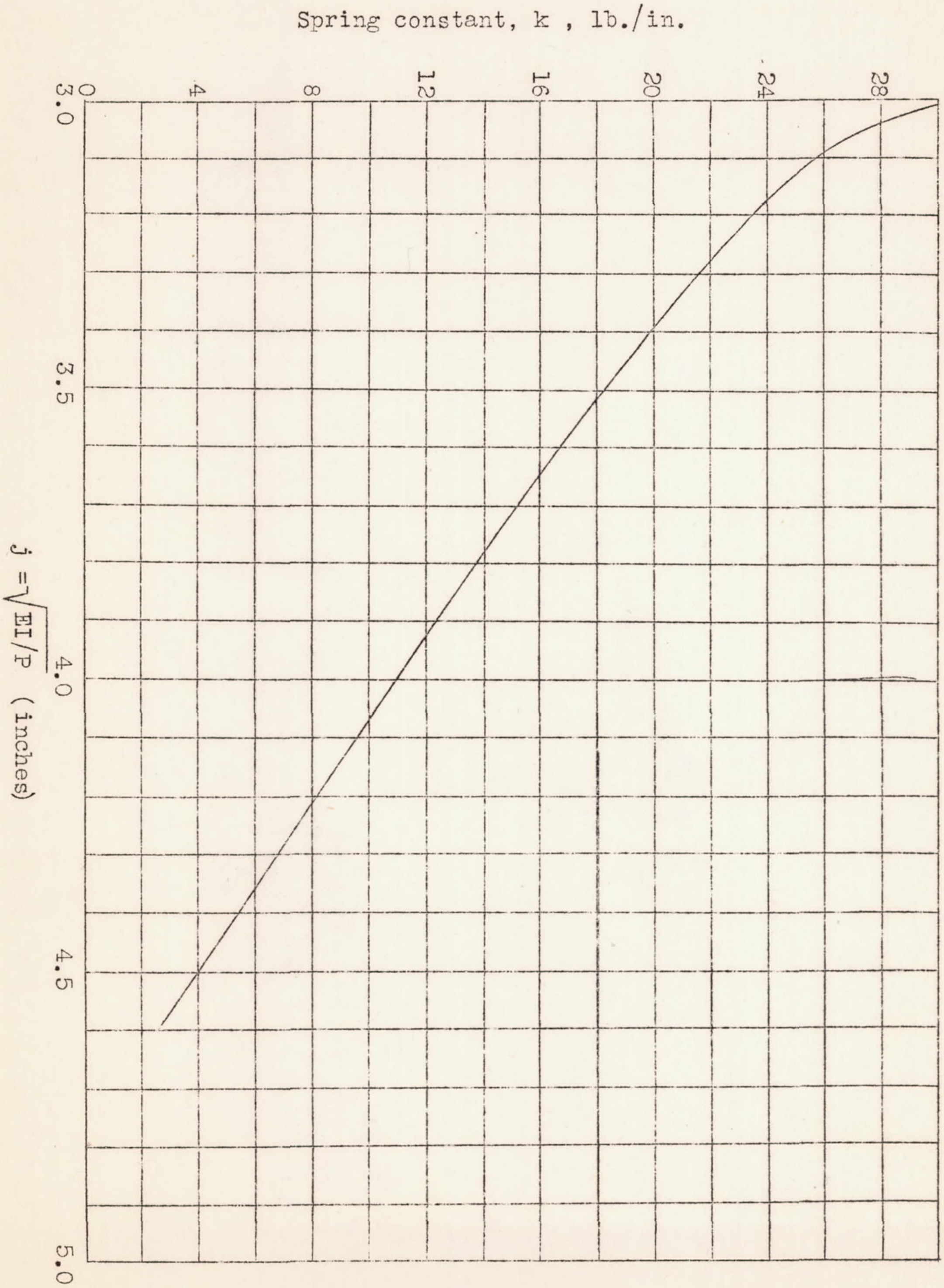


Figure 6. k against j for strut of examples I, III.

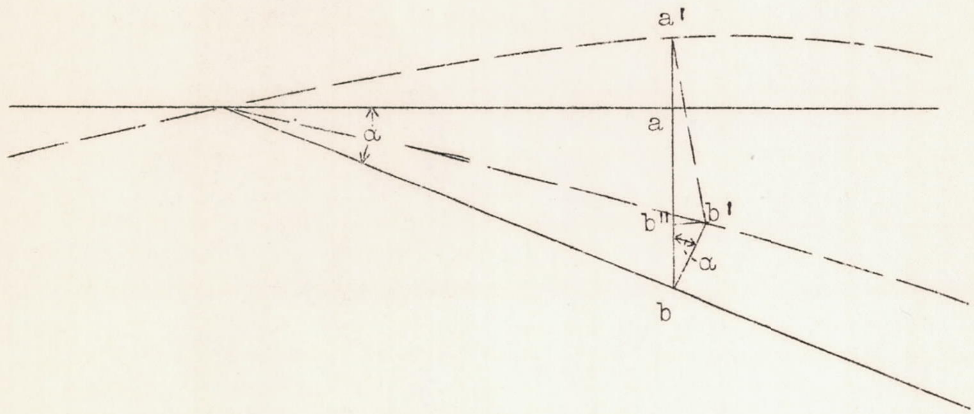


Figure 9.

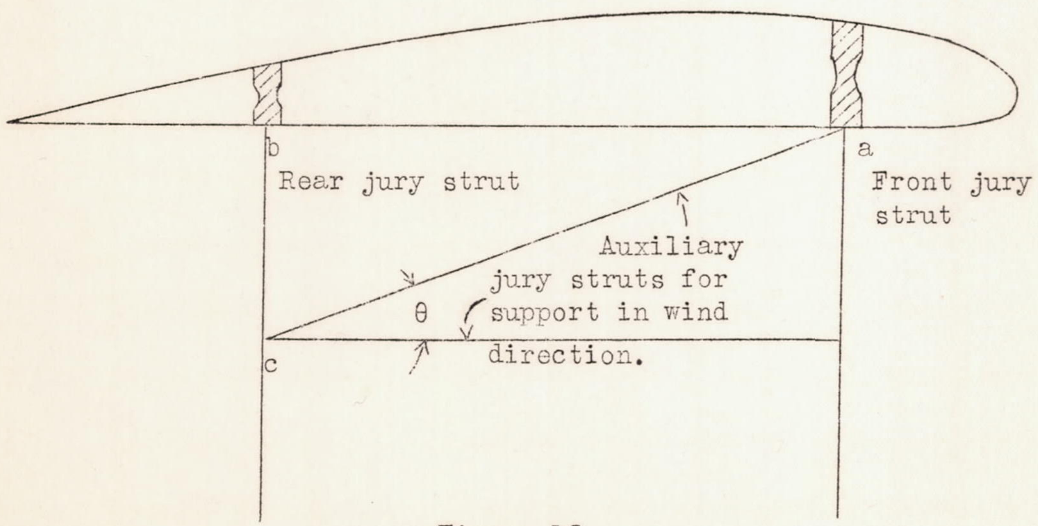


Figure 12.

