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A SIMPLIFIED METHOD OF ELASTIC-STABILITY ANALYSIS FOR THIN CYLINDRICAL SHELLS

I - DONNELL'S EQUATION

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A SIMPLIFIED METHOD OF ELASTIC-STABILITY ANALYSIS FOR THIN CYLINDRICAL SHELLS I - DONNELL'S EQUATION

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SUMMARY

The equation for the equilibrium of cylindrical shells introduced by Donnell in NACA Report No. 479 to find the critical stresses of cylinders in torsion is applied to find critical stresses for cylinders with simply supported edges under other loading conditions. It is shown that by this method solutions may be obtained very easily and the results in each case may be expressed in terms of two nondimensional parameters, one dependent on the critical stress and the other essentially determined by the geometry of the cylinder. The influence of boundary conditions related to edge displacements in the shell median surface is discussed. The accuracy of the solutions found is established by comparing them with previous theoretical solutions and with test results.

The solutions to a number of problems concerned with buckling of cylinders with simply supported edges on the basis of a unified viewpoint are presented in a convenient form for practical use.

INTRODUCTION

The recent emphasis on aircraft designed for very high speed has resulted in a trend toward thicker skin and fewer stiffening elements. As a result of this trend, a larger fraction of the load is being carried by the skin and thus ability to predict accurately the behavior of the skin under load has become more important. Accordingly, it was considered desirable to provide the designer with more information on the buckling

of curved sheet than has been available in the past. In carrying out a theoretical research program for this purpose, a method of analysis was developed which is believed to be simpler to apply than those generally appearing in the literature. The specific problems solved as a part of this research program are treated in detail in other papers. The purpose of the present investigation, which is discussed in two papers, is to present the method of analysis that was developed to solve these problems. In the present paper the method is briefly outlined and applied to a number of the simpler problems in the buckling of cylindrical shells. In reference 1 the method is generalized for application to more complicated problems.

THEORETICAL BACKGROUND

In most theoretical treatments of the buckling of cylindrical shells (see references 2 to 4) three simultaneous partial differential equations have been used to express the relationship between the components of shell median-surface displacement u, v, and w in the axial, circumferential, and radial directions, respectively. No general agreement has been reached, however, on just what these equations should be. In 1934 Donnell (reference 5) pointed out that the differences in the various sets of equations arose from the inclusion or omission of a number of relatively unimportant terms (referred to in the present paper as higher-order terms), and proposed the use of simpler equations in which only the most essential terms (first-order terms) were retained. The omitted terms were shown to be small, and thus the simplified equations to be applicable, if the cylinders have thin walls and if the square of the number of circumferential waves is large compared with unity. Donnell further showed that the three simplified equations can be transformed into a single eighth-order partial differential equation in w (see appendix A of the present paper) in which the effects of the displacements u and v are properly taken into account; this equation will hereinafter be referred to as Donnell's equation.

When higher-order terms are included in the three partial differential equations previously mentioned, the resulting theoretical buckling stresses are usually very complicated functions of the cylinder dimensions and the

elastic properties of the material. A family of curves is ordinarily drawn giving the critical stress as a function of the length-diameter ratio for specified values of the radius-thickness ratio and for given elastic properties (references 3, 4, and 6). When the higher-order terms are omitted from the equations and the requirement of an integral number of circumferential waves is removed, new parameters can be introduced which combine the cylinder dimensions and material properties in such a way that the results can be given in terms of a single curve. These parameters have been used, with slight variations in detail, by Donnell, Kromm, Leggett, and Redshaw (references 5 and 7 to 10). The omission of the higher-order terms also greatly simplifies the calculations, and the calculations are simplest if Donnell's equation, rather than the set of three simultaneous equations, is employed. Donnell's equation, or an equivalent equation, may therefore be presumed to be the most promising for use in solving hitherto unsolved problems in the stability of cylindrical shells.

In spite of the fact that it was introduced some time ago, Donnell's equation has not achieved the wide acceptance for use in the stability analysis of cylindrical shells which it appears to merit. Some investigators have continued to use simultaneous differential equations in which higher-order terms appear, presumably on the assumption that the errors arising from neglect of these terms might be undesirably large. Others have dropped second-order terms but have continued to employ simultaneous equations, probably in order to specify directly edge-restraint conditions having to do with displacements in the axial and circumferential directions, which cannot be done with Donnell's equation.

The purposes of the present paper are to establish the accuracy of the equation by comparing the results found by the use of Donnell's equation with the results found by other methods and with experimental results and to investigate the question of boundary conditions on u and v. The additional purpose is achieved of presenting the solutions of a number of problems concerned with buckling of cylinders with simply supported edges on the basis of a unified viewpoint and in a convenient form for practical use. In reference 1 Donnell's equation is modified to facilitate solution of problems concerned with shells having clamped edges.

SYMBOLS

a	length of curved panel
b	width of curved panel
d	diameter of cylinder
m, n	integers
p	lateral pressure
r	radius of cylindrical shell
t	thickness of cylindrical shell
u	displacement in axial (x-) direction of point on shell median surface
v	displacement in circumferential (y-) direction of point on shell median surface
w	displacement in radial direction of point on shell median surface; positive outward
x	axial coordinate
y	circumferential coordinate
a_{mn}, b_{mn}, c_{mn}, d_{mn}	numerical coefficients
k_s	shear-stress coefficient $\left(\frac{\tau t L^2}{D\pi^2}\right.$ for cylinder or $\left.\frac{\tau t b^2}{D\pi^2}\right.$ for infinitely long curved strip$\left.\right)$
k_x	axial compressive-stress coefficient $\left(\frac{\sigma_x t L^2}{D\pi^2}\right.$ for cylinder or $\left.\frac{\sigma_x t b^2}{D\pi^2}\right.$ for infinitely long curved strip$\left.\right)$
k_y	circumferential compressive-stress coefficient $\left(\frac{\sigma_y t L^2}{D\pi^2}\right.$ for cylinder or $\left.\frac{\sigma_y t b^2}{D\pi^2}\right.$ for infinitely long curved strip$\left.\right)$

C_p hydrostatic-pressure coefficient $\left(\frac{prL^2}{D\pi^2}\right)$

w_o amplitude of deflection function

D plate flexural stiffness per unit length $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$

E Young's modulus

F Airy's stress function for the median surface stresses produced by the buckle deformation

L length of cylinder

Z curvature parameter $\left(\frac{L^2}{rt}\sqrt{1 - \mu^2}\right.$ for cylinder or $\frac{b^2}{rt}\sqrt{1 - \mu^2}$ for infinitely long curved strip$\left.\right)$

β L/λ for cylinder or b/λ for infinitely long curved strip

λ half wave length of buckles; measured circumferentially in cylinders and axially in infinitely long curved strips

ξ dimensionless axial coordinate (x/b)

η dimensionless circumferential coordinate (y/b)

