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SOME RECENT CONTRIBUTIONS TO THE STUDY OF TRANSITION AND TURBULENT BOUNDARY LAYERS

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SOME RECENT CONTRIBUTIONS TO THE STUDY OF TRANSITION AND TURBULENT BOUNDARY LAYERS[1]

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SUMMARY

The first part of this paper reviews the present state of the problem of the instability of laminar boundary layers which has formed an important part of the general lectures by von Kármán at the first and fourth Congresses and by Taylor at the fifth Congress. This problem may now be considered as essentially solved as the result of work completed since 1938. When the velocity fluctuations of the free-stream flow are less than 0.1 percent of the mean speed, instability occurs as described by the well-known Tollmien-Schlichting theory. The Tollmien-Schlichting waves were first observed experimentally by Schubauer and Skramstad in 1940. They devised methods of introducing controlled small disturbances and obtained measured values of frequency, damping, and wave length at various Reynolds numbers which agreed well with the theoretical results. Their experimental results were confirmed by Liepmann. Much theoretical work was done in Germany in extending the Tollmien-Schlichting theory to other boundary conditions, in particular to flow along a porous wall to which suction is applied for removing part of the boundary layer.

The second part of this paper summarizes the present state of knowledge of the mechanics of turbulent boundary layers, and of the methods now being used for fundamental studies of the turbulent fluctuations in turbulent boundary layers. A brief review is given of the semi-empirical method of approach as developed by Buri, Gruschwitz, Fediaevsky, and Kalikhman. In recent years the National Advisory Committee for Aeronautics has sponsored a detailed study at the National Bureau of Standards of the turbulent fluctuations in a turbulent boundary layer under an adverse pressure gradient sufficient to produce separation. The aims of this investigation and its present status are described.

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[1]Paper presented at the Sixth International Congress for Applied Mechanics, Paris, September 22-29, 1946.

INTRODUCTION

Since the last meeting of the International Congress for Applied Mechanics in 1938, there have been important advances in our understanding of some of the most fundamental aspects of fluid mechanics, in spite of the fact that the major attention of scientists was devoted to war research during most of the period. The results of this basic research have been available to a limited number of workers in confidential or secret reports. These reports have now been declassified for the most part and the exchange of information between scientific workers is increasing as evidenced by the convening of this Congress. Nevertheless, it would be presumptuous of me to assume that I am familiar with all the scientific work in fluid mechanics in all countries, even in the relatively narrow field which forms the subject of this lecture. Apologies are therefore made to any colleague who may find his accomplishments overlooked. Of necessity a large part of the paper describes work done in my own country and in particular at my own laboratory with the cooperation and financial assistance of the National Advisory Committee for Aeronautics. Acknowledgment is made to G. B. Schubauer, H. K. Skramstad, P. S. Klebanoff, and W. Squire, who carried out the experimental work at the National Bureau of Standards, and to Dr. George W. Lewis, Director of Research of the National Advisory Committee for Aeronautics, who gave permission to use some data not hitherto published.

This paper reviews the present state of knowledge of the mechanics of boundary-layer flow. As is well known, L. Prandtl (reference 1) in 1904 introduced the concept of boundary layer, a region of small thickness near the surface of an object immersed in a fluid stream, or moving through a fluid, within which the speed of the fluid relative to the surface rises within a comparatively short distance from zero at the surface to a value comparable with the relative speed of the body and the fluid at a great distance. The rate of change of speed with distance within the layer in the direction of the normal to the surface is at least an order of magnitude higher than in the flow outside the layer. Beginning about 25 years ago it has been possible to study the distribution of speed and the character of motion within the boundary layer.

It was soon realized that the state of flow in the boundary layer may be either laminar or turbulent corresponding to the two states of flow in a pipe which had been known since the time of Osborne Reynolds. Every layman has observed these two states of flow in the smoke rising from a cigarette in a quiet room, that is, the smooth flow in layers, with lazy oscillation of the stream as a whole near the cigarette and the sudden transition to a confused series of whirls and eddies at some distance above it. The essential distinction between these two types of flow has nevertheless been the subject of long and profound discussion at many of our previous meetings.

The key to many problems in the mechanics of fluids is to be found in the state of motion in the boundary layer. In a lecture at Toulouse in June 1943 M. J. Kampé de Fériet (reference 2) reviewed the significance of boundary layer studies to practical aeronautics under the title "Un probleme—clé de l'aéronautique: l'étude de la couche limite," giving special attention to the reduction in the drag of an airplane wing by smoothing the surface and maintaining laminar flow. The whole question of scale effect, the effect of wind tunnel turbulence on wind tunnel measurements of aerodynamic forces and moments on aircraft models and on control surfaces of aircraft models, flow separation, and maximum lift coefficients — all these and many similar questions can be adequately understood only in terms of knowledge of the state of flow in the boundary layer.

Even at speeds approaching the speed of sound, the recent work of Ackeret (reference 3) and Liepmann (reference 4) has shown that the flow pattern, including the location and form of shock waves, depends markedly on the state of flow in the boundary layer. There is thus a scale effect even at high Mach numbers which can be understood only in the light of the state of flow in the boundary layer.

It is however not the intention in this paper to pursue in detail the effects of boundary layer flow on the general flow pattern and the consequent effects on aerodynamic forces and moments. We shall restrict our attention to the boundary layer itself with a view to understanding the phenomena occurring within it.

The equations of laminar flow in a boundary layer were given in the original paper of Prandtl (reference 1) in 1904 and a solution for the special case of flow along a thin flat plate parallel to the direction of flow and without pressure gradient was given by Blasius (reference 5) in 1908. Experimental data on the velocity distribution were obtained by Burgers and Van der Hegge Zijnen (reference 6) in 1925 and later by others which confirmed the theoretical computations. It is certain that the equations are correct and hence that the fundamental mechanical principles are fully understood. For that reason no attempt is made here to review the many papers relating to laminar boundary layers. Reference is made to the book "Modern Developments in Fluid Dynamics" by S. Goldstein and others (Oxford, 1938) especially to chapter IV, and to the papers listed in the bibliography under the heading, Laminar Boundary Layer.

Thus more specifically this paper deals with two problems: (1) Stability of Laminar Flow in a Boundary Layer, and (2) Mechanics of the Turbulent Boundary Layer. The general plan of procedure will be first to review briefly the status of each problem at the time of the last Congress in 1938 and then to present the important developments since that time.

STABILITY OF LAMINAR FLOW IN A BOUNDARY LAYER

Status at 1938 Congress

Cause of transition.- The problem of transition to turbulence in boundary layers received much attention at the last Congress. G. I. Taylor devoted nearly half of his general lecture to the subject and four additional papers had some bearing on the problem. There were at that time two schools of thought. One school thought that transition is the result of a definite instability of the laminar boundary layer in which infinitesimal disturbances grow exponentially. The other thought that the laminar flow is stable for infinitesimal disturbances but that transition occurs as a result of external disturbances of finite magnitude.

Many investigators have attempted to solve the theoretical problem of the stability of laminar flow by determining what conditions are necessary to cause small disturbances in the form of velocity variations to increase with time. This work dates from that of Lord Rayleigh (reference 7) in 1880. About 17 years ago the specific problem of the stability of the laminar flow in a boundary layer near a thin flat plate without pressure gradient was studied by W. Tollmien (reference 8) at Göttingen. The problem was idealized by assuming a layer of constant thickness extending to an infinite distance in both directions within which the distribution of mean velocity was that computed by Blasius. According to the results of Tollmien's computation small disturbances in velocity of wave length lying in a certain region would be amplified whereas disturbances of shorter or longer wave length would be damped, provided the Reynolds number of the boundary layer was greater than a certain value. The calculations were repeated and extended by H. Schlichting (references 9 and 10) in 1933 and 1935. The amplified disturbance was assumed to grow until it caused a breakdown of the laminar flow. These were the views of the theoretical workers of the Göttingen school in 1938.

The experimental workers could find no evidence in 1938 to support the Tollmien-Schlichting theory. In 1935 my colleagues and I had studied the flow near a flat plate experimentally in continuation of the work described at the 1934 Congress for Applied Mechanics and failed to find any evidence of the development of the amplified disturbances described by Tollmien and Schlichting. Experiments by the groups at the California Institute of Technology, Massachusetts Institute of Technology, and Cambridge University, England, likewise failed to show the amplified disturbances. Hence the views of these groups were that the experimental evidence was definitely against the instability theory.

Effect of free-stream turbulence.- An alternative view was that the presence of finite disturbances in the free stream was the principal factor producing transition. A theory was given at the 1938 Congress by G. I. Taylor of the effect of free-stream turbulence. He assumed that transition occurred as the result of momentary separation in the regions of adverse pressure gradient associated with the turbulent fluctuations at sufficiently large values of the Kármán-Pohlhausen parameter $\delta^2/\mu U \times dp/dx$, where δ is the thickness of the boundary layer, μ the viscosity of the fluid, U the speed of the fluid in the free stream, and dp/dx the pressure gradient. On this assumption Taylor found that the Reynolds number at transition was a function of $\frac{u'}{U}\left(\frac{D}{L}\right)^{1/5}$ where $\frac{u'}{U}$ is the intensity of the turbulence, L is its scale, and D is a reference dimension of the body. This relation had been experimentally verified in the Cambridge laboratory for flat plates, and in our laboratory for spheres and for an elliptic cylinder.

L. Schiller presented at the 1938 Congress a study of transition in a pipe, attributing transition to the eddies originating at the entrance of the pipe and giving a quantitative theory.

Effect of curvature.- In addition to the experiments of G. I. Taylor (references 11 and 12) on the flow between rotating cylinders confirming the greater stability of flow over a convex as compared with a concave surface, the results of Clauser (reference 13) on the flow past a curved plate were available. These agreed qualitatively with the rotating cylinder result but there were some quantitative differences.

Effect of pressure gradient.- The discussion of the effect of pressure gradient at the 1938 Congress dealt mainly with the possible destabilizing effect of adverse pressure gradients. The information then available indicated that the boundary layer might be non-turbulent until separation occurs and that turbulence set in at this point at high Reynolds numbers and at some distance downstream from it at low Reynolds numbers.

Fluctuations in the laminar boundary layer.- It was suggested that the slow fluctuations in the laminar boundary layer reported at the 1934 Congress were probably fluctuations in thickness of the layer. Since the components of the fluctuations are not correlated, there are no shear stresses, and the distribution of mean velocity is unaffected by the presence of the fluctuations.

Discovery of Tollmien-Schlichting Oscillations

In 1940 a research program was undertaken at the National Bureau of Standards with the cooperation and financial assistance of the National Advisory Committee for Aeronautics to investigate the effectiveness of damping screens in reducing wind-tunnel turbulence and to study transition on a flat plate down to the lowest attainable turbulence level. Turbulence levels as low as 0.02 percent $\left(\frac{u'}{U} = 0.0002\right)$ were obtained. The Reynolds numbers for transition increased steadily as the turbulence was reduced to 0.08 percent and thereafter remained constant, as shown in figure 1.

In the course of this work it was decided to repeat the earlier observations of fluctuations in the laminar layer under conditions of low turbulence and in August 1940 my colleagues, G. B. Schubauer and H. K. Skramstad, obtained the records of the velocity fluctuations in the boundary layer of a flat plate by means of a hot wire anemometer as shown in figure 2. The frequency of these spontaneously occurring almost-sinusoidal oscillations agreed very well with the value predicted by the Tollmien-Schlichting theory. Their amplitude increased downstream until turbulent motion ensued, at first intermittently and then continuously. Since the occurrence of these regular oscillations was unexpected, numerous tests were made to rule out the possibility that they were effects of vibration or the result of disturbances traveling upstream from the turbulent part of the boundary layer or the result of acoustic phenomena. All these causes were ruled out.

The reason why these phenomena had not previously been observed was made clear by increasing the stream turbulence tenfold to 0.2 percent. The oscillations were found, but if the turbulence had been much larger they would have been difficult to identify because of the presence of irregular fluctuations and the near coincidence of their point of appearance with the transition point. In all known previous experiments the stream turbulence was of the order of 0.5 to 1.0 percent. Under these conditions transition was controlled by the magnitude and scale of the free stream turbulence in accordance with Taylor's theory.

Systematic Experimental Study at National Bureau of Standards

The next step was to produce waves in the laminar layer under controlled conditions and study their behavior. I will pass over the many schemes tried before completely satisfactory results were obtained with only a mention of sound, both pure notes and random noise, from a loud speaker in the tunnel wall and from a small headphone behind a hole in the plate. These attempts are described in the original NACA advance confidential report issued in April 1943, now declassified and available from the NACA as a War Time Report. Excitation by intense sound waves does produce transition nearer the leading edge of the plate, and in the lowest turbulence conditions in the wind tunnel the fluctuations were shown to be mainly wind tunnel noise. In the free atmosphere it may well be that the principal initial disturbances are sound waves from the engine or propeller.

Schubauer and Skramstad finally devised an ingenious method of introducing small disturbances of known frequency as illustrated in figure 3. The disturbances were produced by vibrating a very thin (0.002 inch) flat ribbon whose mean position was about 0.006 inch from the plate. A magnetic field was produced in the vicinity of the ribbon by electromagnets on the opposite side of the plate, and an alternating current of the desired frequency passed through the ribbon. The amplitude of the disturbance could be controlled by varying the current. The ensuing fluctuations of speed in the boundary layer were measured by a hot wire anemometer. Frequency, damping, and wave length could be independently measured.

The effect of the ribbon on the mean flow was exceedingly small and could not be detected 2 inches downstream. The results did not depend on the exact dimensions or position of the ribbon. The usual procedure was to measure u' at various distances for a fixed wind speed and frequency, then repeat at several different frequencies. From such a series of curves the neutral curve showing frequencies of waves which are neither damped or amplified could be determined and the damping and amplification coefficients could be determined for the region of Reynolds numbers covered. The wave lengths were determined by connecting the input to the ribbon to one pair of plates of a cathode-ray oscillograph and the hot-wire output to the other. The distance through which the hot wire was moved to change the phase by 180° is one-half the wave length. The wave velocity is equal to the frequency times the wave length.