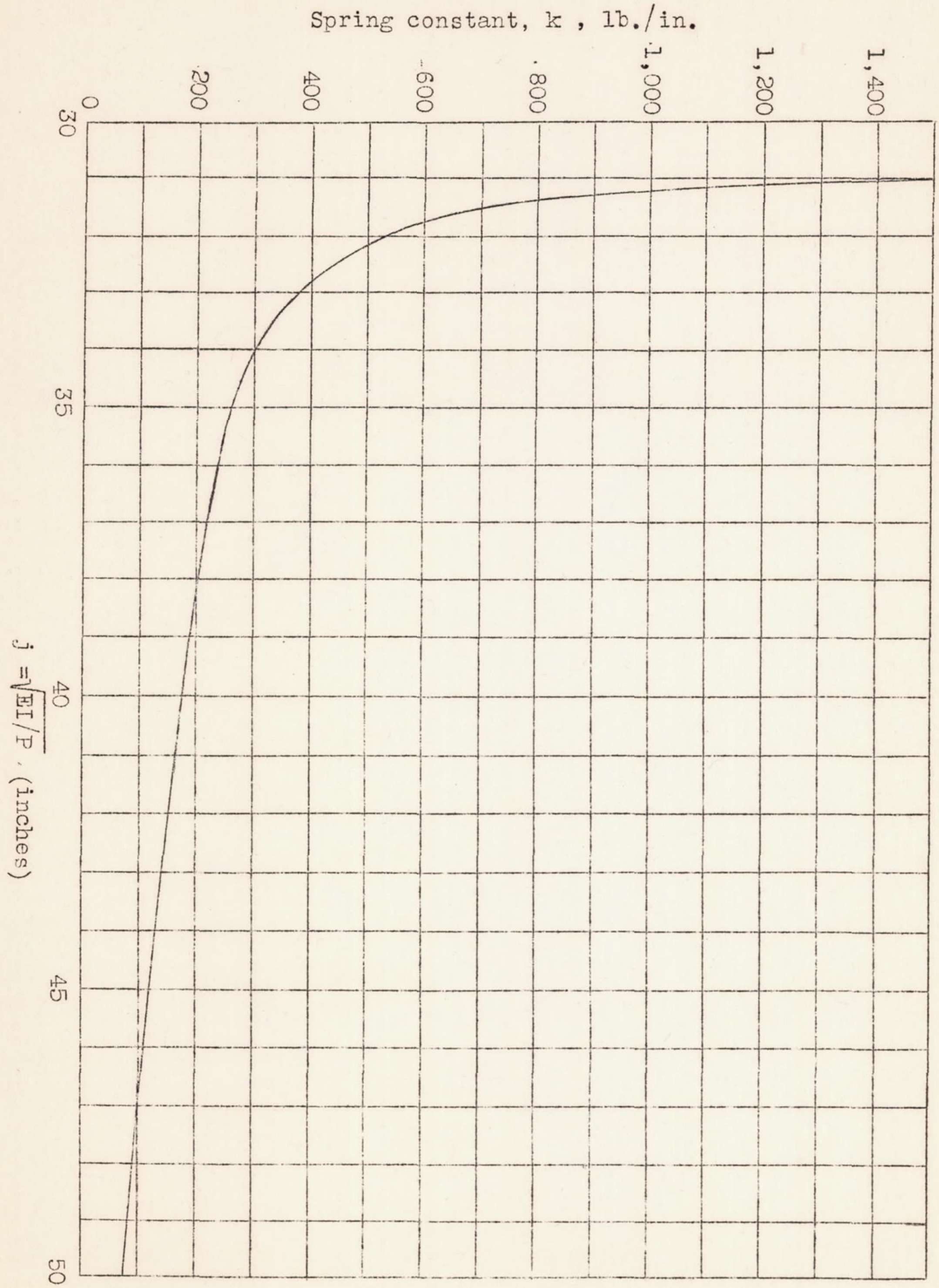


Figure 10. k against j for strut of example IV.

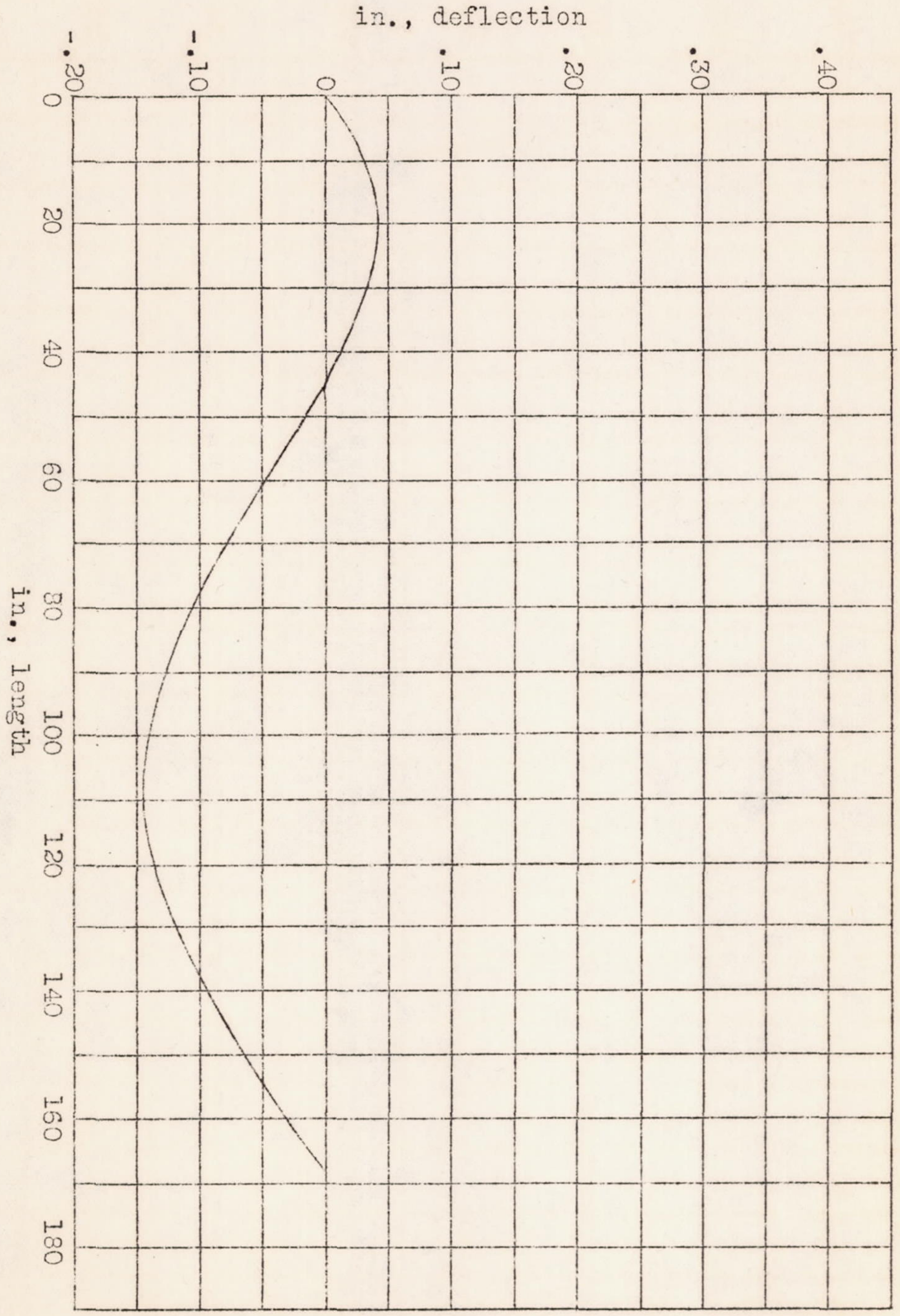


Figure 11.-Deflection of spar in example IV. (Due to external loads only).

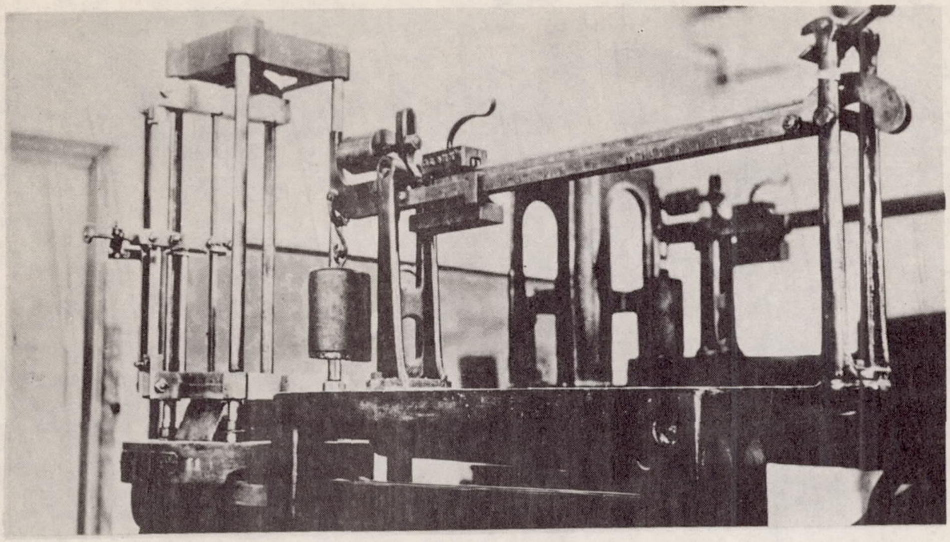


Figure 13.