μ Poisson's ratio

τ applied shear stress

σ_x applied axial stress, positive for compression

σ_y applied circumferential stress, positive for compression

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla_G^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$$

$$\nabla^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}$$

$$\nabla_G^4 = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^2$$

$$\nabla^8 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^4$$

$$\nabla_G^8 = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^4$$

BUCKLING STRESSES OF CYLINDERS WITH SIMPLY SUPPORTED EDGES

Lateral pressure.- The theory for the lateral pressure (uniform external pressure applied to walls only) at which a cylinder will buckle is given in appendix B in which it is assumed that the lateral pressure causes the buckling by producing a circumferential stress σ_y and that it affects the buckling in no other way. The results are shown in a logarithmic plot in figure 1. The ordinate in this figure is the stress coefficient k_y which appears in the flat-plate buckling equation (see, for example, reference 4, p. 339)

$$\sigma_y = k_y \frac{\pi^2 D}{L^2 t}$$

(The discussion given in the section of the present paper entitled "Parameters Appearing in Buckling Curves" shows the relationship between a cylinder of length L and an infinitely long flat plate of width b = L.) The abscissa

$$Z = \frac{L^2}{rt}\sqrt{1 - \mu^2} = \left(\frac{L}{r}\right)^2 \frac{r}{t}\sqrt{1 - \mu^2}$$

may be regarded either as a measure of the curvature, or, for any given ratio of radius to thickness, as a measure of the length-radius ratio of the cylinder. Figure 1 shows that for small curvature k_y approaches the value 4, which applies in the case of simply supported long flat plates in longitudinal compression (reference 4, p. 327). As the curvature parameter Z increases, the stress coefficient k_y also increases. For large values of Z, the curve approaches a straight line of slope 1/2. This straight line is expressed by the formula

$$k_y = 1.04Z^{1/2}$$

As the length-radius ratio increases, for a given value of r/t, the number of circumferential waves n diminishes. Although n must be an integer, the curves of figure 1 were obtained on the assumption that n is free to vary continuously. Only small conservative errors are involved in this assumption. Because n = 1 corresponds merely to a lateral displacement of the entire circular cross section, the minimum value of n is 2, which corresponds to deformation of the section into an ellipse. This limitation on n results in splitting the curve of figure 1 into a number of curves for different values of r/t when Z becomes large. Thick-walled cylinders may reach n = 2 at moderate lengths, but thin-walled cylinders reach n = 2 only when much longer than they are likely to be in practical construction.

In figure 2 the curve of figure 1 is compared with results based on more complicated calculations given in reference 4 and in reference 6. At fairly large values

of Z the results given in reference 4 and in reference 6 are in good agreement with the results of the present paper. At small values of Z the curve based on reference 4 (Timoshenko) is definitely too low, because k_y should approach the flat-plate value of 4 as Z approaches zero. An interesting feature of the comparison is that one calculation gives results below, and the other calculation results above, those given herein. The test data, taken from reference 6, are in reasonable agreement with and show more scatter than the theoretical curves.

In the case of cylinders so long that $n = 2$, the requirement for the validity of Donnell's equation that $n^2 \gg 1$ is no longer satisfied and appreciable error is to be expected. Indeed it may be shown that for very long cylinders when $n = 2$ Donnell's equation gives $4D/r^3$ as the critical value of the applied lateral pressure, whereas the accepted theoretical result is $3D/r^3$ (by use of the formula given on p. 450 of reference 4). The curves for $n = 2$ will probably not often be needed, however, since they apply only when $\left(\frac{L}{r}\right)^2 > \left(\frac{5r}{t}\right)$, which in the case of thin cylinders corresponds to a very large length-radius ratio, and if needed, the curves for $n = 2$ can be applied in conjunction with a correction factor 0.75.

Axial compression.- The theory for the axial stress at which a cylinder will buckle is given in appendix B, and the results are shown in figure 3. The ordinate is analogous to, and the abscissa identical with, the corresponding coordinates used in figure 1. Figure 3 shows that for small values of Z, k_x approaches the value 1, which applies in the case of long flat plates in transverse compression with long edges simply supported (reference 4). For large values of Z, the curve becomes a straight line of slope 1. This straight line is expressed by the formula

$$k_x = \frac{4\sqrt{3}}{\pi^2} Z = 0.702Z$$

For any fixed value of r/t some value of Z always exists above which L/r is so large that the cylinder fails as an Euler strut rather than by buckling of the cylinder walls. Pin-ended Euler buckling of cylinders is indicated in figure 3 by means of dashed curves.

The result just given for the critical-stress coefficient for a cylinder in axial compression leads to the following expression for the critical stress:

$$\sigma_x = \frac{1}{\sqrt{3(1 - \mu^2)}} \frac{Et}{r} \tag{1}$$

The value given in equation (1) for the critical stress of a moderately long cylinder in axial compression by use of Donnell's equation is identical with the value found by a number of investigators using other equations as starting points (references 2 to 4). In the case of cylinders under axial compression the errors involved in dropping the second-order terms are therefore concluded to be small.

The buckling stresses given by equation (1) are nevertheless in serious disagreement with the buckling stresses obtained by experiment (reference 11). For a discussion of the degree of correlation that can be found between theory and experiment for cylinders under axial compression, see reference 12.

Hydrostatic pressure on closed cylinders.- When closed cylinders are subjected to external pressure, both axial and circumferential stress are present. The theory for buckling under these combined loads is given in appendix B. The results are shown in figure 4. The ordinate C_p used in this figure is a nondimensional measure of the pressure p defined as follows:

$$C_p = \frac{prL^2}{\pi^2 D}$$

The coefficient C_p can be directly related to the corresponding stress coefficients k_x and k_y. By definition

$$k_y = \frac{\sigma_y t L^2}{\pi^2 D}$$

and, according to the hoop-stress formula,

$$\sigma_y = \frac{pr}{t}$$

It follows from the three preceding equations that C_p is numerically equal to k_y. Similarly C_p can be shown to be numerically equal to $2k_x$.

At low values of Z, C_p approaches the value 2, which implies that $k_x = 1$ and $k_y = 2$. That these values of k represent a critical combination of stresses for an infinitely long flat plate was shown in reference 13. At large values of Z, the curve approaches the curve given in figure 1 for buckling under lateral pressure alone and, like that curve, has branches representing buckling into two circumferential waves.

In figure 5 the computed values of the pressure coefficient C_p at which the cylinder would buckle if only the axial pressure were acting and if only lateral pressure were acting are compared with the results when both are acting because of hydrostatic pressure. At large values of Z the circumferential stress at which buckling occurs under hydrostatic pressure is substantially the same as it would be if no axial stress were present, as in the case of lateral pressure. The reason that the circumferential stress appears as the main factor in buckling at high values of Z presumably is that at these values of Z the axial stress required to produce buckling is many times the circumferential stress required, whereas under hydrostatic pressure the axial stress actually present is only one-half the circumferential stress.

