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METHODS FOR DETERMINATION AND COMPUTATION OF FLOW PATTERNS OF A COMPRESSIBLE FLUID

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SUMMARY

A well-known method of generating stream functions of an incompressible fluid flow is that of taking the imaginary part of an analytic function of a complex variable. In previous publications of the author this method was generalized to the case of subsonic flows of a compressible fluid. Flow patterns, which until the present, have proved impossible to obtain by existing methods, were, however, obtained by this procedure; for example, flows around an obstacle the boundary of which is a closed curve,[1] as well as around nonsymmetric profiles. The procedure can be extended to the case of partially supersonic flows. As this method for obtaining flow patterns of compressible fluid from analytic functions of a compressible fluid requires rather lengthy computations, the present paper is devoted to a detailed discussion of performing these computations. The operations are divided into two groups: namely, those which need only be carried out once and for all and then can be tabulated (or put on master cards) and those which have to be repeated in every individual case. A detailed description is given concerning the performance of necessary computations on punch card machines. This description is illustrated by an example.

In the appendixes some theoretical questions, which to a certain extent complete the results of NACA Technical Note No. 972, are considered. For instance, in appendix II, some questions which arise in connection with the determination of flow patterns around a nonsymmetric profile and the use of linear integral equations for constructing flow patterns are discussed.

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[1]The method of Von Kármán and Tsien yields a flow around a closed curve. However, this method assumes a linear pressure-specific volume relation, i.e., that $p = A + \sigma/\rho$, where A and σ are constant instead of the actual relation $p = \sigma\rho^k$ (adiabatic case).

In appendix III those alterations are indicated which are necessary in order that the operator which has been introduced for subsonic flows may be transformed into an operator which generates stream functions of supersonic flows from two functions of one real variable.

INTRODUCTION

The mathematical theory of steady two-dimensional flows of an incompressible fluid is based essentially on the fact that a stream function of a flow of this kind can be obtained by taking the imaginary part of a conveniently chosen function of one complex variable $s = \theta + i \log v$, where v is the speed (at the point) and θ the angle which the velocity vector (at the point) forms with a fixed direction.[1]

The success of this method in dealing with problems of the theory of an incompressible fluid, seems to suggest the possibility of generalizing this approach to the case of a compressible fluid. An attempt, in this direction, has been made by the author in previous publications. To this end, instead of $\log v$, there is introduced $\lambda(M)$, a function of the local Mach number M.[2] Further, instead of taking the imaginary part of an arbitrary analytic function (i.e., applying the operator Im(≡Imaginary part of)) as in the case of an incompressible fluid, it is necessary to apply a generalization of this procedure to obtain from $f(\theta + i\lambda(M))$ the desired stream function.

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[1]The introduction of functions of the variable s (instead of the customarily employed functions of $x + iy$, (x,y) being Cartesian coordinates in the plane) causes some difficulties of a mathematical nature; however, in contrast to the latter method, the former, more complicated approach (often called the hodograph method), admits of <u>direct generalization</u> to the case of a compressible fluid.

[2]The function $\lambda(M)$ is real if $M < 1$, and purely imaginary if $M > 1$. Thus, $s = \theta + i\lambda(M)$ is a complex variable in the subsonic case and a real variable in the supersonic case. Note that in the body of the text $\lambda(M) - i\theta = -i(\theta + i\lambda(M))$ is employed.

One of the advantages of this approach is that it manifests a far-reaching analogy[1] with the case of an incompressible fluid, and is capable of yielding flow patterns which have not been obtained until the present - for example, flows around a closed profile, and so forth.[2] This approach makes it possible to determine a flow pattern corresponding to any given function. In general, the actual construction of the flow leads to a considerable amount of computation; consequently, the use of special computational devices such as the differential analyzer, punch cards, and so forth, would seem necessary as well as the preparation of certain tables which are independent of the specific flow pattern, and therefore need be prepared only once.

The most efficient means of accomplishing this is not at all evident, and it is necessary to analyze the needed computations from this point of view. The present report has been prepared in an effort to answer this question, especially as regards punch card machines.[3]

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DESCRIPTION OF METHOD

In the author's previous report a new approach in the two-dimensional theory of a compressible fluid was developed.

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[1]This analogy often serves as an indication of the proper method for obtaining results in the theory of compressible fluid which are similar to the case of an incompressible fluid.

[2]In the only case of a flow around a closed body which has heretofore been considered, Von Kármán and Tsien have assumed a linear pressure-specific volume relation, $p = A + \sigma/\rho$, A, σ being constant, instead of the actual relation $p = \sigma\rho^k$, $k = 1.4$, which is used in papers of the author.

[3]The author would like to point out that other devices, in particular, the differential analyzer, are also of considerable importance for many of the above computations. (See reference 1.)

This method of attack is a generalization of a procedure ordinarily employed in the theory of an incompressible fluid: namely, the generating of stream functions of flows from analytic functions of a complex variable.[1]

For the convenience of the reader the general idea of this method will be described in the following. The stream function ψ of an incompressible fluid flow is a harmonic function - that is, it satisfies Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{1}$$

x, y being Cartesian coordinates in the plane of the flow. Conversely, every function which satisfies equation (1) may be interpreted as a stream function of a suitable flow. Thus, if the imaginary part of an analytic function $f(z)$ of the complex variable $z = x + iy$ is taken, a stream function of a possible flow of an incompressible fluid is obtained. As noted before, the method of generating stream functions in this simple form cannot be extended to the case of a compressible fluid, since in the latter case the partial differential equation which $\psi(x,y)$ satisfies is a very complicated nonlinear one. This situation makes it necessary to use an alternate method, the so-called "hodograph method," - that is, to consider the stream function ψ not as a function of z, but as a function of the velocity vector.

If v_1, v_2, and (v,θ) denote the Cartesian and polar coordinates, respectively, of the velocity vector $\vec{v}$ - that is, if $\vec{v} = v_1 + iv_2 = ve^{i\theta}$ and if the stream function ψ is considered as a function of (v_1,v_2) or of $(\log v,\theta)$, then ψ is in each case a harmonic function of the given variables. That is, if $\psi(x,y)$ is transformed by means of the substitution

$$x = x(v_1,v_2), \quad y = y(v_1,v_2), \qquad \frac{\partial(x,y)}{\partial(v_1,v_2)} \neq 0 \tag{2a}$$

---

[1]In order to make possible this generalization, it is, however, necessary to consider the stream function in the so-called "hodograph" plane (i.e., in the plane the Cartesian coordinates of which are the components of the velocity vector) instead of considering it in the "physical" plane (i.e., in the plane of the flow).

or

$$x = x(\log v, \theta), \quad y = y(\log v, \theta), \qquad \frac{\partial(x,y)}{\partial((\log v),\theta)} \neq 0 \tag{2b}$$

then the functions ψ obtained by the transformation (2a), (2b) satisfy the equation

$$\frac{\partial^2\psi}{\partial v_1{}^2} + \frac{\partial^2\psi}{\partial v_2{}^2} = 0 \tag{3a}$$

in the first case and

$$\frac{\partial^2\psi}{\partial(\log v)^2} + \frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{3b}$$

in the second case. (Note that these ψ's are different functions of their respective arguments, although the notation does not indicate this.)

By writing

$$\psi = \mathrm{Im}[g(v_1 - iv_2)] \tag{4a}$$

or

$$\psi = \mathrm{Im}[h(\log v - i\theta)] \tag{4b}$$

if g and h are arbitrary functions of the complex variable $v_1 - iv_2$, and $\log v - i\theta$, respectively, then the stream functions of possible flows of an incompressible fluid are obtained.

Since the flow pattern in the physical plane is of primary interest, it is necessary in this case, to carry out the transition to the physical plane; that is, to determine ψ as a function of x,y.

It is this second method which, though more complicated than the first, has the advantage of being capable of generalization to the case of a compressible fluid flow for which the equation of state, $p = A + \sigma\rho^k$, holds (A, σ, k are constants, p the pressure, and ρ the density). In the equation of state for an adiabatic process $A = 0$, however, this additional constant does not entail any theoretical difficulties.

As has been indicated previously, the stream function of a compressible fluid flow, considered as a function of (x,y) - that is, in the physical plane - satisfies a nonlinear partial differential equation. If, however, ψ is considered in the hodograph or in the logarithmic plane (i.e., as a function of (v_1,v_2) and of $(\log v,\theta)$, respectively), then ψ satisfies, in each of these planes, a linear partial differential equation.

In order to simplify this equation it is expedient to introduce, instead of log v, a new variable λ,

$$\lambda = \frac{1}{2} \log \left[\frac{1 - (1 - M^2)^{1/2}}{1 + (1 - M^2)^{1/2}} \left(\frac{1 + h(1 - M^2)^{1/2}}{1 - h(1 - M^2)^{1/2}}\right)^{1/h}\right] \qquad (5)$$

where

$$M = \frac{\frac{v}{a_0}}{\left[1 - \frac{1}{2}(k - 1)\frac{v^2}{{a_0}^2}\right]^{1/2}}$$

and

$$h = \left[\frac{k - 1}{k + 1}\right]^{1/2}, \quad k > 1;$$

here k is the ratio of specific heats of the gas ($k = 1.4$ for air), and a_0 the velocity of sound at a stagnation point. The equation which ψ satisfies then assumes a particularly simple form: namely,[1]

$$L_0(\psi) \equiv \frac{1}{4}\left(\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\theta^2}\right) + N\frac{\partial\psi}{\partial\lambda}$$

$$\equiv \left(\frac{\partial^2\psi}{\partial Z\partial\overline{Z}}\right) + N\left(\frac{\partial\psi}{\partial Z} + \frac{\partial\psi}{\partial\overline{Z}}\right) = 0 \qquad (6)$$

---

[1]In the following, instead of λ and θ the complex variables $Z = \lambda - i\theta$, $\overline{Z} = \lambda + i\theta$ will frequently be used. The derivatives with respect to Z and $\overline{Z}$ have the following meaning

$$\frac{\partial}{\partial Z} = \frac{1}{2}\left(\frac{\partial}{\partial\lambda} + i\frac{\partial}{\partial\theta}\right), \quad \frac{\partial}{\partial\overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial\lambda} - i\frac{\partial}{\partial\theta}\right), \quad \frac{\partial^2}{\partial Z\partial\overline{Z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial\lambda^2} + \frac{\partial^2}{\partial\theta^2}\right)$$

where

$$N = -\frac{(k+1)M^4}{8(1-M^2)^{3/2}} \tag{7}$$

In order to obtain a generalization of the representation (4b), the author in the previous report derived the following result:

From the function N (see (7)), certain other functions $H(2\lambda)$, $Q_n^{(m)}(2\lambda)$, $n = 1,2,\ldots$, $m = 1,2,\ldots$ were determined, and it is proved that the expression

$$\psi(\lambda,\theta) = \mathrm{Im}\left\{H(2\lambda)\left[g(Z) + \lim_{m\to\infty}\sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q_m^{(n)}(2\lambda-2\alpha)\int_0^Z\cdots\int_0^{\zeta_{n-1}} g(\zeta_n)d\zeta_n\ldots d\zeta_1\right]\right\} \tag{8}$$ [1]

(where $g(Z)$ is an arbitrary analytic function α, an arbitrary non-negative constant) is a solution of (6). Thus, from an arbitrary analytic function it is possible to derive a function $\psi[\lambda(v),\theta]$, which represents the stream function of a possible (subsonic) flow of a compressible fluid.[2] Formula (8) can also be written in another form which is suitable for certain purposes: namely,

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[1]It has been proved, subsequently, that it is possible to interchange the summation and the passage to the limit in equation (8) to obtain

$$\psi(\lambda,\theta) = \mathrm{Im}\left\{H(2\lambda)\left[g(Z) + \sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q^{(n)}(2\lambda)\int_0^Z\cdots\int_0^{\zeta_{n-1}} g(\zeta_n)d\zeta_n\ldots d\zeta_1\right]\right\} \tag{8a}$$

However, as it is desired to make no reference to unpublished results, all the computations in this report are presented in such fashion that the use of (8) instead of (8a) entails no additional computation; equation (8) is almost always employed throughout the following.

[2]Formula (8) can be considered as a direct generalization of (4b), since by choosing $H = 1$, $Q_m^{(n)} = 0$, for all n and m, and $Z = \log v - i\theta$, (8) becomes (4b).

$$\psi(\lambda,\theta) = \mathrm{Im}\left\{ H(2\lambda) \left[\int_{-1}^{+1} f\left(\frac{Z(1-t^2)}{2}\right) \frac{dt}{\sqrt{1-t^2}} + \lim_{m\to\infty} \int_{t=-1}^{+1} E_m(\lambda,t)\, f\left(\frac{Z(1-t^2)}{2}\right) \frac{dt}{\sqrt{1-t^2}} \right] \right\} \qquad (9)$$

$$E_m(\lambda,t) = 1 + \sum_{n=1}^{\infty} t^{2n} Q_m^{(n)} (2\lambda - 2\alpha)$$

where $f(Z)$ is again an arbitrary analytic function of Z.[1]

It should be emphasized that the functions $H(2\lambda)$, $Q_m^{(n)}(2\lambda)$ are independent of the function g, and hence once computed (for a given value of k) may be employed in all other stream computations without change.

Once $\psi(\lambda,\theta)$ (corresponding to a given function g) has been computed, the transition to the physical plane - that is, the determination of the corresponding flow pattern in the physical plane - does not involve any theoretical difficulty.

Two problems immediately arise in connection with this method of attack.

I. How to determine function g, in (8), so as to obtain, in the physical plane, a flow around a given obstacle or in a channel whose boundary curves are given.

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[1]Function $f(Z)$ is connected with $g(Z)$ by the following relation:

$$f(Z) = \frac{2}{\pi} \int_0^{\pi/2} Z \sin\vartheta \, \frac{dg(2Z \sin^2\vartheta)}{d(Z \sin^2\vartheta)} \, d\vartheta + \frac{g(0)}{\pi}$$

II. Assume that $g(Z)$ is known,[1] to develop a procedure which would permit the determination of the corresponding flow pattern in the physical plane with the minimum of computation. Naturally, the flow patterns in which the aerodynamicist is primarily interested are partially supersonic ones. Since the subsonic case serves as a basis for further developments, as outlined in reference 3,[2] the author will limit himself in the present report primarily to this case.[3]

Although problem II does not entail any theoretical difficulty, it does involve a very considerable amount of numerical computations for applications, as can be seen from the example described in reference 4, section 3, a fact which represents a serious obstacle for the application of the method.

Since the determination of various flow patterns is one of the purposes of the theory, the above-described situation suggests two possible modifications of the procedure for generating flow patterns.

1. The modification of the method so that a substantial part of the computation is independent of the particular choice of g; thus these computations can be carried out and tabulated once and for all.[4]

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[1]It may be remarked here that often a first approximation to the desired flow pattern of a compressible fluid is obtained by substituting for $g(Z)$ in (8) that analytic function the imaginary part $G(\log v,\theta) = \text{Im}\, g(\log v - i\theta)$ of which gives the desired flow pattern in the physical plane for an incompressible fluid. The corrections which are necessary in obtaining a better approximation, as well as other methods of determining $g(Z)$, will be discussed in future reports. (See also reference 2.)

[2]In sec. 16 of reference 3 a procedure is described which makes it possible to generalize this method to the case of partially supersonic flows.

[3]The author intends in a succeeding report to consider analogous questions for the case of a mixed flow in the light of methods described in sec. 17 of reference 3.

[4]The need of tabulating various functions which appear in the theory of compressible fluids has been emphasized by some authors. (See, for example, Garrick and Kaplan (reference 5), where the Chaplygin solutions have been tabulated.)

2. The rearrangement of the remaining computations (which must be repeated in every particular case) in such a form that they can be carried out with a minimum amount of labor using a punch card machine.[1]

The main purpose of the present report is the development of a method of determining the flow patterns according to requirements 1 and 2.

In four additional notes certain problems considered in reference 3 are developed further; these are of a more theoretical nature.

In appendix II, the author shows that by employing results obtained from a consideration of the singularities of the solutions of (8) and applying the theory of linear integral equations, it is possible to determine a flow for a given hodograph. In certain cases, solutions of this kind can be considered as a first approximation to the solution of boundary value problems.

In appendix III methods are given for the construction of purely supersonic flows, which methods employ various integral operator representations.

The derivation of the complex potential for a Joukowski profile is given in appendix IV, while appendix I is devoted to the question of determining the $Q_m^{(n)}$ and $L_m^{(n)}$.

NOTATION

The following list of notation is to serve the double purpose of being both an index of symbols used in the present report and a collection of some of the formulas, used in previous reports, to which reference is made in the text; however, no claim to completeness is made in this respect.

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[1]As has been emphasized by Kraft and Dibble, certain aspects of this theory may be successfully treated by use of the differential analyzer. (See reference 1.)

$$u_z = \frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y}\right); \qquad u_{\bar{z}} = \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\,\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{1}{4}\Delta u; \quad z = x + iy, \quad \bar{z} = x - iy$$

$a = \left[a_0{}^2 - \frac{1}{2}(k-1)v^2\right]^{1/2}$ speed of sound (reference 3, equation (28))

a_0 speed of sound at a stagnation point

a_n coefficients in the series expansion of T in powers of x (formula (45))

b_n coefficients in the series expansion of T^{-1} in powers of x (formula (46))

e base of Naperian logarithms

$\exp(x) = e^x$

f_1, f_2 arbitrary twice continuously differentiable functions of their arguments

$g(Z)$ an analytic function of the complex variable Z

$$g^{(-1)}(Z) = \frac{dg(Z)}{dZ} = \frac{dg^{(o)}(Z)}{dZ}$$

$g^{(o)}(Z) = g(Z)$

$g^{(n)}(Z)$ nth iterated integral of $g(Z)$ (formula (10))

$h = \left(\frac{k-1}{k+1}\right)^{1/2}$, for $k > 1$

k constant in the equation of state $p = A + \sigma\rho^k$: The ratio of specific heats at constant pressure to constant volume; $k = 1.4$ for air (reference 3, formula (22))

$l(H) = \left(\frac{\rho_o}{\rho(H)}\right)^2 (1-M^2(H)) = \left(\frac{\partial \Lambda}{\partial H}\right)^2$ (reference 3, formula (45))

p pressure

p_0 pressure at a stagnation point

schlicht ≡ univalent

v magnitude of the velocity vector $\vec{v}$; occasionally, the reduced speed v/a_0

$\vec{v}$ velocity vector; that is, $\vec{v} = ve^{i\theta}$

v_1, v_2 Cartesian components of $\vec{v}$; that is, $\vec{v} = v_1 + iv_2$

(x, y) Cartesian coordinates in the physical plane

$x = e^{a\lambda}$ note that in reference 3, $X = e^{a\lambda}$

$x_1 = \left(\frac{1-h}{1+h}\right)^{1/h} x$

A constant in the equation of state $p = A + \sigma\rho^k$ (reference 3, formula (22))

$A = \frac{1}{8}(k+1)M^4(M^2-1)^{-3/2}$ (formula (67))

A_n coefficients in the series expansion of T in powers of x_1 (formula (45))

$A_{n,m}$ coefficients in the series expansion of $g(Z)$ as a function of ζ, of the fractional powers of ζ (formula (17))

$A^+_{n,m} = \max(A_{n,m}, 0)$, $A^-_{n,m} = \max(-A_{n,m}, 0)$

B_n coefficients in the series expansion of T^{-1} in powers of x_1 (formula (46))

$C_{n,m}$ coefficients in the series expansion of $g(Z)$ as a function of ζ, of the integral powers of ζ (formula (17))

$C^+_{n,m} = \max(C_{n,m}, 0)$; $C^-_{n,m} = \max(-C_{n,m}, 0)$

E_1, E_2, E_1^*, E_2^* (See appendix III, sec. 3); (reference 3, theorem (53).) Note that in reference 3

$$E^* = \exp\left(\int_{-\infty}^{\bar{\zeta}+\zeta} N d(\bar{\zeta}+\zeta)\right) E,$$ differing from the usage here.

$F = -\left(\frac{1}{2}N_\zeta + N^2\right) = \dfrac{(k+1)M^4}{64(1-M^2)^3}\left[-(3k-1)M^4 - 4(3-2k)M^2 + 16\right]$; (formula (42)); (reference 3, formula (71))

$F_m(2\lambda)$ polynomial approximation of mth degree in x_1 to F

G operation a computation which, since it is independent of the flow, can be computed and tabulated once and for all

$$H = \exp\left(-\int_{-\infty}^{\bar{\zeta}+\zeta} N d(\bar{\zeta}+\zeta)\right) = \frac{1}{(1-M^2)^{1/4}}\left[\frac{1}{1+\frac{1}{2}(k-1)M^2}\right]^{1/2(k-1)}$$

for the subsonic case (reference 3, formula (111)); and

$$H = \exp\left(\int^{\xi+\eta} A(s)ds\right)$$ for the supersonic case; (formula (68));

in this sense H is used only in the series expansion of ψ, as formula (8)

$H = \int_0^V \frac{\rho}{V} dv$ (formula (52)); (reference 3, formula (42)); in this sense H is used as an independent variable.

Im imaginary part of

$L_0(\psi) \equiv \frac{1}{4}\left(\frac{\partial^2\psi}{\partial\lambda^2} + \frac{\partial^2\psi}{\partial\theta^2}\right) + N\frac{\partial\psi}{\partial\lambda}$ (formula (6)); (reference 3, formula (46))

$L^{(n)}(2\lambda) = \dfrac{(2n)!}{2^n n!} H(2\lambda) Q^{(n)}(2\lambda)$ (formula (22))

$L_m^{(n)} = \frac{(2n)!}{2^n n!} H(2\lambda) Q_m^{(n)}(2\lambda)$

M local Mach number; $M = v/a = \frac{v}{\left[a_o^2 - \frac{1}{2}(k-1)v^2\right]^{1/2}}$
(formula (5)); (reference 3, formula (31))

$N = -\frac{(k+1)}{8}\frac{M^4}{(1-M^2)^{3/2}}$ (formula (7)); (reference 3, formula (47))

$Q^{(1)} = -4\int_{-\infty}^{\lambda} F\,d\lambda$ (formulas (49) and (8)); (reference 3, formula (107))

$Q^{(2)} = \frac{4}{3} F + \frac{1}{6}\left(Q^{(1)}\right)^2$ (formulas (49) and (8)); (reference 3, formula (108))

$Q^{(n)}$ functions, independent of the flow, which occur in the series expansion of ψ (See formulas (49) and (8); reference 3, formula (84).)

$Q_m^{(n)}$ $Q^{(n)}$ computed employing F_m instead of F (See $Q^{(1)}$, etc.)

$R^{(o)} = H \frac{d\lambda}{dv}$ (formula (23)); (reference 3, formula (114), ff)

$R^{(n)}$ functions, independent of the flow, which occur in the series expansion of ψ (formula (23)); (reference 3, formula (114), ff)

$R_m^{(n)}$ $R^{(n)}$ computed employing F_m instead of F

Re real part of

S operation a computation which must be repeated for each individual flow pattern to be computed

$S = 1 - T$ (formula (44)); note in reference 3, formula (161), s is used for $1 - T$.