Figure 4 shows the experimental values of the wave lengths of disturbances which are neither damped nor amplified corresponding to the Reynolds numbers indicated. Disturbances lying within the open loop are amplified; those outside it are damped. For Reynolds numbers below 420, disturbances of all wave lengths are damped.

Lin's Revision of Theory

After the experimental work had been completed, C. C. Lin, then a Chinese student at the California Institute of Technology and now professor at Brown University, undertook a revision of the mathematical theory of the stability of two-dimensional parallel flows and a clarification of some features of the Tollmien-Schlichting theory which had been adversely criticized. Lin made a brief report to the National Academy of Sciences (reference 14) and later published a complete account (reference 15). Lin's calculation gave slightly different results from those of Schlichting. Both curves are shown in figure 4. The agreement between theory and experiment is remarkable and similar agreement is found for the independently determined frequency locus (fig. 5). Further checks were made of the distribution of the u-fluctuations across the boundary layer. These showed the node and the phase reversal in the outer part of the layer as computed by Schlichting and with good quantitative agreement. The validity and applicability of the theories of Tollmien, Schlichting, and Lin are beyond question.

Confirmation of Experiments at Guggenheim Aeronautical Laboratory, California Institute of Technology

Early progress reports on this work were made available to H. W. Liepmann at the California Institute of Technology, who was engaged in the study of transition on the convex and concave sides of a curved plate. Liepmann was able to observe and study the Tollmien-Schlichting waves on the convex side of the plate. Schlichting had computed the effect of convex curvature on the neutral curve and found a slightly stabilizing effect. Liepmann used the oscillating ribbon technique of Schubauer and Skramstad as well as methods using an artificial regular roughness of variable wave length, and acoustic excitation. The results were found to agree closely with those obtained at the National Bureau of Standards, the influence of convex curvature being extremely small. The results were originally issued as NACA advance confidential report ACR No. 3H30 in August 1943 (now declassified and available as an NACA War Time Report).

Relation of Tollmien-Schlichting Waves to Transition

The mechanism of the instability of the laminar boundary layer can be said to be fully understood. Whatever small disturbances are initially present are selectively amplified until large sinusoidal oscillations are present. The small disturbances may arise from noise, free-stream turbulence, or surface roughness. The large oscillations do not, however, constitute turbulent flow.

Some experiments were made to study the passage from waves to turbulent flow as described in the report of Schubauer and Skramstad. The regular waves grow in amplitude, then become very distorted, then bursts of high frequency fluctuations occur. In records at some distance from the surface the bursts appear in the low velocity part of the cycle and in some records very near the surface there is evidence of intermittent separation.

The general belief is that the small scale irregular eddies of the turbulent motion arise from some dynamic instability like the breaking of water waves or the rolling-up of a vortex sheet. These dynamically unstable vortex sheets may arise from intermittent separation during a part of the wave cycle, or from modification of the velocity distribution by the shearing stress existing because of the $u'v'$ correlation in the Tollmien-Schlichting waves. These stresses increase with increasing amplitude of the wave.

It seems clear that further theoretical and experimental research is needed on this difficult non-linear problem.

Effect of Free-Stream Turbulence

The effect of free-stream turbulence at low turbulence levels has already been mentioned and illustrated in figure 1. Liepmann made a few measurements of the effect of turbulence levels from 0.0006 to 0.003 ($\frac{u'}{U}$ values, v' and w' not measured) on transition on the concave side of a curved plate. Otherwise, no work in this field has come to my attention.

Effect of Curvature

Liepmann completed an extensive study of the effect of curvature on transition. The work on a convex surface has already been mentioned as well as the result - the boundary layer on convex surfaces, at least up to values of momentum thickness equal to 0.001 times the radius of curvature, exhibits the same Tollmien-Schlichting instability as the flat plate and the effect of curvature is so small as to lie within the precision of measurement. Transition on the upper surface of an airfoil will therefore not be noticeably influenced by the curvature.

The experimental work on concave surfaces was described by Liepmann in NACA advance confidential report ACR No. 4J28 (now declassified and available as an NACA War Time Report). When the momentum thickness θ is greater than 0.0005 times the radius of curvature r the laminar flow is dynamically unstable due to three-dimensional disturbances as studied theoretically by Görtler (reference 16). In a stream of the lowest turbulence the Görtler parameter $R_\theta \sqrt{\theta/r}$, where R_θ is the Reynolds number based on momentum thickness, is equal to 9.0. This value decreases to about 6.0 at a turbulence level of 0.003 (value of u'/U).

If the value of θ/r is less than 0.00005 the flow is unstable because of Tollmien-Schlichting waves like the flat plate and the convex surface and transition occurs at a constant Reynolds number whose value depends on the free stream turbulence. For values of θ/r between 0.00005 and 0.0005 there appears to be a more or less continuous change from transition due to Görtler vortices to transition caused by Tollmien-Schlichting waves.

Effect of Pressure Gradient

Schubauer and Skramstad made a few measurements of the effect of pressure gradient. For moderate gradients, that is, 2 or 3 percent of the velocity pressure per foot acting on boundary layers less than 1/2 inch in thickness the favorable gradient had no noticeable effect on the neutral curve whereas the adverse gradient expanded the region of amplification. For gradients about five times as great, comparable to those which might be found on airfoils, the favorable gradient gave damping up to the highest Reynolds number reached (about 2600 based on displacement thickness) and the adverse gradient gave amplification over the region studied, which was limited by the onset of transition. These results are in agreement with the effects commonly observed on transition, namely, that a falling pressure delays or prevents transition whereas a rising pressure brings about early transition. This is the phenomenon used in the design of laminar flow airfoils.

Liepmann made some measurements of the effect of pressure gradient on the critical Reynolds numbers for the boundary layer on the convex surface of a plate with 20-foot radius of curvature. These measurements showed changes of the critical Reynolds number for this plate of about 20 percent for quite moderate pressure gradients, adverse gradients (decelerated flow) reducing and favorable gradients (accelerated flow) increasing the critical Reynolds number. The rate of change was most rapid near zero pressure gradient. For the concave surface of a plate with 2.5 foot radius of curvature the effect of pressure gradient appeared to be smaller than the errors of measurement which were for this case rather large.

Effect of Suction Applied to Boundary Layer

J. Ackeret (reference 17) and his colleagues at the Aerodynamic Institute of the Federal Institute of Technology at Zürich have demonstrated that the transition from laminar to turbulent flow in the boundary layer can be prevented by removing a part of the boundary layer air by suction through a small number of slots. He was in fact able to restore a turbulent boundary layer to the laminar condition. By careful attention to the detailed design of the internal ducting of the suction system he has obtained extremely low drag coefficients.

Mr. E. F. Relf (reference 18) gives a general account of work carried out in England during the war, on boundary layer suction as applied to the maintanance of laminar flow and reduction of drag of rather thick airfoils. A single slot was used on each surface of an airfoil especially designed to suit the suction, an idea due to A. A. Griffith and followed up by S. Goldstein and by Lighthill on the theoretical side and presumably by Relf and his collaborators on the experimental side. Reference is also made to the use of distributed suction acting through a porous surface with theoretical work by J. H. Preston and a flight test by F. G. Miles. No experimental results are given and no references to detailed reports.

The information now available on the aeronautical research work carried out in Germany during the war (reference 19) shows that the subject of boundary layer suction as a means of preserving laminar flow received attention. Various theoretical studies were made of the stability of laminar flow with pressure drop and with homogeneous suction. I have not had the opportunity of examining any of the reports in detail. The theoretical calculations indicate that the stability limit can be increased from its value of about 420 without suction to about 70,000 for the asymptotic velocity distribution finally attained at a great distance from the leading edge with uniform suction.

It is hoped that details of all of this work may soon be generally available.

MECHANICS OF THE TURBULENT BOUNDARY LAYER

Status at the 1938 Congress

While the 1938 Congress demonstrated a considerable advance in the theoretical and experimental aspects of isotropic turbulence and the effect of free stream turbulence on transition, there was no corresponding advance in the theory of fully developed turbulent motion involving shear flow. At the 1934 Congress Von Kármán had reviewed the turbulence problem and summarized the various physical concepts underlying the several theories of turbulent interchange, including Reynolds apparent stresses, the Boussinesq eddy viscocity, the Prandtl momentum transfer theory, the Taylor vorticity transfer theory, and the Von Kármán theory. Mention was made of Burgers' attempt to apply the principles of statistical mechanics and of the measurements of the turbulent fluctuations by Wattendorf and Reichardt in a straight channel. In the interval between 1934 and 1938 Taylor developed his statistical theory of turbulence which was so fruitful in treating the problem of isotropic turbulence. Von Kármán had made a generalization of this theory and attempted to treat the problem of shear flow. Otherwise, there were no developments which seem at this time to be of outstanding importance.

At the 1938 Congress itself C. B. Millikan gave a critical discussion of various formulae proposed for the velocity distribution and friction in channels and circular tubes attempting to bring them into a consistent system. I described a method devised by H. K. Stramstad for direct measurement of the turbulent shearing stress by means of a hot wire anemometer. Von Kármán considered the problem of parallel shear motion from the viewpoint of the theory of correlation coefficients. In the course of his paper he discussed the measurements of Wattendorf and Reichardt previously mentioned and remarked that over the middle region of half of the channel $\overline{u'^2}$ was approximately proportional to the shearing stress which in turn equals $-\overline{u'v'}$. This might, he said, indicate the existence of a kind of statistical similarity in this region. Von Kármán also pointed out the difficulty of reconciling Reichardt's measurements of the plane stress tensor with the picture that a scalar mixing length determines the transfer of momentum. In the discussions Tollmien and Prandtl introduced the idea that the turbulent fluctuations might consist of two components, one derivable from a harmonic function and the other satisfying an equation of the heat-conduction type, that is, a non-diffusive and a diffusive component or a viscosity-independent and a viscosity dependent type. These were the only contributions bearing on the mechanics of the turbulent boundary layer.

Development Arising from the Methods of Buri and Gruschwitz

Practically all of the methods available for the computation of the flow in turbulent boundary layers are of the "one dimensional" type involving the application of the Von Kármán integral equation for the momentum and the use of empirical relations obtained from a few experimental studies of flow in convergent and divergent passages. The earliest procedure was that suggested by Buri (reference 20). He assumed that the boundary layer at any section could be characterized by the momentum thickness $\theta = \int_0^\delta \left(1 - \frac{\bar{u}}{U}\right)\left(\frac{u}{U}\right) dy$ and by a single parameter Γ fixing the shape of the velocity distribution curve. The Von Kármán momentum equation was then applied to give an ordinary differential equation connecting θ and Γ. Buri first suggested a parameter $\Gamma_a = \frac{\partial(\tau/\tau_w)}{\partial(y/\theta)}$ where τ is the shearing stress at distance y and τ_w is the shearing stress at the wall, that is, Γ_a is the non-dimensional slope of the shearing stress curve. From the equations of motion $\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} = -\rho UU'$ at the wall. Hence $\Gamma_a = -\frac{\rho UU'\theta}{\tau_w}$. The shearing stress at the wall was assumed to be of the form

$$\tau_w/1/2\ \rho U^2 = f(U\theta/\nu, \Gamma_a)$$

But for $\Gamma_a = 0$, $f \sim (U\theta/\nu)^{-1/4}$. Then Buri assumed that in the general case $f = \xi(\Gamma_a)(U\theta/\nu)^{-1/4}$, that is, that the variables are separated. With this empirical assumption

$$\Gamma_a = \frac{-\rho UU'\theta}{1/2\rho U^2 \xi(\Gamma_a)(U\theta/\nu)^{-1/4}}$$

It is then more convenient to use $\Gamma = -1/2\Gamma_a\xi(\Gamma_a) = \frac{U'\theta}{U}\left(\frac{U\theta}{\nu}\right)^{1/4}$ as the form parameter. The introduction of these assumptions in the momentum relation gives the differential equation

$$\frac{d}{dx}\left[\left(\frac{U\theta}{\nu}\right)^{1/4}\theta\right] = \frac{5}{4}\left\{\xi - \left(\frac{9}{5} + H\right)\Gamma\right\}$$

in which the symbol H represents the ratio of the displacement thickness $\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$ to θ. On Buri's assumptions the right-hand side is a function of Γ only. From his own experiments on accelerating flows and from Nikuradse's experiments on both accelerating and decelerating flows, it appeared that the right-hand side was roughly a linear function and the equation was integrated on that assumption. Howarth (reference 21) discusses Buri's assumptions and gives an alternative method of solution. Separation occurs when Γ is −0.07 according to Buri, −0.06 according to Howarth, and between −0.05 and −0.09 according to Prandtl.

At about the same time that Buri proposed the method just described, Gruschwitz (reference 22) also suggested that the non-dimensional velocity distribution curves, u/U versus y/θ, belong to a one-parameter family. He selected as the parameter $\eta = 1 - \left(\frac{u}{U}\right)^2_{y=\theta}$. Then

$$\int_0^{\delta/\theta} \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\theta}\right) = \frac{\delta^*}{\theta} = H$$

should be a function of η. From experimental results, Gruschwitz found empirically for $\eta < 0.8$

$$-\frac{\theta}{1/2\rho U^2} \frac{d}{dx} \left(1/2\rho U^2 \eta\right) = 0.00894\eta - 0.00461$$

The momentum equation gives the relation

$$\frac{\tau_w}{\rho U^2} = \frac{d\theta}{dx} + \frac{\theta U'}{U} (2 + H)$$

If some assumption is made for τ_w, the equations can be solved for θ and η. Gruschwitz made the very rough assumption that τ_w is constant and suggested that separation takes place at or soon after $\eta = 0.8$.

Stüper (reference 23) found fair agreement with Gruschwitz's equations for observations in free flight under conditions where no separation occurred, but Peters (reference 24) found greater and greater departures as separation was approached.

Von Doenhoff and N. Tetervin (reference 25) gave a comprehensive summary and analysis of experimental data and suggested a modified procedure. They selected H as an appropriate shape parameter, plotting the values of u/U for a series of constant values of y/θ against H as abscissa.

A compilation of data from many sources gave a reasonably close approximation to a single family of curves. The value of H for separation was found to lie between 1.8 and 2.6 as compared with Gruschwitz's value of 1.85.