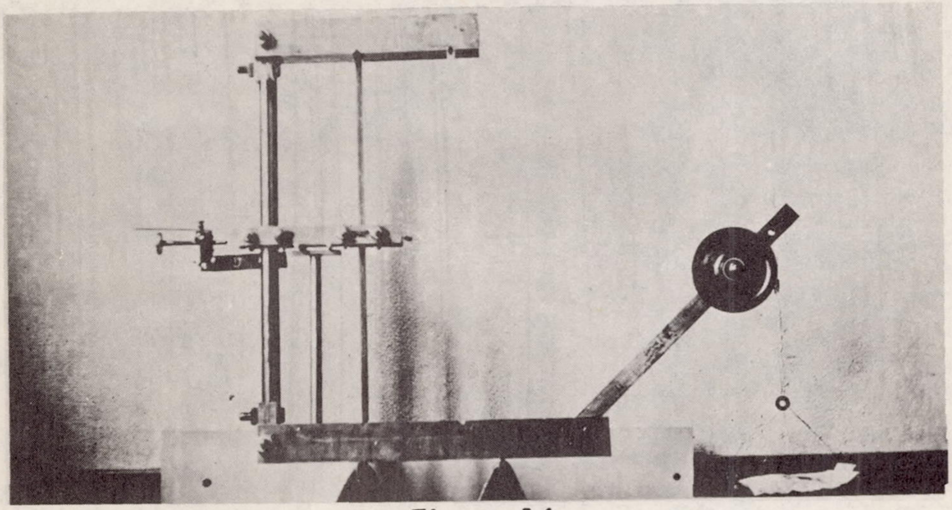


Figure 14.

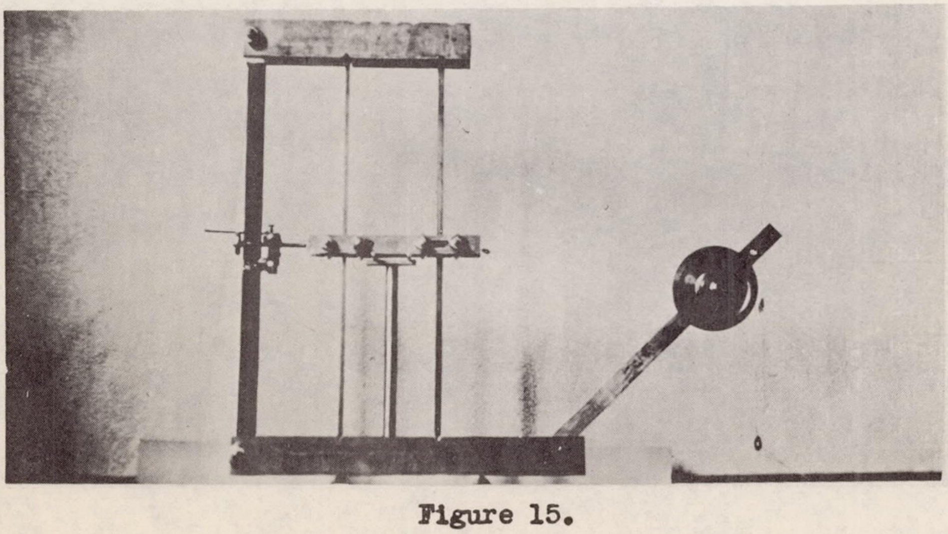
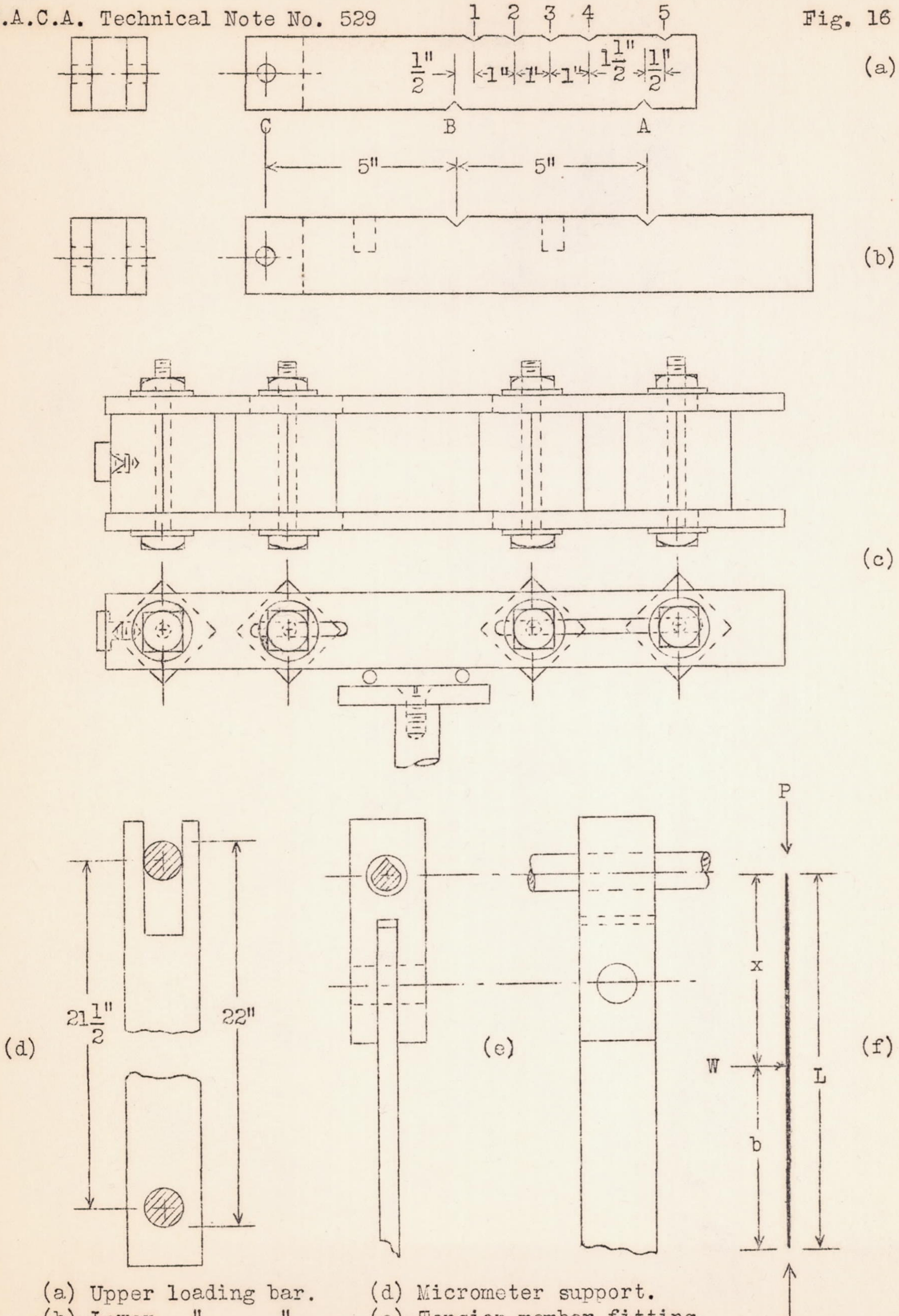
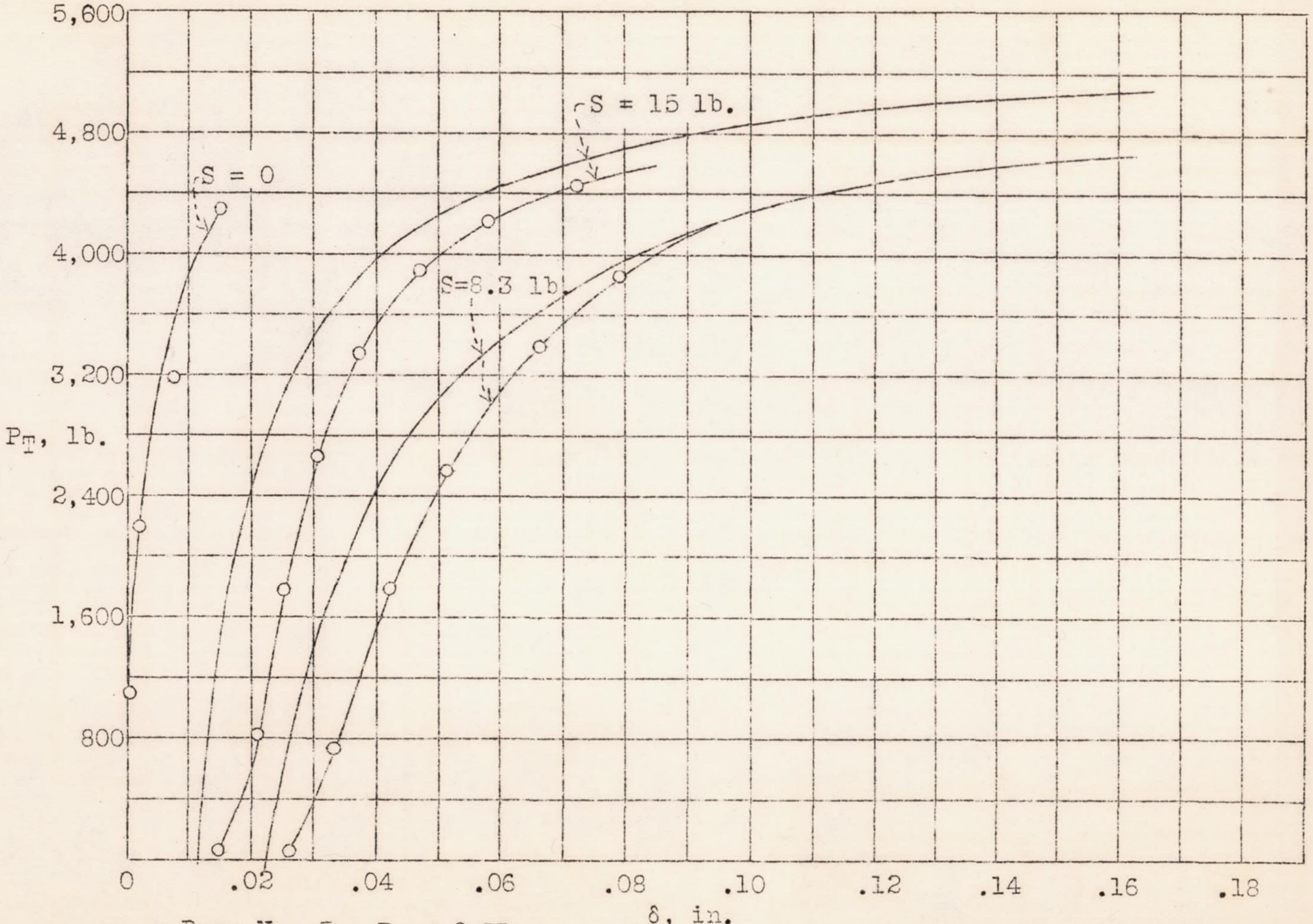