In figure 6 the curve of figure 4 is compared with curves representing Sturm's theoretical results (reference 6) and with a curve based on the following formula developed at the U. S. Experimental Model Basin (reference 14, equation (9)):

$$p = \frac{2.42E}{(1 - \mu^2)^{3/4}} \frac{\left(\frac{t}{d}\right)^{5/2}}{\left[\frac{L}{d} - 0.45\left(\frac{t}{d}\right)^{1/2}\right]}$$

This formula is an approximation based on theoretical results obtained by von Mises (reference 4, p. 479) which are identical with the results in the present paper for buckling under hydrostatic pressure. Figure 6 shows that Sturm's theoretical results (reference 6) are in reasonable agreement with those of the present paper and that the formula from the U. S. Experimental Model Basin practically coincides with the present results except at very low values of Z.

Test results from references 6 and 14 are included in figure 6. The test data are in good agreement with the theoretical results except at low values of the curvature parameter Z at which the theoretical results are appreciably above those obtained experimentally. A possible explanation of the discrepancy between the theoretical and experimental results at low curvature is suggested by the relative importance of axial and circumferential stress in causing buckling. The axial stress becomes important only at low values of the curvature parameter Z. It is known experimentally that buckling under axial stresses may occur far below the theoretical value of the critical stress. At low values of Z cylinders under hydrostatic pressure may therefore be expected to buckle well below the theoretical critical load just as cylinders do under axial compression.

Torsion.- The problem of the determination of the buckling stresses of cylinders in torsion was solved by Donnell (reference 5) who gave an approximate solution of the equation of equilibrium. A somewhat more accurate solution of this equation is given in reference 15. The

essential results are shown in figure 7 taken from reference 15. At low values of Z the buckling-stress coefficient k_s approaches the value 5.34 appropriate to infinitely long flat plates loaded in shear (reference 16). At higher values of Z the curve approaches a straight line given by

$$k_s = 0.85Z^{3/4}$$

At very high values of the curvature parameter the curve splits up into a number of other curves, depending on the value of r/t. The curves for various r/t values at high values of Z represent buckling into two circumferential waves. As mentioned before, Donnell's equation is not reliable for the case $n = 2$ (a case which occurs for cylinders in torsion when $\left(\frac{L}{r}\right)^2 > 10\frac{r}{t}$). A solution for this case given by Schwerin and discussed in reference 5 results in critical stresses about 20 percent below those of the present paper. Because Schwerin's solution does not satisfy the condition $w = 0$ at the end of the cylinder, however, it is probable that the error in the present solution for $n = 2$ is less than 20 percent.

In experimental investigations of cylinders in torsion the maximum rather than the critical loads have usually been reported. Because these maximum loads usually exceed the critical loads by only a small margin, it is common practice to check theoretical buckling stresses by comparison with the average stresses at maximum load. Such a comparison is provided in figure 8 which incorporates test data from references 5, 11, 17, and 18. For this figure the test results average about 15 percent below those given by theory.

DISCUSSION

Parameters appearing in buckling curves.- The fact that the buckling of a cylinder under axial compression, lateral pressure, hydrostatic pressure, or torsion involves substantially the same parameters is not a mere

coincidence but is a direct consequence of the differential equation. The differential equation implies that when the requirement of an integral number of circumferential waves is removed the six variables L, r, t, E, μ, and the load may be combined into two nondimensional parameters, one (k_x, k_y, k_s, or C_p) describing the stress condition, and the other (Z) essentially determined by the geometry. (See appendix C.) It is also shown in appendix C that the buckling of a curved rectangular plate of any given length-width ratio may be represented in terms of these parameters. The critical stress of a cylinder or a curved plate of given length-width ratio may therefore be given by a single curve relating the two parameters provided that the number of circumferential waves may be regarded as continuously variable. This restriction becomes important at very large values of Z, for which the curves may split into a number of curves for cylinders of different values of r/t buckling into two circumferential waves.

Except for hydrostatic pressure, each type of loading considered results in a single uniform stress in the cylinder, and the nondimensional parameter k describing this stress is defined as follows in analogy to the parameter used in describing the buckling of a flat plate:

$$k = \frac{\sigma \text{ (or } \tau)}{\frac{\pi^2 D}{L^2 t}}$$

As the radius of the cylinder increases toward infinity (the other dimensions remaining constant), the cylinder approaches an infinitely long flat plate of the same thickness as the cylinder, having a width b equal to the length L of the cylinder. Accordingly, as the radius approaches infinity, the critical-stress coefficient k for the cylinder approaches the value of the corresponding stress coefficient for an infinitely long flat plate under the appropriate loading condition.

The other nondimensional parameter Z is defined by the equation

$$Z = \frac{L^2}{rt}\sqrt{1 - \mu^2} = \left(\frac{L}{r}\right)^2 \frac{r}{t}\sqrt{1 - \mu^2}$$

If the small correction due to Poisson's ratio is neglected, a direct physical significance can be assigned to Z when its magnitude is small. The maximum distance from a slightly curved arc of length L and radius r to its chord can be shown to be given by the expression $L^2/8r$, which is called the "bulge" by some writers (see references 9 and 10). Accordingly, in the case of a curved strip of length L in the circumferential direction, $L^2/8rt$ is the bulge divided by the thickness and is thus a nondimensional measure of the deviation from flatness of the strip. As applied to a short cylinder, $L^2/8rt$ is the deviation from flatness of a square panel of the cylinder, each side of which is equal to the length of the cylinder. For cylinders having a length greater than a few tenths of the diameter, the parameter Z loses this simple physical significance and is perhaps best regarded as a nondimensional measure of the length of the cylinder. Some indication of the variety of cylinder shapes corresponding to a fixed value of Z is given in figure 9.

Boundary conditions.- When problems in the stability of cylindrical shells are solved by the use of Donnell's equation, boundary conditions on u and v cannot be imposed directly because only w appears in the equations. The method of solution, however, may in some cases imply boundary conditions on u or v. In appendix D it is shown that for simply supported cylinders the method used in the present paper (a solution using one or more terms of a Fourier series satisfying the boundary conditions on w term by term) implies that at both ends of the cylinder the circumferential displacement v is zero, but that the cylinder edges are free to warp in the axial direction ($u \neq 0$). For a simply supported rectangular curved panel, the present method implies (with regard to displacements within the panel median surface) zero displacement along the four edges of the panel and free warping normal to the edges. These edge conditions on u and v are appropriate to cylinders or panels bounded by light bulkheads or deep stiffeners which are stiff in their own planes but may be readily warped out of their planes.