Von Doenhoff and Tetervin then proposed to use the parameter $\frac{2\rho UU'\theta}{\tau_w}$ $\left(\text{in their notation } \frac{\theta}{q}\frac{dq}{dx}\frac{2q}{\tau_o}\right)$ as characteristic of the external forces acting on the boundary layer. This parameter is the same except for the sign and the factor 2 as the Γ_a originally proposed by Buri. The parameter is the ratio of the non-dimensional pressure gradient to the shearing stress coefficient at the wall. It is then assumed that this parameter determines not H itself, that is, the particular velocity distribution curve of the one-parameter family, but the rate of change of H. Thus $\theta\frac{dH}{dx}$ is plotted against $\frac{2\rho UU'\theta}{\tau_w}$, τ_w being determined from the Squire and Young formula (reference 26)

$$\frac{\rho U^2}{\tau_w} = \left[5.890 \log_{10}\left(4.075 \frac{U\theta}{\nu}\right)\right]^2$$

The experimental data are fairly well represented by the empirical equation

$$\theta \frac{dH}{dx} = e^{4.680(H-2.975)}\left[-\frac{2\rho UU'\theta}{\tau_w} - 2.035(H-1.286)\right]$$

The two equations just given taken with the momentum relation give, on elimination of τ_w two simultaneous first order differential equations which can be solved by a step-to-step calculation.

Kalikhman (reference 27) describes a different empirical approach based again on the momentum relation and the Squire and Young formula. He introduces a variable

$$z = \left(\frac{\rho U^2}{\tau_w}\right)\left(\frac{U\theta}{\nu}\right)$$

which reduces the momentum equation to an equation linear in z. The variation of the coefficients is said to be small in a turbulent boundary layer and the solution of the resulting linear equation with constant coefficients is given by the formula

$$z = \frac{C + \frac{1.2}{\nu U_o^3}\int U^4 dx}{U^3/U_o^3}$$

The physical intrepretation of the assumptions made appears to be that the pressure gradient has a considerable influence on the resistance law and on the parameter H but only a negligible influence on $U\theta/\nu$.

Kalikhman suggests that separation occurs when

$$\frac{U'\theta}{U}\left(\frac{U\theta}{\nu}\right)^{-0.08} = 0.0013,$$

this simple criterion being an empirical approximation to a much more complicated expression.

These are the principal developments along this path which have come to my attention. Although of some importance to the practical engineer, they do not satisfy the inquiring mind attempting to construct a rational theory. All rest on more or less arbitrary empirical assumptions, and become more and more unsatisfactory as separation is approached.

The Fediaevsky Method

Fediaevsky (reference 28) called attention to the fact that the previous treatments had assumed either a constant shearing stress or a linear variation of shearing stress through the boundary layer. These distributions were known to be incorrect when a pressure gradient is present. He proposed the polynomial distribution given by

$$\frac{\tau}{\tau_w} = 1 + A_1\,(y/\delta) + A_2\,(y/\delta)^2 + A_3\,(y/\delta)^3 + A_4\,(y/\delta)^4$$

where the coefficients A are determined from the relation at the wall $\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y}$ and the relations at the outer boundary $\tau = 0$, $\partial\tau/\partial y \equiv 0$, $\partial^2\tau/\partial y^2 = 0$. They are found to be

$$A_1 = \frac{\delta}{\tau_w}\frac{\partial p}{\partial x},\quad A_2 = 0,\quad A_3 = -4-3\,\frac{\delta}{\tau_w}\frac{\partial p}{\partial x},\quad A_4 = 3+2\,\frac{\delta}{\tau_w}\frac{\partial p}{\partial x}$$

Fediaevsky proceeds by using the Prandtl assumption

$$\tau = \rho l^2\left(\frac{du}{dy}\right)^2$$

The mixing length l is approximated by the relation

$$\frac{l}{\delta} = 0.14 - 0.08\,(1 - y/\delta)^2 - 0.06\,(1 - y/\delta)^4$$

an empirical formula developed by Prandtl on the basis of experiments on flow in pipes. The solution of the resulting equation in explicit form is a complicated equation of many terms.

Developments Proceeding from Reynolds Theory of Turbulent Stresses

It is not necessary to review here the fundamental equations of turbulent flow developed by Reynolds which differ from the Navier Stokes equation only in the additional stress components

$$-\rho u'^2, \ -\rho v'^2, \ -\rho w'^2, \ \overline{-\rho u'v'}, \ \overline{-\rho v'w'}, \text{ and } -\overline{\rho u'w'}.$$

The great success of the Taylor-von Kármán statistical theory as applied to isotropic turbulence leads to confidence in its fundamental validity. Isotropic turbulence can, however, be described very simply by two parameters, an intensity parameter and a scale parameter. In non-isotropic turbulence the description of the state of the turbulence becomes much more complex. Six quantities instead of one are required to specify the intensity and six scalar functions are required to specify the correlation tensor. As noted by von Kármán at the last Congress the single available experimental measurement of a turbulent stress tensor was inconsistent with the idea of a simple mixing process dependent on one scalar quantity.

A recent paper by Nevzgljadov (reference 29) suggests a type of theory intermediate between a complete theory utilizing a stress tensor and a mixing length tensor and the well-known simple mixing length theories of Prandtl and von Kármán. Nevzgljadov writes down the Reynolds equations, the continuity equation, and the energy equation for the turbulent velocity fluctuations. The unknown functions are the mean velocity, the mean pressure, the turbulent pressure, $\pi = 1/3\ \rho(u'^2+v'^2+w'^2)$ the turbulent stress tensor, the current density of the kinetic energy of the turbulent fluctuations, the energy flux transferred by the pressure fluctuations, and the viscous dissipation. It is proposed to select the mean velocity, the mean pressure, and the turbulent pressure as independent fundamental quantities and to assume that the other unknowns can be related to these fundamental quantities by "equations of state".

This procedure is a generalization of the assumption made by Prandtl as to the relation between the turbulent shearing stress and the mean motion, a relation which corresponds to what Nevzgljadov describes as an equation of state. The difference is that in Nevzgljadov's theory the turbulent shearing stress may depend on the turbulent pressure as well as on the mean motion. In fact, the question is discussed as to whether the shearing stress at a given point <u>can</u> be expressed in terms of the values of the fundamental quantities $\overline{\text{in}}$ the immediate neighborhood of the point. It might depend on their values upstream or even at all points in the field and thus be capable of representation only by an integral equation.

The particular "equation of state" connecting the turbulent shearing stress and the independent quantities proposed by Nevzgljadov is

$$\tau_{ik} = -\epsilon_1 \pi \left(\frac{\partial \overline{u_i}}{\partial x_k} + \frac{\partial \overline{u_k}}{\partial x_i} \right)$$

where ϵ_1 is a numerical constant. This amounts to assuming that the Prandtl mixing length is proportional to the turbulent velocity fluctuation. I will report at once that our experiments to be described in the next section do not support this assumption and our main interest in the paper at this time is its general philosophical background. These same experiments show that the turbulent shearing stress depends much more on the local turbulent pressure than on the local mean velocity gradient.

National Bureau of Standards Experimental Program

The experiments referred to are part of a long range study of the mechanics of a separating turbulent boundary layer conducted by the National Bureau of Standards in cooperation with, and with the financial support of, the National Advisory Committee for Aeronautics. Emphasis is placed on the measurement of the turbulent stress tensor by the use of the hot-wire anemometer. The problem is a difficult one because the size of the hot-wire configurations cannot be reduced below about 2 or 3 millimeters and hence a very thick boundary layer is required. Furthermore, the curvature of the surface must be sufficiently small to avoid appreciable pressure difference across the thick boundary layer. After several unsatisfactory experimental arrangements the desired conditions were obtained on a partition of airfoil-like section in the 10-foot wind tunnel of the National Bureau of Standards. This partition was 27.9 feet long and 2 feet thick at its maximum thickness extending in a diametral plane across the wind tunnel. The leading edge has a radius of 1 inch and is joined tangentially to cylindrical surfaces of 23-foot radius forming the nose. The trailing portion of the partition is flat on one side and has the form of a circular cylinder of 31-foot radius on the other. This unsymmetrical shape was found necessary to secure separation. A blister was added at the tunnel wall to modify further the pressure distribution to cause separation to occur well upstream from the trailing edge. By various tricks two-dimensional flow was obtained over the central region with separation occurring uniformly at 25.7 feet from the nose.

Figure 6 shows the variation of the velocity just outside the boundary layer along the length of the plate. Because of the unsymmetrical shape of the plate at the trailing edge, the flow around the nose is very turbulent and produces early transition. The pressure distribution resembles somewhat that on an airfoil. These and all other measurements were made at a speed of about 161 feet per second at the 17-foot position, the Reynolds number being held as constant as possible from day to day. The boundary layer was about $2\frac{1}{3}$ inches thick at the 17-foot position and about 9 inches thick at separation. The thickness of the boundary layer at 17 feet was equal to that which would have prevailed on a flat plate 14.3 feet long with fully turbulent boundary layer and no pressure gradient, the equivalent flat plate Reynolds number for this position being 14,300,000. The turbulence in the free stream of the tunnel is about 0.5 percent.

A contour map of the mean speed distribution is shown in figure 7 and the displacement thickness, momentum thickness, and their ratio, the parameter H, are plotted in figure 8. The velocity U in figure 7 varies with x as shown in figure 6. The value of H at the separation point is approximately 2.7. Figure 9 shows the distribution of mean speed plotted in the manner suggested by von Doenhoff and Tetervin. To avoid confusion no comparison with their results is shown on the figure. The agreement is in general good, but there are some systematic differences slightly greater than the experimental dispersion, which are perhaps to be ascribed to differences in Reynolds number and pressure gradient or to other differences in the experimental conditions.

The National Bureau of Standards experimental program contemplates an exhaustive study of the turbulent fluctuations within the boundary layer whose general characteristics have just been described. The first attention was given to the direct measurement of turbulent shearing stress by the method described at the last Congress. A detailed discussion of the technique is outside the scope of this paper. In brief, the hot-wire head for shearing stress measurement consists of two tungsten wires about 0.00031 inch in diameter and $1\frac{1}{2}$ to 2 millimeters long, set approximately 90° to each other and each making an angle of 45° to the flow direction. The instantaneous voltage on one wire is proportional to $Au + Bv$; on the other to $Au - Bv$. Hence the root mean square voltage a of the first wire is proportional to $A^2u'^2 + 2AB\,\overline{u'v'} + B^2v'^2$; for the second wire the root mean square voltage b is $A^2u'^2 - 2AB\,\overline{uv} + B^2v'^2$. Hence $a - b = 4AB\,\overline{uv}$ and the shearing stress τ is found from

$$\tau = -\rho\,\overline{uv} = -\frac{\rho\,(a - b)}{4AB}$$

The root mean square voltages are measured by suitably compensated amplifiers.

The measured distributions at $22\frac{1}{2}$ feet and 25 feet are shown in figures 10 and 11, in which the ordinate is the coefficient of shearing stress defined by

$$C_\tau = \tau/(1/2)(\rho U^2)$$

The curves given by Fediaevsky's polynomial approximation are also shown. For the 17-foot station (curves not shown), where there is no pressure gradient, the agreement is fairly good, and for the 25-foot station, where the shearing stress at the wall is nearly zero, the agreement is excellent. At intermediate stations the agreement is poor.

These curves give a fair idea of the scatter of the experimental data which are perhaps not yet so accurate as one might desire. The measurements are very tedious. It may be noted that the viscous shearing stresses are wholly negligible in the regions studied because of the large Reynolds number. The laminar sublayer is extremely thin and is never approached in any of the measurements. At the 17-foot position at 0.1 inch from the surface the turbulent shearing stress is about 190 times the viscous shearing stress.

The distribution of the turbulent shearing stress has been determined at some 15 sections between the 14-foot station and a station just beyond the separation point at 25.7 feet. A contour map of C_τ is shown in figure 12. The contours show some irregularities whose causes are not definitely known. Beginning at the region of minimum pressure (maximum velocity outside the boundary layer), the maximum value of the shearing stress is found at greater and greater distances from the wall until at separation it is located near the middle of the boundary layer. The value of the maximum stress increases as the distance along the wall increases, although the actual stress does not increase anywhere near as much as the stress coefficient which is defined in terms of the maximum velocity just outside the boundary layer at the station in question. This velocity decreases by a factor of about 0.7 in passing from the 17-foot to the 25-foot station (see fig. 6), and hence a stress coefficient based on the velocity at 17 feet as a constant reference velocity would be equal for the 25-foot station to approximately one-half the values plotted in figure 12 for that station. The contours plotted on this basis are shown in figure 13. The contours are thus those of equal absolute values of the stress. Comparison of this diagram with figure 7 shows some similarity between the two. The contour for $u/U = 0.8$ is almost identical with the outer contour for $C_{\tau_o} = 0.003$; the contour for $u/U = 0.6$ lies approximately along the locus of maximum shearing stress.

The estimated value of the shearing stress coefficient at the wall is shown in figure 14. These values were obtained from curves like those shown in figures 10 and 11 by extrapolation, use being made of the fact that at the wall $\partial\tau/\partial y = \partial p/\partial x$. The Squire-Young formula is not a good representation of this curve.

The distribution of the turbulent shearing stress coefficient for the $17\frac{1}{2}$-, 20-, and $22\frac{1}{2}$-foot stations is shown in figure 15.

The Prandtl mixing length l can be computed from the relation

$$l = \frac{\sqrt{\frac{C_\tau}{2}}}{\frac{1}{U}\frac{du}{dy}}$$

Typical values for the $17\frac{1}{2}$, 20, and $22\frac{1}{2}$-foot stations are shown in figure 16. These curves indicate that l is a definite function of the distance from the wall for y/θ up to about 1.0.

Figure 17 shows the distribution of mean speed at these three stations determined by pitot tube measurements.

Figure 18 shows the stress coefficient plotted against the mean velocity gradient. These curves certainly do not suggest a very definite relation between the turbulent shearing stress and the mean velocity gradient. Over extended regions the velocity gradient is nearly constant; yet the shearing stress changes by a factor of 4 or 5.