Figure 15.



(a) Upper loading bar.
(b) Lower " "
(c) Tie rod
(d) Micrometer support.
(e) Tension member fitting.
(f) Strut dimensions.

Figure 16.-Details of test apparatus.



Beam No. 5 $P_b = 0.3P_T$
Strut No. 3 $P_s = 0.7P_T$

o———— Experimental
———— Theoretical

Figure 17. Test A3.

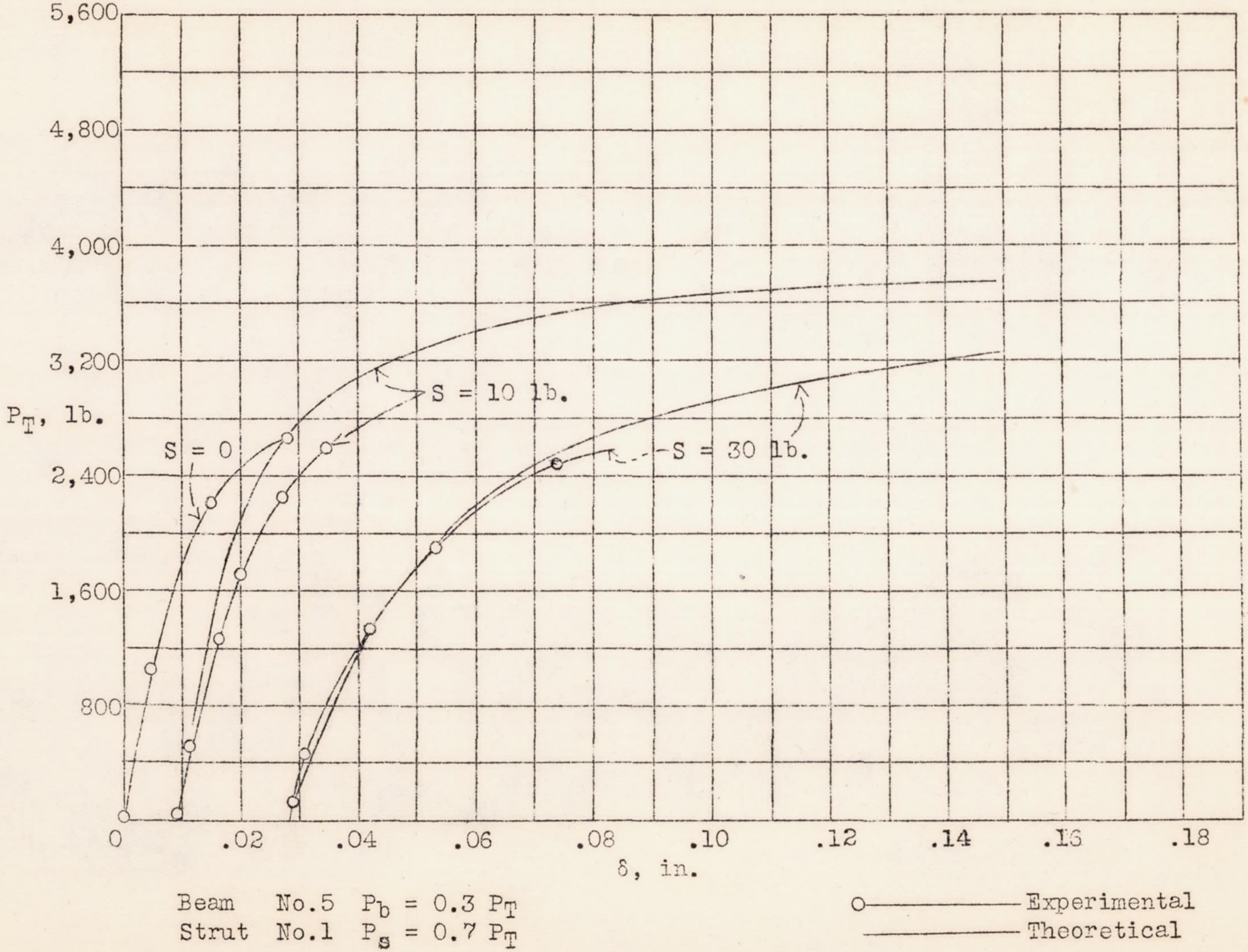


Figure 18.-Test B3.