Relatively few calculations of the stability of a cylinder take into account the boundary conditions on u and v. A calculation for the case of torsion, however, was recently made by Leggett (reference 19). The results of this calculation, computed for $u = v = 0$ at the edges of the cylinder, are given only for $Z < 50$. Throughout the range for which they are given, however, they agree very closely with the results found by the method employed in the present paper, which implies that at the edge of the cylinder $v = 0$ and $u \neq 0$. Restraining the ends of the cylinder from warping in the axial direction may therefore be assumed to have a negligible effect upon the buckling stress. This assumption receives added support from the form of the equation of equilibrium given in appendix A

$$D\nabla^4 w + p + t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x\, \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial x^2}\frac{1}{r}\right) = 0$$

In this equation, σ_x, σ_y, and τ are the stresses present just before buckling and $\frac{\partial^2 F}{\partial x^2}$ is the circumferential stress produced by the buckling itself. The equation indicates that the only difference between the buckling behavior of a cylindrical sheet and that of a flat plate (found by omitting the last term in the foregoing equation) is due to the effect of the circumferential stresses caused by the buckling deformations. Because the restraint against warping in the axial direction requires the application of axial rather than circumferential stresses, this restraint might be expected to have only small effects on buckling stresses. Circumferential stresses would have to be applied to the straight sides of a curved strip to prevent warping normal to these edges during buckling. Because the circumferential stress due to buckling appears explicitly in the equation of equilibrium, the imposition of the restraint $v = 0$ to the straight sides of a panel should have an appreciable effect on the buckling stress (except when the straight sides of the panel are short compared with the curved sides).

Theoretical results on the buckling of curved strips infinitely long in the axial direction are available to

test the foregoing conclusion. In figure 10 the critical axial compressive stress for an infinitely long curved strip with u and v both zero along the edges (reference 9) is compared with the critical axial compressive stress when u is zero along the edges, and the edges are free to warp in the circumferential direction. (See appendix B for solution.) The critical axial stress is appreciably increased by the constraint $v = 0$ in a certain range of small curvature. In figure 11 the critical shear stresses are compared under the same sets of edge conditions (references 7 and 8). The critical shear stress is conspicuously increased by the constraint $v = 0$ except near the limiting case of flat plates.

It appears from the foregoing discussion that the effect on the buckling stresses of preventing free warping normal to the curved edges of a cylinder or panel is very small but that the effect on the buckling stresses of a similar restraint on the straight edges of a panel may be quite important.

Simplicity of results.- The theoretical results based on Donnell's equation for the critical stresses of cylinders under a given loading condition appear particularly simple when presented as a logarithmic plot of buckling coefficient k against the curvature parameter Z. As r approaches infinity, and therefore as Z approaches zero, k approaches the value appropriate to a flat plate. At large values of Z the curve approached a straight line in each of the cases investigated. These straight lines had slopes 0.5, 0.75, and 1 and are given approximately by the following equations which have already been given in the present paper and are reassembled here and provided with upper and lower limits for easy reference:

$$k_y = 1.04Z^{1/2} \qquad 100 < Z < 5\left(\frac{r}{t}\right)^2 (1 - \mu^2)$$

$$k_s = 0.85Z^{3/4} \qquad 50 < Z < 10\left(\frac{r}{t}\right)^2 (1 - \mu^2)$$

$$k_x = 0.702Z \qquad 3 < Z < 6\left(\frac{r}{t}\right)^2 (1 - \mu^2)$$

These equations can also be written (when μ is taken to be 0.316)

$$\sigma_y = 0.926 \frac{Et}{r}\left(\frac{tr}{L^2}\right)^{1/2} = 0.926E\left(\frac{t}{r}\right)^{3/2}\left(\frac{r}{L}\right) \qquad \left(100\frac{t}{r} < \left(\frac{l}{r}\right)^2 < 5\frac{r}{t}\right)$$

$$\tau = 0.747 \frac{Et}{r}\left(\frac{tr}{L^2}\right)^{1/4} = 0.747E\left(\frac{t}{r}\right)^{5/4}\left(\frac{r}{L}\right)^{1/4} \qquad \left(50\frac{t}{r} < \left(\frac{l}{r}\right)^2 < 10\frac{r}{t}\right)$$

$$\sigma_x = 0.608 \frac{Et}{r} \qquad \left(3\frac{t}{r} < \left(\frac{l}{r}\right)^2 < 6\frac{r}{t}\right)$$

CONCLUDING REMARKS

The use of Donnell's equation to find the buckling stresses of cylindrical shells leads to simpler results and involves less labor than the use of equations in which second-order terms are retained. The buckling stresses found by use of Donnell's equation are in reasonable agreement with results based on other theoretical calculations. Except for the case of axial loading, they are also in reasonable agreement with test results.

Boundary conditions having to do with axial and circumferential displacements cannot be handled directly by use of Donnell's equation. This disadvantage is not considered serious, however, because the boundary conditions on axial and circumferential displacement, which are implied by the simple solutions given, correspond approximately to those that are most likely to occur in practical construction and because in many cases the buckling stress is not very sensitive to these boundary conditions.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., March 20, 1947

APPENDIX A

SIMPLIFIED EQUATIONS OF EQUILIBRIUM FOR CYLINDRICAL SHELLS

The principal sets of simplified equations currently in use for the equilibrium of cylindrical shells are listed for convenient reference. The various sets of equations are equivalent. The reference papers in which the equations are derived are also listed. The equations given are generally not identical with those in the reference papers but are modified in certain respects to include all the loading conditions studied in the present paper or to put them in the notation of the present paper.

The three following simultaneous equations in displacements u, v, and w (reference 4) are derived from the conditions of static equilibrium:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1 - \mu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1 + \mu}{2}\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\mu}{r}\frac{\partial w}{\partial x} = 0 \qquad (A1)$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{1 - \mu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1 + \mu}{2}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{1}{r}\frac{\partial w}{\partial y} = 0 \qquad (A2)$$

$$D\nabla^4 w + \frac{Et}{r(1 - \mu^2)}\left(\frac{\partial v}{\partial y} + \mu\frac{\partial u}{\partial x} + \frac{w}{r}\right) + t\left(\sigma_x\frac{\partial^2 w}{\partial x^2} + 2\tau\frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y\frac{\partial^2 w}{\partial y^2}\right) + p = 0 \qquad (A3)$$

Two simultaneous equations in deflection w and stress function F (reference 7) are as follows:

$$\nabla^4 F + \frac{E}{r}\frac{\partial^2 w}{\partial x^2} = 0 \qquad (A4)$$

$$D\nabla^4 w + t\left(\sigma_x\frac{\partial^2 w}{\partial x^2} + 2\tau\frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y\frac{\partial^2 w}{\partial y^2} + \frac{1}{r}\frac{\partial^2 F}{\partial x^2}\right) + p = 0 \qquad (A5)$$

A single equation in deflection w (Donnell's equation, reference 5) is

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + t\nabla^4\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x\,\partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) + \nabla^4 p = 0 \qquad (A6)$$

The relationships between u and w and between v and w are (reference 5)

$$r\nabla^4 u = -\mu \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\,\partial y^2} \qquad (A7)$$

$$r\nabla^4 v = -(2+\mu)\frac{\partial^3 w}{\partial x^2\,\partial y} - \frac{\partial^3 w}{\partial y^3} \qquad (A8)$$

APPENDIX B

THEORETICAL SOLUTIONS

Donnell's equation for the equilibrium of cylindrical shells is used to investigate the stability of simply supported cylinders subject to lateral pressure, axial compression, and hydrostatic pressure, and of simply supported curved strips long in the axial direction subject to axial compression.