Surveys of u', v', and w' are in progress. Values for the $17\frac{1}{2}$-, 20-, and $22\frac{1}{2}$-foot stations are shown in figures 19, 20, and 21. The turbulence is three-dimensional, even though the mean flow is two-dimensional. The turbulence is strongly non-isotropic, v' being considerably less than u', and w' being intermediate in value but closer to u'. Isotropy is reached in the free stream.

In order to test Nevzgljadov's assumption that the shearing stress τ is proportional to $(u'^2 + v'^2 + w'^2)\frac{du}{dy}$, the ratio of τ to $u'^2 + v'^2 + w'^2$ was plotted against $\frac{du}{dy}$. The results are shown in figure 22. Far from being proportional to $\frac{du}{dy}$, the ratio is almost independent of $\frac{du}{dy}$ except in the outer part of the layer, where τ falls to zero and the turbulence decreases to that of the free stream. Perhaps if the free stream were sufficiently free from turbulence, both quantities would fall to zero together, retaining a constant ratio. It is evident that Nevzgljadov's assumption is not a good approximation, but confirmation is given to Von Kármán's assumption of statistical similarity between the fluctuations at different points in the same cross section. The ratio of τ to $u'^2 + v'^2 + w'^2$ is more nearly constant than its ratio to $u'^2 + v'^2$ or to u'^2 or to v'^2.

From the data given, it is possible to compute the direction of the principal axis of the turbulent stress tensor by the relation

$$\tan 2\alpha = \frac{2\,\overline{u'v'}}{u'^2 - v'^2} = \frac{C_\tau U^2}{u'^2 - v'^2}$$

where α is the angle between the principal axis and the direction of flow. The results are plotted in figure 23 along with Reichardt's result for two-dimensional flow under pressure between two plates as given by Von Kármán. It is seen that the results agree in general with those of Reichardt. The variation across the boundary layer is of the order of 5° to 10°. These results emphasize the difficulty pointed out by Von Kármán, namely, that the principal axis of dilatation is at 45° to the mean flow as compared with 10° to 30° for the principal axis of the turbulent shearing stress. Thus there are directions for which there is a shearing stress but no rate of shear of the mean flow and vice versa. I suggest that the data are to be interpreted as meaning that the turbulent shearing stress is not determined by the local mean velocity and its derivatives but by the local non-isotropic turbulence, and that this local turbulence cannot be dependent solely on the local mean velocity and its derivatives. The turbulence must depend on conditions upstream.

The process of turbulent separation is seen not to depend on processes within the laminar sublayer, for the turbulent stress near the wall falls regularly and smoothly as separation is approached. At these high Reynolds numbers the laminar sublayer serves only as a medium for transferring the stress from the fluid to the wall. A further study of the conditions near the separation point must await the completion of the u', v', and w' measurements in this region. The difficulties are great as the u' component becomes a large fraction of the mean speed, and intermittent low-frequency disturbances are much more frequent.

This is the point to which our present study of the data has brought us. The conclusions so far drawn are mostly negative in character; Fediaevsky's approximation is not entirely satisfactory, and Nevzgljadov's assumptions are not supported by the experimental data. The Prandtl and Von Kármán formulas relating the turbulent shearing stress to the local mean motion may be fundamentally in error not only in the difficulty of reconciling the experimental data with the concept of a scalar mixing length varying from point to point but also in its fundamental philosophy. The stress may not be determined by the local mean motion; the local non-isotropic turbulence, which is directly related to the shear stress, is itself not determined by the local mean motion but by the generation of turbulence upstream, and its convection, diffusion, and dissipation on the path to the point under study.

There is, however, considerable support for Von Kármán's concept of the statistical similarity of the turbulent fluctuations.

The objective of the National Bureau of Standards long range experimental program is to obtain sufficient detailed information on the turbulence itself to permit the test of various theoretical assumptions and to suggest others which have an experimental foundation. When the u', v', and w' measurements are completed, the correlation tensor of second order will be studied. Unfortunately, we have not yet developed methods of measuring a large number of quantities at a large number of points in the flow field simultaneously. The measurements require months to complete. They are therefore not suitable for the analysis of differential changes from point to point with high accuracy. They are intended rather to give an over-all survey of relations between quantities directly measured with moderate accuracy. It is hoped that this progress report will prove suggestive to other workers in this important field.

National Bureau of Standards,
Washington, D. C., September 1946.

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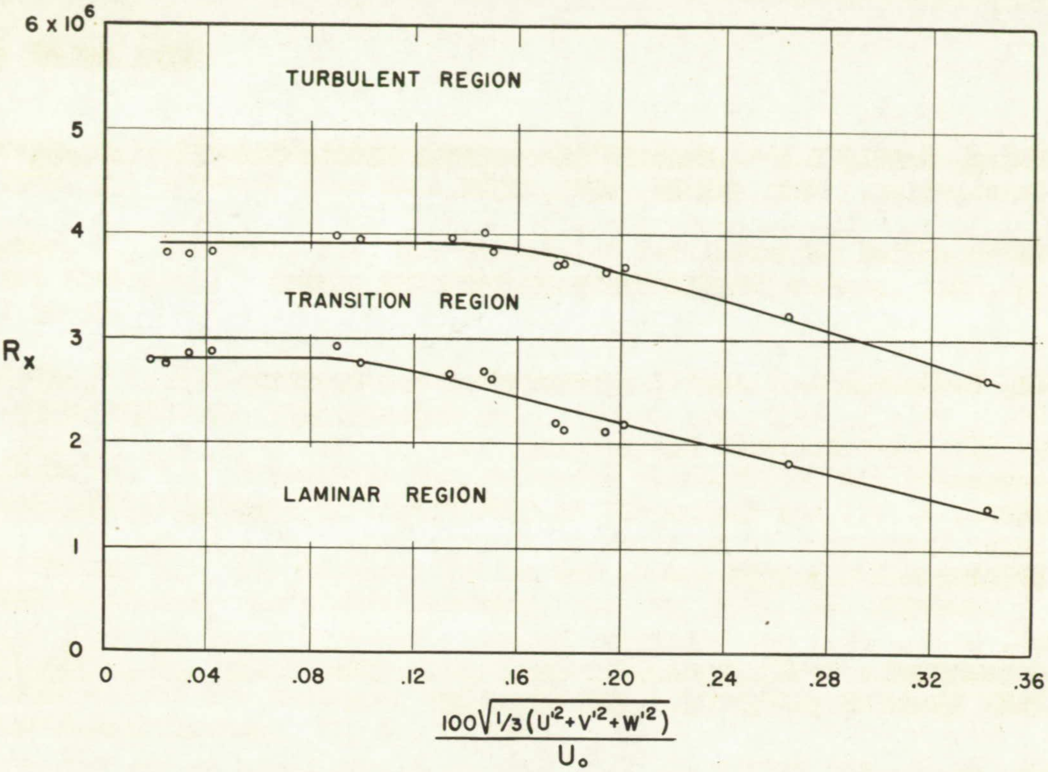

Figure 1.- Reynolds number of transition as a function of free stream turbulence for a flat plate in a flow without pressure gradient.

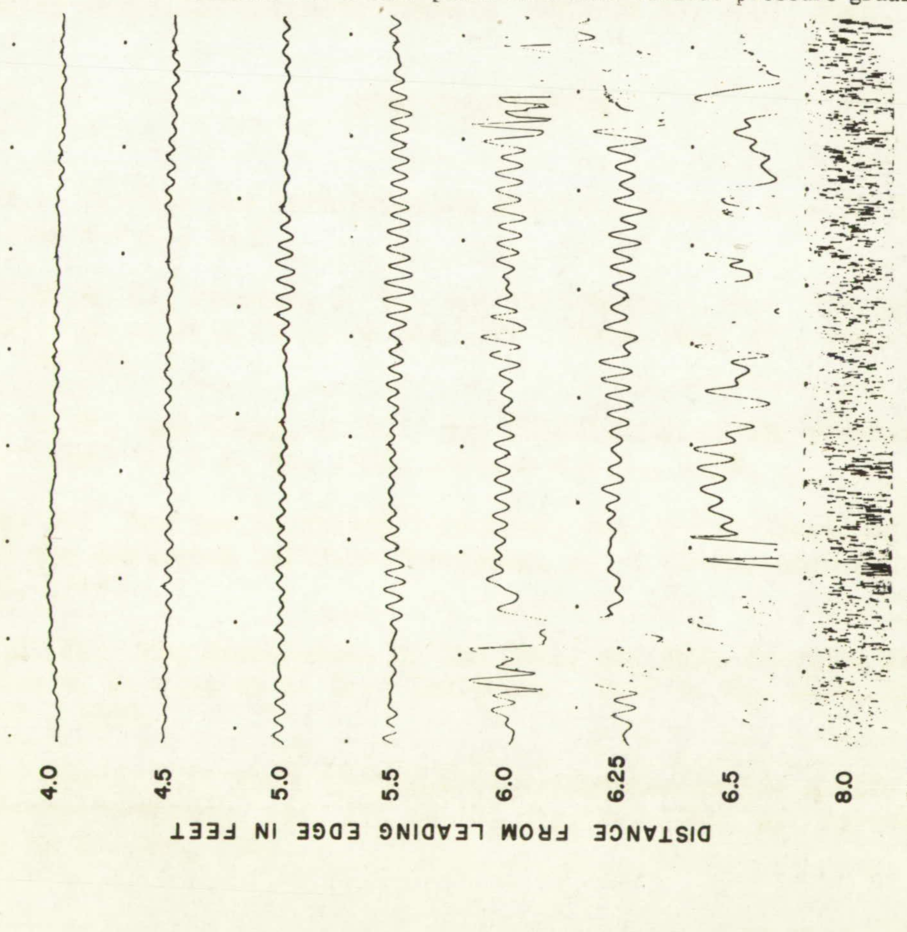

Figure 2.- Oscillograms showing laminar boundary layer oscillations in the boundary layer of a flat plate in a flow without pressure gradient. Distance from surface = 0.025 in., speed of free stream = 80 ft/sec, time interval between dots = 1/30 sec.

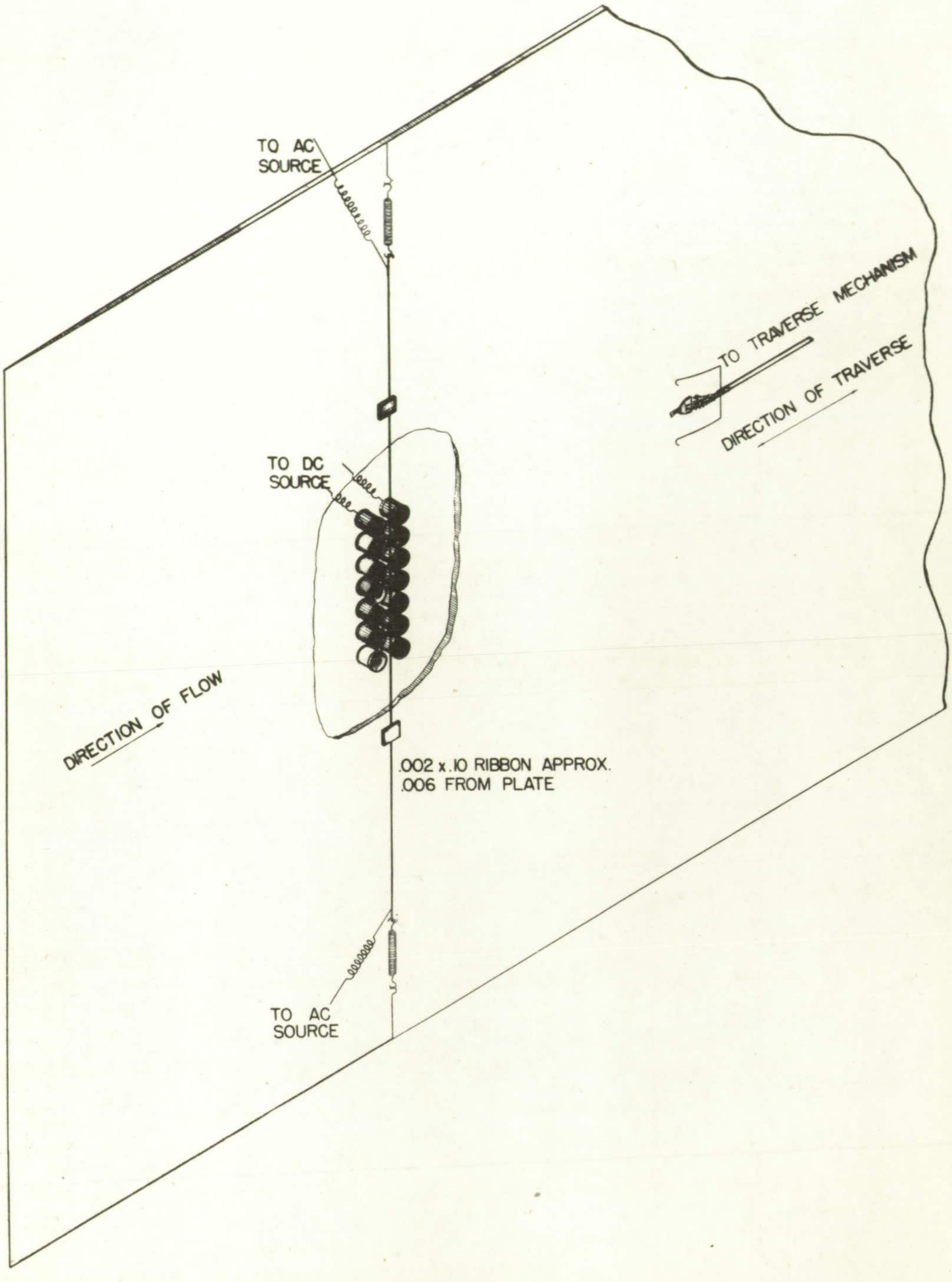

Figure 3.- Experimental arrangement of Schubauer and Skramstad for exciting controlled disturbances in the laminar boundary layer of a plate by means of a vibrating ribbon.

