TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 667

OPERATIONAL TREATMENT OF THE NONUNIFORM-LIFT THEORY

IN AIRPLANE DYNAMICS

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Washington
October 1938

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SUMMARY

The method of operators is used in the application of
nonuniform-lift theory to problems of airplane dynamics.
The method is adapted to the determination of the lift under
prescribed conditions of motion or to the determination of
the motions with prescribed disturbing forces.

INTRODUCTION

Problems in airplane dynamics are usually treated on
the assumption that the air forces are instantly adjusted
to each motion of the airplane. Since the development of
recent theories for the nonuniform motion of airfoils, it
has become possible to consider more exact laws for the ad-
justment of the lift.

The nonuniform-lift theory has already been applied to
certain dynamical problems, notably to the problem of flut-
ter. These applications have, however, been confined either
to approximate solutions or to cases in which the type of
motion is prescribed beforehand. The more usual problem,
in which the resulting motion is unknown, requires the so-
lution of integral equations. The present paper shows how
solutions of these equations may be obtained fairly simply
by operational methods.

SUPERPOSITION OF LIFTS

In nearly every aerodynamic problem, the approximations
that must be made to effect solutions are such as to lead to
linear relations. Thus, in the case of the unsteady lift of
a wing, Laplace's equation combined with the assumption of

an undistorted wake leads to a linear relation between the lift and the angle of attack. Such a relation means that the lift due to the sum of two variable motions is equal to the sum of the lifts for the two motions taken independently.

In particular, if the lift following a sudden unit jump of angle of attack is known (see reference 1), then the lift for any variable motion is easily obtained by breaking the given motion down into a succession of small jumps or steps and adding the lifts incident to each one. The case treated by Wagner thus becomes the key to the calculation of lift for any variable motion.

Wagner's function (reference 1) giving the lift after a sudden unit jump of angle of attack (two-dimensional case) may be denoted by $c_{l_1}(s)$. The superposition of lifts for any variable motion $\alpha(s)$, as previously explained, is accomplished by the integration of Duhamel's integral

$$\int_0^s c_{l_1}(s - s_0)\, \alpha'(s_0)\, ds_0 \qquad\qquad (1)$$

(See reference 2.)

OPERATIONAL SOLUTION OF INTEGRAL EQUATIONS

It is evident that, in order to take account of unsteady air-flow phenomena in the theory of airplane dynamics (including stability and related problems) the customary instantaneous equations of motion must be replaced by equations involving the integral (1). The equations of motion then become linear integral equations. Solutions of these equations may be conveniently obtained by operational methods.

Let D represent the operator d/ds and let $1 = 1(s)$ represent the unit jump fuction, that is, a function of s having the value 1 at $s > 0$ and having the value 0 at $s < 0$. Then a function of s may be represented by a combination of operations on the unit jump function

$$\varphi(s) = \bar{\varphi}(D)\, 1(s) \qquad\qquad (2)$$

The combination of operations $\overline{\varphi}(D)$ on 1 necessary to reproduce the function $\varphi(s)$ is called the "operational equivalent" of the function $\varphi(s)$. The operational equivalent $\overline{\varphi}$ of a given function φ may be found by the infinite-integral theorem (reference 2)

$$\overline{\varphi}(a) = a \int_0^\infty \varphi(x) \, e^{-ax} \, dx \tag{3}$$

A general operational equivalent is

$$s^n = \Gamma(1 + n) \, D^{-n} \, 1(s) \tag{4}$$

(See Peirce's table, p. 63, no. 493.)

The operational treatment of integral equations is based on the proposition that an integral of the form

$$\varphi(s) = Z(s) \, X(0) + \int_0^s Z(s - s_0) \, X'(s_0) \, ds_0 \tag{5}$$

may be regarded as the solution of a linear differential equation. As such, its operational equivalent is

$$\varphi(s) = \overline{Z}(D) \, X(s) = \overline{Z}(D) \, \overline{X}(D) \, 1(s) \tag{6}$$

where $\overline{Z}$ and $\overline{X}$ are the operational equivalents of the functions Z and X.