Cylinder under Lateral Pressure

If bending of the cylinder wall is neglected, constant lateral pressure on a cylinder causes only circumferential stresses. Donnell's equation (equation (A6)) then reduces to

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \sigma_y t \nabla^4 \frac{\partial^2 w}{\partial y^2} = 0 \qquad (B1)$$

where

$$\sigma_y = \frac{pr}{t}$$

and p is the pressure applied. (The term $\nabla^4 p$ appearing in equation (A6) is zero when p is constant.) Division of equation (B1) by D results, with proper substitutions, in the following equation:

$$\nabla^8 w + \frac{12Z^2}{L^4}\frac{\partial^4 w}{\partial x^4} + k_y \frac{\pi^2}{L^2}\nabla^4 \frac{\partial^2 w}{\partial y^2} = 0 \qquad (B2)$$

The boundary conditions corresponding to simply supported edges (no deflection and no moment along the edges) are

$$w(0, y) = w(L, y) = 0$$

$$\frac{\partial^2 w}{\partial x^2}(0, y) = \frac{\partial^2 w}{\partial x^2}(L, y) = 0$$

A solution of equation (B2) satisfying the boundary conditions for simple support is

$$w = w_0 \sin \frac{\pi y}{\lambda} \sin \frac{m\pi x}{L} \tag{B3}$$

where λ is the half wave length in the circumferential direction. Combining equation (B3) and equation (B2) yields the following equation:

$$\left(m^2 + \beta^2\right)^4 + \frac{12z^2m^4}{\pi^4} - k_y\beta^2\left(m^2 + \beta^2\right)^2 = 0 \tag{B4}$$

The solution of equation (B4) for k_y is

$$k_y = \frac{\left(m^2 + \beta^2\right)^2}{\beta^2} + \frac{12z^2m^4}{\pi^4\beta^2\left(m^2 + \beta^2\right)^2} \tag{B5}$$

where

$$\beta = \frac{L}{\lambda}$$

The critical value for k_y is found by minimizing the right-hand side of equation (B5) with respect to m and β. If the numerator and denominator of the last term in equation (B5) are divided by m^4, it becomes evident that under the restriction of integral values

of m, k_y will be a minimum when $m = 1$. Equation (B5) therefore becomes

$$k_y = \frac{(1 + \beta^2)^2}{\beta^2} + \frac{12Z^2}{\pi^4\beta^2(1 + \beta^2)^2} \qquad (B6)$$

The results found by minimizing this expression for k_y with respect to β (considered continuously variable) is shown in figure 1 by the curve independent of r/t.

At low values of Z, buckling is characterized by a large number of circumferential waves. As Z increases, the number of circumferential waves decreases until it finally becomes two $\left(\lambda = \frac{\pi r}{2}\right)$, corresponding to buckling into an elliptical cross section. The curves for buckling into two circumferential waves are shown in figure 1 as the curves for various values of $\frac{r}{t}\sqrt{1 - \mu^2}$. The equations for these curves are found by substituting in equation (B5) the last of the following expressions for β:

$$\beta = \frac{L}{\lambda} = \frac{2L}{\pi r} = \frac{2}{\pi}\sqrt{\frac{Z}{\frac{r}{t}\sqrt{1 - \mu^2}}}$$

Cylinder in Axial Compression

When only axial stress is present, equation (A6) becomes

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \sigma_x t\frac{\partial^2 w}{\partial x^2} = 0 \qquad (B7)$$

Division by D results, with proper substitutions, in the following equation:

$$\nabla^8 w + \frac{12Z^2}{L^4} \frac{\partial^4 w}{\partial x^4} + k_x \frac{\pi^2}{L^2} \nabla^4 \frac{\partial^2 w}{\partial x^2} = 0 \qquad (B8)$$

Combination of the deflection equation (B3) with equation (B8) yields the following equation:

$$\left(m^2 + \beta^2\right)^4 + \frac{12Z^2 m^4}{\pi^4} - k_x m^2 \left(m^2 + \beta^2\right)^2 = 0 \qquad (B9)$$

The solution of equation (B9) for k_x is

$$k_x = \frac{\left(m^2 + \beta^2\right)^2}{m^2} + \frac{12Z^2 m^2}{\pi^4 \left(m^2 + \beta^2\right)^2}$$

The critical value of k_x for a given value of Z may be found by minimizing k_x with respect to the parameter

$$\frac{\left(m^2 + \beta^2\right)^2}{m^2}$$

If no restrictions are placed on the value that this parameter can take, the minimum value of k_x is found to be

$$k_x = \frac{4\sqrt{3}}{\pi^2} Z = 0.702Z \qquad (B10)$$

which coincides with the results generally given for the buckling of long cylinders.

For values of Z below 2.85, however, the straight-line formula (equation (B10)) cannot be used, since it implies either imaginary values of the circumferential wave length λ or the number of axial half waves m below unity. The critical stress coefficient k_x for $Z < 2.85$ is found by substituting the limiting values $\beta = 0$ and $m = 1$ in equation (B9). The results are shown in figure 3.

Cylinder under Hydrostatic Pressure

Hydrostatic pressure applied to a closed cylinder produces the following axial and circumferential stresses:

$$\sigma_x = \frac{pr}{2t}$$

$$\sigma_y = \frac{pr}{t}$$

The equation of equilibrium (equation (A6)) when both circumferential and axial stress are present is (since $\nabla^4 p = 0$)

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \sigma_x t\nabla^4 \frac{\partial^2 w}{\partial x^2} + \sigma_y t\nabla^4 \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{(B11)}$$

By use of the definition

$$C_p = \frac{prL^2}{D\pi^2}$$

equation (B11) can be written

$$\nabla^8 w + \frac{12Z^2}{L^4}\frac{\partial^4 w}{\partial x^4} + C_p \frac{\pi^2}{L^2}\nabla^4\left(\frac{1}{2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad \text{(B12)}$$

If the deflection equation (equation (B3)) is combined with equation (B12), the following expression results for C_p:

$$C_p = \frac{(m^2 + \beta^2)^2}{\frac{m^2}{2} + \beta^2} + \frac{12Z^2m^4}{\pi^4 (m^2 + \beta^2)^2 \left(\frac{m^2}{2} + \beta^2\right)} \qquad (B13)$$

The critical value of C_p is found by minimizing the right-hand side of equation (B13) with respect to m and β, with due regard to the values which m and β may assume. It can be shown that the minimum value of C_p is found by taking m equal to 1, so that equation (B13) becomes

$$C_p = \frac{(1 + \beta^2)^2}{\frac{1}{2} + \beta^2} + \frac{12Z^2}{\pi^4 (1 + \beta^2)^2 \left(\frac{1}{2} + \beta^2\right)} \qquad (B14)$$

Equation (B14) is equivalent to an equation derived by von Mises (reference 4, p. 479). The results of minimizing C_p with respect to β are shown in figure 4. (The curves given for various values of $\frac{r}{t}\sqrt{1 - \mu^2}$ have the same significance as in the case of a cylinder buckling under lateral pressure alone.)