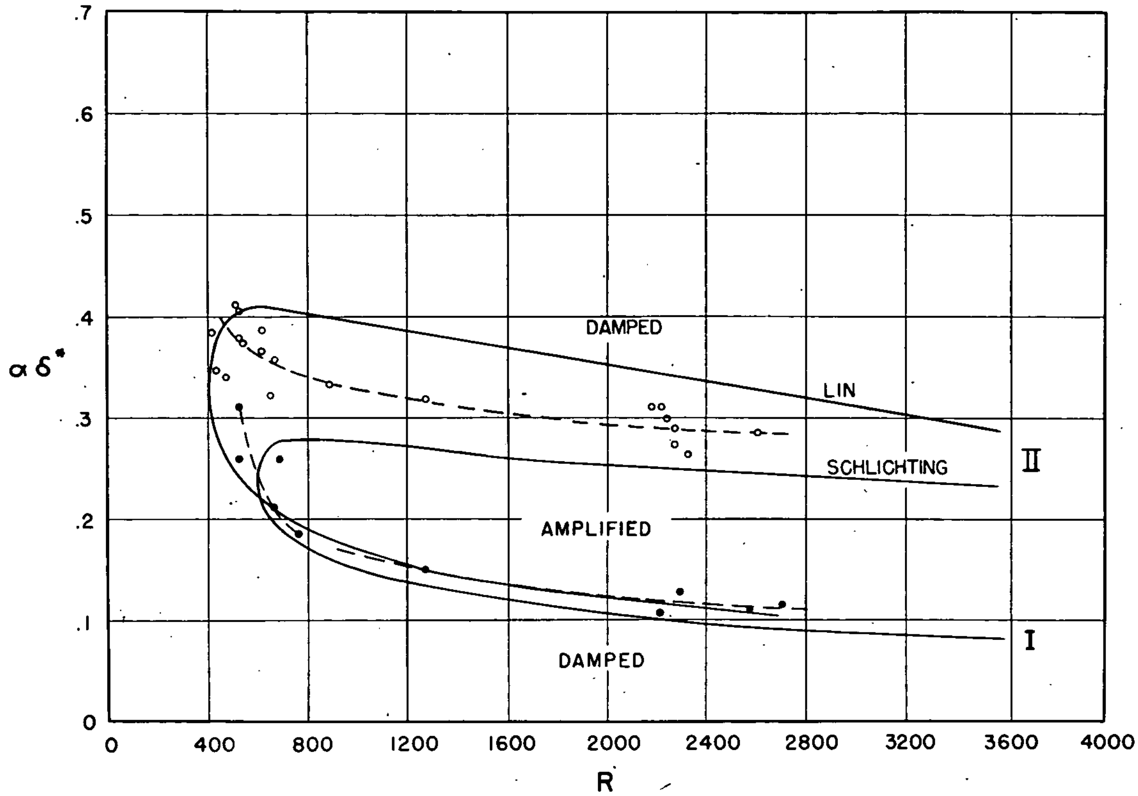

Figure 4.- Neutral curve (instability boundary) obtained from observed wave lengths for zero damping or amplification compared with the theoretical curves of Schlichting and Lin. Wave length = $2\pi/\alpha$; δ^* = displacement thickness of boundary layer; R = Reynolds number of boundary layer based on δ^*.

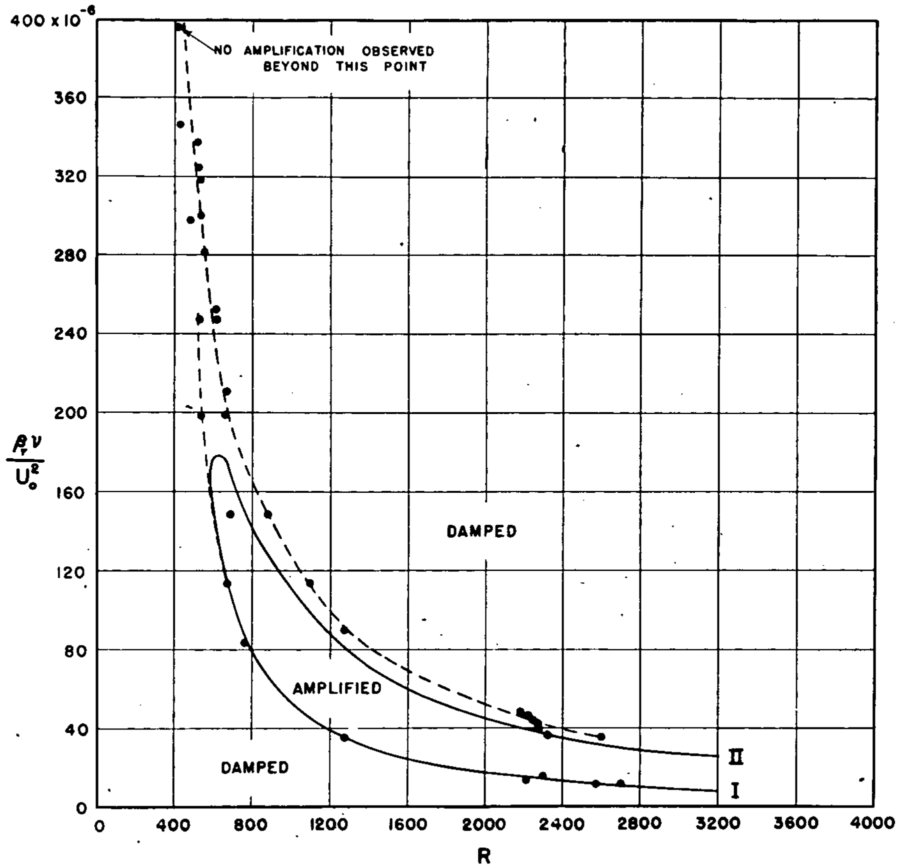

Figure 5.- Neutral curve (instability boundary) obtained from observed frequencies for zero damping or amplification compared with the theoretical curve of Schlichting.

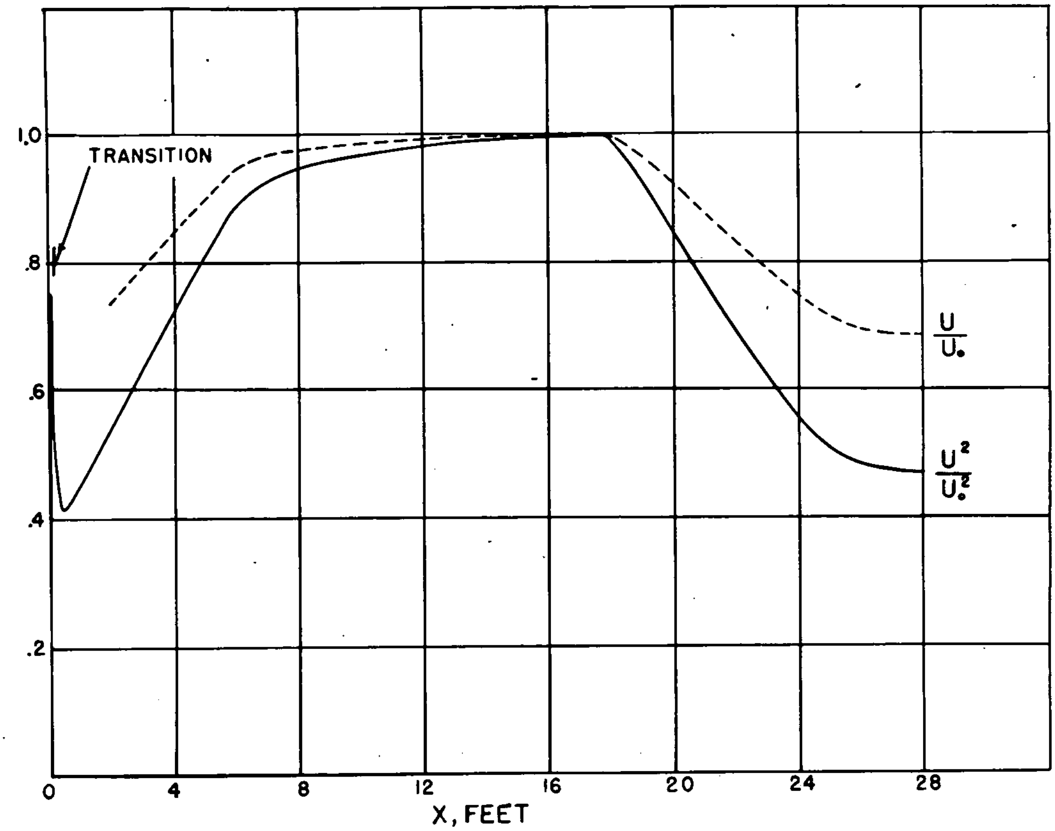

Figure 6.- Velocity and pressure distribution just outside the turbulent boundary layer studied at NBS. All measurements are referred to maximum speed at point of minimum pressure at 17 ft from the leading edge.

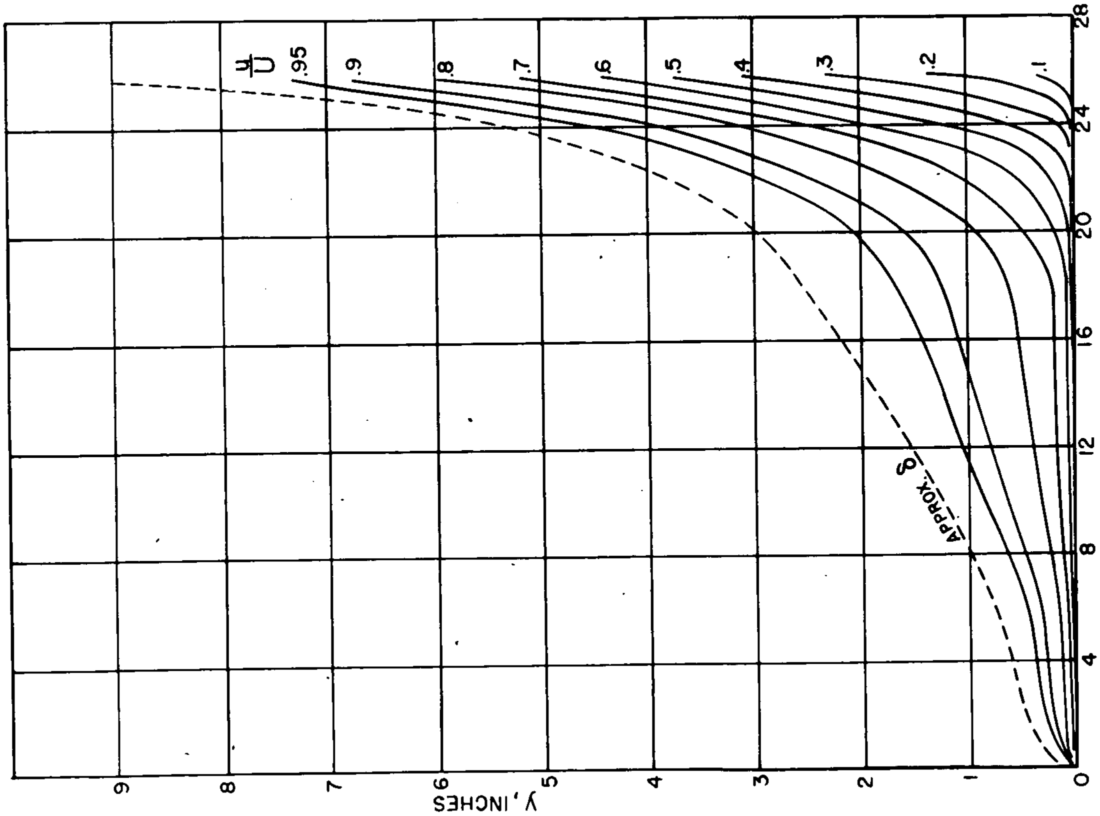

Figure 7.- Contour map of mean speed distribution in the turbulent boundary layer studied at NBS. u = local speed at point x,y; U = speed just outside the boundary layer.

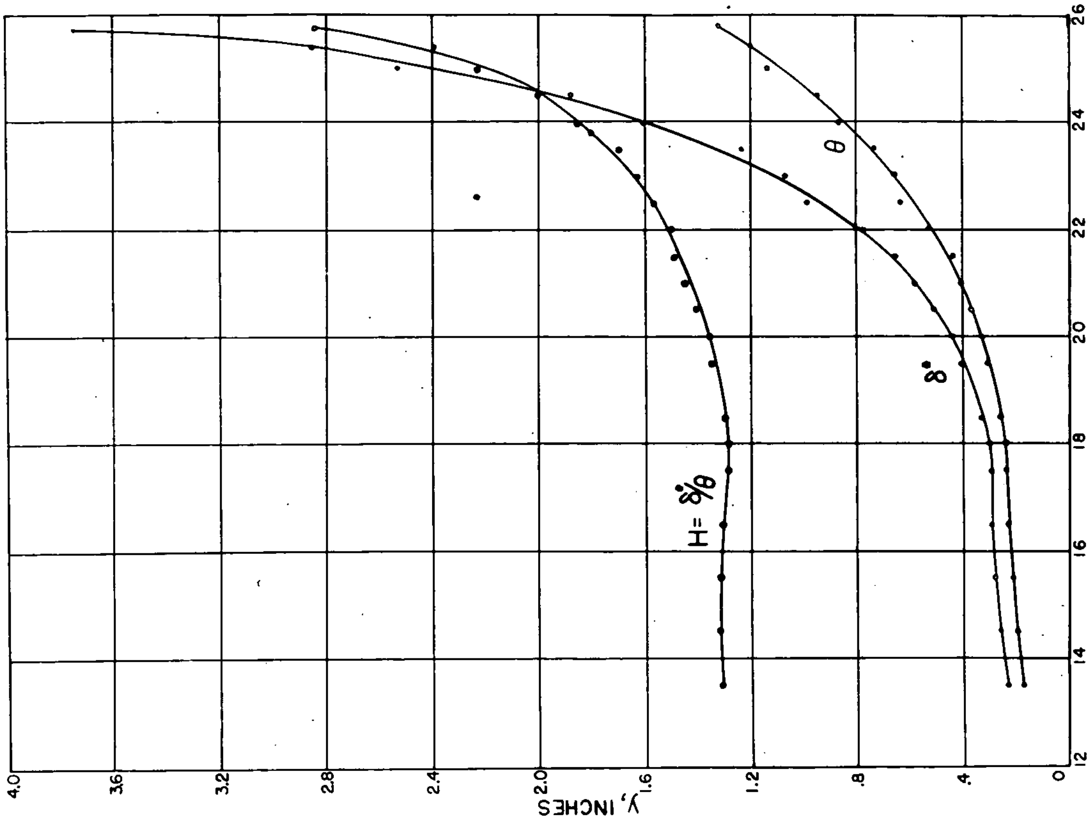

Figure 8.- Displacement thickness δ^*, momentum thickness θ, and parameter $H = \delta^*/\theta$ for NBS turbulent boundary layer.