In order to illustrate the operational solution, let it be required to find the function $X(s)$ from

$$\int_0^s \frac{X'(s_0)}{\sqrt{s - s_0}} \, ds_0 - 2X(s) = \frac{4}{3} s^{3/2} - s^2 \tag{7}$$

assuming that $X(0) = 0$. Here the function corresponding to $Z(s)$ in equation (5) is $1/\sqrt{s}$. With the aid of equation (4), the various componenets are writton in operational form

$$Z(s) = \Gamma(1/2) \sqrt{D}\ 1\ (s) \tag{8}$$

$$-4/3 s^{3/2} + s^2 = -\left(\frac{\Gamma(1/2)}{D\sqrt{D}} - \frac{2}{D^2} \right)\ 1\ (s) \tag{9}$$

Equation (6) becomes

$$\Gamma(1/2)\ \sqrt{D}\ X(s) - 2X(s) = \left(\frac{\Gamma(1/2)}{D\sqrt{D}} - \frac{2}{D^2} \right)\ 1\ (s)$$

or

$$X(s) = \frac{\dfrac{\Gamma(1/2)}{D\ \sqrt{D}} - \dfrac{2}{D^2}}{\Gamma(1/2)\sqrt{D} - 2}\ 1(s)$$

Simplifying:

$$X(s) = \frac{1}{D^2}\ \frac{\Gamma(1/2) - \dfrac{2}{\sqrt{D}}}{\Gamma(1/2) - \dfrac{2}{\sqrt{D}}}\ 1 = \frac{1}{D^2}\ 1 = \frac{1}{2}\ s^2 \tag{10}$$

   It will be helpful to review certain aspects of the theory of unsteady lift before procoeding to the application of this theory in airplane dynamics. It is found convenient to think of the lift on the airfoil as composed of three parts: (1) A part due to instantaneous acceleration of the noncirculatory potential flow. This lift is equal to the virtual additional mass of the wing

($\pi\ \dfrac{c^2}{4}\ \rho$ per unit span, for infinite aspect ratio) times

the rate of increase of the relative wind velocity normal to the chord. (2) A part due to the circulatory flow and dependent on the angle-of-attack variation, i.e., the lift given by $c_{l_1}(s)$. (3) A part due to the circulatory

$c_{l_1}(s)$

flow and ascribed to a relative curvature or camber of the airfoil in pitching motion. The third component will be automatically included with the second if the angle of attack is obtained by resolving velocities at the 75-percent-chord point.

With these provisions, the instantaneous lift of an airfoil in combined pitching and vertical motion may be written (see fig. 1)

$$c_l(s) = \mu \left[\alpha'(s) + l_{50}\, \theta''(s) \right]$$

$$+ \left[\alpha(0) + l_{75}\theta'(0) \right] c_{l_1}(s) + \int_0^s c_{l_1}(s - s_0)\left[\alpha'(s_0) + l_{75}\theta''(s_0) \right] ds_0 \tag{11}$$

where μ is a coefficient for the virtual additional mass of the wing ($\mu = \pi$ for infinite aspect ratio). Now let $\bar{c}_{l_1}$ be the operational equivalent of Wagner's function c_{l_1}:

$$c_l(s) = \mu \left[D\alpha + l_{50}D^2\theta \right] + \bar{c}_{l_1}(D)\left[\alpha(s) + l_{75}D\theta(s) \right] \tag{12}$$