Long Curved Strip in Axial Compression

Because it merely describes equilibrium at a point, equation (B1) applies to the buckling of a long curved strip as well as to cylinder buckling. In modifying this equation to obtain nondimensional coefficients as in equation (B2), however, it is convenient to define k_x and Z in terms of the width of the strip b rather than in terms of the axial length L, which applied in the case of the cylinder. Accordingly, equations (B1) and (B2) for a cylinder in axial compression may be applied also to the buckling of a curved strip, long in

the axial direction, subjected to axial compression, provided the curved width b is everywhere substituted for the axial length L. Substitution of the deflection

$$w = w_0 \sin \frac{\pi x}{\lambda} \sin \frac{n\pi y}{b}$$

into equation (B2) (modified by substitution of b for L) gives

$$k_x = \frac{(n^2 + \beta^2)^2}{\beta^2} + \frac{12Z^2\beta^2}{\pi^4 (n^2 + \beta^2)^2} \qquad \text{(B15)}$$

where

$$\beta = \frac{b}{\lambda}$$

Equation (B15) is very similar to equation (B5) and each equation yields the same critical value for k_x at large values of Z. At small values of Z, the minimum value of k_x is found by taking n = 1 in equation (B15) and minimizing with respect to β the resulting expression for k_x. The results are given in figure 10 together with results found by Leggett (reference 9).

APPENDIX C

PARAMETERS

It is shown that Donnell's equation implies that under certain limitations the buckling coefficient k, familiar from flat-plate theory, can be expressed in terms of the curvature parameter Z alone in the case of a complete cylinder or a curved rectangular panel of given length-width ratio.

Donnell's equation (A6) is (when p is constant or zero)

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + t\nabla^4\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x\, \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad (C1)$$

Let

$$\frac{x}{b} = \xi$$

$$\frac{y}{b} = \eta$$

and

$$\nabla_G^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$$

Then

$$\nabla_G^2 = b^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Multiplication of equation (C1) by b^8 and substitution of the dimensionless coordinates ξ and η gives

$$D\nabla_G^8 w + \frac{Etb^4}{r^2}\frac{\partial^4 w}{\partial \xi^4} + b^2 t \nabla_G^4\left(\sigma_x \frac{\partial^2 w}{\partial \xi^2} + 2\tau \frac{\partial^2 w}{\partial \xi\, \partial \eta} + \sigma_y \frac{\partial^2 w}{\partial \eta^2}\right) = 0$$

Division by D results in

$$\nabla_G^8 w + \frac{Etb^4}{Dr^2}\frac{\partial^4 w}{\partial \xi^4} + \frac{b^2 t}{D}\nabla_G^4\left(\sigma_x \frac{\partial^2 w}{\partial \xi^2} + 2\tau \frac{\partial^2 w}{\partial \xi\, \partial \eta} + \sigma_y \frac{\partial^2 w}{\partial \eta^2}\right) = 0$$

or, since $D = \dfrac{Et^3}{12(1 - \mu^2)}$,

$$\nabla_G^8 w + 12Z^2 \frac{\partial^4 w}{\partial \xi^4} + \pi^2 \nabla_G^4\left(k_x \frac{\partial^2 w}{\partial \xi^2} + 2k_s \frac{\partial^2 w}{\partial \xi\, \partial \eta} + k_y \frac{\partial^2 w}{\partial \eta^2}\right) = 0 \qquad (C2)$$

where

$$Z = \frac{b^2}{rt}\sqrt{1 - \mu^2}$$

$$k_x = \frac{\sigma_x t b^2}{D\pi^2}$$

$$k_s = \frac{\tau t b^2}{D\pi^2}$$

$$k_y = \frac{\sigma_y t b^2}{D\pi^2}$$

Even without solving this equation it is clear that w must be a function of the independent variables ξ and η,

and also the parameters Z, k_x, k_s, and k_y, and the derivatives of w will be functions of the same variables and parameters. Thus, if only one type of loading (represented by the buckling coefficient k) is present, equation (C2) may be written

$$f_1(\xi, \eta, Z, k) + 12Z^2 f_2(\xi, \eta, Z, k) + \pi^2 k f_3(\xi, \eta, Z, k) = 0 \tag{C3}$$

where f_1, f_2, and f_3 are definite, though unknown, functions. The variables ξ and η may now be eliminated by integration of both sides of this equation over the entire range of ξ and η. In the case of a curved panel of circumferential dimension a and axial dimension b the resulting equation is

$$\int_0^1 d\xi \int_0^{\frac{a}{b}} d\eta \left[f_1(\xi, \eta, Z, k) + 12Z^2 f_2(\xi, \eta, Z, k) + \pi^2 k f_3(\xi, \eta, Z, k) \right] = 0 \tag{C4}$$

The integrals of the functions f_1, f_2, and f_3 depend only upon Z, k, and the value of the ratio a/b. Accordingly, equation (C4) implies that a relationship of the following type exists:

$$f_4\left(k, Z, \frac{a}{b}\right) = 0 \tag{C5}$$

Equation (C5) indicates that for any given value of the panel aspect ratio a/b, the critical-stress coefficient k depends only upon Z.

If a complete cylinder of length L rather than a panel of length b is under consideration, and the

deflection w is periodic with wave length 2λ in the circumferential coordinate, the integration

$$\int_0^{\frac{a}{b}} d\eta$$

appearing in equation (C4) may be replaced by

$$\int_0^{\frac{2\lambda}{L}} d\eta$$

where ξ and η are now defined as x/L and y/L, respectively. The result then becomes

$$f_5\left(k, Z, \frac{2\lambda}{L}\right) = 0$$

or

$$k = f_6\left(Z, \frac{2\lambda}{L}\right) \tag{C6}$$

The actual buckling stress is found by minimizing k with respect to $2\lambda/L$.

Theoretically, λ must satisfy the equation

$$\pi r = n\lambda \tag{C7}$$

where n is the number of circumferential waves and therefore an integer. When many circumferential waves

are present, however, this restriction does not significantly affect the buckling stress, and the minimization of k with respect to $\frac{2\lambda}{b}$ (considered continuously variable) leads to the result

$$k_c = f_7(Z) \qquad (C8)$$

Equation (C8) indicates that provided the number of circumferential waves is not too small the critical-stress coefficient for a cylinder depends for practical purposes only upon the curvature parameter Z.