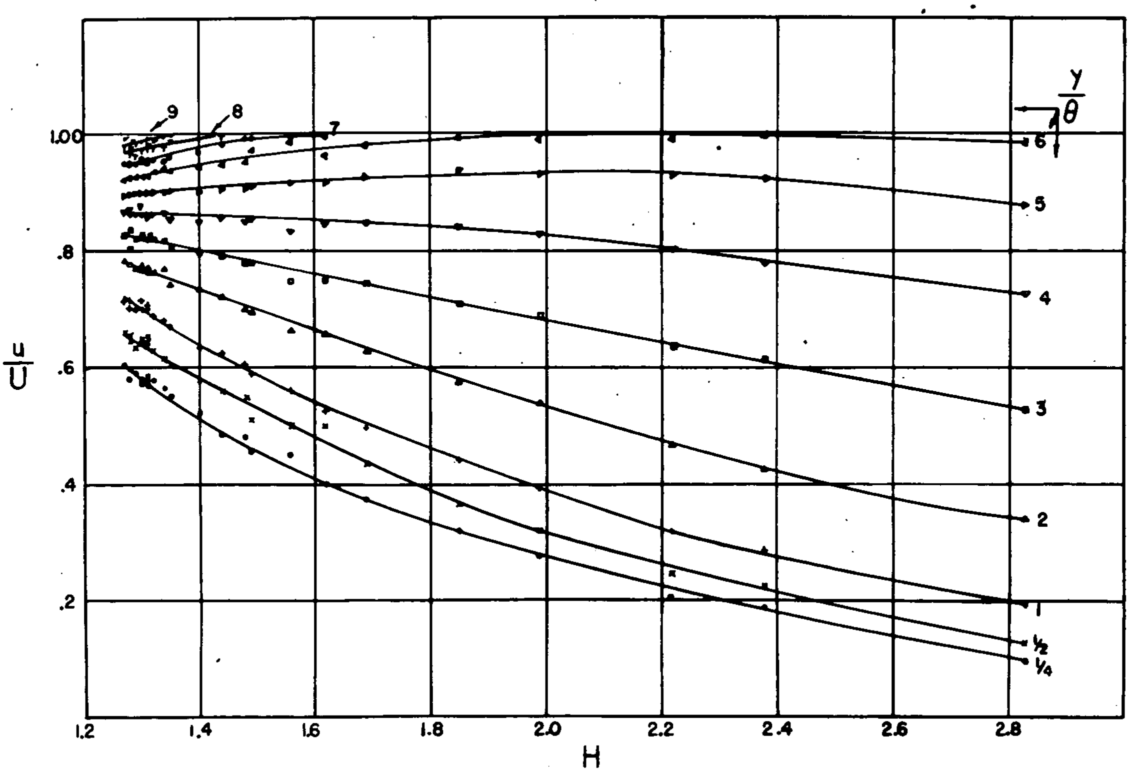

Figure 9.- Variation of u/U with H for various values of y/θ.

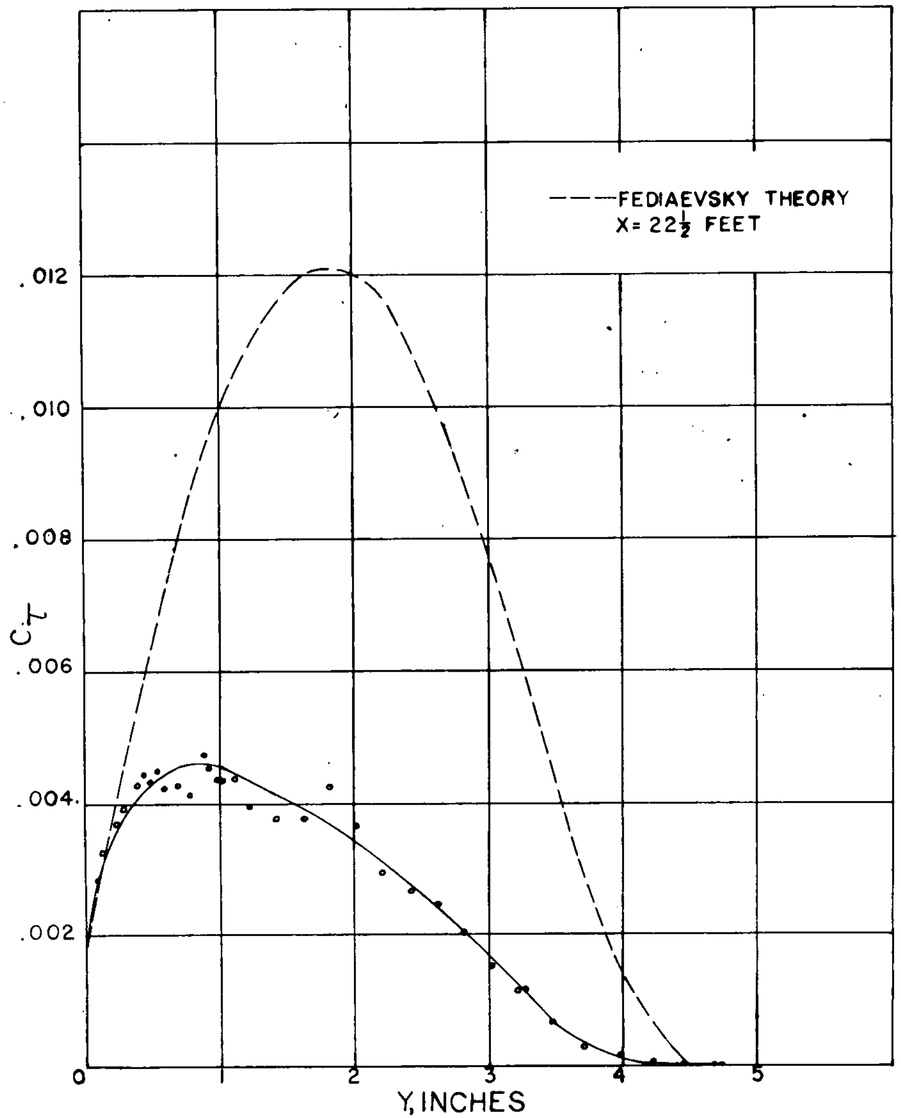

Figure 10.- Distribution of turbulent shearing stress coefficient at the 22 1/2 ft station.

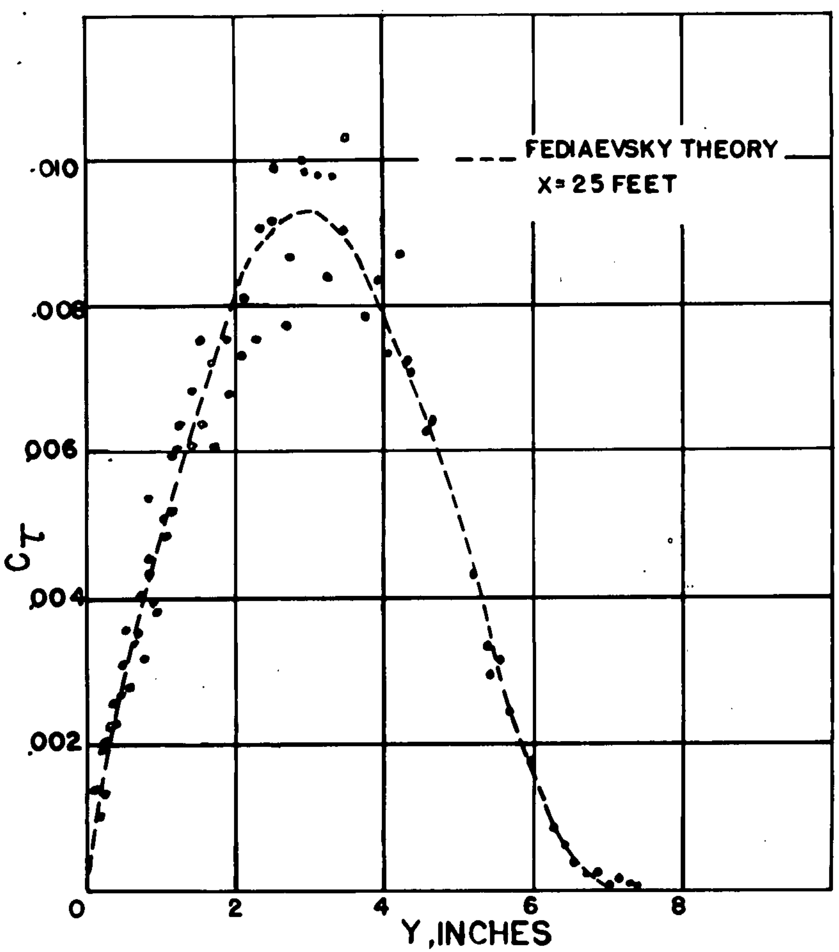

Figure 11.- Distribution of turbulent shearing stress coefficient at the 25 ft station.

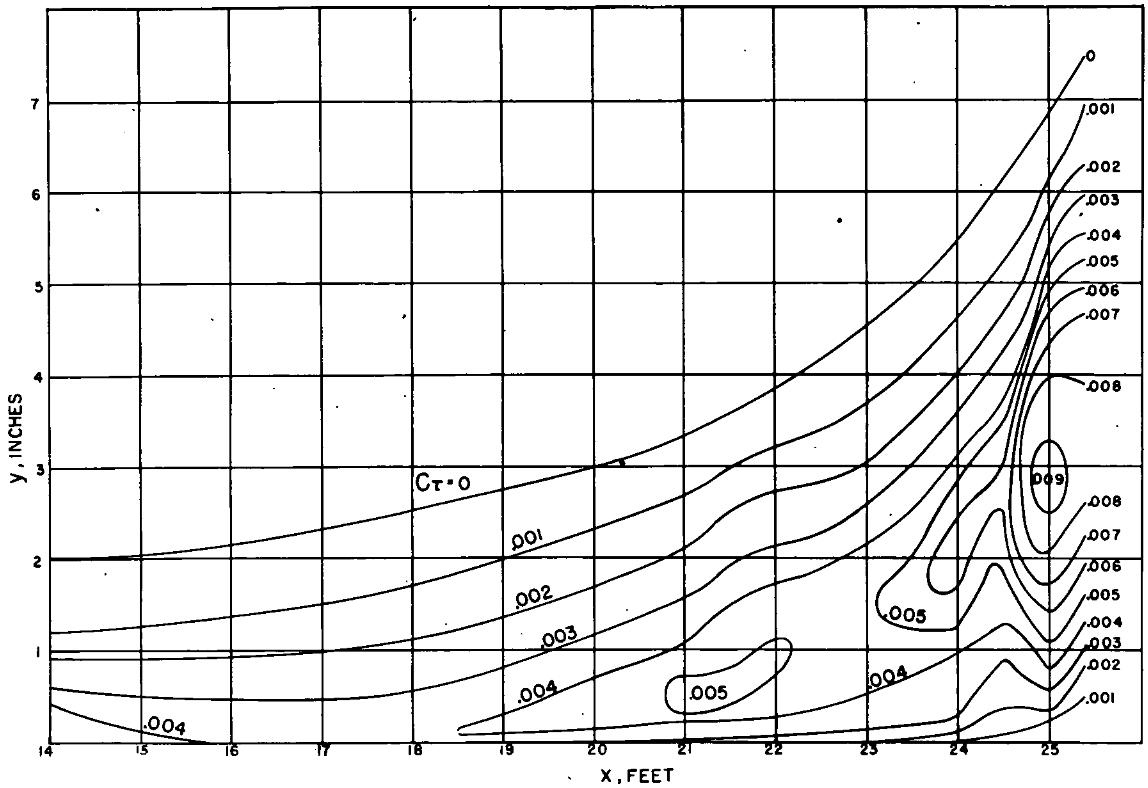

Figure 12.- Contour map of distribution of turbulent shearing stress coefficient for NBS turbulent boundary layer. $C_\tau = \tau/1/2\rho U^2$; τ = turbulent shearing stress at point x,y; U = speed just outside boundary layer varying with x as shown in figure 6.