$S^{(n)}$ real part of $g^{(n)}$, that is, $g^{(n)} = S^{(n)} + iT^{(n)}$

$T^{(n)}$ imaginary part of $g^{(n)}$

$T = \sqrt{1-M^2}$

T_m polynomial approximation of the mth degree in x_1 to T

$W^*(\lambda, \theta; \lambda^{(o)}, \theta^{(o)})$; a fundamental solution of formula (6), possessing logarithmic singularity, at $\zeta^{(o)} = \lambda^{(o)} + i\theta^{(o)}$. (See also reference 3, section 13.)

$X_m^{(p)}(v, \theta)$ (See formula (35).)

$Y_m^{(p)}(v, \theta)$ (See formula (35).)

$\alpha_{2p+1}, \alpha_{2p}$ real and imaginary parts, respectively, of the coefficients of $Z^{(p)}$ in the power series expansion of $g(Z)$; that is, $g(Z) = \sum_{p=0}^{\infty} (\alpha_{2p+1} + i\alpha_{2p}) Z^{(p)}$ (See formula (36).)

$\alpha_n^+ = \max(\alpha_n, 0)$, $\alpha_n^- = \max(-\alpha_n, 0)$

$\beta(M) = -\left[\tan^{-1}\sqrt{M^2-1} - \frac{1}{h}\tan^{-1}\left(h\sqrt{M^2-1}\right)\right]$ (formula (65))

$\zeta = Z + \log 2$ (formula (16)). In the appendixes $\zeta = \lambda - i\theta$.

$\eta = -\theta + \beta(M)$ (formula (64))

θ angle which $\vec{v}$ makes with the real axis

$\theta^{(o)}, \theta^{(1)}, \ldots$ values of θ at mesh points for a "lattice" computation (See sec. 2.)

$\lambda = \frac{1}{2}\log\left[\left(\frac{1-\sqrt{1-M^2}}{1+\sqrt{1-M^2}}\right)\left(\frac{1+h\sqrt{1-M^2}}{1-h\sqrt{1-M^2}}\right)^{1/h}\right]$ (formula (5)); (reference 3, formula (48))

λ_o λ corresponding to the maximum Mach number of a flow in D_2

$\lambda^{(o)}, \lambda^{(1)}, \ldots$ values of λ at mesh points for a "lattice" computation (See sec. 2.)

$\zeta = \theta + \beta(M)$ (formula (64))

ρ density; $\rho = \rho_o \left[1 - \frac{k-1}{2a_o^2} v^2\right]^{1/k-1}$ (reference 3, formula (25))

ρ modulus of ζ; that is, $\zeta = \rho e^{i\varphi}$ (formula (19) ff)

ρ_o density at a stagnation point

σ constant in the equation of state: $p = A + \sigma\rho^k$ (reference 3, formula (22))

φ potential function

φ argument of ζ; that is, $\zeta = \rho e^{i\varphi}$ (formula (19) ff)

ψ stream function

$\psi^* = \exp\left(\int_{-\infty}^{\bar{\zeta}+\zeta} N d(\bar{\zeta}+\zeta)\right)\psi$, for the subsonic case (formula (41)); (reference 3, formula (69)); and

$\psi^* = \exp\left(\int^{\xi+\eta} A(s)ds\right)\psi$ for the supersonic case (formula (68))

Δ Laplace operator $\Delta\psi(x,y) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 4\left(\frac{\partial^2\psi}{\partial z\partial\bar{z}}\right)$

$Z = \lambda - i\theta$

$\bar{Z} = \lambda + i\theta$

$\Lambda = \int^H \sqrt{l(H)}\, dH$ that is, $\frac{d\Lambda}{dH} = \sqrt{l(H)}$ (reference 3, formula (45)); (See sec. 5.)

$\Lambda_H = \frac{\partial\Lambda}{\partial H}$

$\chi^{(n)}$ (See formula (33) ff.)

$\underline{\psi}$ (See sec. 5 of appendix III.)

Remark: Observe that quite frequently functions will be considered in different planes although the notation may not, in general, indicate this. Thus, given $f(x,y)$, let

$$x = x(x^1, x^2), \quad y = y(x^1, x^2), \quad \frac{\partial(x^1, x^2)}{\partial(x, y)} \neq 0$$

and obtain $f(x(x^1,x^2),\ y(x^1,x^2)) = f^1(x^1,x^2)$. The superscript will be omitted and only $f(x^1,x^2)$ written, since the meaning will be clear from the context.

ANALYSIS

1. An Outline of the Method to be Developed in the Present Report

A method of determining the stream function (in the physical plane) corresponding to a given function g, the basis of which method is equation (8), was given in section 3 of reference 4, together with a numerical example. An outline of this method has been given in the introduction.

However, a good deal of numerical work is entailed by this approach, and the amount of labor involved increases considerably for a flow the maximum velocity of which is close to the velocity of sound. For this reason, a modification of the method which would cut down the amount of computation is desirable. A description of the proposed modification follows:

The domain, D (in the (λ, θ)-plane), in which the function $g(Z)$ is to be considered, can be divided into two distinct parts D_1 and D_2, defined as the subdomains in which $\lambda_0 < \lambda < 0$ and $\lambda < \lambda_0$, respectively. (See fig. 1.) The number λ_0 is a preassigned number which can be altered to suit the case[1] although, in general, it will lie somewhere

[1]The choice of λ_0 will depend upon the conditions in each case; for the most part, λ_0 must be larger than the maximal λ coordinate of the regular points of $g(Z)$ in D.

between $\lambda = -0.4$ and $\lambda = -0.1$, corresponding to local Mach numbers $M = 0.65$ and $M = 0.85$, respectively (M as defined in equation (5)).

In D_1 the argument λ varies over values which are near zero, and, as a consequence the series (8) will converge very slowly, necessitating taking into account a great number of terms in order to obtain a reasonable degree of accuracy. On the other hand, g of equation (8) and therefore f is regular in D_1 and can be represented there by a series development.[1]

In the domain D_2 the values of λ are much smaller and therefore only a smaller number of terms of equation (8) need be taken into account. On the other hand, in D_2 the behavior of g may be considerably more complex; for example, g may have singularities and be many-valued. Therefore, it will be assumed that g is given by its numerical values on a sufficiently fine lattice, or by a number of series, each of which converges in some subdomain of D_2.

In D_1 the function g can be represented by a power series development, and since the operator (8) is linear it is hence possible to prepare tables once and for all, which will facilitate, to a very large extent, the determination of the flow pattern (in the physical plane). This will be explained in detail in section 4.

In order to determine the flow in the domain corresponding to D_2, the procedure of section 3 of reference 4 may be applied. Since the computations are rather extensive, it is expedient to employ mechanical devices. This requires a certain modification of the above procedure, which modification will be described in section 2. Thus, two methods for determining the flow corresponding to a given $g(Z)$ will be described in sections 2 and 3. Both methods employ punch card machines; in addition, the second method presupposes that the computor has certain tables available which are independent of the particular flow and hence can be computed once and for all.

<u>Remark</u>: The division of D into two subdomains D_1 and D_2 is not necessarily to take place along the line $\lambda = \lambda_0$. It will often be more convenient to subdivide D as

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[1] Or, if more convenient, by a polynomial approximation.

indicated in figure 2, so that the tables which have been prepared may be used for the largest feasible part of the domain D.

Remark: In order to emphasize the character of a computation which is being performed - that is, whether it is one of that large class which need be computed and tabulated only once since they are independent of the particular flow, or whether the computation involved holds good only for an individual flow - to every description of a computation will be added the characterization "(G operation)" or "(S operation)" according to its membership in the former or latter class of operations.

2. Description of the First Method for the Construction of the Stream Function of a Compressible Fluid Flow by Use of Punch Card Machines

In this section the computation of a subsonic compressible flow by means of punch card machines will be described. This procedure is a modification of the method of section 3 of reference 4.[1]

As indicated in that report, the procedure was divided into three separate stages.

I. Computation of the integrals

$$g^{(n)}(Z) = \int_0^Z \cdots \int_0^{Z_{n-1}} g(Z_n)dZ_n\, dZ_{n-1} \ldots dZ_1 \qquad (10)$$

and the derivative $\dfrac{dg^{(o)}}{dZ}$

where $g^{(o)}(Z) \equiv g(Z)$, $Z = \lambda - i\theta$, is an analytic function

II. Construction of the flow in the (λ,θ)-plane - that is, evaluation of expression (8)

[1]In sec. 2 of reference 4, it was assumed that only an ordinary computing machine was to be used in performing the operations described there.

III. Transition from the logarithmic plane to the physical plane

Step I.- Three different methods of evaluation of $g^{(n)}(Z)$, $n = 0, 1, 2, \ldots$ and of $(dg^{(o)}(Z)/dZ)$ will be given in the following; two of these methods employ punch card machines; the third uses graphical means.

The first method is to be applied if the real and imaginary parts of $g(Z)$ are given numerically on a sufficiently dense set of points (λ_k, θ_k) of the lattice.

The second method can be used when the function $g^{(o)}(Z)$ is given analytically and can be represented in the whole region concerned by several series developments[1] around conveniently chosen points.

The third method is much less exact; it can be used in order to check the results obtained by one of the above-described methods.

$$g^{(n+1)}(Z) = \int_0^Z g^{(n)}(Z_1)dZ_1 \tag{11}$$

may be written in the form

$$g^{(n+1)}(Z) = \int_0^{(\lambda,\theta)} (S^{(n)}d\lambda + T^{(n)}d\theta) + i\int_0^{(\lambda,\theta)} (T^{(n)}d\lambda - S^{(n)}d\theta) \tag{12}$$

where

$$g^{(n)} = S^{(n)} + iT^{(n)} \; ^{2}$$

(See equation (20) of reference 4.) The right-hand side of equation (12) may then be replaced by the approximating sum (13).

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[1]These series developments are not necessarily power series since g can have singularities in D_2, i.e., branch points, poles, etc.

[2]Observe that $\varphi^{(n)}$, $\psi^{(n)}$ of sec. 3 of reference 4 are replaced by $S^{(n)}$, $T^{(n)}$, respectively.

$$g^{(n+1)}(z) = \sum_{k=1}^{s} S^{(n)}\left[\lambda_o + (k-1)\Delta\lambda, \theta_o\right]\Delta\lambda + i \sum_{k=1}^{s} T^{(n)}\left[\lambda_o + (k-1)\Delta\lambda, \theta_o\right]\Delta\lambda + \sum_{k=1}^{s} T^{(n)}\left[\lambda_o, \theta_o + (k-1)\Delta\theta\right]\Delta\theta - i \sum_{k=1}^{s} S^{(n)}\left[\lambda_o, \theta_o + (k-1)\Delta\theta\right]\Delta\theta \qquad (13)$$

(See (21) of reference 4.) The terms $\Delta\lambda, \Delta\theta$ denote the directed distances between the meshes of the lattice (see fig 3); that is, they are positive if the integration proceeds in a positive direction, otherwise negative.

As indicated above, the (approximate) integration $\int_{(0,0)}^{(\lambda,\theta)}$ is to be carried at first from (0,0) to $(\lambda, 0)$ along $\theta = 0$ (or if more convenient from $(0,\theta)$ to (λ,θ) along $\theta = \theta_o$), and then from $(\lambda, 0)$ to (λ, θ) along λ = constant.

A. (All computations of A are (S operations).) The sums $\sum_{k=1}^{s} S^{(0)}\left[\lambda_o + (k-1)\Delta\lambda, \theta_o\right]\Delta\lambda$, $s = 1, 2, 3, \ldots$ can be computed on punch card machines by the following procedure. Every number[1] $\left|S^{(0)}[\lambda_o + (k-1)\lambda, \theta_o]\right|$, $k = 1, 2, \ldots, n$, is to be punched, say in columns 1 to 6, into a single card of a set N_1. With every entry on this card an extra column c_1 (say, col. 7) is employed in which a number, say 1, is punched if $S^{(0)}[\lambda_o + (k-1)\Delta\lambda, \theta_o]$ is negative, and nothing is punched if the above number is positive. Then the cards are set for progressive totaling; $\left|S^{(0)}[\lambda_o + (k-1)\Delta\lambda, \theta_o]\right|$ will be added if nothing is punched in the column c_1 and subtracted if 1 is punched in this column. The machine stops after each addition (or subtraction) punches the absolute value of the progressive total, in a new card s_k, $k = 1, 2, \ldots, n$, say in

[1]The symbol "| |" indicates that sign of $S^{(0)}$ has to be disregarded.

columns 1 to 6, and in an extra column c_2, punches 1 if the total is negative and nothing if it is positive.

Now the absolute value of $\Delta\lambda$ is punched in an extra card M and, as before, 1 is entered in an extra column, c_3 if $\Delta\lambda$ is negative, and nothing if it is positive. Now, in a multiplying machine every number on the card s_k is multiplied with the number of M. In order to obtain the right sign, an extra column, c_4, is provided in the new card. If the columns c_2 and c_3, are both empty or both have 1 punched, then the machine will punch nothing into the column c_4. If, however, in one of the columns c_2 (or c_3) the number 1 is punched and the other column, c_3 (or c_2), is empty, the machine will punch 1 in column c_4.

The obtained results then have to be printed. In analogous manner the remaining sums are to be evaluated.

The obtained cards can then be used for evaluation of $S^{(1)}$ and $T^{(1)}$, and so forth.

Remarks: Clearly, the approximate summation can replace the integration, only if the integrand is uniformly continuous. Since, in general, the integrand has singularities, it is necessary to replace the approximate summation in the neighborhood of these points by the exact formula. This can be done, for instance, using series developments around the considered singularity, (for details, see method B) or by other methods.

The derivatives of first order, $dg^{(o)}/dZ = dg^{(o)}/d\lambda$, may be obtained by replacing differentials by finite differences. (See method C.)

B. A method, employing the example

$$g(Z) = \frac{1}{2}\left[(1 - 2e^{Z})^{1/2} + (1 - 2e^{Z})^{-1/2}\right] \qquad (14)$$

considered in reference 4, which may be successfully applied when $g(Z)$ is given by series developments, will now be described.

Remark: If the function $g(Z)$ is given in an analytic form, then it is always possible to represent it by finitely many series developments.[1]

In the case of the function given by the right-hand side of equation (14), the series given in reference 4, equation (25) may be used in order to represent g in the domain D_2, $\lambda < -0.691$.

Another series development of equation (14) which is more suitable for the present purposes can be obtained in the following manner.

The above function $g(Z)$ possesses singularities (branch points of the second order) at the points

$$Z = -\log 2 + i\kappa\pi, \quad \kappa = 0, \pm 1, \pm 2, \ldots \tag{15}$$

only. By classical results of the theory of functions $g(Z)$ can be expanded in series in powers of $\zeta^{1/2}$

$$\zeta = Z + \log 2 \tag{16}$$

which series will converge for $|\zeta| < 2\pi$ and therefore will represent g in a large part of the domain, $D_1 + D_2$, which is of interest.[2] A formal computation yields

$$\left.\begin{aligned} g \equiv g^{(o)} &= \frac{1}{2}\left[(1 - e^{\zeta})^{1/2} + (1 - e^{\zeta})^{-1/2}\right] = i \sum_{n=o}^{\infty} A_{0,m}\zeta^{m-1/2} \\ g^{(-1)} &\equiv \frac{dg^{(o)}}{dZ} = \frac{dg^{(o)}}{d\zeta} = i \sum_{m=o}^{\infty} A_{-1,m}\zeta^{m-3/2} \\ g^{(n)} &= i\left[\sum_{n=o}^{\infty} A_{n,m}\,\zeta^{n+m-1/2}\right] + i \sum_{m=o}^{n-1} C_{n,m}\,\zeta^{m} \end{aligned}\right\} \tag{17}$$

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[1] A derivation of the analytic expression for the complex potential in the hodograph plane for a flow of an incompressible fluid around a Joukowski profile is given in appendix IV. By using this formula together with classical results of the theory of functions, the series developments for the above case can be derived.

[2] Note that in example under consideration ψ is determined not merely in D_2, but also in D_1 by the method described in the present section.

The values of $A_{n,m}$ and $C_{n,m}$ are given in tables 5 and 6, respectively.

By writing

$$g^{(n)} = S^{(n)} + iT^{(n)}, \quad n = -1,\ 0,\ 1,\ 2,\ .\ .\ . \tag{18}$$

there is obtained

$$\left.\begin{aligned} S^{(n)} &= -\sum_{m=0}^{\infty} A_{n,m}\rho^{n+m-1/2}\sin\left[\left(n+m-\frac{1}{2}\right)\varphi\right] - \sum_{m=0}^{n-1} C_{n,m}\rho^{m}\sin m\varphi \\ T^{(n)} &= \sum_{m=0}^{\infty} A_{n,m}\rho^{n+m-1/2}\cos\left[\left(n+m-\frac{1}{2}\right)\varphi\right] + \sum_{n=0}^{m-1} C_{n,m}\rho^{m}\cos m\varphi \end{aligned}\right\} \tag{19}$$

where

$$\zeta = \rho e^{i\varphi}$$

The evaluation of the $S^{(n)}$ and $T^{(n)}$ on a punch card machine proceeds as follows:

The values of $\rho^{k/2}$, $k = \pm 1, \pm 2, \pm 3, \ldots$, $\rho^{1/2} = 0.1$, $0.2, \ldots$, of $\cos\left(\frac{k}{2}\varphi\right)$, and of $\sin\left(\frac{k}{2}\varphi\right)$, $k = \pm 1, \pm 2, \ldots \mu$, $\varphi = 0^{\circ}, 30^{\circ}, 60^{\circ}, \ldots, 330^{\circ}$ can easily be computed (see tables 7 and 8) and entered on three sets of punch cards A, B, C, respectively (G operation). By using set A, two new sets, D and E are then prepared (the following are all (S operations)). On every punch card of the set D the values of $A^{+}_{n,m}\rho^{n+m-1/2}$ and of $C^{+}_{n,m}\rho^{m}$ for a fixed n and fixed ρ are entered, say $A^{+}_{n,o}\rho^{n-1/2}$ are punched in columns 1 to 6, $A^{+}_{n,1}\rho^{n+1/2}$ in columns 7 to 12, $A^{+}_{n,2}\rho^{n+3/2}$ in columns 13 to 18, and so forth. Here $A^{+}_{n,m}$ denotes $A_{n,m}$ if $A_{n,m}$ is positive, and 0 if $A_{n,m}$ is zero or negative; $C^{+}_{n,m}$ has an analogous meaning. In a similar manner $A^{-}_{n,m}\rho^{n+m-1/2}$ and of $C^{-}_{n,m}\rho^{m}$ are entered on the cards of set E. (Again $A^{-}_{n,m} = 0$ if $A_{n,m} > 0$, and equals $-A_{n,m}$ if $A_{n,m} \leqq 0$; the same holds for $C^{-}_{n,m}$.) By using the sets C and D,

$$\sum_{m=0}^{\mu} A_{n,m}^{+}\rho^{n+m-1/2} \sin\left[\left(n+m-\frac{1}{2}\right)\varphi\right] + \sum_{m=0}^{n-1} C_{n,m}^{+}\rho^{m} \sin m\varphi \quad (20)$$

is evaluated, and by using the sets C and E there may be computed

$$\sum_{m=0}^{\mu} A_{n,m}^{-}\rho^{n+m-1/2} \sin\left[\left(n+m-\frac{1}{2}\right)\varphi\right] + \sum_{m=0}^{n-1} C_{n,m}^{-}\rho^{m} \sin m\varphi \quad (21)$$

By subtracting (20) from (21) $S^{(n)}$ is obtained. Similarly, $T^{(n)}$ can be determined. By interpolation, the values of $S^{(n)}(\lambda,\theta)$ and $T^{(n)}(\lambda,\theta)$ may be determined at intermediate points. [Note that $\rho e^{i\varphi} = (\lambda - i\theta) + \log 2$, which yields the relation between (ρ,φ) and (λ,θ). Alternately, the expressions (19) may be evaluated by adding on cards of the sets B and C an extra column, say, column 7, in which nothing is punched if the corresponding sine or cosine is positive and, say, 1 is punched if it is negative. In columns 1 to 6 the absolute value of the sine or cosine is entered.

Analogously, on cards of the set D an additional column is provided in which 1 or nothing is punched according to the sign of $A_{n,m}$ or $C_{n,m}$. The actual multiplication of the two factors proceeds similarly to that of method A.

Since in the future it will be necessary to have values of $S^{(n)}$ and $T^{(n)}$ along lines λ = constant, these values for various values of θ and for $\lambda = -0.02, -0.06, -0.10$, and so forth, were computed.[1] (See table 9.)

C. The method described below is essentially the same as that described in method A; however, now, instead of punch card methods, graphical means are employed.

On millimeter paper the values of $S^{(o)}(\lambda,\theta)$ and $T^{(o)}(\lambda,\theta)$ at first for some fixed values of λ, say, for $\lambda = -0.02, -0.06$, and so forth, and then for some fixed values of θ, are drawn. (All operations of C are (S operations).) (See figs. 8 to 15.)

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[1]A portion of these values had already been computed (much less exactly) and presented in table II or reference 4, where the symbol T_n was used instead of $T^{(n)}$.

If $S_\lambda^{(o)}$, $T_\lambda^{(o)}$, $S_\theta^{(o)}$, $T_\theta^{(o)}$ are replaced by $(\Delta S^{(o)}/\Delta\lambda)$, $(\Delta T^{(o)}/\Delta\lambda)$, $(\Delta S^{(o)}/\Delta\theta)$, $(\Delta T^{(o)}/\Delta\theta)$, respectively, approximate values for ψ_λ and ψ_θ are obtained. (See table 10.) An integraph may be used to determine $\int S^{(o)}(\lambda,\theta)d\lambda$, $\int T^{(o)}(\lambda,\theta)d\lambda$, $\int S^{(o)}(\lambda,\theta)d\theta$, $\int T^{(o)}(\lambda,\theta)d\theta$, and so forth (approx.), and so obtain, using equation (12), $g^{(1)}(Z) = S^{(1)}(\lambda,\theta) + iT^{(1)}(\lambda,\theta)$. Similarly, $g^{(n)}(Z)$, $n = 2, 3, \ldots$ can be computed.