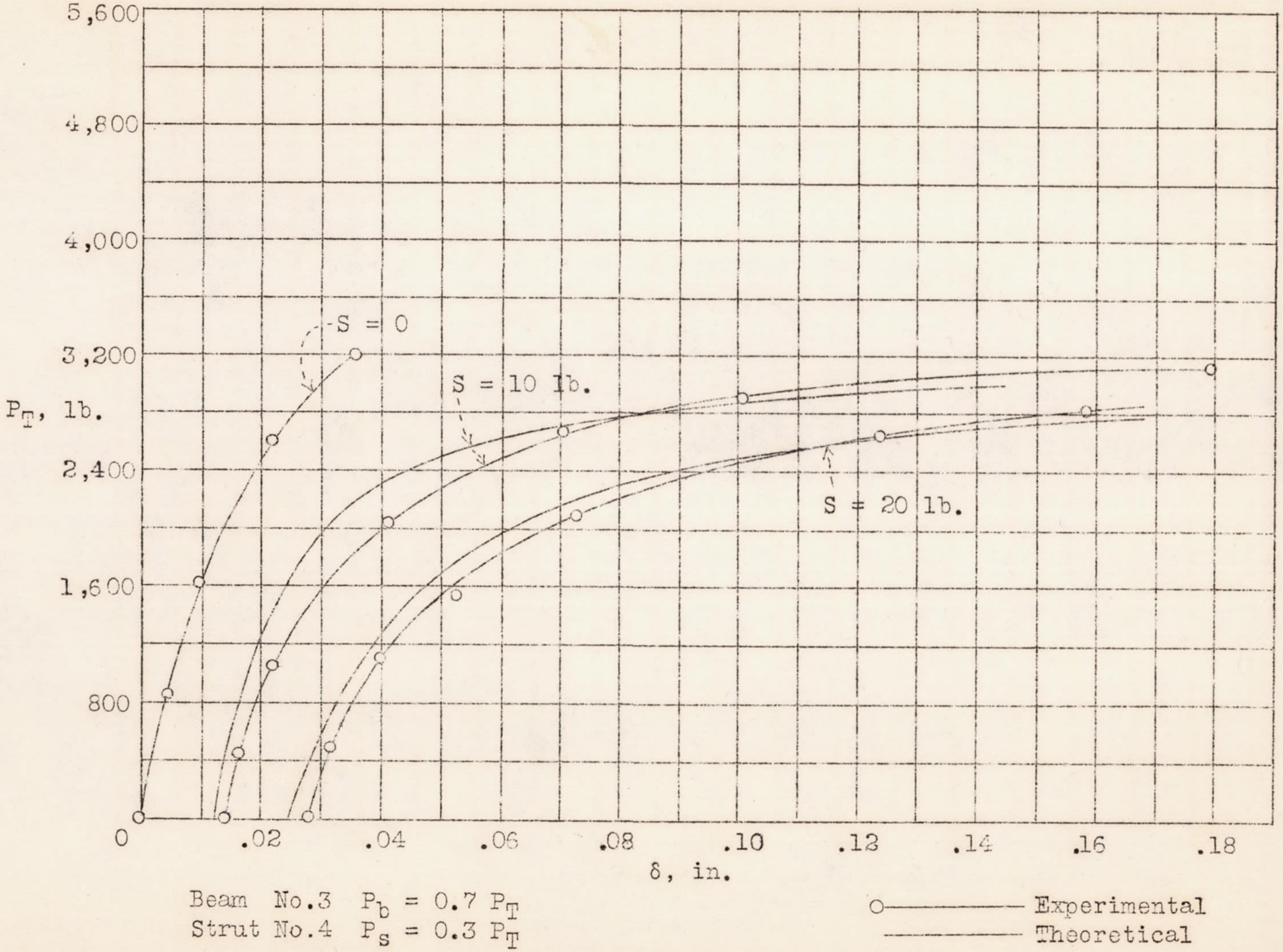
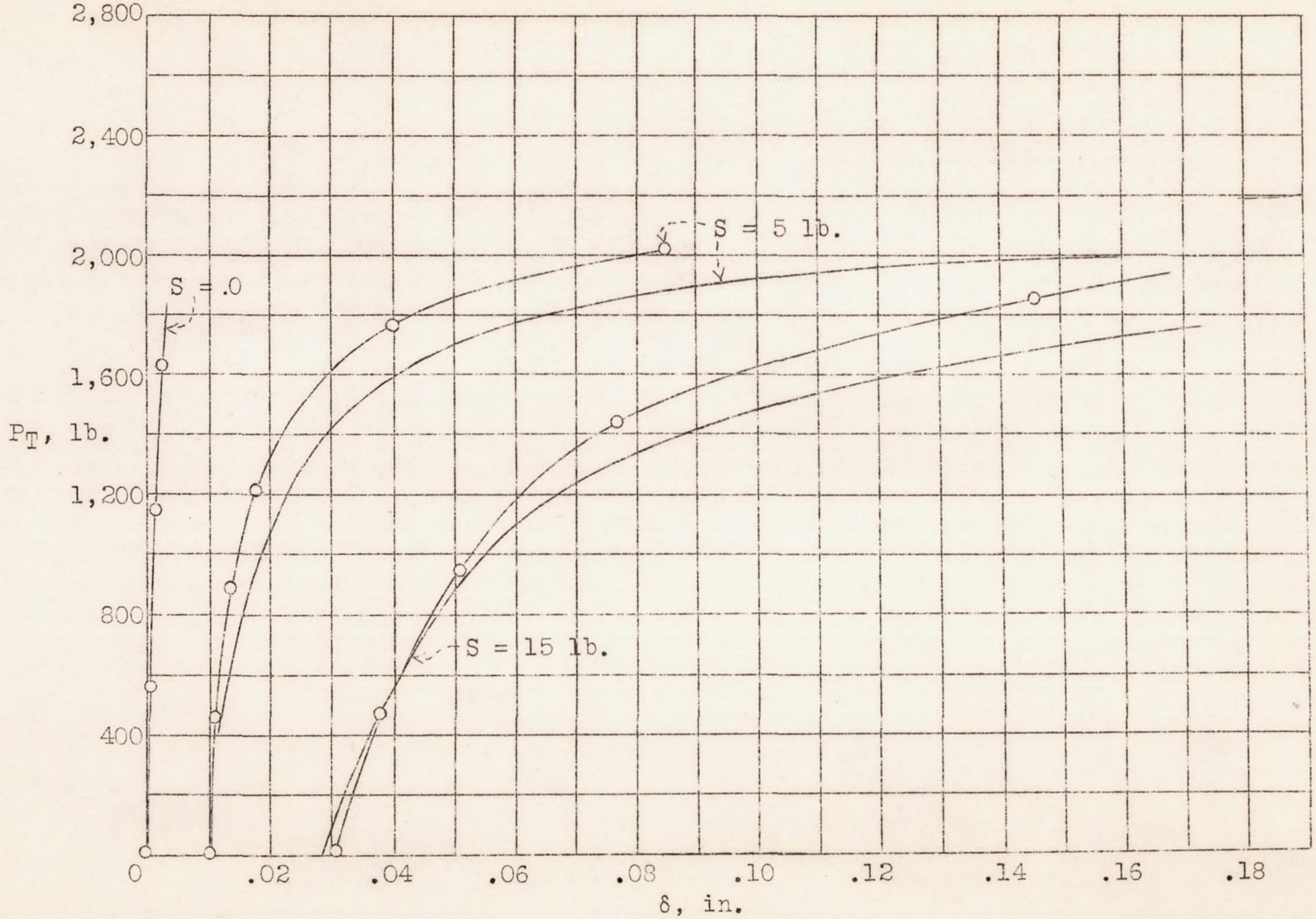