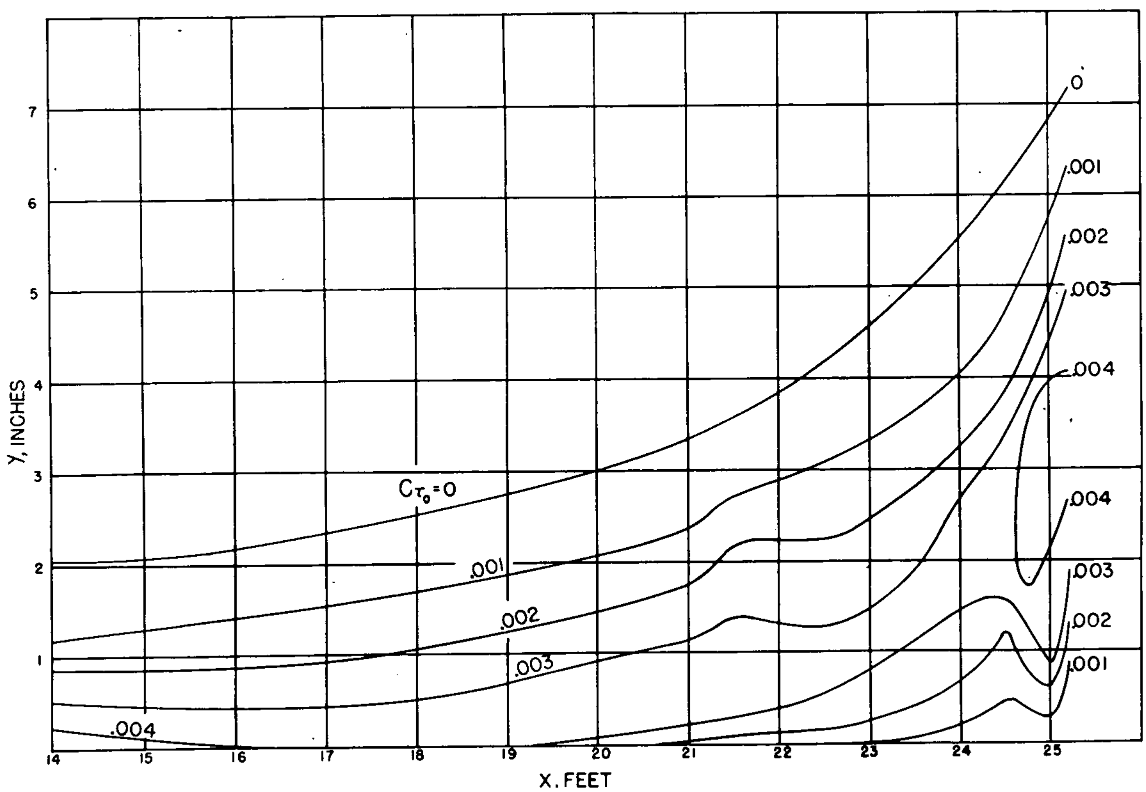

Figure 13.- Contour map of distribution of turbulent shearing stress for NBS turbulent boundary layer. $C_{\tau_0} = \tau/1/2\rho U^2$; U_0 = constant reference speed (speed just outside boundary layer at 17 1/2 ft station).

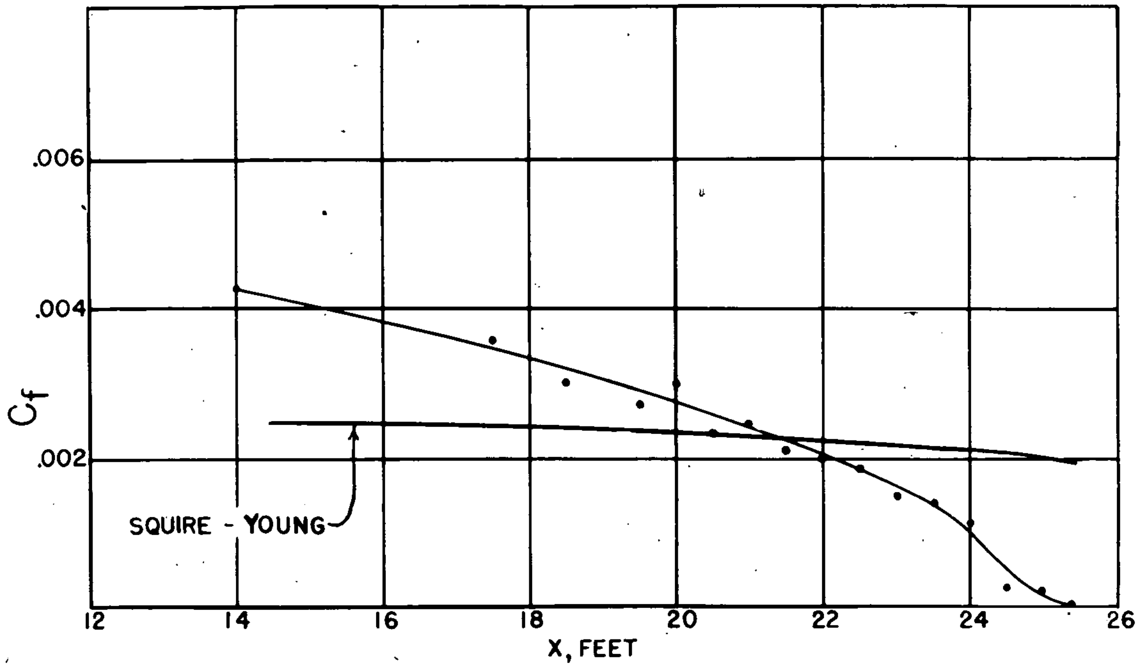

Figure 14.- Estimated shearing stress coefficient at the wall for NBS turbulent boundary layer.

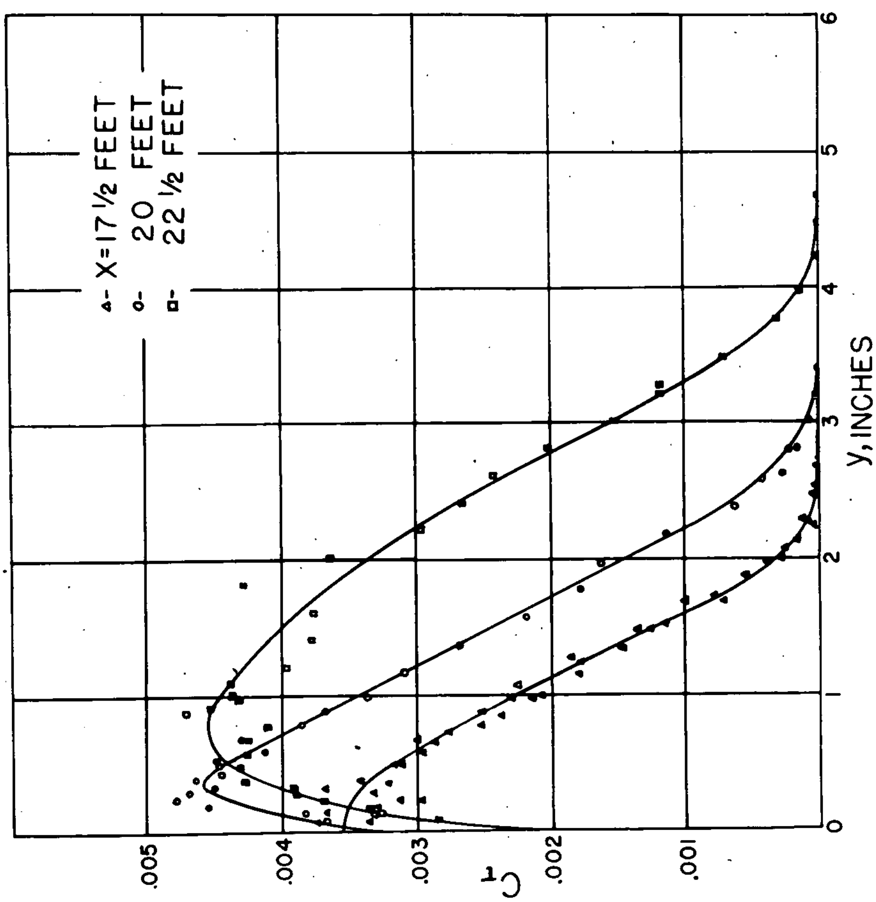

Figure 15.- Distribution of turbulent shearing stress coefficient at 17 1/2, 20, and 22 1/2 ft stations.

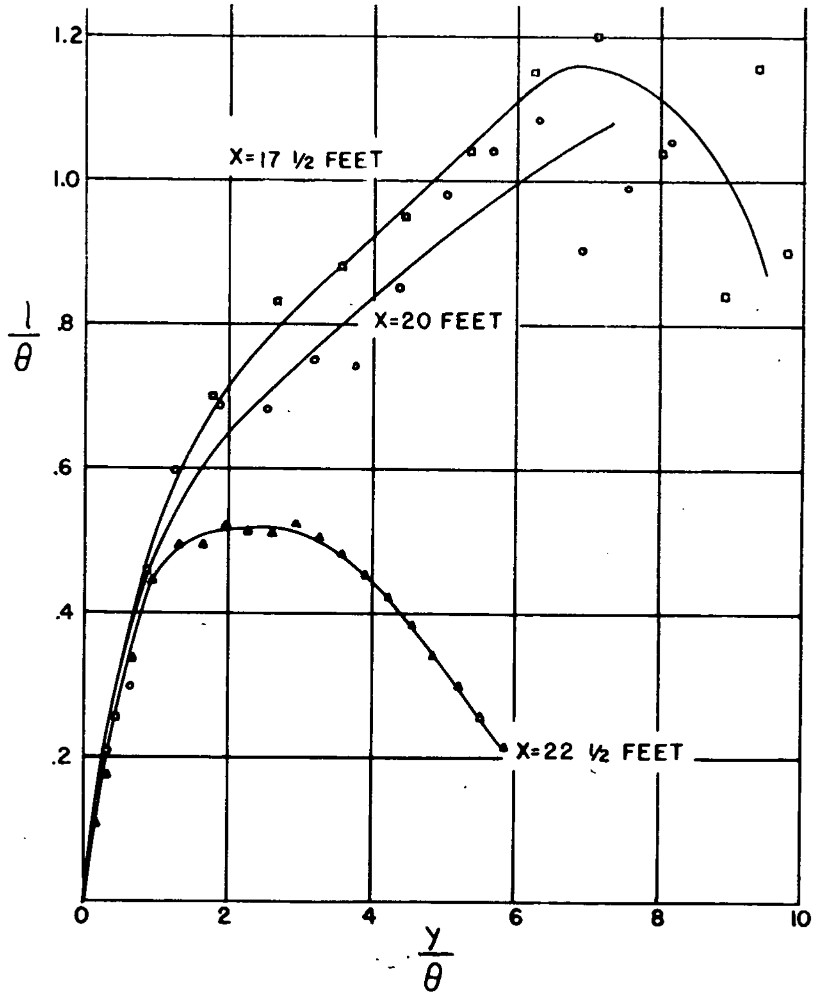

Figure 16.- Prandtl mixing length l at 17 1/2, 20, and 22-1/2 ft stations.

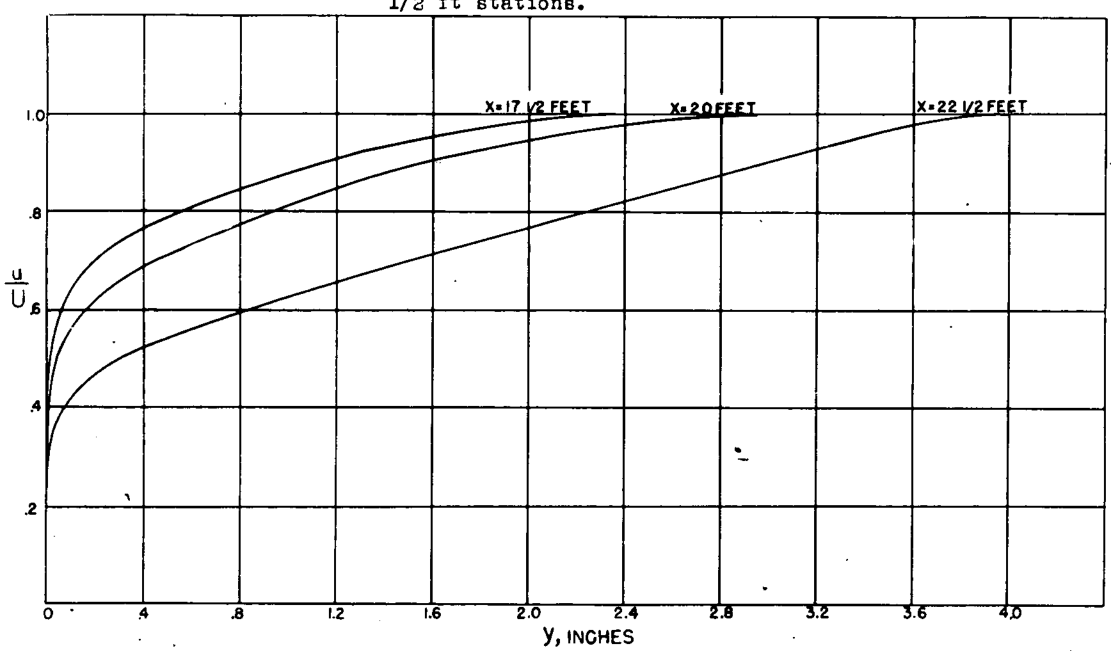

Figure 17.- Distribution of mean speed at 17 1/2, 20, and 22 1/2 ft stations.

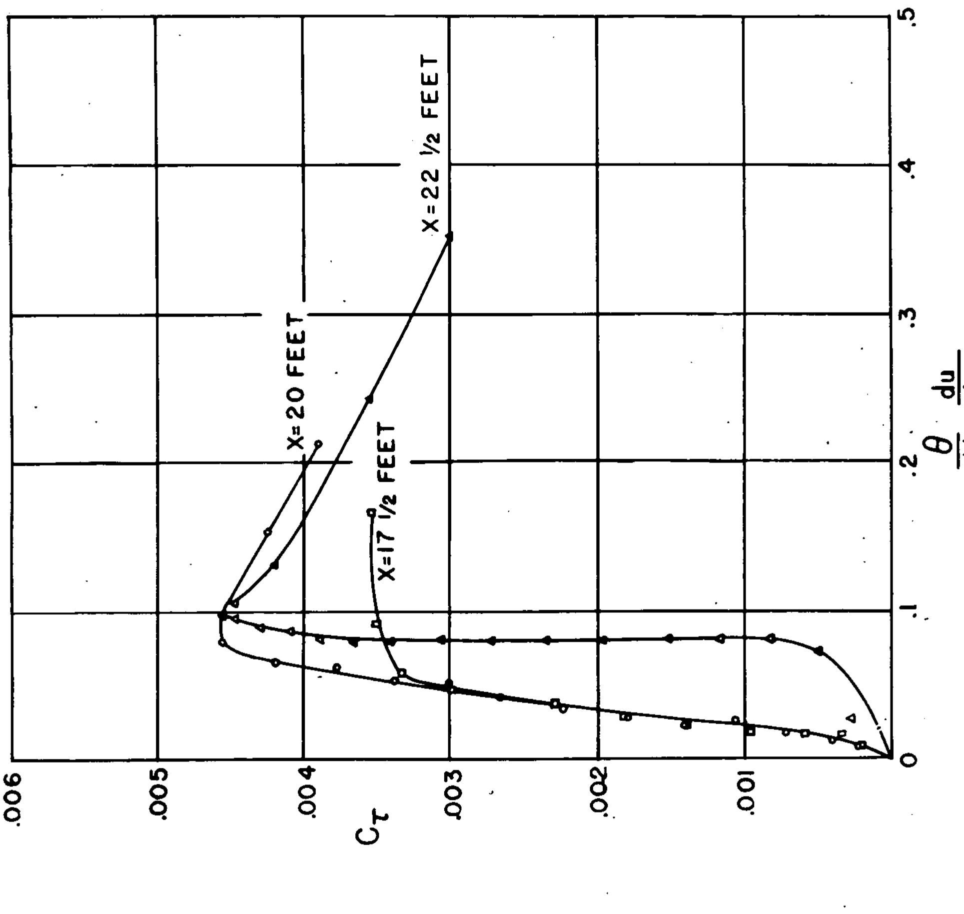

Figure 18.- Relation between shearing stress coefficient and the mean velocity gradient for 17 1/2, 20, and 22 1/2 ft stations.