Step II.- The second stage of the method is then to obtain the values of the stream function and its derivatives in the (λ,θ)-plane - that is, to evaluate the expressions[1]

$$\left.\begin{aligned}
\psi(\lambda,\theta) &= L^{(o)}(2\lambda)T^{(o)}(\lambda,\theta) + L^{(1)}(2\lambda)T^{(1)}(\lambda,\theta) + \ldots \\
&\qquad + L^{(n)}(2\lambda)T^{(n)}(\lambda) + \ldots \\
L^{(o)}(2\lambda) &= H(2\lambda), \quad L^{(1)}(2\lambda) = \frac{1}{2}H(2\lambda)Q^{(1)}(2\lambda), \ldots \\
L^{(n)}(2\lambda) &= \frac{(2n)!}{2^n n!}H(2\lambda)Q^{(n)}(2\lambda)
\end{aligned}\right\} \quad (22)$$

$$\begin{aligned}
\psi_v(\lambda,\theta) &= R^{(o)}(2\lambda)\operatorname{Im} g_z + R^{(1)}(2\lambda)T^{(o)}(\lambda,\theta) + \ldots \\
&\qquad + R^{(n)}(2\lambda)T^{(n-1)}(\lambda,\theta) + \ldots
\end{aligned} \quad (23)$$

$$\begin{aligned}
\psi_\theta(\lambda,\theta) &= L^{(o)}(2\lambda)\operatorname{Re} g_z + L^{(1)}(2\lambda)S^{(o)}(\lambda,\theta) + \ldots \\
&\qquad + L^{(n)}(2\lambda)S^{(n-1)}(\lambda,\theta) + \ldots
\end{aligned} \quad (24)$$

---

[1]Since it is assumed in this case that the speed at every point of D_2 is considerably smaller than that of sound, the expression (8) is replaced here by

$$\psi(\lambda,\theta) = \operatorname{Im} H(2\lambda)\left[g(Z) + \sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}n!}Q^{(n)}(2\lambda)g^{(n)}(Z)\right]$$

See equations (30), (31), and (33) of reference 4. Since the $L^{(s)}(2\lambda)$, $s = 0,1,2,\ldots$ are independent of g, they can be entered on master cards once and for all, for different values of s and different values of λ; that is, they are (G operations). For instance, on master card No. 1 in columns 1 to 6, the value of $L^{(0)}(2\lambda)$ for a fixed value λ, say λ_0, is entered while nothing is punched in column 7 if $L^{(0)}$ is positive; in columns 8 to 14 the absolute value of $L^{(1)}(2\lambda)$ is entered, and in column 15 the number 1 is punched, if $L^{(1)}$ is negative, and so forth. Similarly, on master card No. 2 the corresponding values of $L^{(s)}(2\lambda^{(1)})$ are punched, and so forth.

The remainder of step II consists of (S operations). From previous computations the values of $T^{(\kappa)}(\lambda,\theta)$, $\kappa = 0, 1, 2, \ldots$ for $\lambda = \lambda^{(0)}$ and $\theta = \theta^{(0)}, \theta^{(1)}, \theta^{(2)}$, and so forth, for $\lambda = \lambda^{(1)}$, $\theta = \theta^{(0)}, \theta^{(1)}, \theta^{(2)}$, and so forth, are obtained; both sets of cards, that is, the $L^{(s)}$ and $T^{(s)}$ are then put into the multiplier, which then yields the values of (22) for the set of points $(\lambda^{(0)}, \theta^{(0)})$, $(\lambda^{(0)}, \theta^{(1)}), \ldots$, $(\lambda^{(1)}, \theta^{(0)})$, $(\lambda^{(1)}, \theta^{(1)}), \ldots$, $(\lambda^{(2)}, \theta^{(0)})$, $(\lambda^{(2)}, \theta^{(1)}), \ldots$ And ψ_v and ψ_θ may be obtained in similar fashion. The values of ψ, ψ_v, ψ_θ obtained for the case under consideration are given in tables 11 and 12.

Now, the values of $\psi(\lambda^{(0)}, \theta^{(\kappa)})$, $\kappa = 0,1,2,\ldots$, $\psi_v(\lambda^{(0)}, \theta^{(\kappa)})$, $\psi_\theta(\lambda^{(0)}, \theta^{(\kappa)})$ are plotted on graph paper along the abscissa of which the values of θ are given. By using this diagram, the values of θ can be determined for which $\psi(\lambda^{(0)}, \theta)$ = constant, say, 0 ± 0.1, ± 0.2, and so forth. The values of $\psi_v(\lambda^{(0)}, \theta)$ and of $\psi_\theta(\lambda^{(0)}, \theta)$ corresponding to $\psi(\lambda^{(0)}, \theta)$ = constant may then be determined. This procedure is then repeated for different values of λ.

See table 13 and figure 13.

<u>Step III</u>.- To every value $\lambda^{(\kappa)}$, $\kappa = 0,1,2,\ldots$, the values of $\theta^{(\kappa\tau)}$ were determined for which

$$\psi(\lambda^{(n)}, \theta^{(n\tau)}) = \tau = \text{constant} \tag{25}$$

as well as the corresponding values of $\psi_\lambda(\lambda,\theta)$ and $\psi_\theta(\lambda,\theta)$.

Tables (or figures) of v^2, $1 - M^2$, $\frac{1}{\rho v^2}$, $\frac{d\lambda}{dv}$, can be prepared, which, since these quantities are functions of λ alone, have to be computed only once, that is, they are (G operations).

The image of a streamline (25) in the physical plane is given by

$$\left.\begin{aligned} x &= x(v) = -\int_0^v \frac{\rho_0 \cos\theta}{\rho v^2} \left[\frac{\psi_\theta^2(1 - M^2) + v^2 \psi_v{}^2}{\psi_\theta}\right] dv \\ y &= y(v) = -\int_0^v \frac{\rho_0 \sin\theta}{\rho v^2} \left[\frac{\psi_\theta^2(1 - M^2) + v^2 \psi_v{}^2}{\psi_\theta}\right] dv \end{aligned}\right\} \tag{26}$$

(See equation (19) of reference 4.)

The integrals (26) will be approximated by the sums

$$\left.\begin{aligned} x &= x(v_l) = \sum_{s=0}^{s=l} \Delta x_s(\tau) \\ y &= y(v_l) = \sum_{s=0}^{s=l} \Delta y_s(\tau), \qquad l = 1,2,3,\ldots \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} \Delta x_s &= \frac{\rho_0}{\rho_s v_s{}^2}\left[\psi_\theta{}^{(s\tau)^2}(1 - M_s{}^2) + v_s{}^2 \psi_v{}^{(s\tau)^2}\right] \frac{\cos\theta^{(s\tau)}}{\psi_\theta{}^{(s\tau)}} \Delta v_s \\ \Delta y_s &= \frac{\rho_0}{\rho_s v_s{}^2}\left[\psi_\theta{}^{(s\tau)^2}(1 - M_s{}^2) + v_s{}^2 \psi_v{}^{(s\tau)^2}\right] \frac{\sin\theta^{(s\tau)}}{\psi_\theta{}^{(s\tau)}} \Delta v_s \\ \Delta v_s &= v_{s+1} - v_s \end{aligned}\right\} \tag{28}$$

The remainder of step III consists of (S operations). By using tables for squares and the reciprocal, the values of $\psi_\theta^{(s\tau)^2}$, $\psi_v^{(s\tau)^2}$ and $\frac{1}{\psi_\theta^{(s\tau)}}$ are determined, together with the previously described tables for $\frac{1}{\rho v^2}$, $1 - M^2$, and so forth. The quantity

$$\frac{\rho_o}{\rho_s v_s^2}\left[\psi_\theta^{(s\tau)^2}(1 - M_s)^2 + v_s^2\psi_v^{(s\tau)^2}\right]\frac{\Delta v_s}{\psi_\theta^{(s\tau)}} \qquad (29)$$

is determined with the use of punch cards. Equation (29) is then multiplied by $\cos\theta^{(s\tau)}$ and $\sin\theta^{(s\tau)}$ to yield the first and second terms of equations (28), respectively.

Since the cosine and sine may vary in sign, an extra column must be provided with each term of the product as described previously. The cards are put in the multiplier which is set for progressive totaling, the values (27), which correspond to (25) then resulting.

3. Description of the Second Method for the Construction of a Compressible Fluid Flow

As indicated in section 1, this method will often be applied if the local Mach number is nearly 1. For this reason, in contrast to the considerations of section 2, it is now necessary to use the exact formula[1] (8), that is,

$$\left.\begin{aligned}
\psi(\lambda,\theta) &= \lim_{m\to\infty}\psi_m(\lambda,\theta) = \operatorname{Im}\left\{H(2\lambda)\left[g(Z)+\lim_{m\to\infty}\sum_{n=1}^{\infty}\frac{(2n)!}{n!\,2^{2n}}Q_m^{(n)}g^{(n)}(Z)\right]\right. \\
&= \operatorname{Im}\left.\left\{L^{(o)}(2\lambda)g(Z)+\lim_{m\to\infty}\sum_{n=1}^{\infty}L_m^{(n)}(2\lambda)g^{(n)}(Z)\right\}\right. \\
g^{(n)} &= \int_0^Z\cdots\int_0^{\zeta_{n-1}} g(\zeta_n)\,d\zeta_n\ldots d\zeta_1, \\
L^{(o)}(2\lambda) &= H(2\lambda),\quad L_m^{(n)}(2\lambda) = \frac{(2n)!}{2^{2n}n!}H(2\lambda)Q_m^{(n)}(2\lambda)
\end{aligned}\right\} \qquad (30)$$

[1]A method for determining the $L_m^{(n)}$ is given in appendix I.

As indicated in reference 3, sections 9 and 15 (see also appendix I of the present paper), if λ is considered only in a range $\lambda \leq \lambda_0 < 0$, where λ_0 is a fixed negative number, then a fixed m can be determined so that equation (6) can be replaced by[1]

$$\frac{1}{4}(\psi_{\lambda\lambda} + \psi_{\theta\theta}) + N_m\psi = 0 \tag{31}$$

The solutions of (31) are given by

$$\psi_m(\lambda, \theta) = \operatorname{Im}\left\{L^{(o)}(2\lambda)g(Z) + \sum_{n=1}^{\infty} L_{(m)}^{(n)}(2\lambda)g^{(n)}(Z)\right\} \tag{32}$$

In the following it will be assumed that λ_0 is a very small number, say $\lambda_0 = -0.01$ (i.e., that the flows with local Mach number $= M_0 = 0.99$ can be considered). Then m will be a very large <u>but fixed</u> number.

<u>Remark</u>: In order to avoid confusion, all quantities which depend on m will have a subscript m; however, it is necessary to bear in mind that in this section m is a very large but <u>fixed</u> quantity, which remains unchanged in all considerations of this section.

As indicated in section 1, in this method, certain tables can be employed which are independent of the flow and which, therefore, can be computed once and for all, and used in all subsequent computations.

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[1]Since N does not satisfy the hypothesis of theorem (83), equation (6) was replaced by equation (31), where N_m does satisfy the conditions of the above theorem and differs only slightly from N for values of λ smaller than $\lambda_0 < 0$. And λ_0 can be taken as near zero as desired.

In appendix I a method is given for determining N_m for a given λ_0 with any prescribed degree of accuracy.

Note that instead of $\frac{1}{4}(\psi_{\lambda\lambda} + \psi_{\theta\theta}) + N_m\psi_\lambda = 0$ in appendix I, the equation $\frac{1}{4}(\psi^*_{\lambda\lambda} + \psi^*_{\theta\theta}) + F_m\psi^* = 0$ is employed. This last equation is obtained from (31) by means of the transformation (41).

A. Description of Two Kinds of Tables

1. If a sufficiently large number of the[1] $L_m^{(n)}$ are computed, the functions $\chi_m^{(2p)}(v,\theta) + i\chi_m^{(2p+1)}(v,\theta)$ which correspond to $g(Z) = Z^p$; that is,

$$\chi_m^{(2p)}(v,\theta) + i\chi_m^{(2p+1)}(v,\theta) = H(2\lambda)Z^p$$

$$+ \sum_{n=1}^{\infty} L_m^{(n)}(2\lambda)Z^{p+n}\Big/(p+1)\ldots(p+n) \quad \text{(G operation)} \qquad (33)$$

where $Z = \lambda - i\theta$ and λ is given by (5), may be determined.

Remark: In the case of an incompressible fluid, where $\lambda = \log v$, $H(2\lambda) = 1$, and $L_m^{(n)}(2\lambda) = 0$, $n = 1,2,\ldots$ the corresponding functions are

$$\left.\begin{aligned} \chi^{(2p)} &= \operatorname{Re}\,(\log v - i\theta)^{p-1} \\ \chi^{(2p+1)} &= \operatorname{Im}\,(\log v - i\theta)^{p-1} \end{aligned}\right\} \qquad (34)$$

Analogously, as every function of (34) is a solution of (3), every function $\chi_m^{(p)}$, $p = 0.1,\ldots$ is a solution of (31), and since for $\lambda \leq \lambda_0$, N_m practically equals N, every one of these functions is a solution[2] of (6).

2. To every function $\chi_m^{(p)}(v,\theta)$, $p = 0, 1,\ldots$, two real functions are determined:

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[1]See appendix I.

[2]Exactly speaking: An approximate solution of (6). It, however, does not differ essentially from the corresponding exact solution of (6).

$$\left.\begin{aligned} X_m^{(p)}(v,\theta) &= \int_{(o,o)}^{(v,\theta)} \frac{\rho_o}{\rho}\left\{\left[\frac{-(1-M^2)\cos\theta}{v^2}\frac{\partial\chi_m^{(p)}}{\partial\theta} - \frac{\sin\theta}{v}\frac{\partial\chi_m^{(p)}}{\partial v}\right]dv + \left[\cos\theta\,\frac{\partial\chi_m^{(p)}}{\partial v} - \frac{\sin\theta}{v}\frac{\partial\chi_m^{(p)}}{\partial\theta}\right]d\theta\right\} \\ Y_m^{(p)}(v,\theta) &= \int_{(o,o)}^{(v,\theta)} \frac{\rho_o}{\rho}\left\{\left[\frac{-(1-M^2)\sin\theta}{v^2}\frac{\partial\chi_m^{(p)}}{\partial\theta} + \frac{\cos\theta}{v}\frac{\partial\chi_m^{(p)}}{\partial v}\right]dv + \left[\sin\theta\,\frac{\partial\chi_m^{(p)}}{\partial v} + \frac{\cos\theta}{v}\frac{\partial\chi_m^{(p)}}{\partial\theta}\right]d\theta\right\} \end{aligned}\right\} \quad (35)$$

__Remark__: Since the above integrands are complete differentials, the values of the integrals are independent of the path of integration (G operation).

__Remark__: In the case of an incompressible fluid, there is obtained for the corresponding functions $X^{(p)}$, $Y^{(p)}$ the expressions:

$$\begin{aligned} X^{(2p)} &= (p-1)^{-1}\,pv^{(p-1)}\sin\left((p-1)\theta\right) \\ Y^{(2p)} &= (p-1)^{-1}\,pv^{(p-1)}\cos\left((p-1)\theta\right) \\ X^{(2p+1)} &= (p-1)^{-1}\,pv^{(p-1)}\cos\left((p-1)\theta\right) \\ Y^{(2p+1)} &= (p-1)^{-1}\,pv^{(p-1)}\sin\left((p-1)\theta\right) \end{aligned}$$

In the following it is assumed that the above-described functions, $\chi_m^{(p)}(v,\theta)$ and $X_m^{(p)}(v,\theta)$ $Y_m^{(p)}(v,\theta)$ are computed for a sufficiently large number of values of p and tabulated for a large number of values of (v,θ).

B. Determination of Flow Using the Above-Described Tables

__Step I__: Determination of streamlines $\psi_m(v,\theta)$ = constant in the hodograph plane: According to the assumption of section 1, the function $g(Z)$ can be represented in the domain D_1, in which it will be considered in this section, in the form of a power series

$$g(Z) = \sum_{p=0}^{\infty} (\alpha_{2p+1} + i\alpha_{2p}) Z^p \qquad (36)$$

where α_p are real constants. For s sufficiently large, the power series (36) can be replaced in D_1 by the polynomial

$$\sum_{p=0}^{s} (\alpha_{2p+1} + i\alpha_{2p}) Z^p \qquad (37)$$

By substituting (37) into (30) and by observing (33), there is obtained for the stream function ψ_n corresponding to (37)

$$\psi_m(v,\theta) = \mathrm{Im}\left\{\sum_{n=0}^{\infty} L_m^{(p)}(2\lambda) \sum_{p=0}^{s} (\alpha_{2p}+i\alpha_{2p+1}) \frac{Z^{n+p}}{(n+1)\ldots(n+p)}\right\}$$

$$= \mathrm{Im}\left[\sum_{p=0}^{s} (\alpha_{2p+1}+i\alpha_{2p}) \left(\chi_m^{(2p)}+i\chi_m^{(2p+1)}\right)\right]$$

$$= \sum_{p=0}^{s} \left(\alpha_{2p}\chi_m^{(2p)} + \alpha_{2p+1}\chi_m^{(2p+1)}\right) = \sum_{p=0}^{p=2s+1} \alpha_p \chi_m^{(p)} \qquad (38)$$

Since, as a rule, it is necessary to determine the values of $\chi_m(v,\theta)$ at many points, it is convenient to use punch cards. For every point (v,θ) a master card is prepared, and in this, in columns 1 to 6, the value of $\chi_m^{(o)}$ at the considered point (v,θ) is entered and, in general, in columns 6p+1 to 6(p+1) the value of $\chi_m^{(p)}$ (G operations).

Since α_p can be positive and negative, (38) will be represented[1] by

$$\psi_m(v,\theta) = \sum_{p=0}^{2s+1} \alpha_p^{+} \chi_m^{(p)}(v,\theta) - \sum_{p=0}^{2s+1} \alpha_p^{-} \chi_m^{(p)}(v,\theta) \qquad (39)$$

Each of the sums in the right-hand expression of (39) can be easily evaluated for a large number of points using punch card machines (S operations). The curves $\psi_m(v,\theta)$ = constant can then be determined by interpolation.

[1] $\alpha^+ = \max(\alpha, 0)$, $\alpha^- = \max(-\alpha, 0)$

Step II.- Transition to the physical plane.- To every point (v,θ) of the hodograph plane there corresponds a point (x,y) of the physical plane, which is obtained by writing

$$\left.\begin{aligned} x = x(v,\theta) &= \int_{(o,o)}^{(v,\theta)} \frac{\rho_o}{\rho}\left\{\left[-\frac{(1-M^2)\cos\theta}{v^2}\frac{\partial\psi_m}{\partial\theta} - \frac{\sin\theta}{v}\frac{\partial\psi_m}{\partial v}\right]dv \right. \\ &\quad \left. + \left[\cos\theta\frac{\partial\psi_m}{\partial v} - \frac{\sin\theta}{v}\frac{\partial\psi_m}{\partial\theta}\right]d\theta\right\} = \sum_{p=o}^{2s+1}\alpha_p X_m^{(p)}(v,\theta) \\ y = y(v,\theta) &= \int_{(o,o)}^{(v,\theta)} \frac{\rho_o}{\rho}\left\{\left[-\frac{(1-M^2)\sin\theta}{v^2}\frac{\partial\psi_m}{\partial\theta} + \frac{\cos\theta}{v}\frac{\partial\psi_m}{\partial v}\right]dv \right. \\ &\quad \left. + \left[\sin\theta\frac{\partial\psi_m}{\partial v} + \frac{\cos\theta}{v}\frac{\partial\psi_m}{\partial\theta}\right]d\theta\right\} = \sum_{p=o}^{2s+1}\alpha_p Y_m^{(p)}(v,\theta) \end{aligned}\right\} \quad (40)$$

See reference 3, equation (136); also equations (35) and (38) of this paper.

Let $\psi_m(v,\theta) = c =$ constant be a streamline (in the hodograph plane). To every value of v on $\psi_m(v,\theta) = c$, there corresponds a value[1] of θ, say $\theta(v)$, which can be easily determined by interpolation or directly from the diagram for the $\psi_m(v,\theta) =$ constant.

By interpolation (and the use of the tables described under II) the values of $X_m^{(p)}(v,\theta(v))$, $Y_m^{(p)}(v,\theta(v))$ are determined. Substituting these values into (40) gives the coordinates (x,y) of the streamline $\psi_m = c$ in the physical plane (S operations).

Remark: Clearly, in order to apply this method, it is sufficient that the function $g(Z)$ can be approximated in the domain under consideration by a polynomial

$$\sum_{p=1}^{s}\left[\alpha_{2p}^{(s)} + i\alpha_{2p+1}^{(s)}\right] Z^p$$

On the other hand, by Runge's theorem, an analytic function can be approximated by a polynomial in every simply covered

[1] Or several values, say, $\theta_1, \theta_2, \ldots, \theta_n$.

and simply connected domain. Because of this fact, the method described in this section may be applied not only when D_1 is some region which lies inside of the circle of convergence of (36) but for a much larger class of domains.

CONCLUDING REMARKS

A method of obtaining a subsonic flow pattern of a compressible fluid from a given analytic function $g(Z)$ is described in this report.[1] The amount of time and labor needed for this method is reasonably small once certain tables have been prepared.[2] These tables are completely independent of the flow, and consequently once prepared, the problem of determining the flow pattern may be regarded as solved, not only from a theoretical but from a practical point of view as well.

The present method yields only subsonic flow patterns,[3] but by combining these with those described in section 17 of reference 3, it will then be possible to construct mixed, (i.e., partially supersonic) flow patterns, from a given function $g(Z)$.

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[1]The method described in this report, and references 8, 2, and 3, is a generalization of the determination of flow patterns of an incompressible fluid from the complex potential $g(\zeta) = \varphi(\log v,\theta) + i\psi(\log v,\theta)\ \zeta = \log v - i\theta$, which potential is given in the logarithmic plane.

Assuming that the necessary auxiliary tables have been prepared and that punch card machines, are available, the amount of labor needed in determining the pattern of a subsonic flow corresponding to a given function $g(Z)$ will only slightly exceed that needed for determining the flow pattern of an incompressible fluid from a given $g(\log v - i\theta)$.

[2]The author would like to emphasize that the tables of sec. 9 of reference 3, and those of the present report (the former are only an approximation to those of appendix I) serve merely to illustrate the procedure. The functions are computed for comparatively few values of the arguments, and hence by using them it is possible to obtain only a rather inaccurate picture of the flow pattern.

[3]Note that a similar method can be developed for purely supersonic flows. See appendix III.

A method for determining various flow patterns is, of course, only the initial step in the study of compressible fluid flows, since the aerodynamicist is, in the main, interested in determining the influence of different factors such as the shape of the profile, the maximum Mach number, and so forth, on the flow pattern.

By the choice of suitable functions for g, it should be possible to obtain many cases of flows which are of considerable practical interest and value in studying various phenomena in the theory of compressible fluids.[1]

<u>Remark</u>: As has been emphasized in the introduction, it is frequently of considerable importance to solve the "direct" problem determining the flow in the physical plane around a profile, which flow behaves in a prescribed fashion at infinity (i.e., far from the profile). Although in many instances it is possible[2] to determine the function $g(Z)$ so there is obtained a flow around a profile approximating the given one, it seems desirable to have a method of solving the "direct" problem, and to determine when solutions to this "direct" problem do or do not exist. The author hopes to return to this question in a future report.

Brown University,
Providence, R. I., September 6, 1945.

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[1]As has been indicated previously, the examples obtained which correspond to the Chaplygin solutions cannot, in general, yield the entire flow pattern (in the physical plane) around a closed profile. An exception to this has been the work of Kármán-Tsien (references 6, 7) but in order to accomplish this they have substituted for the true adiabatic pressure - specific volume relation, a linear approximation to it.

[2]Once a sufficiently large number of flows corresponding to various functions g have been "catalogued."