When n is so small that its integral character must be taken into account, it appears from equations (C6) and (C7) that k depends upon both Z and r/L. Since, however,

$$\left(\frac{r}{L}\right)^2 = \frac{1}{Z}\frac{r}{t}\sqrt{1 - \mu^2}$$

k for small values of n can alternatively be expressed in terms of Z and $\frac{r}{t}\sqrt{1 - \mu^2}$, as in figures 1, 4, and 7.

By a similar analysis, it can be shown that when the buckling of a cylinder under hydrostatic pressure is represented by plotting the pressure coefficient C_p against Z, a single curve is obtained except where the small number of circumferential waves requires splitting the curve into a series of curves for different values of $\frac{r}{t}\sqrt{1 - \mu^2}$.

APPENDIX D

BOUNDARY CONDITIONS ON EDGE DISPLACEMENTS WITHIN THE MEDIAN SURFACE

The solution of Donnell's eighth-order partial differential equation for the stability of cylindrical shells is not unique under the imposition of the ordinary boundary conditions for simply supported or clamped edges. Two more boundary conditions at each edge, for example, one condition for u and one for v, are required to define completely the physical problem and are therefore needed to make the solution unique. Because only w appears in the equation, boundary conditions on u and v cannot be imposed directly; they may, however, be implied by the method of solution. The purpose of this appendix is to show what boundary conditions on u and v are implied by the method of solution used in the present paper. In order to simplify the discussion, the analysis will first be made for the case when only axial compression is present and will then be extended to other cases.

When only axial stress is present, Donnell's equation (equation (A6)) becomes

$$D\nabla^8 w + \frac{Et}{r^2}\frac{\partial^4 w}{\partial x^4} + \sigma_x t \nabla^4 \frac{\partial^2 w}{\partial x^2} = 0$$

If the shell described by this equation is a curved panel with the origin of coordinates in one corner of the panel, a solution satisfying the usual boundary conditions for simple support is

$$w = w_o \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (D1)$$

where m and n are integers. This solution is also the solution to the problem of the buckling of an infinite two-dimensional array of panels identical to the one under

consideration. (See fig. 12.) When such an array buckles, the displacements u, v, and w as well as the stresses, described by the stress function F, may be presumed to be periodic over the interval 2a in the axial direction and 2b in the circumferential direction.

Any function u(x, y) that is periodic with a wave length 2a in the x-direction and with a wave length 2b in the y-direction may be expanded as follows (see, for example, reference 20):

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \qquad (D2)$$

The relationship which must exist between u and w is (equation (A7))

$$r\nabla^4 u = - \mu \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \, \partial y^2}$$

Substitution into this equation of the expressions for u and w from equations (D2) and (D1), respectively, and use of the orthogonality of the functions in equation (D2) leads to the result

$$u = \frac{w_o \left[\mu \left(\frac{m\pi}{a} \right)^3 - \frac{m\pi}{a} \left(\frac{n\pi}{b} \right)^2 \right]}{r \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Accordingly, the boundary conditions on u are

$$u(x, 0) = 0 \tag{D3}$$

$$u(x, b) = 0 \tag{D4}$$

$$\frac{\partial u}{\partial x}(0, y) = 0 \tag{D5}$$

$$\frac{\partial u}{\partial x}(a, y) = 0 \tag{D6}$$

Similarly by use of equation (A8) instead of equation (A7) it can be shown that the boundary conditions on v are

$$v(0, y) = 0 \tag{D7}$$

$$v(a, y) = 0 \tag{D8}$$

$$\frac{\partial v}{\partial y}(x, 0) = 0 \tag{D9}$$

$$\frac{\partial v}{\partial y}(x, b) = 0 \tag{D10}$$

The boundary conditions of equations (D5), (D6), (D9), and (D10) may be combined to give four boundary conditions on the stresses induced by buckling. These boundary conditions, which are also derivable from equation (A4) by a method analogous to that just used to derive the conditions relating to u, are

$$\frac{\partial^2 F}{\partial y^2}(0, y) = 0 \tag{D11}$$

$$\frac{\partial^2 F}{\partial y^2}(a, y) = 0 \tag{D12}$$

$$\frac{\partial^2 F}{\partial x^2}(x, 0) = 0 \tag{D13}$$

$$\frac{\partial^2 F}{\partial x^2}(x, b) = 0 \tag{D14}$$

where $\frac{\partial^2 F}{\partial y^2}$ and $\frac{\partial^2 F}{\partial x^2}$ are, respectively, the median-surface axial and circumferential stresses caused by buckling. The eight boundary conditions given by equations (D3), (D4), (D7), (D8), and equations (D11) to (D14), plus the eight boundary conditions on w for simple support of the four panel edges taken together uniquely determine the buckling stress.

Although the preceding discussion of boundary conditions started with the assumption of axial stress only, the only use made of this assumption was in obtaining equation (D1) as the solution for the buckling deformation. The same deformation, and hence the same arguments, apply when circumferential stress is present. When shear is present, a series of terms of the type in equation (D1) must be used to represent the deflection surface, and hence series of terms occur in the expressions for u, v, and F. Since the boundary conditions derived in

the preceding analysis apply to each of the terms individually, by the principle of superposition they must also apply for the sum, so that equations (D11) to (D14) represent the boundary condition no matter what the applied stresses are.

In summary it may be stated that the substitution of one or more terms of a double-sine-series expansion for w into Donnell's equation and solution of the resulting equation for the buckling stress gives the solution corresponding to the following boundary conditions:

(1) Each edge of the panel (or cylinder) is simply supported; that is, the displacement normal to the surface of the panel and the applied moments are zero at the edges.

(2) Motion parallel to each edge during buckling is prevented entirely.

(3) Motion normal to each edge in the plane of the sheet occurs freely.