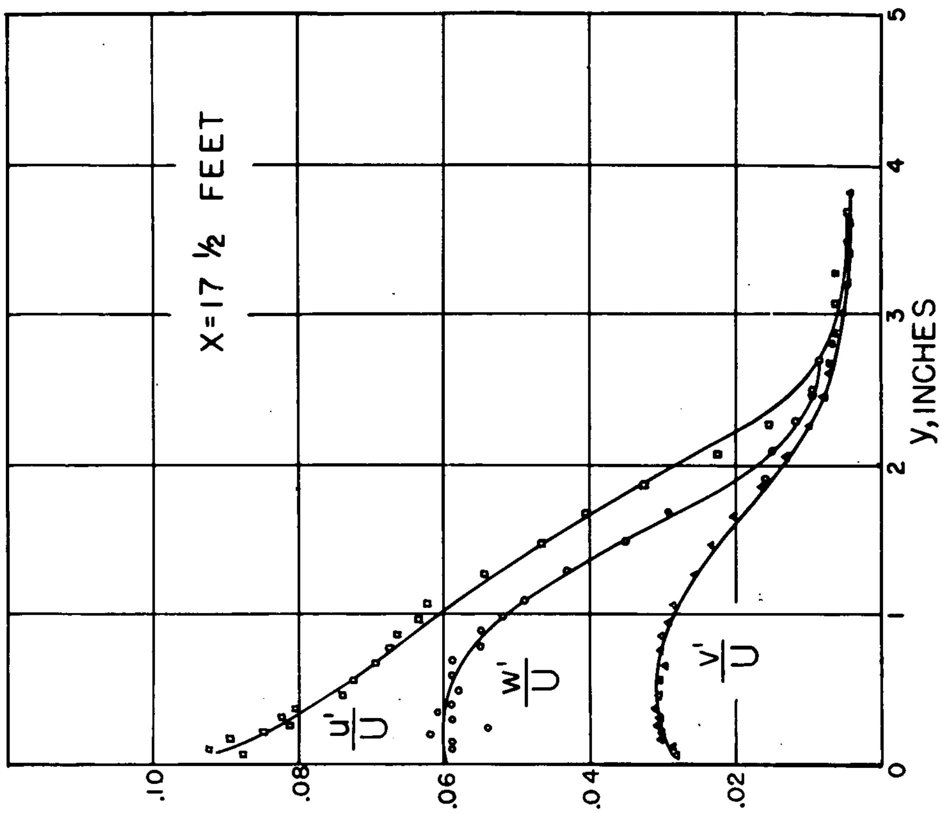

Figure 19.- Values of u', v', and w' at 17 1/2 ft station.

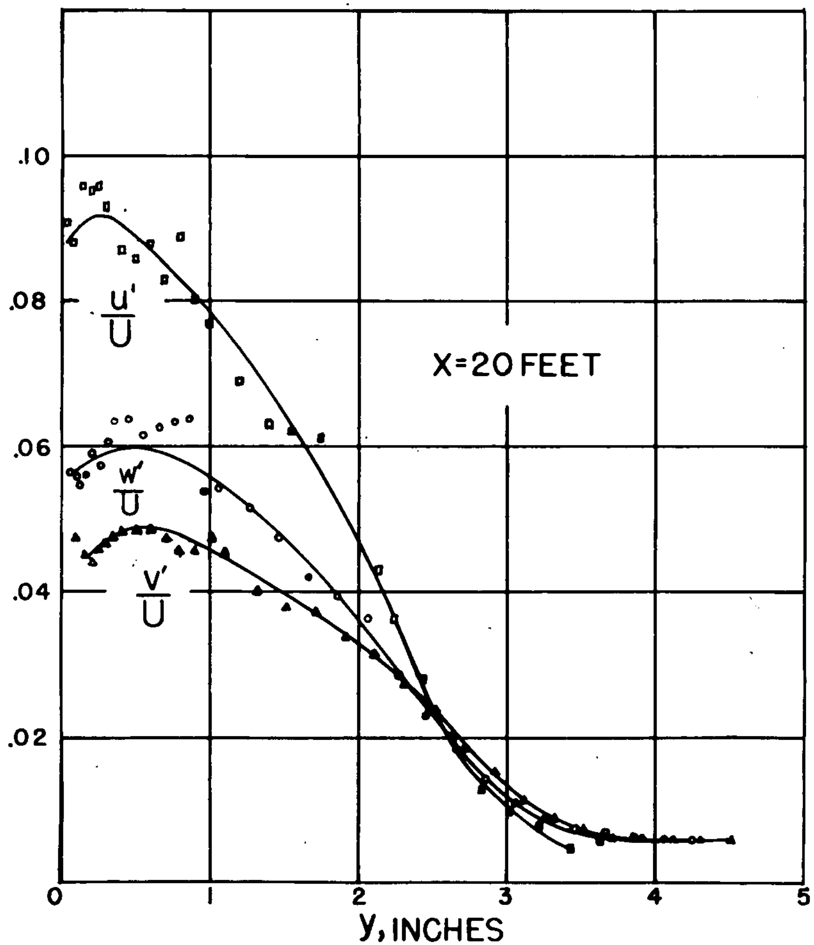

Figure 20.- Values of u', v', and w' at 20 ft station.

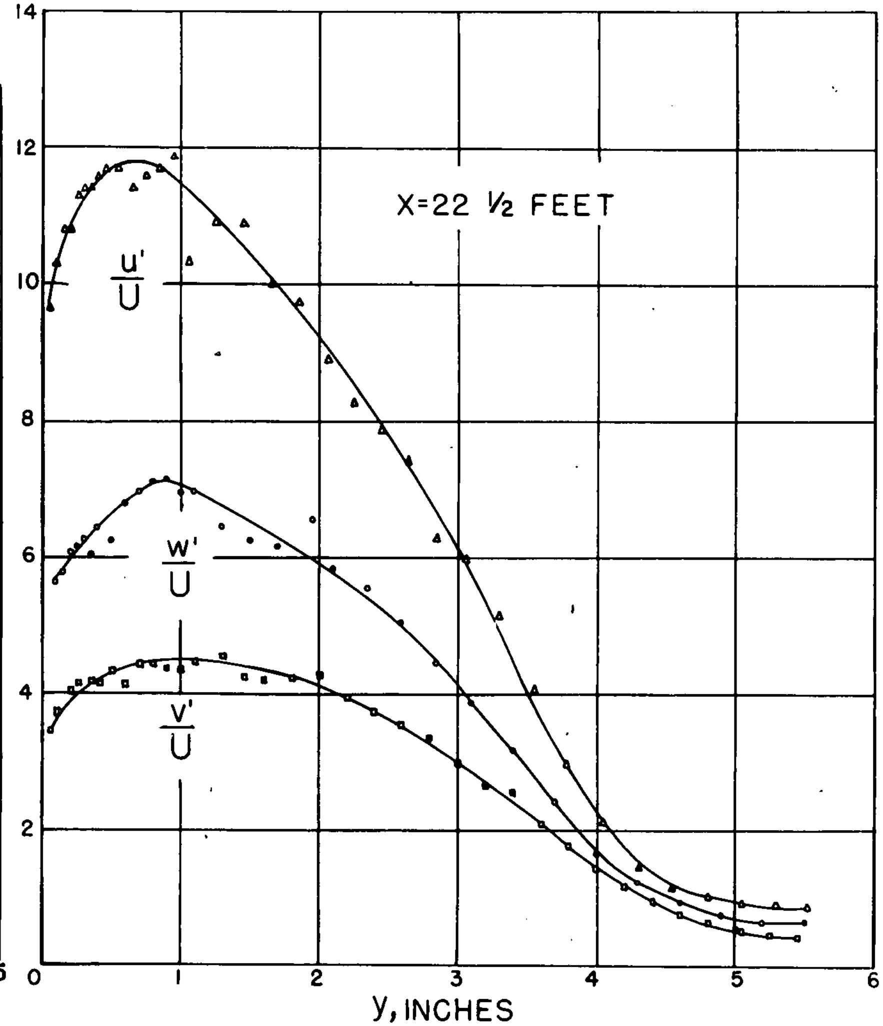

Figure 21.- Values of u', v', and w' at 22 1/2 ft station.

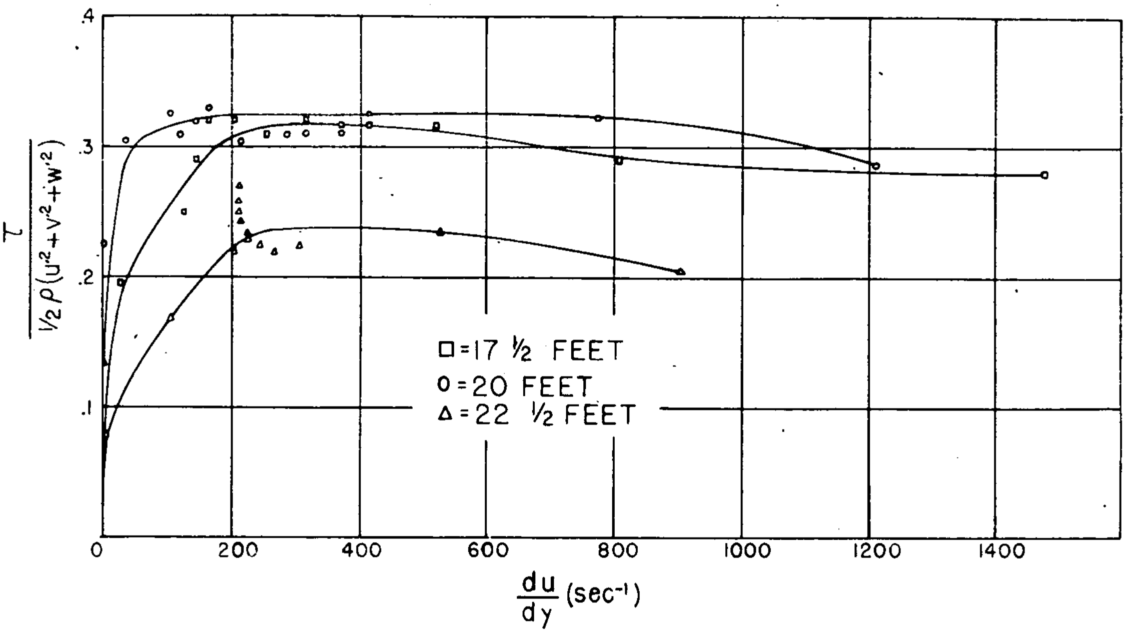

Figure 22.- Ratio of shearing stress to mean energy of turbulence as a function of the mean velocity gradient.

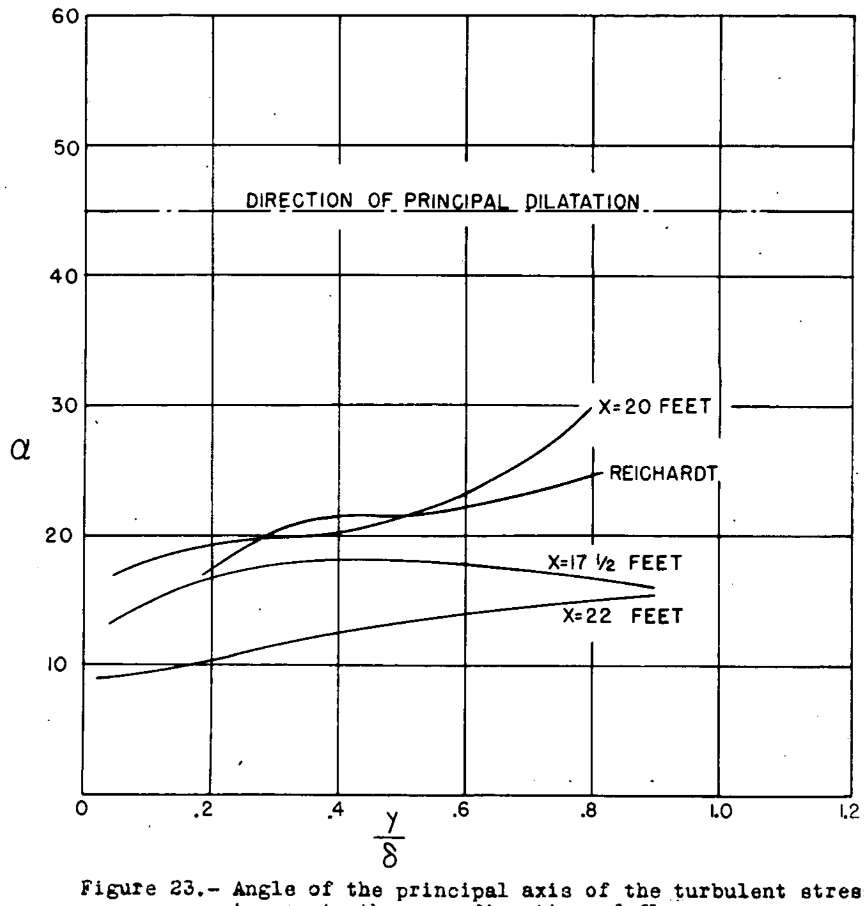

Figure 23.- Angle of the principal axis of the turbulent stress tensor to the mean direction of flow.