APPENDIX I

THE DETERMINATION OF THE $Q_m^{(n)}$ AND $L_m^{(n)}$

The operator (8) (see also secs. 9 to 11 of reference 3) was obtained in the following manner: As was proved in reference 3, the function $\psi(\lambda,\theta)$ satisfies equation (6) where N is given by (7); λ and M are connected by relation (5). Every solution ψ of (6) can be written in the form

$$\psi = H(2\lambda)\psi^* \tag{41}$$

where H is given by (reference 3, (111)) and ψ^* satisfies the equation

$$\frac{1}{4}\left(\frac{\partial^2\psi^*}{\partial\lambda^2} + \frac{\partial^2\psi^*}{\partial\theta^2}\right) + F(2\lambda)\psi^* = 0 \tag{42}$$

$$F(2\lambda) = -\frac{1}{8}\left[\frac{5(1+k)}{(1-M^2)^3} - \frac{12k}{(1-M^2)^2} + \frac{2(3k-7)}{1-M^2} + 4(k+2) - (3k-1)(1-M^2)\right] \tag{43}$$

In order to determine $F(2\lambda)$ it is necessary to compute M as a function of λ from (5) and then substitute into (43). The obtained function becomes infinite for $\lambda = 0$ (i.e., for $M = 1$), which causes certain difficulties. On the other hand, since only the subsonic case is considered here, and since a small modification of the function $F(2\lambda)$ practically does not change (in the subsonic region) the solution of the equation[1], it is expedient to approximate $F(2\lambda)$, in the range $-\infty \leqq \lambda < \lambda_o$, $|\lambda_o|$ sufficiently small, by a function which remains finite at $\lambda = 0$, for instance, by a polynomial F_m of the m-th degree in $e^{2\lambda}$. As was proved in reference 3, $T = (1 - M^2)^{1/2}$ can be developed in a series in $e^{2\lambda}$, namely,

[1]By using the theory of integral equations, it is possible to prove the following theorem. Let B be a given bounded domain in which $\lambda \leqq \lambda_o$, $\lambda_o < 0$ and in which F_m differs from F by a sufficiently small amount. To every solution of $\psi^+(\lambda,\theta)$ of $1/4\Delta\psi^+ + F_m\psi^+ = 0$ a solution $\psi^*(\lambda,\theta)$ of $1/4\Delta\psi^* + F\psi^* = 0$ can be so determined that $|\psi^*(\lambda,\theta) - \psi^+(\lambda,\theta)| \leqq E(\lambda,\theta)\epsilon B$, and E is a given small positive number.

$$S = 1 - T = x_1 + \frac{1}{4}(2k + 1)x_1^2 + \ldots \tag{44}$$

$$x_1 = 2\left(\frac{(k+1)^{1/2} - (k-1)^{1/2}}{(k+1)^{1/2} + (k-1)^{1/2}}\right)^{\sqrt{\frac{k+1}{k-1}}} e^{2\lambda}$$

This series converges for $-\infty < \lambda < 0$. Substituting $k = 1.4$ into (44) yields

$$T = \sum_{n=0}^{\infty} A_n x_1^{\,n} = \sum_{n=0}^{\infty} a_n x^n \tag{45}$$

and

$$T^{-1} = \sum_{n=0}^{\infty} B_n x_1^{\,n} = \sum_{n=0}^{\infty} b_n x^n \tag{46}$$

$$x_1 = 0.239\ e^{2\lambda}$$

$$x = e^{2\lambda}$$

The values of A_n, a_n, B_n, b_n are given in table 1. Since (45) and (46) converge for $-\infty < \lambda < 0$; for $-\infty < \lambda < \lambda_0$, where $\lambda_0 < 0$ is a fixed quantity, it is possible to approximate (45) and (46) by polynomials

$$T_m = \sum_{n=1}^{m} a_n e^{2\lambda n} \tag{47}$$

and

$$(T^{-1})_m = \sum_{n=1}^{m} b_n e^{2n\lambda} \tag{48}$$

By substituting these polynomials into (43) instead of $\frac{1}{(1-M^2)^{\frac{1}{2}}}$ and $(1 - M^2)^{\frac{1}{2}}$, respectively, polynomials of approximation, $F_m(2\lambda)$ in $e^{2\lambda}$, are obtained. Clearly, if a given degree of accuracy is required, m will increase as λ_0 approaches 0. By plotting T, $1/T$, T_m and $(1/T)_m$ for a given m and comparing the corresponding values, the upper bound λ_0 of the values of λ for which $|F_m(2\lambda) - F(2\lambda)|$ is sufficiently small, may be determined.

TABLE 1

n	$-A_n$	$-a_n$	B_n	b_n
0	-1	-1	1	1
1	1	.2392	1	.2392
2	1.9	.1087	2.9	.1659
3	4.81	.0658	9.61	.1315
4	13.939	.0456	33.869	.1108
5	43.68	.0342	123.696	.0968
6	144.02	.0270	462.39	.0865
7	492.11	.0220		.0786
8		.0185		.0724
9		.0158		.0672
10		.0138		.0629

For instance, in the case under consideration where $m = 10$, the values of T and T_m are given in table 2 and plotted in figure 16. As can be seen from figure 16, $\lambda < \lambda_0 = -0.11$ (i.e., $M = 0.75$), $F_{1n}(2\lambda)$ is practically equal to $F(2\lambda)$. If a good approximation is desired for bigger values of λ, more coefficients[1] a_n, b_n must be computed. In order to check the obtained values of a_n, as function of n, see figure 17.

The coefficients $Q_m^{(n)}(2\lambda)$ of the operators which yield solutions of the equation $\psi^*_{\zeta\zeta} + F_m\psi = 0$ can be obtained in the same way as derived in reference 3, from which reference the results are obtained

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[1]It may be remarked that other methods of obtaining approximating polynomials for F exist. These will not be investigated in the present report, despite the fact that they merit considerable attention.

Table 2

The values of T, M, and T_{10}

-2λ	T_{10}	T	$\sqrt{1-T^2} = M$
0.0160	0.43644	0.300	0.954
0.0195	0.44208	0.320	0.947
0.0230	0.44758	0.336	0.942
0.0265	0.45300	0.350	0.937
0.0300	0.45835	0.365	0.931
0.0335	0.46360	0.380	0.925
0.0370	0.46877	0.390	0.921
0.0405	0.47385	0.401	0.916
0.0440	0.47885	0.412	0.911
0.0475	0.48377	0.421	0.907
0.0510	0.48861	0.430	0.903
0.0545	0.49338	0.439	0.898
0.0580	0.49807	0.448	0.894
0.0615	0.50268	0.455	0.890
0.0650	0.50723	0.463	0.886
0.0685	0.51170	0.470	0.883
0.0720	0.51610	0.477	0.879
0.0755	0.52033	0.484	0.875
0.0790	0.52471	0.491	0.871
0.0825	0.52892	0.497	0.868
0.0860	0.53306	0.502	0.865
0.0895	0.53714	0.507	0.862
0.0930	0.49225	0.512	0.859
0.0965	0.49790	0.520	0.854
0.1000	0.54902	0.525	0.851
0.1035	0.55287	0.530	0.848
0.1070	0.53639	0.535	0.845
0.1105	0.56040	0.540	0.842
0.1140	0.56408	0.545	0.838
0.1175	0.56771	0.550	0.835
0.1210	0.57129	0.554	0.832
0.1245	0.57481	0.559	0.829
0.1280	0.57829	0.563	0.826
0.1315	0.58172	0.567	0.823
0.1350	0.58511	0.571	0.821
0.1385	0.58844	0.575	0.818
0.1420	0.59173	0.579	0.815
0.1455	0.59498	0.583	0.812
0.1490	0.59818	0.587	0.810
0.1525	0.60134	0.591	0.806
0.1560	0.60446	0.594	0.804
0.1595	0.60754	0.598	0.801
0.1630	0.61054	0.601	0.799
0.1665	0.61357	0.604	0.797

$$\left.\begin{aligned} Q_m^{(1)} &= -4\int_{-\infty}^{\lambda} F_m \, d\lambda \\ Q_m^{(2)} &= -\frac{4}{3} F_m + \frac{1}{6} Q_m^{(1)^2} \\ Q_m^{(3)} &= -\frac{4}{15}\frac{\partial}{\partial\lambda} F_{m\lambda} + \frac{4}{15} F_m Q_m^{(1)} - \frac{16}{15}\int_{-\infty}^{\lambda} F_m^2 \, d\lambda + \frac{1}{40} Q_m^{(1)^3} \end{aligned}\right\} \quad (49)$$

APPENDIX II

THE EQUATION (IN THE CANONICAL FORM) FOR THE POTENTIAL FUNCTION

AN APPLICATION OF INTEGRAL EQUATIONS TO THE THEORY OF COMPRESSIBLE FLUIDS

1. In section 6 of reference 8 and section 7 of reference 3 the equation (in canonical form[1]) for the stream function has been derived. See equation (6.6) of reference 8 or (46) of reference 3.

There are instances, however, where it is more convenient to operate with the potential function Φ rather than with the stream function ψ.

In this section the canonical form of the equation for ϕ will be derived.

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[1]By introducing suitable new variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$, every equation $L(\psi) \equiv a\psi_{xx} + 2b\psi_{xy} + c\psi_{yy} + d\psi_x + e\psi_y + g\psi = 0$ of elliptic type can be reduced to the form $\psi_{\xi\xi} + \psi_{\eta\eta} + A\psi_\xi + B\psi_\eta + C\psi = 0$, so-called "canonical form of equation L." (See reference 9.)

In the case considered in the present section $x = H$, $y = \theta$, and $\xi = \lambda$, $\eta = \theta$.

Functions ϕ and ψ satisfy the system of equations

$$(\partial\phi/\partial\theta) = (\partial\psi/\partial H), \quad l(H)(\partial\psi/\partial\theta) = -(\partial\phi/\partial H) \qquad (50)$$

[equation (6.21) of reference 8 and equation (30) of reference 3]

where

$$(dH(v)/dv) = \rho/v, \quad l(H) = (1 - M^2)/\rho^2 \qquad (51)$$

[equations (6.1), (8.18) of reference 8; equations (42), (43) of reference 3].

If, now, the new variable λ, given by

$$(d\lambda/dH) = (1 - M^2)^{\frac{1}{2}}/\rho, \quad \text{that is,} \quad (d\lambda/dv) = (1 - M^2)^{\frac{1}{2}}/v \qquad (52)$$

[equation (6.4) of reference 8 and equation (48) of reference 3], is introduced (50) becomes

$$\phi_\theta = \rho^{-1}(1 - M^2)^{\frac{1}{2}}\psi_\lambda, \quad \rho^{-1}(1 - M^2)^{\frac{1}{2}}\psi_\theta = -\phi_\lambda \qquad (53)$$

Differentiating the first equation (53) with respect to θ and the second with respect to λ yields

$$\phi_{\theta\theta} = \rho^{-1}(1 - M^2)^{\frac{1}{2}}\psi_{\lambda\theta} \qquad (54)$$

$$\rho^{-1}(1 - M^2)^{\frac{1}{2}}\psi_{\lambda\theta} + \left[d(\rho^{-1}(1 - M^2)^{\frac{1}{2}})/d\lambda\right]\psi_\theta = -\phi_{\lambda\lambda}$$

Replacing the first term of the second equation of (54) by $\phi_{\theta\theta}$ and ψ_θ by $-\rho(1 - M^2)^{-\frac{1}{2}}\phi_\lambda$ (see (53)) yields

$$\phi_{\theta\theta} + \phi_{\lambda\lambda} - \rho(1 - M^2)^{-\frac{1}{2}}\left[d(\rho^{-1}(1 - M^2)^{\frac{1}{2}})/d\lambda\right]\phi_\lambda = 0 \qquad (55)$$

Now, by the second relation of (52)

$$\rho(1 - M^2)^{-\frac{1}{2}}\left[d(\rho^{-1}(1 - M^2)^{\frac{1}{2}})/d\lambda\right]$$

$$= \rho^2(1 - M^2)^{-1}(v/\rho)\left[d(\rho^{-1}(1 - M^2)^{\frac{1}{2}})/dv\right] = 4N$$

$$= -\frac{(k + 1)M^4}{2}(1 - M^2)^{-3/2} \qquad (56)$$

[equation (6.6) and errata of reference 8, or equation (47) of reference 3]. Thus, the equation for Φ becomes

$$\Phi_{\lambda\lambda} + \Phi_{\theta\theta} - 4N\Phi_{\lambda} \equiv 4\left[\Phi_{\zeta\bar{\zeta}} - N(\Phi_{\zeta} + \Phi_{\bar{\zeta}})\right] = 0 \qquad (57)$$

$$\zeta = \lambda(v) - i\theta$$

2. A case in which it is more advantageous to consider Φ rather than Ψ is the following:

In section 7 of reference 8 and in section 13 of reference 3 singularities of functions satisfying equation (6) were considered. As was indicated there, a flow with a "vortex-like"[1] singularity at $(\lambda^{o}, \theta^{o})$ is obtained if for the stream function, the so-called fundamental solution

$$W^*(\lambda,\theta;\ \lambda^{(o)},\theta^{(o)})$$

$$= A(\zeta,\bar{\zeta};\ \zeta^{(o)},\bar{\zeta}^{(o)}) \log|\zeta - \zeta^{o}| + B(\zeta,\bar{\zeta};\ \zeta^{(o)},\bar{\zeta}^{(o)}) \qquad (58)$$

[equation (7.1) of reference 8; equation (119) of reference 3]

$$\zeta = \lambda - i\theta, \quad \bar{\zeta} = \lambda + i\theta$$

is taken.

As was explained in section 14 of reference 3, it is important (in connection with the transition to the physical plane) to have (working in the λ,θ-plane) singularities the derivatives of which with respect to λ and to θ are single-valued functions of λ and θ.

The point $\zeta^{(o)}$ corresponds to the point $z = \infty$ of the physical plane, and if for the potential function, Φ, a fundamental solution

$$A^*(\zeta,\bar{\zeta};\ \zeta^{(o)},\overline{\zeta^{(o)}}) \log|\zeta - \zeta^{o}| + B^*(\zeta,\bar{\zeta};\ \zeta^{(o)},\overline{\zeta^{(o)}}) \qquad (59)$$

of (57) is taken, a flow with a "source-like" singularity is obtained. (Expression(59) and, therefore, its derivatives are single-valued functions of λ and θ.)

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[1]The names "vortex-like" and "source-like" are used because in the case of an incompressible fluid (and in the physical plane), in the case of a vortex the stream function is given by $m \log|z - z^{o}|$, and in the case of a source the potential function is given by $\Phi = m \log|z - z^{o}|$, m being a real constant. (See reference 10, pp. 198 and 320.)

A single-valued solution of (57), which is infinite of the first order at $\zeta = \zeta^{(0)}$ may be obtained by taking the derivative with respect to θ of (59).

3. A problem of considerable interest is that of determining a flow of a compressible fluid around a given profile, or at least around a profile the shape of which approximates the given profile. Since, in many instances, by reasoning from the incompressible case, the approximate image in the hodograph or (λ,θ)-plane is known[1], it is possible to consider, instead of the above problem, the question of determining a flow for a <u>given hodograph</u>, and the behavior at the point of the hodograph corresponding to $z = \infty$ [2] is prescribed. Clearly, instead of the image in the hodograph plane the image in the (λ,θ)-plane may be used. If the results of section 7 of reference 8, section 13 of reference 3, and those of section 2 of this appendix are employed, it is possible to determine a function $\psi_1(\lambda,\theta)$ satisfying (6), which possesses the required behavior at $z = \infty$. Naturally, $\psi_1(\lambda,\theta)$ for the point[3] $(\lambda_\infty,\theta_\infty)$ must have a singularity which satisfies the conditions indicated in section 14 of reference 3, in order that the flow in the physical plane will be a flow around a <u>closed</u> curve. (See, in particular, equation (148) of reference 3.) Function $\psi_1(\lambda,\theta)$ is as yet, not the required stream function, since it does not assume constant values on the boundary of the domain. In order to determine this function, it is necessary to find a solution $\psi_2(\lambda,\theta)$ of (6) which is regular in the domain H_1, and which assumes, on the boundary h_1 of H_1, the values[4]

---

[1]The image in the logarithmic plane of an incompressible fluid flow around a profile P is often used as a first approximation of the image in the $\lambda\theta$-plane of the flow of a compressible fluid around a profile similar to P. See figs. 4, 5, and 6, where the boundaries (and some streamlines) of a flow around a Joukowski profile in the physical, hodograph and (pseudo-) logarithmic plane, respectively, are given.

[2]The coordinate z refers to the physical plane.

[3]The point $(\lambda_\infty,\theta_\infty)$ corresponds to the point $z = \infty$ of the physical plane.

[4]Since the domain H_1 extends to infinity and, in general, is multiply covered, it is necessary to alter somewhat the method of attack to be described, by mapping H_1 conformally on a finite and Schlicht domain.

For the sake of brevity this step will be omitted in the following.

$$\psi_2(\lambda_h,\theta_h) = -\psi_1(\lambda_h,\theta_h) \qquad (60)$$

(λ_h,θ_h) being an arbitrary point of h_1.

Function $\psi_2(\lambda,\theta)$ can be determined using the theory of integral equations. (See footnote 8, p. 281 of reference 2.) Indeed, let $\psi_3(\lambda,\theta)$ be that harmonic function which assumes the prescribed values on h_1, then

$$\psi_4 = \psi_2 - \psi_3$$

satisfies the equation

$$\Delta\psi_4 + 4N\frac{\partial\psi_4}{\partial\lambda} = -4N\frac{\partial\psi_3}{\partial\lambda} \qquad (61)$$

and vanishes on the boundary h_1.

By employing classical results ψ_4 can be obtained as the solution of the integral equation:

$$\psi_4(\psi,\theta) = \frac{1}{2\pi}\iint_{H_1}\frac{\partial(4NG)}{\partial\lambda_1}\psi_4(\lambda_1,\theta_1)d\lambda_1\,d\theta_1 + \psi_5$$

$$\psi_5 = -2\pi\iint_{H_1}4N\frac{\partial\psi_3}{\partial\lambda_1}G\,d\lambda_1\,d\theta_1 \qquad (62)$$

where $G \equiv G(\lambda,\theta;\lambda_1,\theta_1)$ is Green's function (of Laplace's equation) with respect to the domain H_1.

APPENDIX III

A METHOD FOR DETERMINATION OF STREAM FUNCTIONS OF PURELY SUPERSONIC FLOWS

1. As indicated in reference 8, section 10 and reference 3, section 16, the approach developed in these papers makes it possible to construct mixed (i.e., partially subsonic and partially supersonic) flows by use of the following procedure:

In preceding papers two methods have been described (one given by Chaplygin, the other by the author[1]), which yield certain types of particular solutions ψ_ν, the stream function of a compressible fluid flow. (See sec. 8 (8.3), (8.6), (8.22) of reference 8 and sec. 2 of reference 2.) The ψ_ν represent stream functions of flows, which, in general, include subsonic and supersonic regions.

As was pointed out in detail in reference 2, section 3 and in the introduction of reference 3, the flow patterns generated by the ψ_ν mentioned above or a linear combination of them $\Sigma\alpha_\nu\psi_\nu$, are of rather special character. In particular, the flow patterns with stream function $\Sigma\alpha_\nu\psi_\nu$, cannot (in general) represent an entire flow around a closed body.

Frequently, in the theory of analytic functions of a complex variable in a similar situation (i.e., when one expression of a certain kind - e.g., power series - does not represent the function, say, f, in the entire domain B in which the function has to be considered), the procedure employed is to decompose B into smaller regions, say, into B_K, $K = 1, 2, \ldots, n$, $\sum_{K=1}^{n} B_K = B_n$ (see fig. 7) such that it is possible to find in every region B_K, another analytic expression, say, f_K, which represents f in that region. Generalizing

[1]Bers and Gelbart, in reference 11, obtained the same solutions independently of the author. They denote the functions $\varphi_\nu + i\psi_\nu$ as Σ-monogenic functions. Here φ_ν is the potential function which corresponds to the stream function ψ_ν.

this method of representation of a function of a complex variable, the author described in section 10 of reference 8 and in section 17 of reference 3 a method for representing the stream function and in a similar manner; that is, in decomposing the domain B into parts B_K, and representing ψ in every B_K by <u>another analytical</u> expression.

In order to apply this method[1] a representation for a purely supersonic flow is frequently required.

A method for generating purely supersonic flows, completely analogous to that developed for the subsonic case, will be given in this appendix.

2. The equation

$$S(\psi) = \left(\frac{\rho_0}{\rho}\right)^2 (1 - M^2) \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial H^2} = 0 \qquad (63)$$

(equations (43) and (6.2) of references 3 and 8, respectively) serves once more as the starting point for the following considerations.

In order to write the equation for ψ in the "canonical form"[2], it is necessary to introduce new variables ξ, η.

$$\xi = \theta + \beta(M), \qquad \eta = -\theta + \beta(M) \qquad (64)$$

where

$$\left.\begin{aligned} \beta(M) &= \int \rho^{-1}\rho_o(M^2 - 1)^{\frac{1}{2}} dH = \frac{1}{2}\int v^{-2}(M^2 - 1)^{\frac{1}{2}} dv^2 \\ &= \left[\frac{1}{h}\tan^{-1}(h(M^2 - 1)^{\frac{1}{2}}) - \tan^{-1}(M^2 - 1)^{\frac{1}{2}}\right] \\ h &= \sqrt{\frac{k - 1}{k + 1}}, \quad k > 1 \end{aligned}\right\} \qquad (65)$$

[1]It may be noted that purely supersonic flow patterns can occur upon considering flows in channels or around a body with a cusp, in which case the flow has no stagnation point.

[2]Since in the supersonic case $M > 1$, equation (63) is of hyperbolic type. By introducing suitable variables ξ, η every equation of hyperbolic type can be transformed into the so-called canonical form $\psi_{\xi\eta} + A\psi_\xi + B\psi_\eta + C\psi = 0$.

Equation (63) then becomes

$$\psi_{\xi\eta} + A(\psi_\xi + \psi_\eta) = 0 \tag{66}$$

where

$$A = \frac{1}{8}(k + 1)M^4(M^2 - 1)^{-3/2} \tag{67}$$

The function

$$\psi^* = H\psi, \quad H = \exp\left[\int^{(\xi+\eta)} A(s)ds\right] \tag{68}$$

satisfies the equation

$$\psi^*_{\xi\eta} - F\psi^* = 0, \quad F = A^2 + (dA/ds), \quad s = \xi + \eta \tag{69}$$

3. By use of considerations similar to those developed in references 8 and 3, the following theorems can be derived:

<u>Theorem I</u>.- Suppose that F_m is a function which possesses a continuous first derivative. Let $E_1^*(\xi,\eta,t)$ and $E_2^*(\xi,\eta,t)$ be solutions of

$$(1 - t^2)\frac{\partial^2 E_1^*}{\partial\eta\,\partial t} - \frac{1}{t}\frac{\partial E_1^*}{\partial\eta} + 2t\xi\left[\frac{\partial^2 E_1^*}{\partial\xi\,\partial\eta} - F_m E_1^*\right] = 0 \tag{70}$$

and

$$(1 - t^2)\frac{\partial^2 E_2^*}{\partial\xi\,\partial t} - \frac{1}{t}\frac{E_2^*}{\partial\xi} + 2t\eta\left[\frac{\partial^2 E_2^*}{\partial\xi\,\partial\eta} - F_m E_2^*\right] = 0 \tag{71}$$

respectively.

Let E_1 and E_2 possess continuous second derivatives, and let $(\partial E_1^*/\partial\xi)/\eta t$ and $(\partial E_2^*/\partial\eta)/\xi t$ be finite for $t = 0$. Then

$$U(\xi,\eta) = \int_{-1}^{+1}\left[E_1^*(\xi,\eta,t)f_1\left(\frac{1}{2}\xi(1 - t^2)\right) + E_2^*\left(\xi,\eta,t)f_2\,\frac{1}{2}\eta(1 - t^2)\right)\right](1 - t^2)^{-\frac{1}{2}}\,dt \tag{72}$$

where f_K, $K = 1, 2$ are two arbitrary, twice continuously differentiable functions of their respective arguments, is a solution of the equation

$$\frac{\partial^2 U}{\partial \xi \partial \eta} - F_m U = 0 \qquad (73)$$

The proof of this theorem is given in reference 12, section 2.