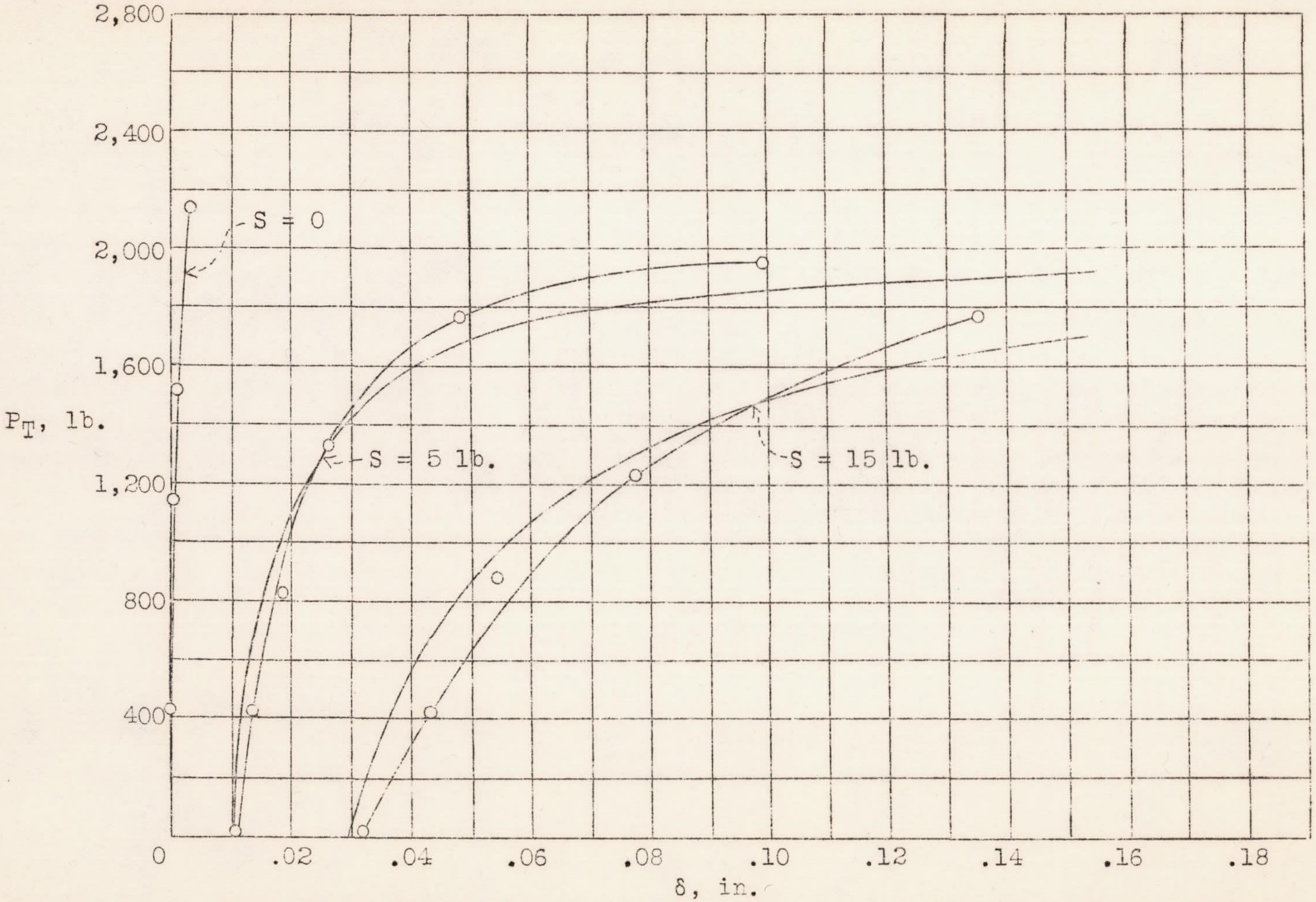
Figure 19.-Test C1.



Beam No.3 $P_b = 0.5\ P_T$
Strut No.1 $P_s = 0.5\ P_T$

o———— Experimental
———— Theoretical

Figure 20.-Test D2.



Beam No.3 $P_b = 0.3\ P_T$
Strut No.1 $P_s = 0.7\ P_T$

o———— Experimental
———— Theoretical

Figure 21.-Test D3.

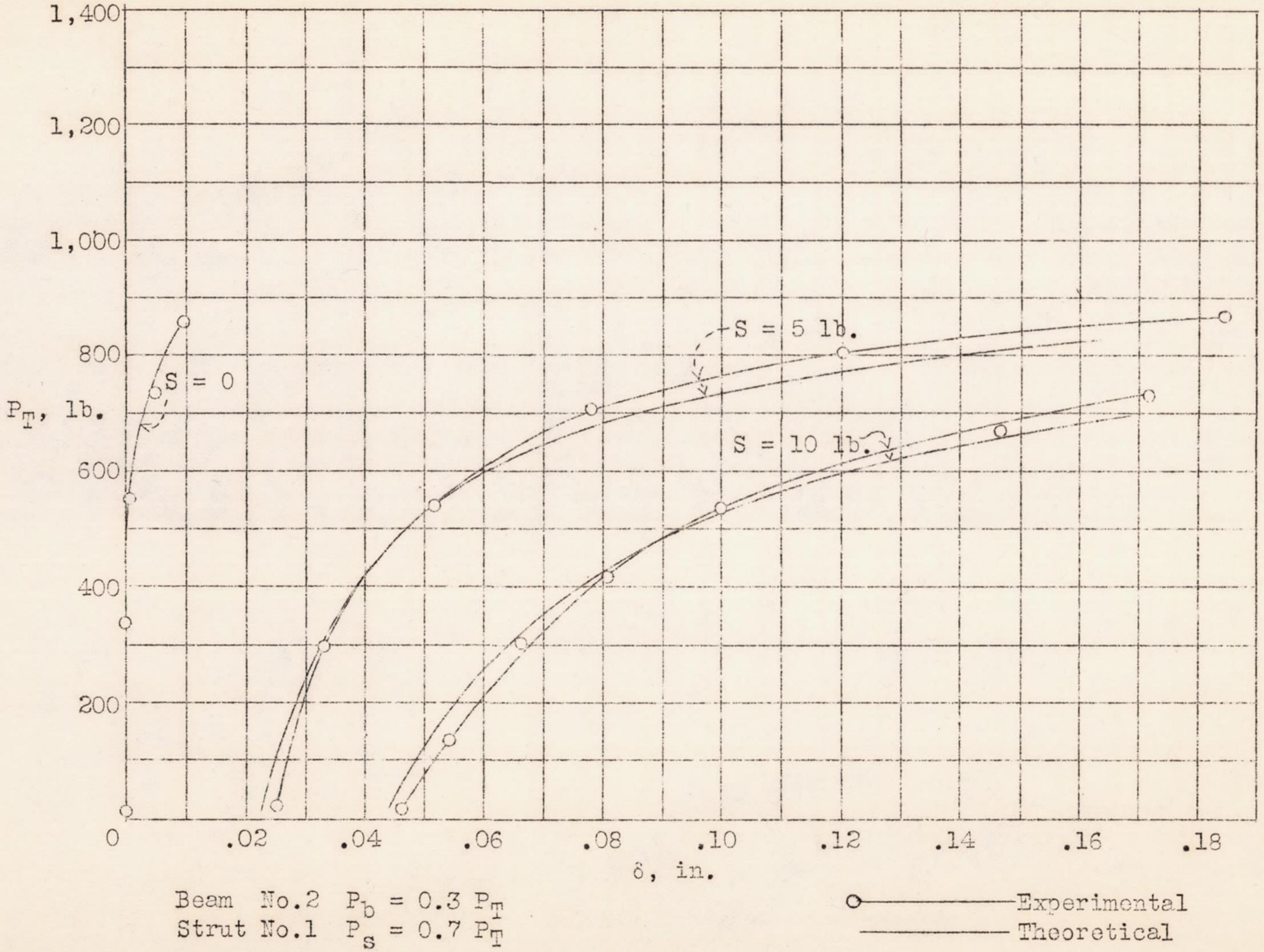
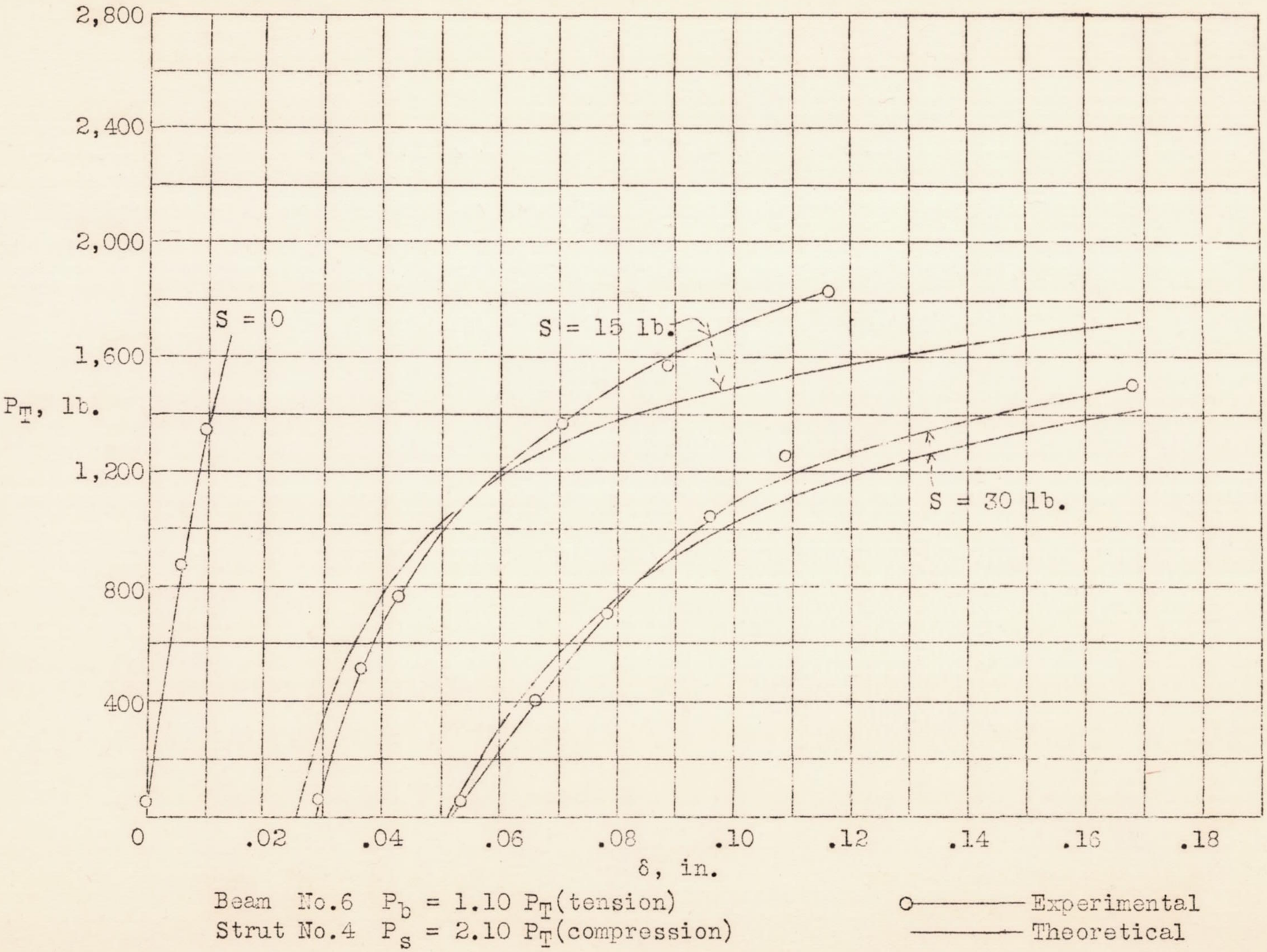