No concise formula for $c_{l_1}(s)$ is known although it is found that Wagner's curve is reproduced almost exactly by the equation

$$c_{l_1}(s) = C_0 + C_1^- e^{\lambda_1 s} + C_2 e^{\lambda_2 s} \tag{13}$$

where

$$C_0 = .2\pi$$

$$C_1 = -0.330\,\pi$$

$$C_2 = -0.670\,\pi$$

$$\lambda_1 = -0.0455$$

$$\lambda_2 = -0.300$$

and where s refers to the half-chord as unit, that is

$$s = \frac{Vt}{c/2}$$

In this form, the operational equivalent is readily found from the relation

$$e^{\lambda s} = \frac{D}{D - \lambda} \cdot 1(s) \qquad \text{(See reference 2)} \qquad (14)$$

whence

$$\bar{c}_{l_1}(D) = C_0 + C_1 \frac{D}{D - \lambda_1} + C_2 \frac{D}{D - \lambda_2} \qquad (15)$$

The calculation of lift under a prescribed variation of angle of attack can be illustrated by assuming that the airfoil is given a sinusoidal motion

$$\alpha(s) \ (\text{or} \ \theta(s)) = \text{R.P. or I.P. of } e^{ins} \qquad (16)$$

This variation is reduced to operational form (see equation (14)):

$$\bar{\alpha}(D) = \frac{D}{D - in} \qquad (17)$$

$$c_{l_n}(s) = \mu D \frac{D}{D - in} 1 + \left(C_0 + C_1' \frac{D}{D-\lambda_1} + C_2 \frac{D}{D-\lambda_2} \right) \frac{D}{D-in} 1' \qquad (18)$$

(See equations (12) and (15).)

The resulting operator may be evaluated by the Heaviside expansion theorem:

$$\frac{f(D)}{F(D)} 1 = \frac{f(0)}{F(0)} + \sum_{\lambda} \frac{f(\lambda)}{\lambda F'(\lambda)} e^{\lambda s} \qquad (19)$$

where the λ's are the roots of $F(D) = 0$.

$$c_{l_n}(s) = \mu \, i n e^{ins} + \left[C_0 + C_1 \frac{in}{in - \lambda_1} + C_2 \frac{in}{in - \lambda_2} \right] e^{ins}$$

$$+ C_1 \frac{\lambda_1}{\lambda_1 - in} e^{\lambda_1 s} + C_2 \frac{\lambda_2}{\lambda_2 - in} e^{\lambda_2 s} \quad (20)$$

The terms involving $e^{\lambda s}$ disappear in time and hence may be disregarded in a continuous oscillation. The terms

$$\left[\mu \lambda n + \left[C_0 + \frac{in}{in - \lambda_1} C_1 + \frac{in}{in - \lambda_2} C_2 \right] e^{ins} = 2\pi \left[F + iG \right] e^{ins} \quad (21) \right.$$

yield approximate expressions of the lift functions for the oscillating airfoil introduced by Theodorsen (reference 3):

$$2\pi F(n) = C_0 + C_1 \frac{n^2}{\lambda_1^2 + n^2} + C_2 \frac{n^2}{\lambda_2^2 + n^2}$$

$$(22)$$

$$2\pi G(n) = - C_1 \frac{\lambda_1 n}{\lambda_1^2 + n^2} - C_2 \frac{\lambda_2 n}{\lambda_2^2 + n^2} + \mu n$$

As pointed out by Garrick (reference 4), Theodrosen's function for sinusoidal motion

$$C(in) = F(n) + iG(n) \qquad (\text{equation } (21))$$

may be regarded as the operational equivalent of Wagner's curve, i.e.,

$$2\pi C(D) 1 = \bar{c}_l (D) 1 = c_{l_1}(s) \qquad (23)$$

This fact may be verified by referring to equation (15). This relation is especially interesting because it shows a connection between the Fourier and the operational analyses.