<u>Theorem II</u>.- Let $F_m(\beta)$ possess derivatives of all orders in the interval $\beta_0 \leqq \beta \leqq \beta_1$, $0 < \beta_0 < \beta_1 < \infty$. If a constant c exists such that the inequalities

$$\left| \frac{d^K F_m}{d\beta^K} \right| \leqq \frac{c(K + 2)}{\beta^{K + 2}}, \quad K = 0,1,2,\ldots, \quad \beta_0 \leqq \beta \leqq \beta_1 \qquad (74)$$

obtain, then there exist solutions $E_1(\xi,\eta,t)$ and $E_2(\xi,\eta,t)$ of (70) and (71), respectively, satisfying the conditions of theorem I.

By substituting the functions E_K^*, $K = 1,2$ into (72) for the E_K, there is obtained a representation for solutions of equation

$$\psi^*_{\xi\eta} - F_m \psi^* = 0 \qquad (75)$$

in terms of two arbitrary, twice differentiable functions.

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4. There exist various other integral representations of solutions of (69) in terms of two arbitrary functions of one variable. One such representation, differing from that given in the preceding section, will be discussed here.

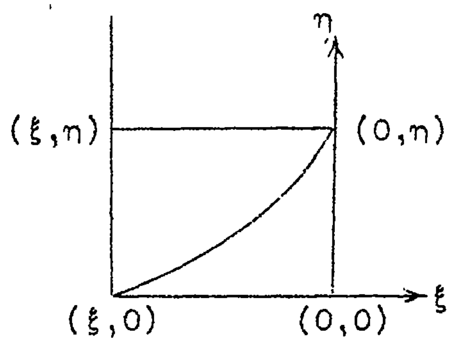
Let $R(\xi,\eta;\xi^*,\eta^*)$ denote the Riemann functions of equations (69) (see reference 9, p. 22) - that is, a function of the four real variables ξ,η,ξ^*,η^*, which satisfies equation (69) for every fixed (ξ^*,η^*), and which further has the properties that

$$\left. \begin{aligned} R(\xi,\eta^*;\xi^*,\eta^*) &= 1 \\ R(\xi^*,\eta;\xi^*,\eta^*) &= 1 \end{aligned} \right\} \qquad (76)$$

This function (for (69)) may be represented in the form

$$R(\xi,\eta;\xi^*,\eta^*) = 1 - \int_{\xi^*}^{\xi}\int_{\eta^*}^{\eta} F(\xi_1,\eta_1)d\xi_1\, d\eta_1$$
$$+ \int_{\xi^*}^{\xi}\int_{\eta^*}^{\eta} F(\xi_1,\eta_1)\int_{\xi^*}^{\xi_1}\int_{\eta^*}^{\eta_1} F(\xi_2,\eta_2)d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2 - \ldots \quad (77)$$

(See reference 8, sec. 7.)



The classical theory of partial differential equations of hyperbolic type yields the following results:

Let $f_K (K = 1, 2)$ be any two arbitrary differentiable functions of one real variable, and if u satisfies the differential equations $u_{\xi\eta} + Fu = 0$, then

$$u(\xi,\eta) = u(0,0)R(\xi,\eta;0,0) + \int_0^{\xi} R(\xi,\eta;\xi^*,0)f_1(\xi^*)d\xi^*$$
$$+ \int_0^{\eta} R(\xi,\eta;0,\eta^*)f_2(\eta^*)d\eta^* \quad (78)$$

(R is, of course, the Riemann function associated with the differential equations satisfied by u.)

<u>Remark</u>: A representation of the form (78) is also valid for the subsonic region.

Indeed, suppose that ξ is replaced by $\zeta = \lambda - i\theta$, η by $\overline{\zeta} = \lambda + i\theta$, ξ^* by ζ^*, η^* by $\overline{\zeta}^*$, in (77), this transformed expression will not differ essentially from the function $\chi(\zeta,\overline{\zeta})$ introduced in (7.4) of reference 8.

If the complex variable $\zeta, \overline{\zeta}$ instead of λ, θ, is used, the equation for the function ψ^*, in the subsonic case, assumes the form

$$\psi^*_{\zeta\overline{\zeta}} + F_m\psi^* = 0$$

(equation (86) of reference 3)

a new representation for ψ^* in terms of two arbitrary analytic functions h_1, h_2 of one complex variable may then be obtained:

$$\psi^*(\zeta,\overline{\zeta}) = \psi^*(0,0)R(\zeta,\overline{\zeta};0,0) + \int_0^{\zeta} R(\zeta,\overline{\zeta};\zeta^*,0)h_1(\zeta^*)d\zeta^* + \int_0^{\overline{\zeta}} R(\zeta,\overline{\zeta};0,\overline{\zeta}^*)h_2(\overline{\zeta}^*)d\overline{\zeta}^* \qquad (79)$$

(R is the Riemann function of the differential equation for ψ^*.)

5. It is of considerable interest to show that both (8) and (72) are different forms of the same operator, the former obtaining in the same subsonic case while the latter holds in the supersonic case. In order to derive this conclusion, it is necessary to develop further the method of attack initiated in sections 6 and 8 of reference 3. The following result is a slight generalization of theorem (53) of reference 3.

Let E_1 be a solution of

$$G_B^{(2)}(\Lambda,\theta,t) \equiv \left[\frac{(1-t^2)^{\frac{1}{2}}}{t(\Lambda+i\theta)}\Lambda_H^2\left(\frac{1}{\Lambda_H}\frac{\partial E_1}{\partial H} + i\frac{\partial E_1}{\partial \theta} + \frac{\Lambda_{HH}E_1}{2\Lambda_H^2}\right)\right]_t + \frac{1}{(1-t^2)^{\frac{1}{2}}}\left(\Lambda_H^2\frac{\partial^2 E_1}{\partial \theta^2} + \frac{\partial^2 E_1}{\partial H^2} + BE_1\right) = 0^{1} \qquad (80)$$

[1]Note that Λ is a function of H alone and that $\Lambda_H = \frac{\partial \Lambda}{\partial H}$, $\Lambda_{HH} = \frac{\partial^2\Lambda}{\partial H^2}$. Indeed, $\Lambda_H^2 = l(H) = \rho_0^2\rho^{-2}(1 - M^2)$ and $\Lambda(H) = \lambda(M)$. (See sec. 8 of reference 3.)

which possesses the property that

$$\left(\frac{1}{\Lambda_H}\frac{\partial E_1}{\partial H} + i\theta\right)\frac{\Lambda_H{}^2\sqrt{1 - t^2}}{t(\Lambda + i\theta)} + \frac{E_1\sqrt{1 - t^2}\,\Lambda_{HH}}{2t(\Lambda + i\theta)} \tag{81}$$

is continuous at $t = 0$, at $\Lambda = 0$, and at $\theta = 0$, then

$$\psi(H,\theta) = \int_{-1}^{+1} E_1(H,\theta,t) f\left[\frac{1}{2}(\Lambda(H) + i\theta)(1 - t^2)\right]\frac{dt}{\sqrt{1-t^2}} \tag{82}$$

where f is an arbitrary, twice differentiable function of one variable will be a solution of

$$\Lambda_H{}^2\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial H^2} + B\psi = 0 \tag{83}$$

The proof of the above theorem follows step-by-step the proof of theorem (53) of reference 3.

Denote by $E_2(H,\theta,t)$ a solution of $G_B^{(2)}(\Lambda,-\theta,t) = 0$ and obtain the following representation for solutions of (83) in terms of two arbitrary, twice differentiable functions f_1, f_2 of one variable.

$$\psi(H,\theta) = \int_{-1}^{+1}\left\{E_1(H,\theta,t) f_1\left[\frac{1}{2}(\Lambda(H) + i\theta)(1 - t^2)\right] + E_2(H,\theta,t) f_2\left[\frac{1}{2}(\Lambda(H) - i\theta)(1 - t^2)\right]\right\}\frac{dt}{\sqrt{1 - t^2}} \tag{84}$$

For $M < 1$, $\Lambda(H) = \Lambda(M)$ (see equation (48) of reference 3) is real and therefore $\zeta = \Lambda - i\theta$, $\overline{\zeta} = \Lambda + i\theta$ represent conjugate complex variables. For $M > 1$, $\Lambda(H) = \lambda(M)$ becomes purely imaginary and therefore $\zeta = \Lambda + i\theta = i\left(\frac{\Lambda}{i} + \theta\right) = i\xi$ and $\zeta = \Lambda - i\theta = i\left(\frac{\Lambda}{i} - \theta\right) = i\eta$ where ξ and η are the real variables introduced in (64). It remains merely to show that (80) can be written in the form (6) for $M < 1$ and in the form (66) for $M > 1$. Suppose first that $M < 1$ noting that $E_\Lambda = \frac{\partial E}{\partial\Lambda} = \frac{E_H}{\Lambda_H} = \frac{\partial E}{\partial H}\Big/\frac{\partial\Lambda}{\partial H}$, (80) can be written in the form

$$\left[\frac{\sqrt{1-t^2}}{t\bar{\zeta}} \Lambda_H{}^2 \left(2 \frac{\partial E_1}{\partial \zeta} + \frac{\Lambda_{HH} E_1}{2\Lambda_H{}^2}\right)\right]_t + \frac{1}{\sqrt{1-t^2}} \left[\Lambda_H{}^2 \frac{\partial^2 E_1}{\partial \theta^2}\right.$$

$$\left. + \Lambda_H{}^2 \frac{\partial^2 E_1}{\partial \Lambda^2} + \frac{\partial E_1}{\partial \Lambda} \Lambda_{HH} + BE_1\right] = 0 \quad (85)$$

Since

$$\frac{\partial^2 E_1}{\partial H^2} = \left(\frac{\partial E_1}{\partial \Lambda} \Lambda_H\right)_H = \frac{\partial^2 E_1}{\partial \Lambda^2} \Lambda_H{}^2 + \frac{\partial E_1}{\partial \Lambda} \Lambda_{HH}$$

equation (85) can be replaced by

$$\frac{2\sqrt{1-t^2} \Lambda_H{}^2}{t\bar{\zeta}}\left(\frac{\partial E_1}{\partial \zeta} + \frac{\Lambda_{HH}}{4\Lambda_H{}^2} \frac{\partial E_1}{\partial t}\right) - \frac{2\Lambda_H{}^2}{t^2\bar{\zeta}\sqrt{1-t^2}}\left(\frac{\partial E_1}{\partial \zeta \partial \bar{\zeta}} + \frac{\Lambda_{HH}}{4\Lambda_H{}^2} E_1\right)$$

$$+ \frac{4\Lambda_H{}^2}{\sqrt{1-t^2}}\left(\frac{\partial^2 E_1}{\partial \zeta \partial \bar{\zeta}} + \frac{\Lambda_{HH}}{\Lambda_H{}^2}\left(\frac{\partial E_1}{\partial \zeta} + \frac{\partial E_1}{\partial \bar{\zeta}}\right) + \frac{B}{4\Lambda_H{}^2} E_1\right) = 0 \quad (86)$$

or introducing $E_1{}^* = E_1 \exp\left(\int^{\zeta+\bar{\zeta}} N(s)ds\right)$ where $N = \dfrac{\Lambda_{HH}}{4\Lambda_H{}^2}$

$$\frac{2\Lambda_H{}^2 \exp\left(-\int^{\zeta+\bar{\zeta}} N(s)ds\right)}{t\bar{\zeta}\sqrt{1-t^2}}\left\{(1-t^2)\frac{\partial^2 E_1{}^*}{\partial \zeta \partial t} - \frac{1}{t} \frac{\partial E_1{}^*}{\partial \zeta}\right.$$

$$\left. 2\bar{\zeta}t\left[\frac{\partial^2 E_1{}^*}{\partial \zeta \partial \bar{\zeta}} + \left(F + \frac{B}{4\Lambda_H{}^2}\right)E_1{}^*\right]\right\} = 0^{1} \quad (87)$$

If $F + \dfrac{B}{4\Lambda_H{}^2}$ is replaced by F_m and divided by a nonvanishing factor, it is seen that (87) is essentially the same as equation (75) of reference 3.

---

[1]Note that $t^{-1}E_{\zeta}{}^*$ of reference 3 should be corrected to $-t^{-1}E_{\zeta}{}^*$ and that (86) is the conjugate of equation (75) of reference 3, i.e., the latter may be obtained from the former by replacing ζ by $\bar{\zeta}$ and $\bar{\zeta}$ by ζ.

Table 3

Title: "β as a function of M"

M	β	M	β	M	β
1.00	0.0000	1.47	0.1921	5.90	1.4703
1.01	0.0008	1.48	0.1974	6.00	1.4827
1.02	0.0020	1.49	0.2026	6.10	1.4944
1.03	0.0039	1.50	0.2078	6.20	1.5061
1.04	0.0061	1.60	0.2590	6.30	1.5172
1.05	0.0085	1.70	0.3108	6.40	1.5279
1.06	0.0110	1.80	0.3618	6.50	1.5388
1.07	0.0140	1.90	0.4116	6.60	1.5491
1.08	0.0169	2.00	0.4602	6.70	1.5591
1.09	0.0200	2.10	0.5076	6.80	1.5689
1.10	0.0235	2.20	0.5535	6.90	1.5785
1.11	0.0268	2.30	0.5983	7.00	1.5877
1.12	0.0304	2.40	0.6413	7.10	1.5966
1.13	0.0339	2.50	0.6827	7.20	1.6054
1.14	0.0376	2.60	0.7229	7.30	1.6143
1.15	0.0415	2.70	0.7613	7.40	1.6225
1.16	0.0455	2.80	0.7983	7.50	1.6308
1.17	0.0494	2.90	0.8340	7.60	1.6388
1.18	0.0535	3.00	0.8682	7.70	1.6466
1.19	0.0577	3.10	0.9013	7.80	1.6542
1.20	0.0618	3.20	0.9329	7.90	1.6617
1.21	0.0663	3.30	0.9637	8.00	1.6688
1.22	0.0707	3.40	0.9930	8.10	1.6761
1.23	0.0754	3.50	1.0213	8.20	1.6829
1.24	0.0798	3.60	1.0487	8.30	1.6898
1.25	0.0845	3.70	1.0748	8.40	1.6964
1.26	0.0887	3.80	1.1002	8.50	1.7028
1.27	0.0932	3.90	1.1243	8.60	1.7093
1.28	0.0981	4.00	1.1481	8.70	1.7155
1.29	0.1027	4.10	1.1707	8.80	1.7216
1.30	0.1075	4.20	1.1926	8.90	1.7276
1.31	0.1124	4.30	1.2136	9.00	1.7335
1.32	0.1172	4.40	1.2339	9.10	1.7391
1.33	0.1220	4.50	1.2537	9.20	1.7447
1.34	0.1268	4.60	1.2724	9.30	1.7502
1.35	0.1319	4.70	1.2908	9.40	1.7557
1.36	0.1369	4.80	1.3088	9.50	1.7608
1.37	0.1417	4.90	1.3257	9.60	1.7659
1.38	0.1467	5.00	1.3424	9.70	1.7710
1.39	0.1519	5.10	1.3585	9.80	1.7760
1.40	0.1567	5.20	1.3740	9.90	1.7809
1.41	0.1619	5.30	1.3891	10.00	1.7858
1.42	0.1668	5.40	1.4038		
1.43	0.1719	5.50	1.4179		
1.44	0.1772	5.60	1.4315		
1.45	0.1819	5.70	1.4451		
1.46	0.1873	5.80	1.4579		

For the case $M > 1$, a similar procedure yields

$$\frac{2\Lambda_H^2 \exp\left(-\int^{\zeta+\bar{\zeta}} N(s)ds\right)}{i^2 t\xi\sqrt{1-t^2}} \left\{(1-t^2)\frac{\partial^2 E_1^*}{\partial\eta\partial t} - \frac{1}{t}\frac{\partial E_1^*}{\partial\eta} + 2\xi t\left[\frac{\partial^2 E_1^*}{\partial\xi\partial\eta} - E_1^*\left(F - \frac{B}{4\Lambda_H^2}\right)\right]\right\} = 0 \quad (88)$$

which up to a constant factor coincides with (70).

APPENDIX IV

THE COMPLEX POTENTIAL IN THE HODOGRAPH PLANE FOR A JOUKOWSKI PROFILE

1. In connection with the second method for the determination of $g^{(n)}(Z)$ it is necessary to have an analytic representation for the complex potential (in the hodograph plane) of an incompressible flow around various profiles.

This problem will be treated in the following for a symmetric Joukowski profile.

2. The function

$$z^+ = \eta a + z^* + \eta z^*, \ \eta > 0 \quad (89)$$

maps the circle $|z^*| = a$ into the circle $|z^+ - \eta a| = a(1+\eta)$. The transformation

$$z = \frac{1}{2}\left(z^+ + \frac{a^2}{z^+}\right) \quad (90)$$

maps $|z^+ - \eta a| = a(1 + \eta)$ into a Joukowski profile. Therefore

$$z = \eta a + (1 + \eta)z^* + \frac{a^2}{\eta a + (1 + \eta)z^*} \quad (91)$$

maps $|z^*| = a$ into a Joukowski profile.

Since the complex potential around $|z^*| = a$ is

$$w(z^*) = -V\left(z^* e^{i\alpha} + \frac{a^2}{z^* e^{i\alpha}}\right) - i\,\frac{\Gamma}{2\pi}\log\frac{z^*}{a} \tag{92}$$

the complex potential $W(z) = w[z^*(z)]$ is obtained by substituting the function

$$z^* = \frac{(z - 2\eta a) + s}{2(1 + \eta)}, \qquad s = \pm(z^2 - 4a^2)^{\frac{1}{2}} \tag{93}$$

(which is the inverse to (91)) into (92).

$$W(z) = -V\left[\frac{(z - 2\eta a + s)e^{i\alpha}}{2(1 + \eta)} + \frac{2a^2(1 + \eta)}{e^{i\alpha}(z - 2\eta a + s)}\right] - \frac{i\Gamma}{2\pi}\log\frac{z - 2\eta a + s}{2(1 + \eta)a} \tag{94}$$

Denoting by q the conjugate to the velocity vector gives

$$q \equiv ve^{-i\theta} = \frac{dW}{dz} = \frac{dW}{dz^*}\,\frac{dz^*}{dz}$$

$$= \left[-Ve^{i\alpha} + V\,\frac{a^2}{e^{i\alpha}z^{*2}} - \frac{i\Gamma}{2\pi}\,\frac{1}{z^*}\right]\left[\frac{+s + z}{2(1 + \eta)s}\right] \tag{95}$$

The aim of this appendix will be to represent W as a function of q. By writing

$$e^{i\alpha}\,\frac{(z - 2\eta a + s)}{2(1 + \eta)} + \frac{2a^2(1 + \eta)}{e^{i\alpha}(z - 2\eta a + s)} = r_1(z, s) \tag{96}$$

$$\frac{z - 2\eta a + s}{2(1 + \eta)a} = r_2(z, s) \tag{97}$$

it is seen that r_1 and r_2 are rational functions of z and s, where z, s, and q are connected by the relation

$$q = -V\,\frac{(s + z)e^{i\alpha}}{2(1 + \eta)s} + V\,\frac{2(1 + \eta)a^2(s + z)}{e^{i\alpha}(z - 2\eta a + s)^2 s} - \frac{i\Gamma}{2\pi}\,\frac{s + z}{(z - 2\eta a + s)s} \qquad (98)$$

and

$$s^2 = z^2 - 4a^2 \qquad (99)$$

Introducing a new variable t, defined by

$$z = \left(t + \frac{1}{t}\right) a$$

gives

$$s = a\left(t - \frac{1}{t}\right)$$

and it is found that r_1 and r_2 become rational functions of t, which will be denoted by R_1 and R_2. A formal computation yields

$$R_1(t) = a\left[\frac{(t - \eta)e^{i\alpha}}{(1 + \eta)} + \frac{(1 + \eta)}{(t - \eta)e^{i\alpha}}\right]$$

$$R_2(t) = \frac{(t - \eta)}{(1 + \eta)}$$

t and q are connected by a relation

$$q - R_3(t) = 0; \quad R_3(t) = \left(\frac{-t^2}{t^2-1}\right)\left[\frac{e^{i\alpha}U}{1 + \eta} - \frac{U(1 + \eta)}{(t-\eta)^2 e^{i\alpha}} + \frac{i\Gamma}{2\pi a}\,\frac{1}{(t-\eta)}\right] \qquad (100)$$

which is obtained by replacing in (98) s and z by $a\left(t+\frac{1}{t}\right)$ and $a\left(t-\frac{1}{t}\right)$, respectively. R_1, R_2, R_3 are rational functions of t; R_1, R_2 are so-called algebraic functions of q.

The determination of singular points of these functions as well as determination of series development of R_1 and R_2 around these points can be achieved using classical methods of theory of functions.

The derivation of the corresponding developments of R_1 and $\log R_2$ as a function of $Z = \log q$ does not involve additional essential difficulties.