Figure 22.-Test El.



Beam No.6 $P_b = 1.10\ P_T$(tension)
Strut No.4 $P_s = 2.10\ P_T$(compression)

Figure 23.-Test F1.

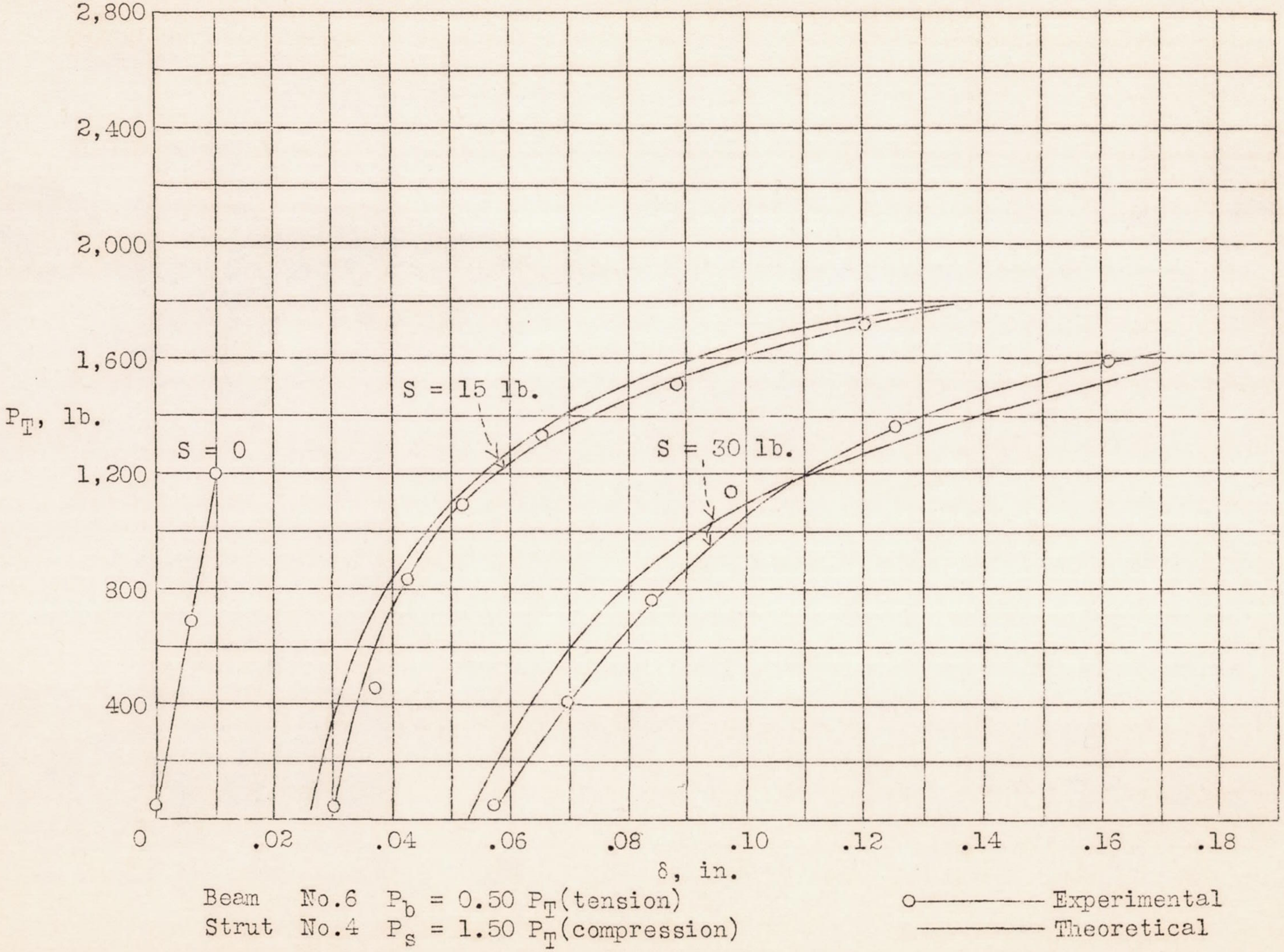


Figure 24.-Test F2.