Thus, if the response of a linear system to a continuous sinusoidal excitation is known,

$$R_n(s) = f(in) \ e^{ins} \qquad (24)$$

Then the function f furnishes immediately the operational equivalent of the unit response so that for any variable excitation $Z(s)$,

$$R(s) = f(D) \ Z(s) = f(D) \ \overline{Z}(D) \ 1 \qquad (25)$$

In general, the motion of the airfoil or airplane will not be prescribed beforehand but must be determined from dynamical equations. This type of problem can be illustrated simply by considering the disturbed vertical motion of the airplane without pitching. The dynamical equation in this case is

$$m \frac{dw}{dt} - \text{resisting force} = \text{impressed force}, \ Z \qquad (26)$$

where w is the vertical velocity of the airplane and m is the mass including the virtual additional mass of the wing. Since

$$\frac{dw}{dt} = \frac{V^2}{c/2} \frac{d\alpha}{ds} \qquad (27)$$

$$m \frac{dw}{dt} = \frac{2m}{S \ \rho/2 \ c} \times S \ \rho/2 \ c/2 \times \frac{V^2}{c/2} \frac{d\alpha}{ds} \qquad (28)$$

Making the substitution

$$\frac{2m}{S \ \rho/2 \ c} = \sigma \qquad (29)$$

and writing the equation in coefficient form,

$$\sigma D\alpha + \overline{c}_{l_1} (D) \ \alpha = c_{l_0} (s) \qquad (30)$$

where c_{l_0} is the lift coefficient of the given disturbing force. The operational solution is

$$\alpha(s) = \frac{1}{\sigma D + \overline{c}_{l_1}(D)} c_{l_0}(s) \qquad (31)$$

Again, as in the case of the lift, the solution for the elementary jump is the key to solutions for variable conditions.

$$\alpha_1(s) = \frac{1}{\sigma D + \overline{c}_{l_1}(D)} 1 \qquad (32)$$

Replacing $\overline{c}_{l_1}(D)$ by (15) and simplifying:

$$\alpha_1(s) = \frac{(D - \lambda_1)(D - \lambda_2)}{aD^3 + bD^2 + cD + d} 1 = \frac{f(D)}{F(D)} 1 \qquad (33)$$

which is in standard form for evaluation by the expansion theroem (19). Finally,

$$\alpha(s) = c_{l_0}(o)\, \alpha_1(s) + \int_0^s \alpha_1(s - s_0)\, c_{l_0}{}'(s_0)\, ds_0 \qquad (34)$$

The extension of this treatment to problems involving a number of degrees of freedom will be evident.

Langley Memorial Aeronautical Laboratory,
   National Advisory Committee for Aeronautics,
      Langley Field, Va., September 12, 1938.

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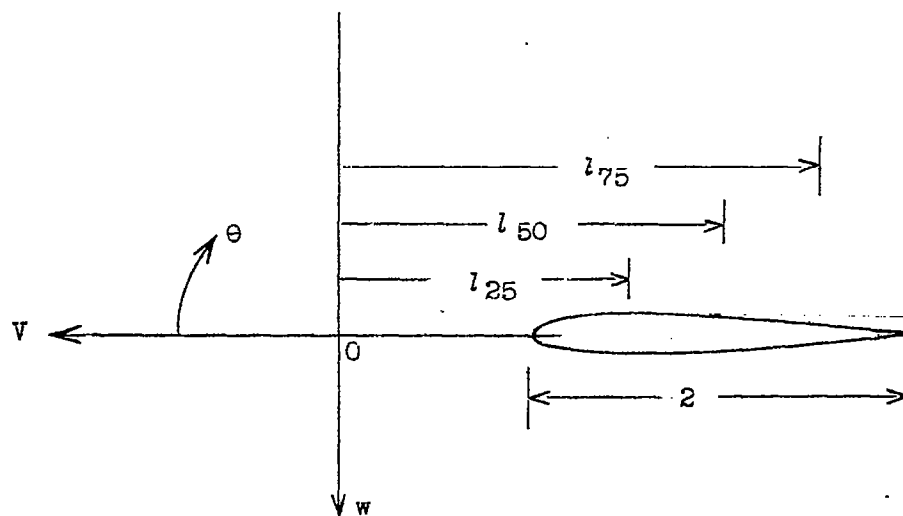   1938.

Figure 1.- Moving axes. $\alpha = w/V$; $s = Vt/c/2$.