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Table 4

The values of $S^{(0)}$ and $T^{(0)}$

(S operation)

	θ=0.0		θ=0.1		θ=0.2		θ=0.3		θ=0.4		θ=0.5	
$-\lambda$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$
0	0.0000	0.0000	-0.0993	0.0049	-0.1950	0.0188	-0.2841	0.0394	-0.3654	0.0639	-0.4388	0.0898
0.1	0.0000	-0.1058	-0.1113	-0.0983	-0.2165	-0.0777	-0.3116	-0.0480	-0.3955	-0.0139	-0.4685	0.0208
0.2	0.0000	-0.2270	-0.1297	-0.2152	-0.2486	-0.1833	-0.3509	-0.1396	-0.4363	-0.0921	-0.5075	-0.0462
0.3	0.0000	-0.3735	-0.1599	-0.3561	-0.2982	-0.3005	-0.4074	-0.2344	-0.4943	-0.1693	-0.5573	-0.1086
0.4	0.0000	-0.5649	-0.2147	-0.5240	-0.3794	-0.4303	-0.4895	-0.3275	-0.5643	-0.2372	-0.6188	-0.1624
0.5	0.0000	-0.8524	-0.3312	-0.7491	-0.5166	-0.5637	-0.6062	-0.4069	-0.6564	-0.2898	-0.6911	-0.2020
0.6	0.0000	-1.4440	-0.6377	-1.0263	-0.7424	-0.6523	-0.7566	-0.4445	-0.7619	-0.3134	-0.7690	-0.2219
0.7	-6.0916	0.0000	-1.3012	-0.9458	-1.0117	-0.5906	-0.9102	-0.4139	-0.8638	-0.2998	-0.8439	-0.2196
0.8	-1.7353	0.0000	-1.4350	-0.3981	-1.1560	-0.3995	-1.0166	-0.3280	-0.9449	-0.2580	-0.9198	-0.2015
0.9	-1.3726	0.0000	-1.2987	-0.1737	-1.1671	-0.2397	-1.0628	-0.2345	-0.9943	-0.2039	-0.9517	-0.1676
1.0	-1.2247	0.0000	-1.2013	-0.0914	-1.1355	-0.1456	-1.0717	-0.1617	-1.0186	-0.1539	-0.9803	-0.1349

	θ=0.6		θ=0.7		θ=0.8		θ=0.9		θ=1.0	
$-\lambda$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$	$S^{(0)}$	$T^{(0)}$
0	-0.5048	0.1150	-0.5643	0.1383	-0.6184	0.1590	-0.6723	0.1765	-0.7136	0.1910
0.1	-0.5326	0.0536	-0.5892	0.0832	-0.6399	0.1090	-0.6859	0.1308	-0.7281	0.1483
0.2	-0.5677	-0.0045	-0.6259	0.0322	-0.6660	0.0629	-0.7076	0.0889	-0.7457	0.1102
0.3	-0.6099	-0.0674	-0.6565	-0.0139	-0.6966	0.0221	-0.7328	0.0517	-0.7660	0.0758
0.4	-0.6619	-0.1016	-0.6985	-0.0525	-0.7310	-0.0125	-0.7683	0.0201	-0.7888	0.0462
0.5	-0.7192	-0.1347	-0.7444	-0.0818	-0.7680	-0.0397	-0.7908	-0.0057	-0.8129	0.0217
0.6	-0.7791	-0.1536	-0.7916	-0.1007	-0.8058	-0.0587	-0.8213	-0.0249	-0.8376	0.0025
0.7	-0.8366	-0.1577	-0.8371	-0.1090	-0.8424	-0.0697	-0.8510	-0.0377	-0.8619	-0.0115
0.8	-0.8866	-0.1490	-0.8777	-0.1080	-0.8759	-0.0736	-0.8788	-0.0449	-0.8848	-0.0209
0.9	-0.9260	-0.1323	-0.9110	-0.1003	-0.9048	-0.0720	-0.9033	-0.0474	-0.9153	-0.0265
1.0	-0.9546	-0.1120	-0.9381	-0.0887	-0.9286	-0.0666	-0.9243	-0.0464	-0.9237	-0.0287

Table 5
(S operation)

$A_{n,m}$

n \ m	0	1	2	3	4	5
0	-0.5	0.62500	0.119797	0.024741	0.003871	0.004286
-1	0.25	0.31250	0.1796955	0.0618525	0.0135485	0.019287
1	-1	0.416667	0.047919	0.007069	0.000860	
2	-0.6666	0.166667	0.013691	0.001571	0.000156	
3	-0.26667	0.047619	0.003042	0.000286	0.000024	
4	-0.076190	0.010582	0.000553	0.000044	0.000003	

Table 6
(S operation)

$C_{n,m}$

n \ m	0	1	2	3	4	5
1	0.5708	0	0	0	0	0
2	-0.0817	+0.5708	0	0	0	0
3	+0.0124	-0.0817	+0.2854	0	0	0
4	-0.0016	+0.0124	-0.0409	+0.0951	0	0

Table 7: The Values of ρ^k (G operation)

$\rho^{-3/2}$	ρ^{-1}	$\rho^{-1/2}$	ρ^0	$\rho^{1/2}$	ρ	$\rho^{3/2}$	ρ^2	$\rho^{5/2}$	ρ^3	$\rho^{7/2}$	ρ^4	$\rho^{9/2}$
1000.00000	100.00000	10.00000	1.	0.1	0.01	0.001	0.0001	0.00001	0.000001	0.(6)*1	0.(7)1	0.(8)1
125.00000	25.00000	5.00000	1.	0.2	0.04	0.008	0.0016	0.00032	0.000064	0.000013	0.000003	0.000001
37.03704	11.11111	3.33333	1.	0.3	0.09	0.027	0.0081	0.00243	0.000729	0.000219	0.000066	0.000020
15.62500	6.25000	2.50000	1.	0.4	0.16	0.064	0.0256	0.01024	0.004096	0.001638	0.000655	0.000262
8.00000	4.00000	2.00000	1.	0.5	0.25	0.125	0.0625	0.03125	0.015625	0.007813	0.003906	0.001953
4.62963	2.77778	1.66667	1.	0.6	0.36	0.216	0.1296	0.07776	0.046656	0.027994	0.016796	0.010078
2.91545	2.04082	1.42857	1.	0.7	0.49	0.343	0.2401	0.16807	0.117649	0.082354	0.057648	0.040354
1.95313	1.56250	1.25000	1.	0.8	0.64	0.512	0.4096	0.32768	0.262144	0.209715	0.167772	0.134218
1.37174	1.23457	1.11111	1.	0.9	0.81	0.729	0.6561	0.59049	0.531441	0.478297	0.430467	0.387420
1.00000	1.00000	1.00000	1.	1.0	1.00	1.000	1.0000	1.00000	1.000000	1.000000	1.000000	1.000000
0.75131	0.82644	0.90909	1.	1.1	1.21	1.331	1.4641	1.61051	1.771561	1.948717	2.143589	2.357948
0.57870	0.69444	0.83333	1.	1.2	1.44	1.728	2.0736	2.48832	2.985984	3.583181	4.299817	5.159780
0.45517	0.59172	0.76923	1.	1.3	1.69	2.197	2.8561	3.71293	4.826809	6.274852	8.157307	10.604499
0.36443	0.51020	0.71429	1.	1.4	1.96	2.744	3.8416	5.37824	7.529536	10.541350	14.757891	20.661047
0.29630	0.44444	0.66667	1.	1.5	2.25	3.375	5.0625	7.59375	11.390625	17.085938	25.628906	38.443359

ρ^5	$\rho^{11/2}$	ρ^6	$\rho^{13/2}$	ρ^7	$\rho^{15/2}$
0.(9)1	0.(10)1	0.(11)1	0.(12)1	0.(13)1	0.(14)1
0.(6)1	0.(7)205	0.(8)410	0.(9)819	0.(9)164	0.(10)328
0.000006	0.000002	0.(6)531	0.(6)159	0.(7)478	0.(7)143
0.000105	0.000042	0.000017	0.000007	0.000003	0.000001
0.000977	0.000488	0.000244	0.000122	0.000061	0.000031
0.006047	0.003628	0.002177	0.001306	0.000784	0.000470
0.028248	0.019773	0.013841	0.009689	0.006782	0.004748
0.107374	0.085899	0.068719	0.054976	0.043980	0.035184
0.348678	0.313811	0.282430	0.254187	0.228768	0.205891
1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2.593742	2.853117	3.138428	3.452271	3.797498	4.177248
6.191736	7.430084	8.916100	10.699321	12.839185	15.407022
13.785849	17.921604	23.298085	30.287511	39.373764	51.185893
28.925465	40.495652	56.693912	79.371477	111.120069	155.568096
57.665039	86.498086	129.747129	194.620493	291.931040	437.896560

(*) Note: The number in parentheses indicates the number of zeros following the decimal point. Thus, 0.(7)205 = 0.0000000205.

Table 8: The Values of $\cos \frac{m}{2}\varphi$ and $\sin \frac{m}{2}\varphi$ (G operation)

φ	sin φ/2	cos φ/2	sin φ	cos φ	sin 3φ/2	cos 3φ/2	sin 2φ	cos 2φ	sin 5φ/2
00°	.000 000	1.000 000	.000 000	1.000 000	.000 000	1.000 000	.000 000	1.000 000	.000 000
30°	.258 819	.965 926	.500 000	.866 025	.707 107	.707 107	.866 025	.500 000	.965 926
60°	.500 000	.866 025	.866 025	.500 000	1.000 000	.000 000	.866 025	-.500 000	.500 000
90°	.707 107	.707 107	1.000 000	.000 000	.707 107	-.707 107	.000 000	-1.000 000	-.707 107
120°	.866 025	.500 000	.866 025	-.500 000	.000 000	-1.000 000	-.866 025	-.500 000	-.866 025
150°	.965 926	.258 819	.500 000	-.866 025	-.707 107	-.707 107	-.866 025	.500 000	.258 819
180°	1.000 000	.000 000	.000 000	-1.000 000	-1.000 000	.000 000	.000 000	1.000 000	1.000 000
210°	.965 926	-.258 819	-.500 000	-.866 025	-.707 107	.707 107	.866 025	.500 000	.258 819
240°	.866 025	-.500 000	-.866 025	-.500 000	.000 000	1.000 000	.866 025	-.500 000	-.866 025
270°	.707 107	-.707 107	-1.000 000	.000 000	.707 107	.707 107	.000 000	-1.000 000	-.707 107
300°	.500 000	-.866 025	-.866 025	.500 000	1.000 000	.000 000	-.866 025	-.500 000	.500 000
330°	.258 819	-.965 926	-.500 000	.866 025	.707 107	-.707 107	-.866 025	.500 000	.965 926
360°	.000 000	-1.000 000	.000 000	1.000 000	.000 000	-1.000 000	.000 000	1.000 000	.000 000

φ	cos 5φ/2	sin 3φ	cos 3φ	sin 7φ/2	cos 7φ/2	sin 4φ	cos 4φ	sin 9φ/2	cos 9φ/2
0°	1.000 000	.000 000	1.000 000	.000 000	1.000 000	.000 000	1.000 000	.000 000	1.000 000
30°	.258 819	1.000 000	.000 000	.965 926	-.258 819	.866 025	-.500 000	.707 107	-.707 107
60°	-.866 025	.000 000	-1.000 000	-.500 000	-.866 025	-.866 025	-.500 000	-1.000 000	.000 000
90°	-.707 107	-1.000 000	.000 000	-.707 107	.707 107	.000 000	1.000 000	.707 107	.707 107
120°	.500 000	.000 000	1.000 000	.866 025	.500 000	.866 025	-.500 000	.000 000	-1.000 000
150°	.965 926	1.000 000	.000 000	.258 819	-.965 926	-.866 025	-.500 000	-.707 107	.707 107
180°	.000 000	.000 000	-1.000 000	-1.000 000	.000 000	.000 000	1.000 000	1.000 000	.000 000
210°	-.965 926	-1.000 000	.000 000	.258 819	.965 926	.866 025	-.500 000	-.707 107	-.707 107
240°	-.500 000	.000 000	1.000 000	.866 025	-.500 000	-.866 025	-.500 000	.000 000	1.000 000
270°	.707 107	1.000 000	.000 000	-.707 107	-.707 107	.000 000	1.000 000	.707 107	-.707 107
300°	.866 025	.000 000	-1.000 000	-.500 000	.866 025	.866 025	-.500 000	-1.000 000	.000 000
330°	-.258 819	-1.000 000	.000 000	.965 926	.258 819	-.866 025	-.500 000	.707 107	.707 107
360°	-1.000 000	.000 000	1.000 000	.000 000	-1.000 000	.000 000	1.000 000	.000 000	-1.000 000

Table 9 (S operation)

Computation of the stream function (in the logarithmic plane) of a compressible flow generated by the analytic function (2.5)

$-\lambda$	θ	$S^{(0)}$	$S^{(1)}$	$S^{(2)}$	$S^{(3)}$	$S^{(4)}$	$\frac{\partial S}{\partial \lambda}$	$\frac{\partial S}{\partial \lambda}$
.02	.05	-.0496	.0009	-.0002	.0004	.0001	.0538	.0538
	.2	-.1945	.0026	.0013	.0000	.0001	.1862	.1908
	.4	-.3627	-.0019	.0105	-.0001	-.0001	.2653	.2723
	.6	-.4970	-.0172	.0342	-.0003	-.0009	.2511	.2573
	.8	-.6223	-.1035	.1764	.0003	-.0034	.2125	.2006
.06	.08	-.0840	.0047	.0000	.0000	.0000	.0984	.0999
	.30	-.2961	.0142	.0033	-.0005	.0001	.2792	.2835
	.34	-.3147	.0112	.0058	-.0001	.0000	.2817	.2975
	.40	-.3584	.0094	.0091	-.0004	-.0001	.2906	.3085
	.50	-.4460	.0096	.0200	-.0011	-.0002	.2996	.3056
	.70	-.5640	.0054	.0492	-.0043	-.0020	.2575	.2541
	.90	-.6577	.0322	.1083	-.0020	.0109	.1927	.1832
.10	.15	-.1621	.0148	-.0001	.0000	.0000	.2050	.2082
	.35	-.3490	.0275	.0059	-.0006	-.0000	.3380	.3424
	.50	-.4593	.0282	.0194	-.0018	-.0003	.3362	.3406
	.60	-.5218	.0246	.0338	-.0030	-.0008	.3109	.3134
	.70	-.5770	.0178	.0539	-.0044	-.0016	.2529	.2773
	1.00	-.7085	-.0179	.1507	-.0094	-.0083	.1993	.1608
.20	.22	-.2672	.0458	-.0024	.0000	.0000	.4119	.4150
	.40	-.4311	.0706	.0039	-.0017	.0001	.4701	.4731
	.70	-.6145	.0802	.0623	-.0057	-.0010	.3224	.3549
	.80	-.6556	.0722	.0730	-.0093	-.0017	.2907	.2842
	1.00	-.7324	.0553	.1495	-.0244	-.0059	.2092	.1908
	1.10	-.7664	.0442	.1971	-.0301	-.0101	.1781	.1508
.30	.30	-.4454	.1193	-.0143	.0005	.0001	.9389	.6771
	.75	-.6934	.1791	.0422	-.0197	.0015	.3896	.3593
	.85	-.7274	.1791	.0726	-.0288	.0014	.3244	.2959
	.95	-.7602	.1757	.1102	-.0396	.0006	.2648	.2411
	1.10	-.7158	.1666	.1814	-.0589	-.0025	.2132	.1722
	1.20	-.8317	.1578	.2388	-.0740	-.0065	.1411	.1334
.40	.35	-.5895	.1873	-.0326	.0019	.0001	1.0839	.9059
	.60	-.6962	.2439	-.0128	-.0085	.0021	.5829	.5458
	.85	-.7672	.2587	.0493	-.0351	.0049	.3458	.3240
	.93	-.7883	.2588	.0789	-.0469	.0057	.2980	.2737
	1.05	-.8186	.2553	.1333	-.0673	.0054	.2416	.2109
	1.30	-.8764	.2197	.2881	-.0926	-.0074	.1639	.1122
.60	.40	-.7621	.3008	-.0653	.0074	-.0001	1.0761	
	.45	-.7654	.3149	-.0652	.0055	.0005	.9092	.9117
	.60	-.7808	.3460	-.0532	-.0042	.0036	.5990	.5959
	.75	-.8028	.3628	-.0240	-.0227	.0072	.4250	.4188
	.95	-.8378	.3714	.0434	-.0595	.0133	.2869	.2734
	1.00	-.8479	.3718	.0653	-.0709	.0147	.2623	
	1.15	-.8783	.3694	.1440	-.1108	.0180	.2045	.1785

(continued on next page)

Table 9 (Continued)

$-\lambda$	θ	$S^{(0)}$	$S^{(1)}$	$S^{(2)}$	$S^{(3)}$	$S^{(4)}$	$\frac{\partial S}{\partial \lambda}$	$\frac{\partial S}{\partial \lambda}$
.80	.50	-.9096	.4958	-.1450	.0233	-.0012	.5409	.0425
	.70	-.8844	.5258	-.1238	-.0005	.0076	.3759	.3753
	.90	-.8924	.5407	-.0676	-.0453	.0212	.2666	.2625
	1.05	-.9090	.5448	-.0020	-.0928	.0334	.2089	.1999
	1.10	-.9166	.5449	.0236	-.1129	.0396	.1916	.1819
	1.20	-.9323	.5445	.0832	-.1526	.0457	.1648	.1496
	1.40	-.9721	.5401	.2281	-.2515	.0592	.1242	.0959
1.00	.54	-.9729	.6908	-.2628	.0602	-.0075	.2251	.2292
	.75	-.9428	.7120	-.2373	.0246	.0093	.2156	.2211
	1.00	-.9453	.7247	-.1534	-.0589	.0426	.1612	.1685
	1.10	-.9538	.7270	-.1036	-.1045	.0586	.1319	.1459
	1.20	-.9664	.7277	-.0446	-.1578	.0756	.1191	.1246
	1.35	-.9903	.7262	.0611	-.2522	.1020	.0927	.0949
1.20	.58	-.9925	.8907	-.4198	.1231	-.0231	.0816	.0892
	.80	-.9726	.9049	-.3869	.0722	.0070	.1082	.1230
	1.05	-.9748	.9131	-.2937	-.0397	.0623	.0869	.1108
	1.15	-.9830	.9152	-.2405	-.1001	.0895	.0725	.1000
	1.22	-.9899	.9157	-.1945	-.1495	.1098	.0625	.0919
	1.30	-1.0008	.9149	-.1411	-.2090	.1346	.0508	.0822
	1.40	-1.0171	.9159	-.0653	-.2924	.1662	.0357	.0699

(continued on next page)

Note: Columns 1 - 6_h, i.e. $S^{(0)}$, $S^{(1)}$,.., $\frac{\partial S}{\partial \lambda}$ were computed by means of series as described in method II; column 7 namely , $\frac{\partial S}{\partial \lambda}$ was computed directly from the formula for the stream function, see reference 3.

Table 9 (Continued)

	θ	$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$	$\frac{\partial T}{\partial \lambda}$	$T^{(0)}$	$T^{(1)}$	$-T^{(2)}$	$T^{(3)}$	$\frac{\partial T}{\partial \lambda}$
.02	.05	-.0329	.0057	.0032	-.0007	-.0014	1.0258	-.0188	-.0010	.0000	.0000	1.0146
	.20	.0003	-.0198	.0004	.0001	.0001	.9440	.0001	-.0201	-.0004	.0000	.9412
	.40	.0476	-.0773	.0006	.0012	.0002	.7729	.0484	-.0775	-.0024	.0010	.7756
	.60	.1005	-.1658	-.0010	.0054	.0002	.6123	.1025	-.1659	-.0080	.0051	.6219
	.80	.1503	-.2627	-.0333	.0156	-.0000(01)	.4863	.1487	-.2794	-.0206	.0163	.5105
.06	.08	-.0576	-.0015	.0002	.00001	.0000(4)	1.0534	-.0579	-.0022	-.0001	.0000	1.0528
	.30	-.0129	-.0443	.0018	.00003	.0002	.8775	-.0125	-.0446	-.0029	.0003	.8777
	.34	.0006	-.0533	-.0010	.0012	.0000(5)	.8480	-.0009	-.0574	-.0037	.0004	.8386
	.40	.0195	-.0719	-.0034	.0019	.0000(2)	.7930	.0172	-.0791	-.0052	.0008	.7798
	.50	.0468	-.1204	.0050	.0027	-.0001	.6840	.0482	-.1209	-.0087	.0020	.6893
	.70	.0917	-.2160	.0080	.0086	-.0008	.5382	.1048	-.2247	-.0195	.0083	.5487
	.90	.1390	-.3407	.0020	.0236	-.0014	.4319	.1486	-.3508	-.0377	.0233	.4544
	1.10	.1565	-.4689	-.0008	.0478	-.0070	.3602	.1720	-.5077	-.0670	.0521	
.40	.35	-.3338	.0123	.0188	-.0048	.0006	.6196	-.2808	-.0129			.7482
	.60	-.1228	-.1559	.0725	-.0112	.0001	.5370	-.1017	-.1637			.3921
	.85	-.0100	-.3397	.1366	-.0074	-.0053	.4818	.0045	-.3401			.2979
	.93	.0152	-.4023	.1582	-.0023	-.0086	.4627	.0284	-.4008			.2823
	1.05	.0476	-.4978	.1879	.0106	-.0156	.4349	.0573	-.4950			.2637
	1.30	.1650	-.7081	.2173	.2080	-.0584	.3458	.0970	-.7031			.2338
.10	.15	-.0893	-.0073	.0010	-.00002	.0001	1.0554	-.0895	-.0077	-.0009	.0000	1.0548
	.35	-.0317	-.0598	.0054	.0005	.0001	.8365	-.0312	-.0600	-.0063	.0002	.8379
	.50	.0198	-.1218	.0098	.0023	-.0001	.6776	.0207	-.1223	-.0123	.0014	.6829
	.60	.0519	-.1717	.0119	.0048	-.0003	.5925	.0536	-.1721	-.0181	.0039	.6001
	.70	.0816	-.2282	.0145	.0091	-.0008	.5221	.0832	-.2282	-.0252	.0067	.5343
	1.00	.1488	-.4270	.0147	.0376	-.0039	.3808	.1487	-.4270	-.0693	.0341	.4047
.20	.22	-.1753	-.0091	.0035	-.0014	.0019	1.0753	-.1752	-.0092	-.0035	-.0001	1.0744
	.40	-.0923	-.0730	.0141	-.0006	.0000(4)	.7760	-.0901	-.0734	-.0127	-.0008	.7783
	.70	.0604	-.2573	.0326	.0095	-.0005	.4738	-.0006	-.2677	-.0384	.0022	.4803
	.80	.0626	-.2984	.0453	.0127	-.0030	.4250	.0629	-.2986	-.0465	.0052	.4365
	1.00	.1123	-.4402	.0588	.0347	-.0075	.3485	.1102	-.4395	-.0661	.0156	.3653
	1.10	.1316	-.5165	.0631	.0513	-.0108	.3204	.1272	-.5164	-.0740	.0268	.3396

Table 9 (Concluded)

$-\lambda$	θ	$T^{(0)}$	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$	$T^{(4)}$	$\frac{\partial T}{\partial \lambda}$	$T^{(0)}$	$T^{(1)}$	$-T^{(2)}$	$T^{(3)}$	$\frac{\partial T}{\partial \lambda}$
.30	.30	-.2856	-.0012	.0111	-.0019	-.0108	.9945	-.2344	+.0153	+.0081	-.0025	.9547
	.75	-.0165	-.2681	.0821	-.0014	-.0069	.3566	.0049	-.2682	-.0517	-.0004	.4002
	.85	+.0208	-.3396	.1008	+.0053	-.0056	.3226	.0376	-.3379	-.1029	.0030	.3614
	.95	+.0501	-.4142	.1183	+.0144	-.0092	.2983	.0643	-.4111	-.1199	.0125	.3325
	1.10	+.0855	-.5320	.1438	+.0358	-.0169	.2716	.0952	-.5273	-.1245	.0162	.2998
	1.20	+.0985	-.6144	.1600	+.0566	-.0240	.2577	.1104	-.6082	-.1539	.0211	.2823
.60	.40	-.3132	+.0168	.0291	-.0102	+.0019	.0564	-.3198	+.0139			
	.45	-.2639	-.0214	.0447	-.0135	+.0022	.0717	-.2636	-.0211			.0638
	.60	-.1533	-.1369	.0943	+.0224	+.0015	.1147	-.1535	-.1371			.1131
	.75	-.0777	-.2552	.1475	+.0286	+.0004	.1459	-.0786	-.2556			.1431
	.95	-.0076	-.4180	.2212	+.0270	-.0075	.1704	-.0104	-.4182			.1632
	1.00	+.0060	-.4597	.2398	+.0244	-.0107	.1747	+.0025	-.4628			
	1.15	+.0406	-.5872	.2955	+.0088	-.0240	.1847	+.0341	-.5872			.1696
.80	.50	-.1986	-.0163	.0679	-.0297	+.0070	-.2726	-.1985	-.0159			-.2766
	.70	-.1078	-.1939	.1704	-.0571	+.0097	-.0434	-.1079	-.1939			-.0486
	.90	-.0446	-.3692	.2775	-.0767	+.0058	+.0581	-.0449	-.3689			+.0463
	1.05	-.0094	-.5014	.3587	-.0817	-.0041	+.1016	-.0104	-.5020			.0820
	1.10	+.0007	-.5463	.3864	-.0814	-.0090	+.1130	-.0008	-.5467			.0900
	1.20	+.0185	-.6358	.4408	-.0757	-.0216	+.1322	+.0157	-.6362			.1019
	1.40	+.0465	-.8183	.5505	-.0447	-.0596	+.1643	+.0400	-.8187			.1137
1.00	.54	-.0619	+.3863	-.5576	-.0538	+.0168	-.2056	-.1260	-.0218			-.2693
	.75	+.0923	+.3747	-.7101	-.1071	+.0263	+.0015	-.0780	-.2213			-.0939
	1.00	+.2105	+.3884	-.9227	-.1568	+.0235	+.1543	-.0194	-.4525			+.0094
	1.10	+.2556	+.4009	-1.0174	-.1697	+.0157	+.2010	-.0019	-.5451			.0333
	1.20	+.2995	+.4164	-1.1175	-.1767	+.0033	+.2439	+.0007	-.6376			.0506
	1.35	+.3640	+.4450	-1.2776	-.1752	-.0255	+.3041	+.0171	-.7782			.0678
1.20	.58	+.0785	+.4983	-.7305	-.0903	+.0338	-.0975	-.0757	-.0414			-.1764
	.80	+.1662	+.5113	-.8962	-.1796	+.0566	+.0375	-.0511	-.2558			-.0808
	1.05	+.2708	+.5484	-1.1099	-.2664	+.0621	+.1665	-.0213	-.4945			-.0089
	1.15	+.3122	+.5696	-1.2038	-.2934	+.0556	+.2123	-.0107	-.5896			+.0104
	1.22	+.3412	+.5857	-1.2717	-.3110	+.0475	+.2436	-.0040	-.6563			+.0213
	1.30	+.3733	+.6061	-1.3535	-.3215	+.0335	+.2788	+.0029	-.7325			+.0315
	1.40	+.4133	+.6350	-1.4595	-.3322	+.0102	+.3220	+.0106	-.8283			+.0414

Note: Columns 1-6, i.e., $T^{(0)}$, $T^{(1)}$, ..., $\frac{\partial T}{\partial \lambda}$ were computed by means of series as described in method II; columns 7-11, i.e., $T^{(0)}$, $T^{(1)}$, ..., $\frac{\partial T}{\partial \lambda}$ were computed directly from the formula for the stream functions, see reference 3.