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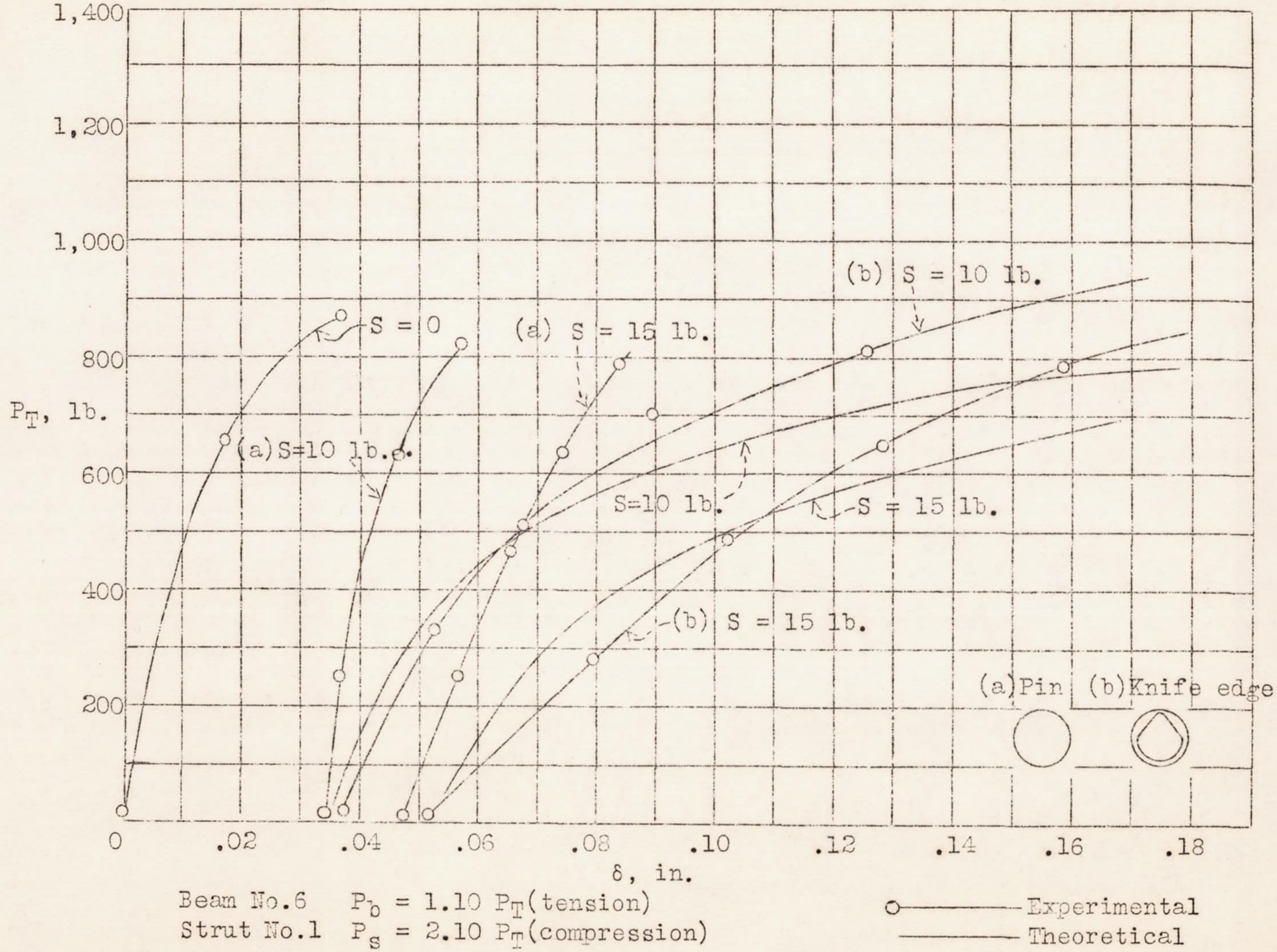


Figure 25.-Test G1.

79

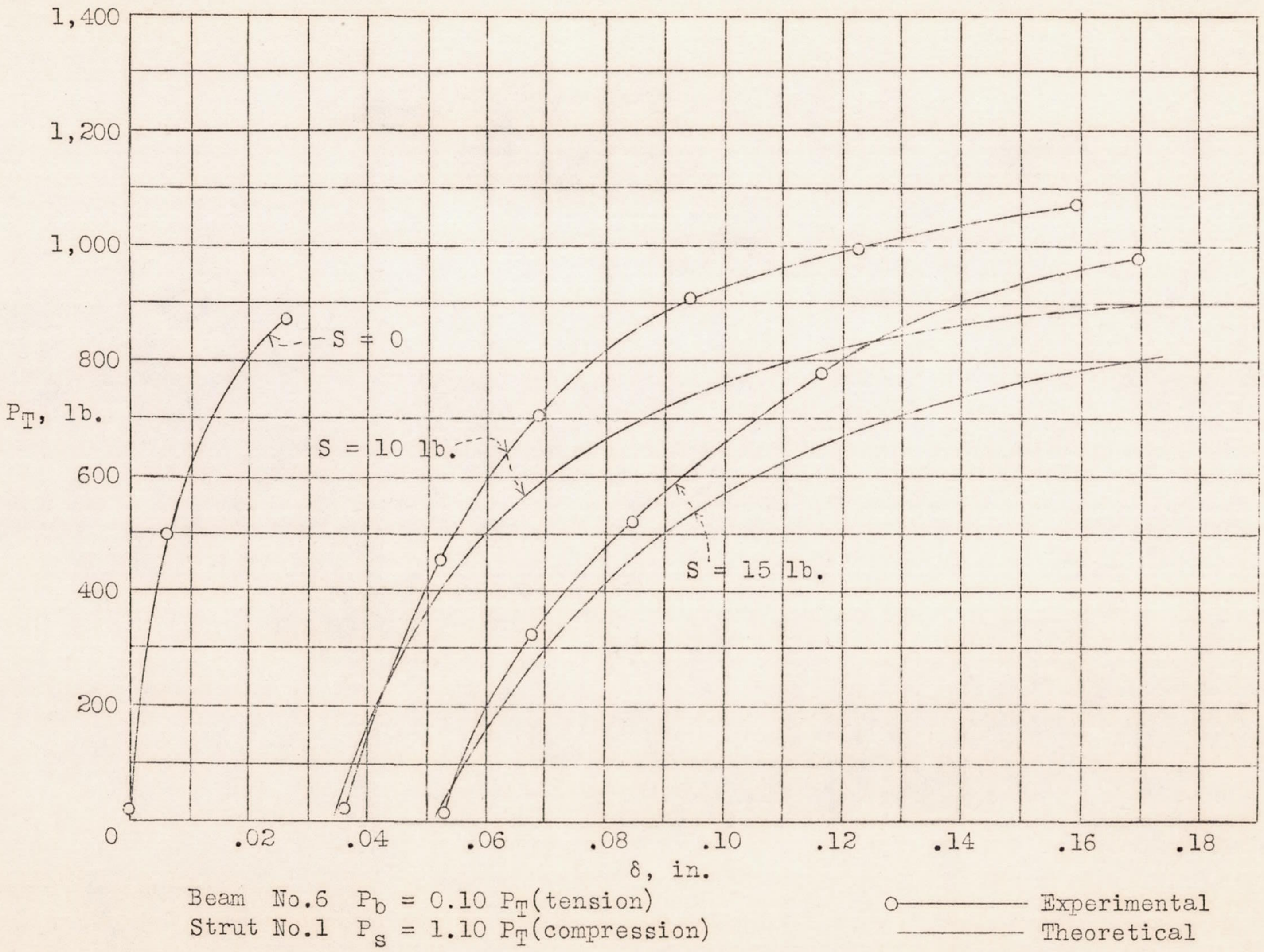


Figure 26.-Test G3.

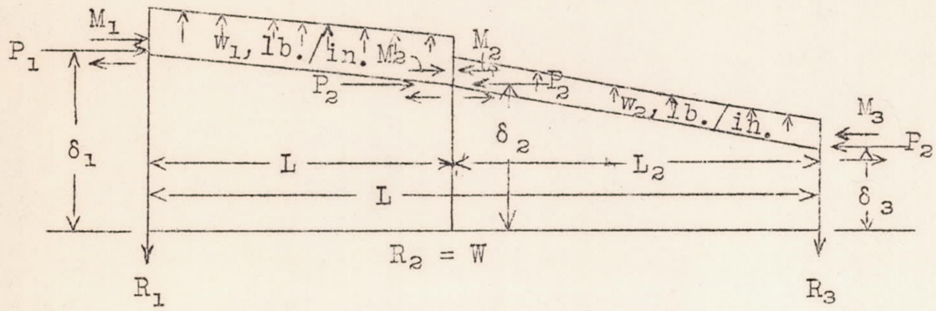
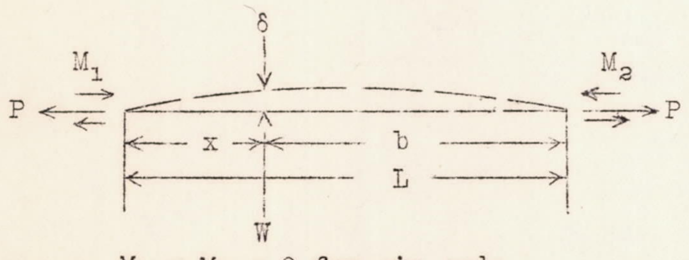


Figure 27.



$M_1 = M_2 = 0$ for pin ends

Figure 28.