Table 10

The values of $\frac{\Delta S^{(0)}}{\Delta\lambda}, \frac{\Delta T^{(0)}}{\Delta\lambda}, \frac{\Delta S^{(0)}}{\Delta\theta}, \frac{\Delta T^{(0)}}{\Delta\theta}$ *

	λ=0		λ=0.1		λ=0.2		λ=0.3		λ=0.4		λ=0.5	
θ	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$
0.0	0.00	0.7	0.00	1.05	0.00	1.40	0.00	1.75	0.00	2.1	0.00	----
0.1	0.13	0.8	0.15	1.05	0.17	1.30	0.46	1.5	0.75	1.7	1.75	3.0
0.2	0.2	0.9	0.33	1.07	0.46	1.24	0.6	1.4	1.2	1.3	1.8	1.2
0.3	0.2	0.9	0.33	0.9	0.46	0.9	0.6	0.9	1.0	0.8	1.4	0.7
0.4	0.3	0.65	0.4	0.70	0.5	0.75	0.6	0.8	0.85	0.625	1.1	0.45
0.5	0.33	0.7	0.42	0.65	0.51	0.59	0.6	0.53	0.675	0.415	0.75	0.3
0.6	0.28	0.6	0.35	0.56	0.42	0.53	0.5	0.5	0.55	0.365	0.6	0.23
0.7	0.23	0.55	0.29	0.49	0.35	0.43	0.4	0.37	0.43	0.31	0.46	0.248
0.8	0.18	0.5	0.22	0.46	0.26	0.42	0.3	0.38	0.327	0.31	0.354	0.247
0.9	0.14	0.44	0.18	0.40	0.22	0.36	0.26	0.33	0.273	0.28	0.286	0.228
1.0	0.1	0.42	0.14	0.385	0.18	0.35	0.21	0.315	0.25	0.28	0.246	0.22

	λ=0.6		λ=0.7		λ=0.8		λ=0.9		λ=1.0	
θ	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$	$\frac{\Delta S^{(0)}}{\Delta\lambda}$	$\frac{\Delta T^{(0)}}{\Delta\lambda}$
0.0	0.00	----	----	0.00	-5.0	0.00	-3.0	0.00	-1.0	0.00
0.1	4.2	0.3	2.1	-2.3	0.0	-3.2	-1.1	-1.5	-0.7	-0.4
0.2	2.8	0.0	2.5	-1.3	0.7	-1.8	0.05	-1.25	-0.6	-0.7
0.3	1.6	-0.1	1.4	-0.5	0.3	-1.1	0.175	-0.85	0.05	-0.6
0.4	0.95	0.05	0.9	-0.3	0.66	-0.333	0.43	-0.366	0.2	-0.4
0.5	0.8	0.05	0.7	-0.17	0.56	-0.213	0.43	-0.256	0.3	-0.3
0.6	0.6	0.1	0.5	-0.03	0.416	-0.103	0.333	-0.176	0.25	-0.25
0.7	0.5	0.18	0.4	0.0	0.35	-0.043	0.30	-0.086	0.25	-0.13
0.8	0.38	0.18	0.35	0.02	0.32	-0.003	0.29	-0.026	0.26	-0.05
0.9	0.3	0.17	0.27	0.10	0.25	0.057	0.23	0.014	0.21	-0.03
1.0	0.243	0.16	0.24	0.10	0.23	0.08	0.115	0.05	0.0	0.02

* Note that $\frac{\Delta S^{(0)}}{\Delta\lambda} = \frac{\Delta T^{(0)}}{\Delta\theta}$, $\frac{\Delta T^{(0)}}{\Delta\lambda} = -\frac{\Delta S^{(0)}}{\Delta\theta}$.

(S operation)

ψ(λ, θ) in the vicinity of the curve ψ = 0.
(S operation)

λ	θ	ψ	λ	θ	ψ
- .02	.10	-.014	.60	.90	-.018
	.20	.006		.95	-.003
	.30	.038		1.00	.009
				1.05	.020
- .06	.20	-.036			
	.30	.003	- .80	.90	-.040
	.40	.038		1.05	-.004
				1.10	.007
- .10	.30	-.043		1.20	.025
	.40	-.001		1.40	.055
	.45	.017			
	.50	.036	-1.00	1.00	-.026
				1.05	-.018
- .20	.40	-.088		1.10	-.010
	.50	-.038		1.15	-.003
	.55	-.014		1.20	.004
	.60	.007			
	.65	.026	-1.20	1.10	-.014
				1.20	-.004
- .30	.65	-.024		1.30	.005
	.70	.001		1.40	.013
	.75	.018			
	.80	.032	-1.90	1.3	-.0025
				1.4	-.0002
- .40	.70	-.033		1.45	.0010
	.80	-.001		1.5	.0020
	.85	.011			
	.95	.032			

Table 12

The values of $a_o\psi_v$, ψ_λ, ψ_θ along the curve $\psi = 0$.

(S operation)

λ	$a_o\psi_v$	ψ_λ	ψ_θ
- .02	1.024	1.085	.256
- .06	1.055	.990	.377
- .10	1.002	.845	.400
- .20	.852	.580	.431
- .30	.715	.410	.392
- .40	.528	.260	.286
- .50	.483	.208	.274
- .60	.418	.159	.269
- .80	.366	.110	.229
-1.00	.254	.061	.134
-1.20	.166	.031	.090
-1.90	.067	.006	.023

Table 13
(S operation)
θ as a function of v/a_o for one quadrant of the curve $\psi = 0$

v/a_o	M	T	λ	θ
.707	.74523	.66681	-0.12091	.177
.669	.70111	.71305	-0.15905	.307
.633	.66000	.75127	-0.19962	.403
.557	.57513	.81806	-0.30026	.582
.494	.50652	.86223	-0.40123	.698
.440	.44877	.89365	-0.50296	.805
.395	.40131	.91594	-0.60065	.885
.355	.35956	.93312	-0.69930	.962
.288	.29042	.95690	-0.89731	1.067
.234	.23529	.97193	-1.09784	1.168
.183	.18362	.98300	-1.33821	1.245
.089	.08907	.99603	-2.05277	1.406

Table 14
(S operation)

The values of $\frac{a_o}{\rho_o}x$, $\frac{a_o}{\rho_o}y$ along the curve $\Upsilon = 0$, for one quadrant of the curve. The curve is symmetric with respect to both the x and y axes.

v/a_o	M	T	λ	$+\frac{a_o}{\rho_o}\lvert y\rvert$	$+\frac{a_o}{\rho_o}\lvert x\rvert$
.10	0.10010	.99498	-1.93676	.027	.002
.20	0.20081	.97963	-1.25082	.055	.009
.30	0.30274	.95307	-0.85821	.080	.020
.40	0.40656	.91362	-0.58907	.103	.034
.50	0.51299	.85839	-0.39083	.123	.052
.55	0.56743	.82342	-0.31062	.131	.063
.60	0.62284	.78235	-0.24069	.138	.077
.65	0.67933	.73383	-0.17949	.146	.093
.70	0.73704	.67585	-0.12761	.152	.113
.725	0.76640	.64236	-0.10237	.156	.135

Table 15
(S operation)

The values of $-\frac{a_o}{\rho_o}\frac{dy}{dv}$, $-\frac{a_o}{\rho_o}\frac{dx}{dv}$ along the curve $\Upsilon = 0$ for one quadrant of the curve.

M	v/a_o	T	λ	$-\frac{a_o}{\rho_o}\frac{dy}{dv}$	$-\frac{a_o}{\rho_o}\frac{dx}{dv}$
.766	.725	.64284	-0.10477		∞
.745	.707	.66707	-0.12111	.99	5.54
.701	.669	.71316	-0.15921	1.29	4.07
.660	.633	.75127	-0.19961	1.48	3.49
.575	.557	.81816	-0.30044	1.69	2.56
.507	.494	.86195	-0.40029	1.82	2.17
.401	.395	.91608	-0.60128	1.95	1.59
.360	.355	.93295	-0.69820	2.19	1.53
.235	.234	.97200	-1.09880	2.65	1.13
.184	.183	.98292	-1.33648	2.75	.94
.089	.089	.99603	-2.05277	2.80	.45

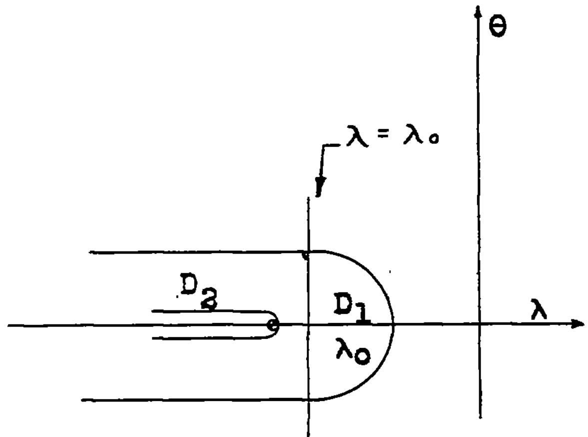

Figure 1

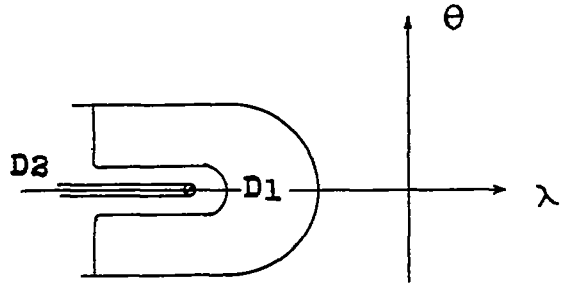

Figure 2

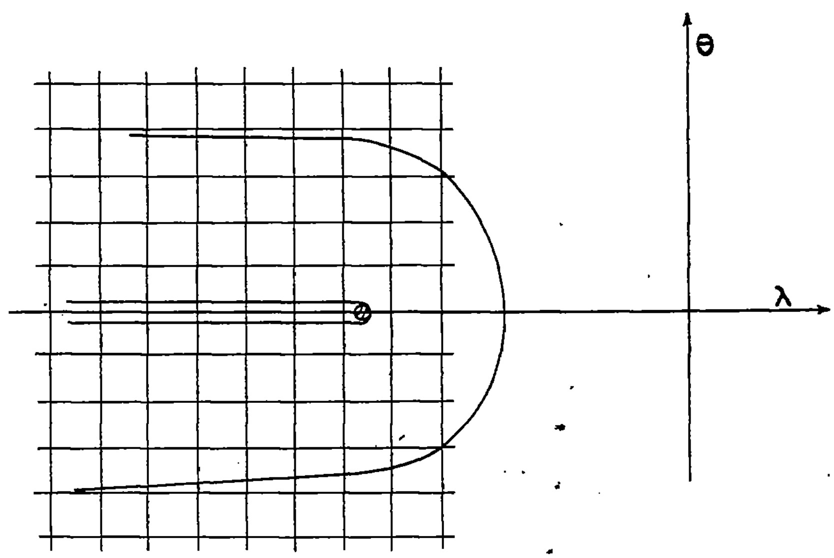

Figure 3

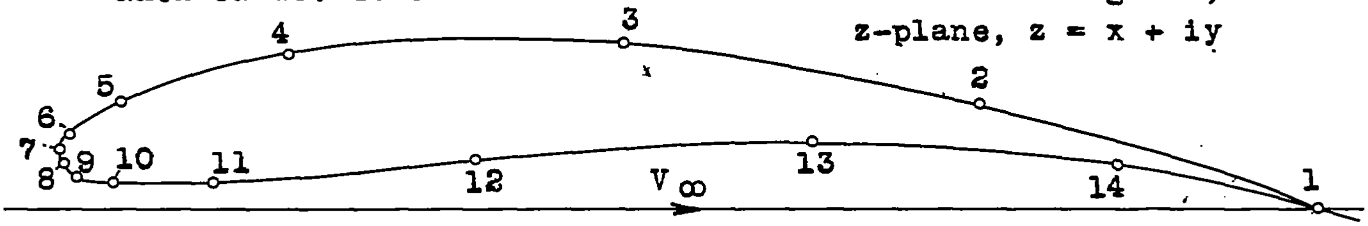

Figure 4.- Joukowski profile.

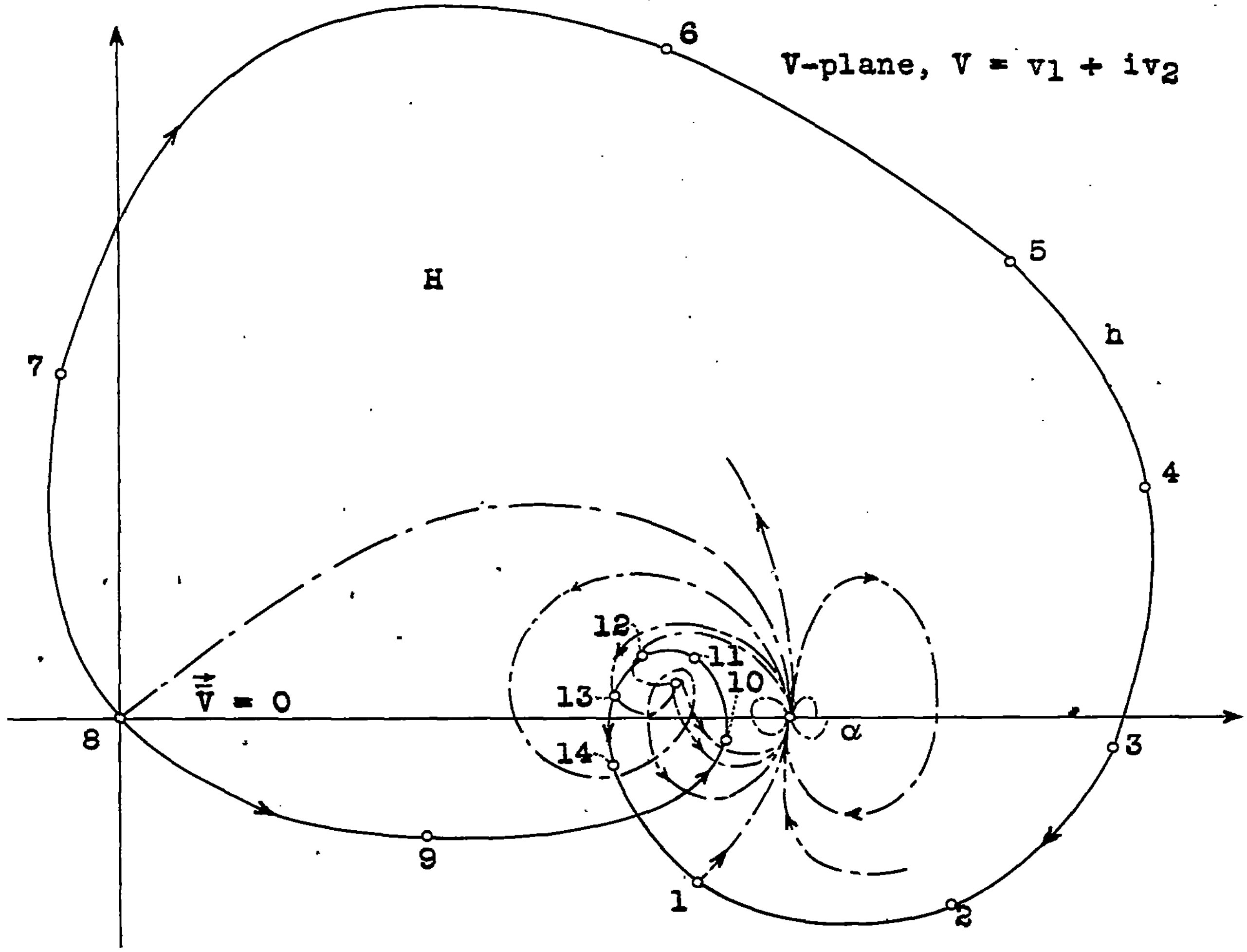

Figure 5.- The hodograph of a flow around the profile in fig. 4.

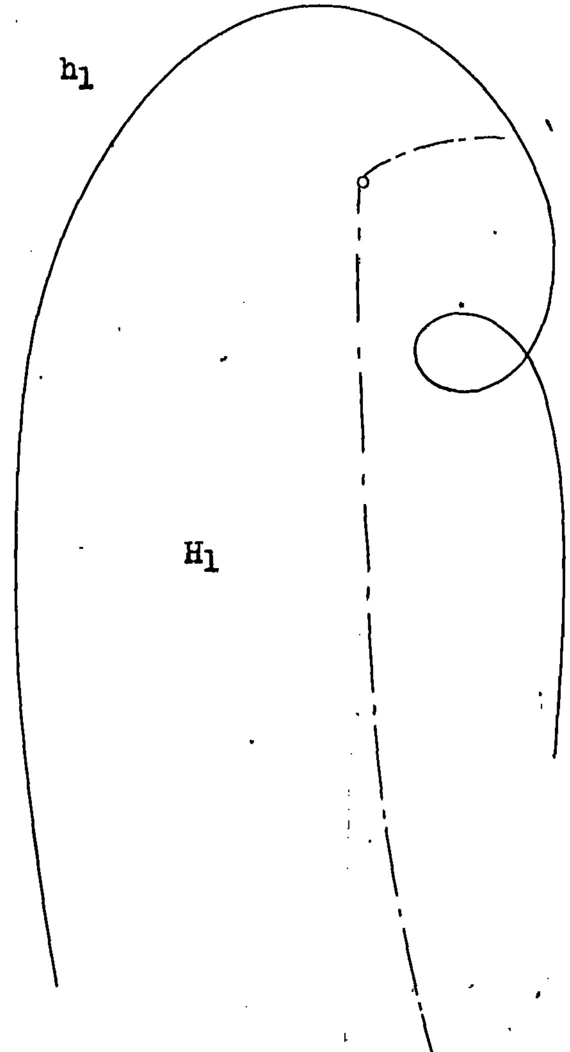

Figure 6.- The image of the hodograph (fig. 5); the (pseudo-)logarithmic plane.

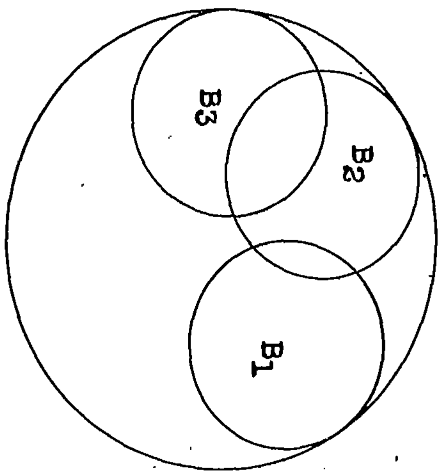

Figure 7.

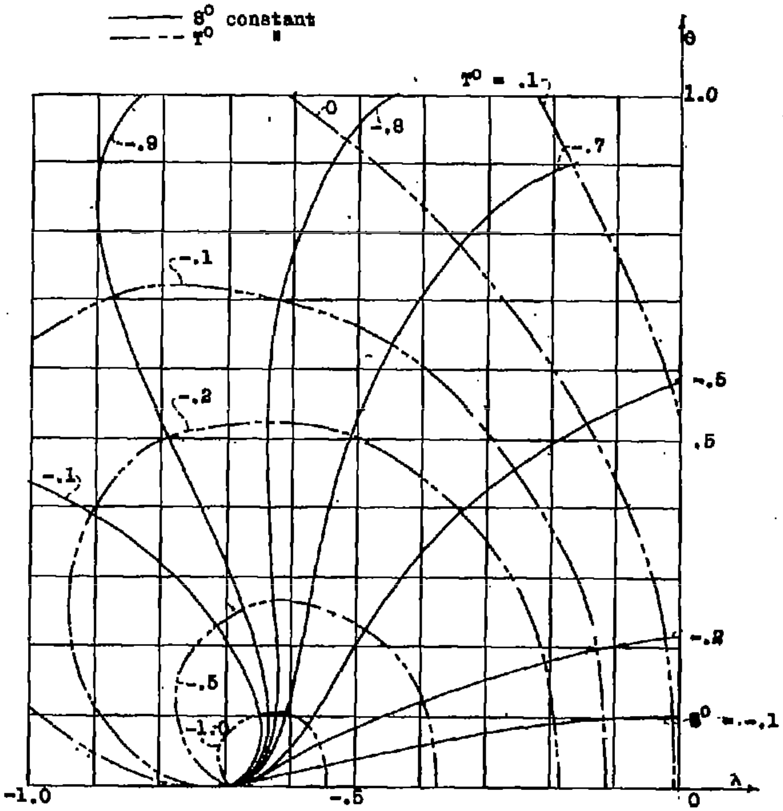

Figure 8.- The lines $S^{(o)}(\lambda,\theta)$ = constant, $T^{(o)}(\lambda,\theta)$ = constant. g(z) given by (14).

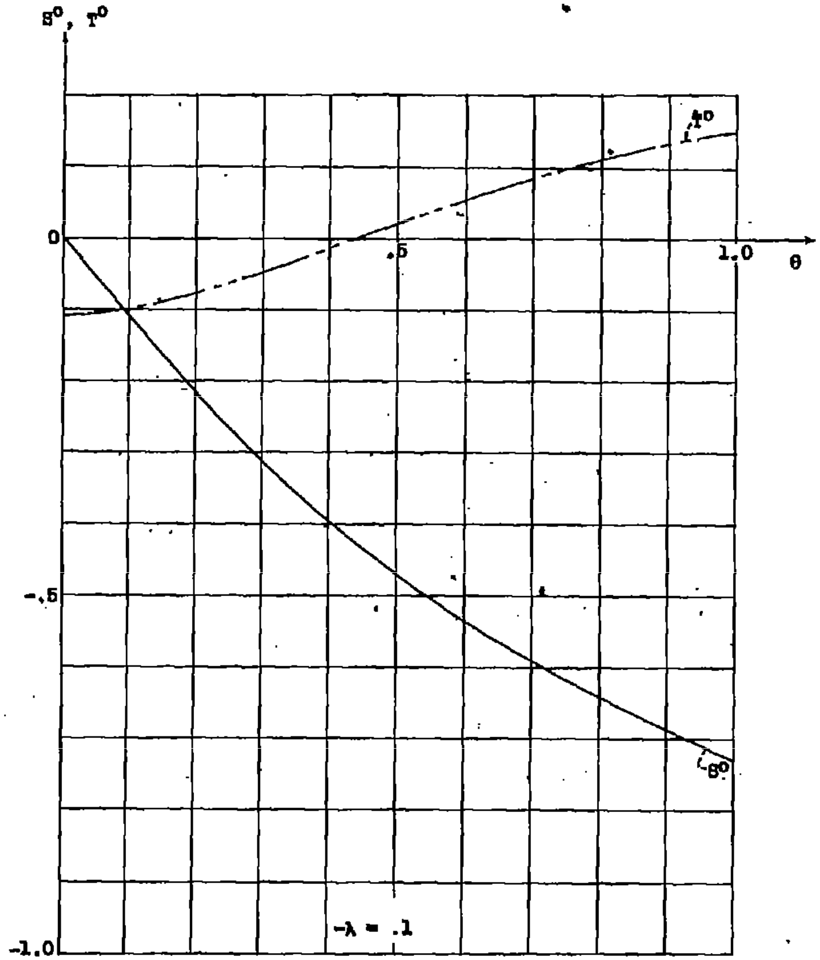

Figure 9.- The values of $S^{(o)}(-.1,\theta)$, $T^{(o)}(-.1,\theta)$.

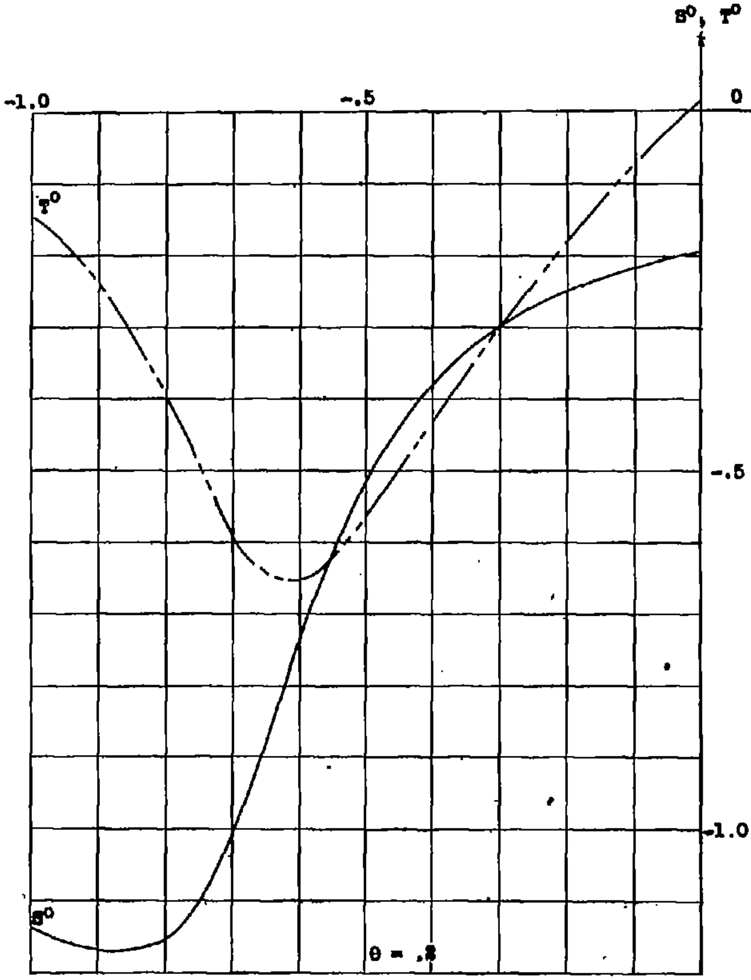

Figure 10.- The values of $S^{(o)}(\lambda, .2)$, $T^{(o)}(\lambda, .2)$.

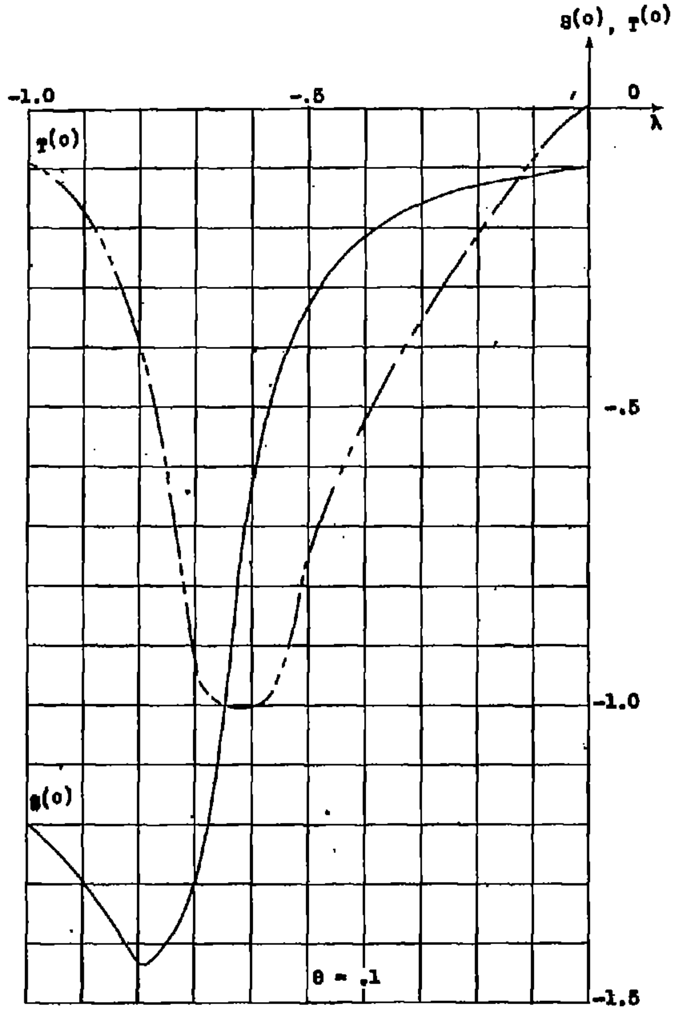

Figure 11.- The values of $S^{(o)}(\lambda, .1)$, $T^{(o)}(\lambda, .1)$.

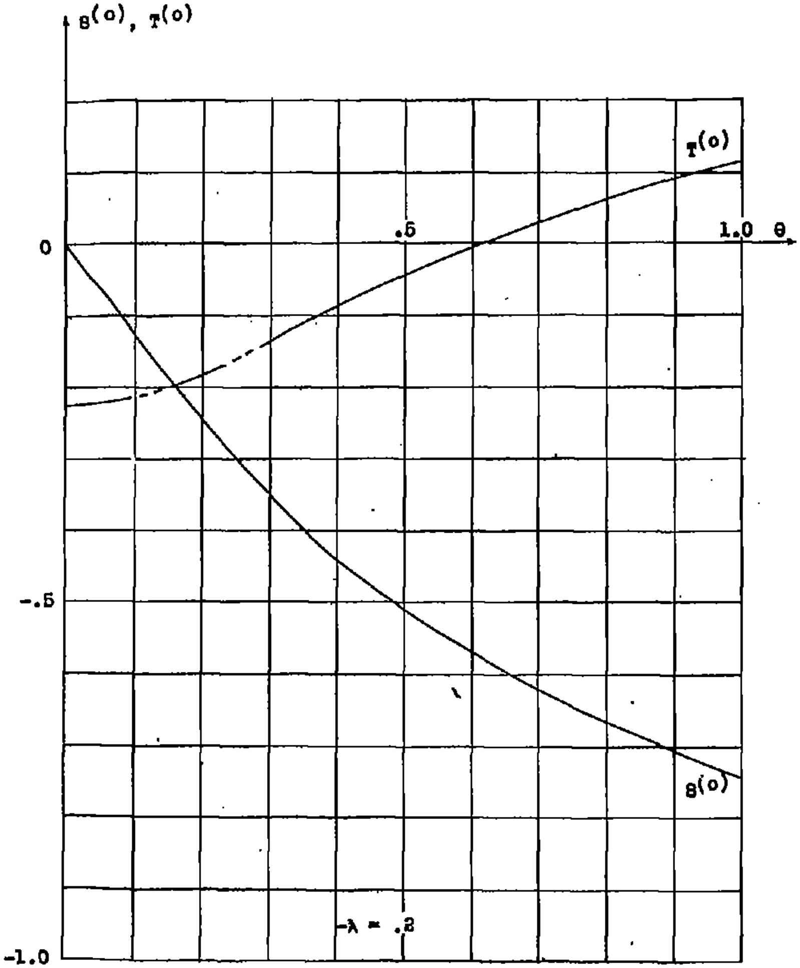

Figure 12.- The values of $S^{(o)}(-.2,\theta)$, $T^{(o)}(-.2,\theta)$.

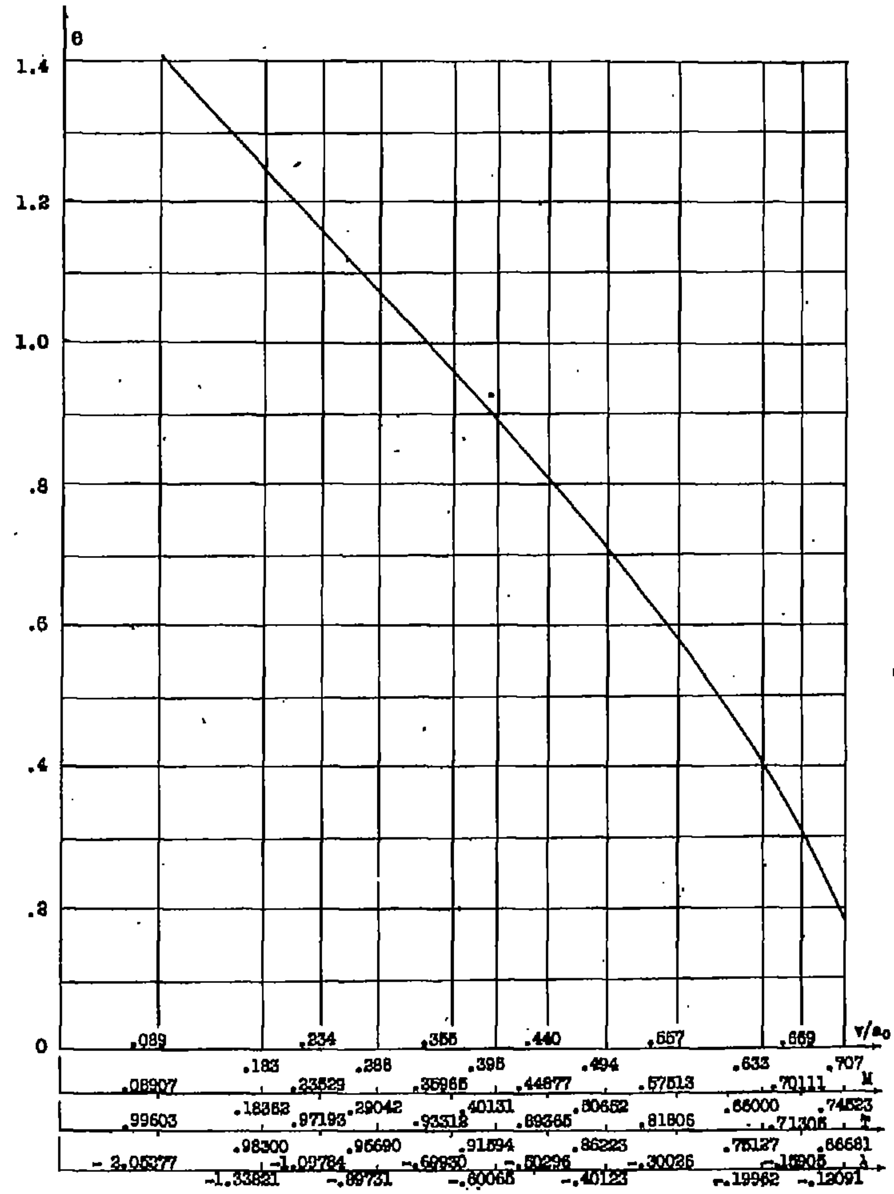

Figure 13.- θ as a function of v for $\psi(\lambda,\theta) = 0$.

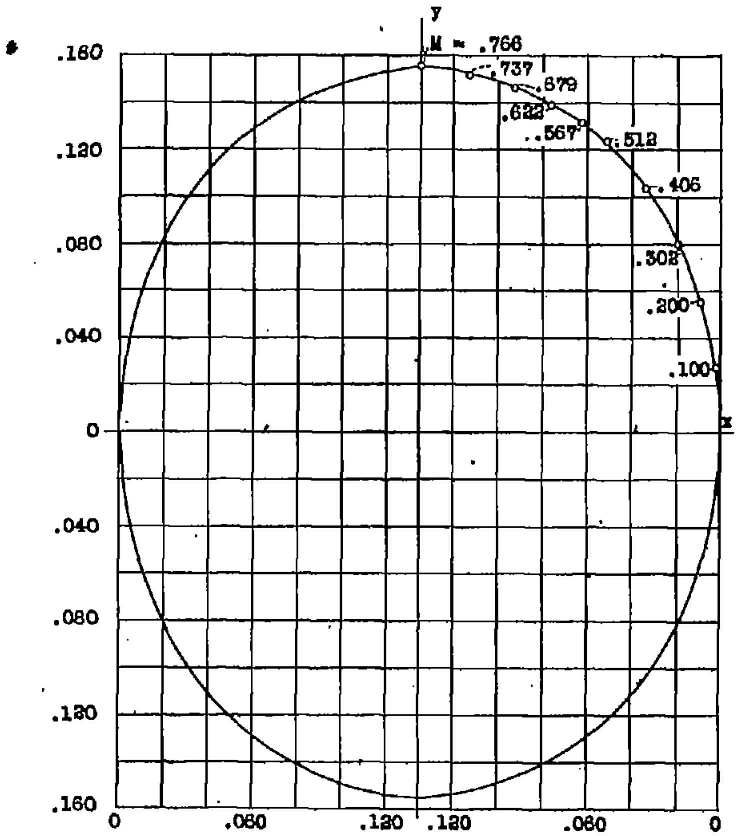

Figure 14.- The image of $\Psi(\lambda,\theta) = 0$ in the physical plane. (The contour of the compressible flow obtained from the function $g(Z) = 1/2\left[(1-2e^{Z})^{1/2} + (1-2e^{Z})^{-1/2}\right]$, $Z = \lambda + i\theta$).

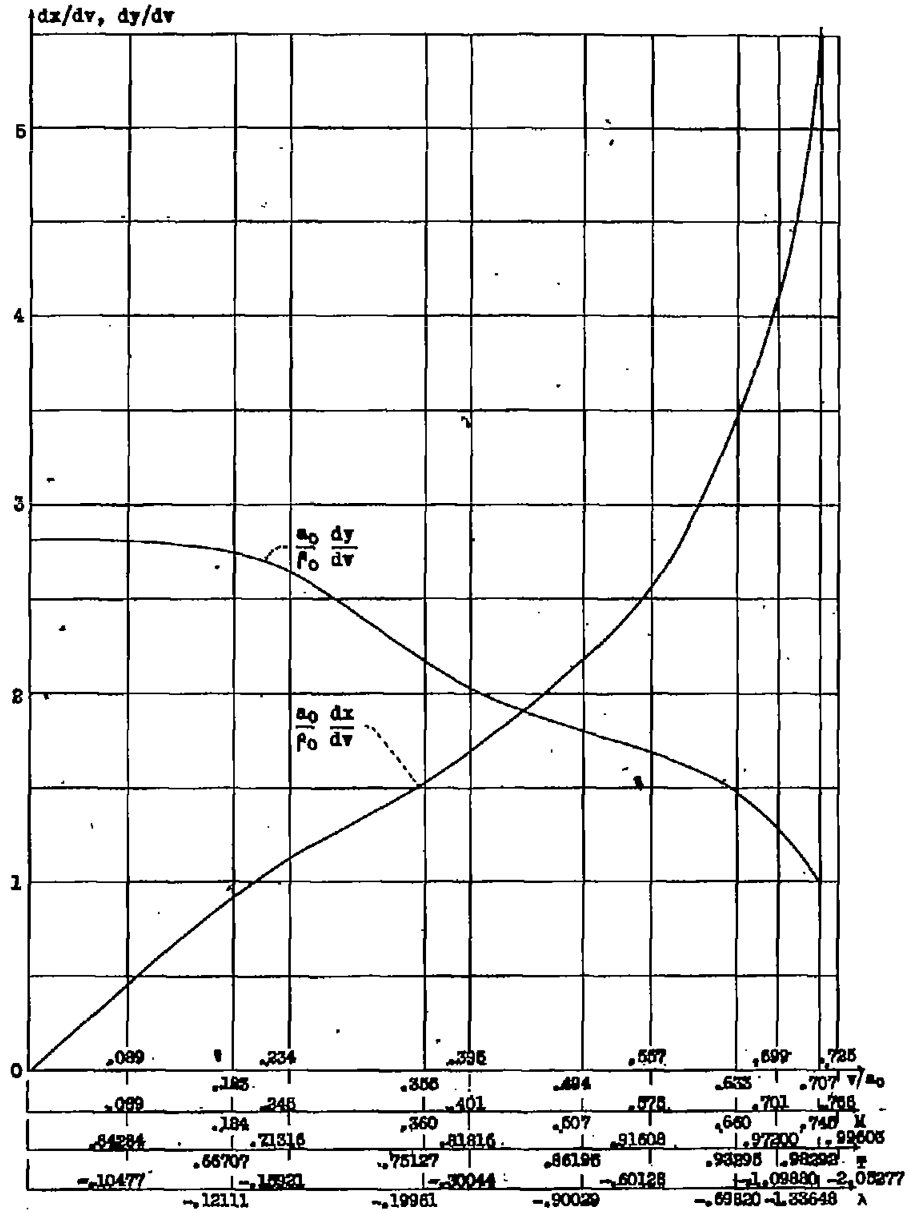

Figure 15.- The values of $\frac{a_0}{\beta_0}\frac{dx}{dv}$, $\frac{a_0}{\rho_0}\frac{dy}{dv}$.

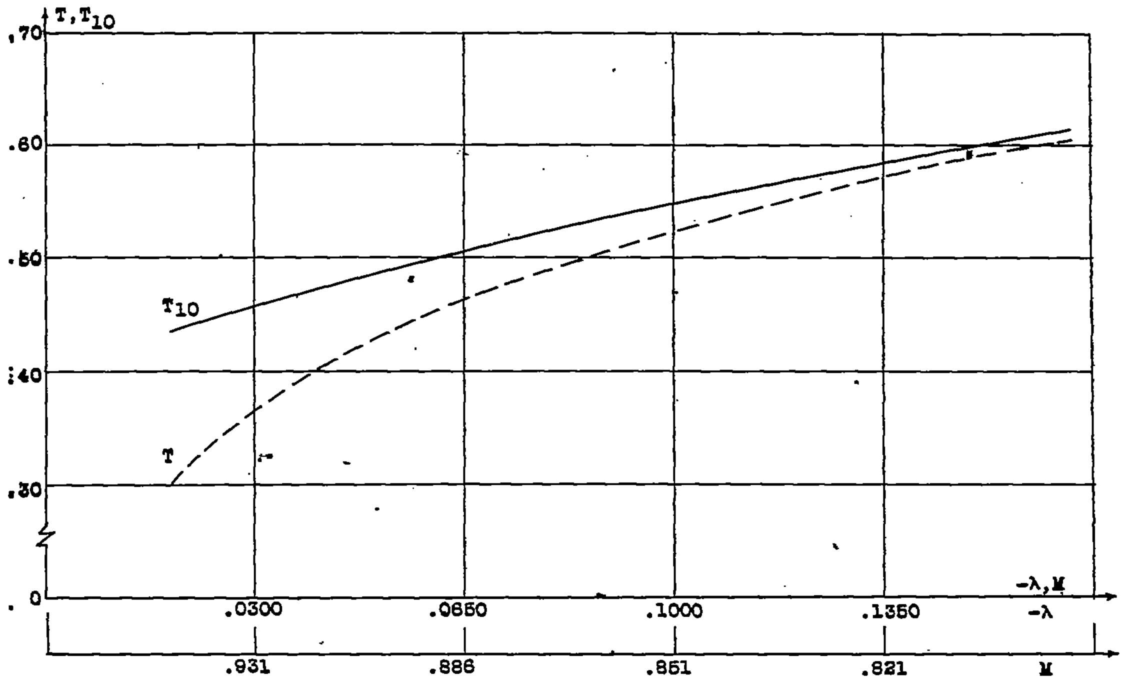

Figure 16.- The functions T and T_{10}.

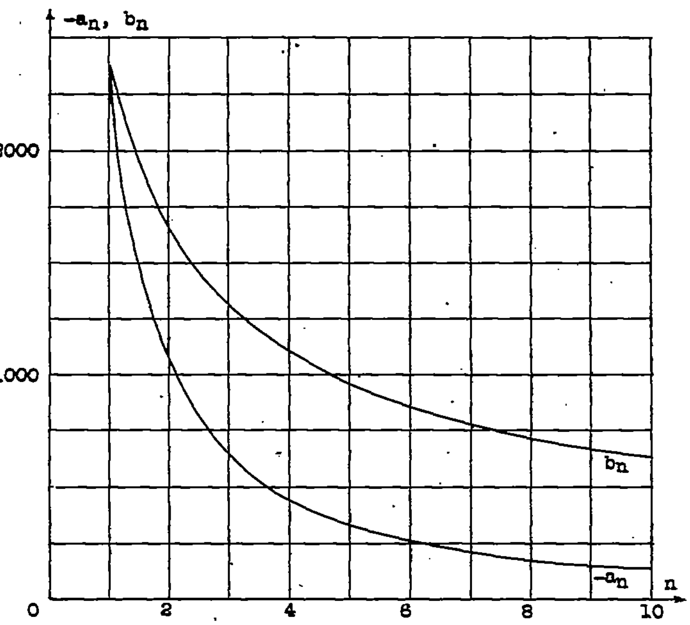

Figure 17.- The coefficients $-a_n$; b_n.