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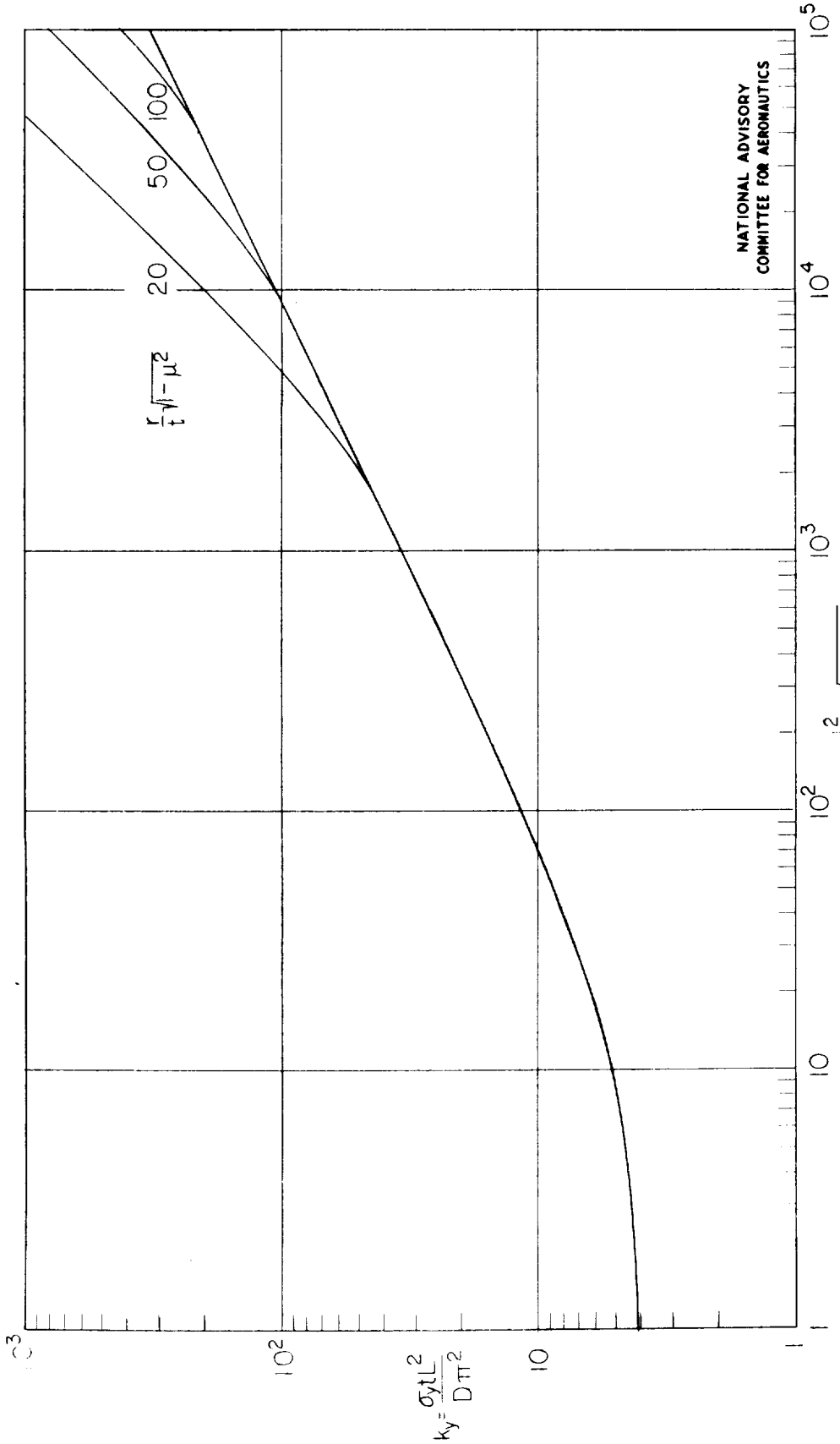

Figure 1.- Critical circumferential-stress coefficients for cylinders with simply supported edges.

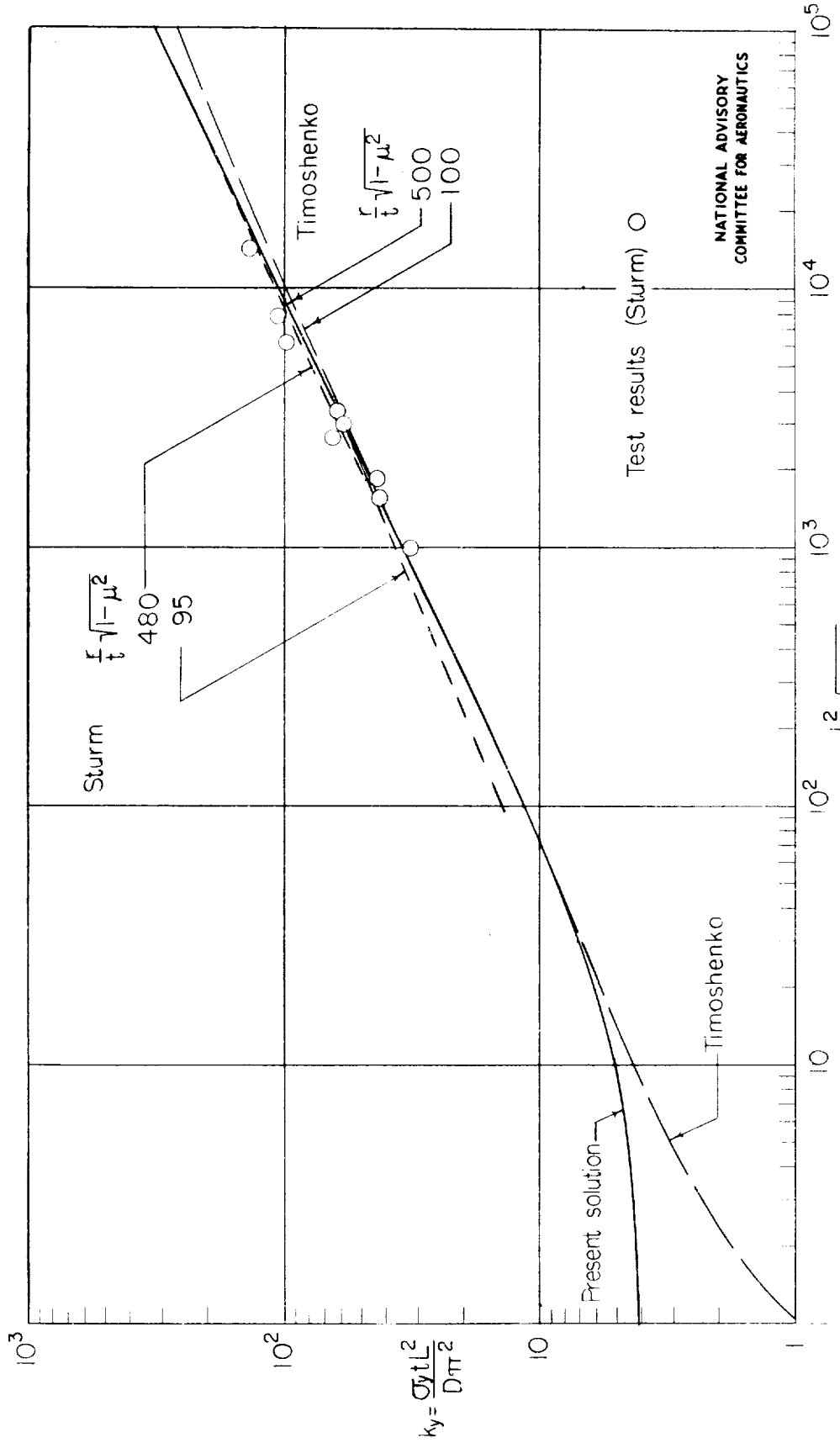

Figure 2.- Comparison of present solution for critical circumferential-stress coefficients for simply supported cylinders with other theoretical solutions and with test results. (Timoshenko's solution is from reference 4 and Sturm's solution is from reference 6.)

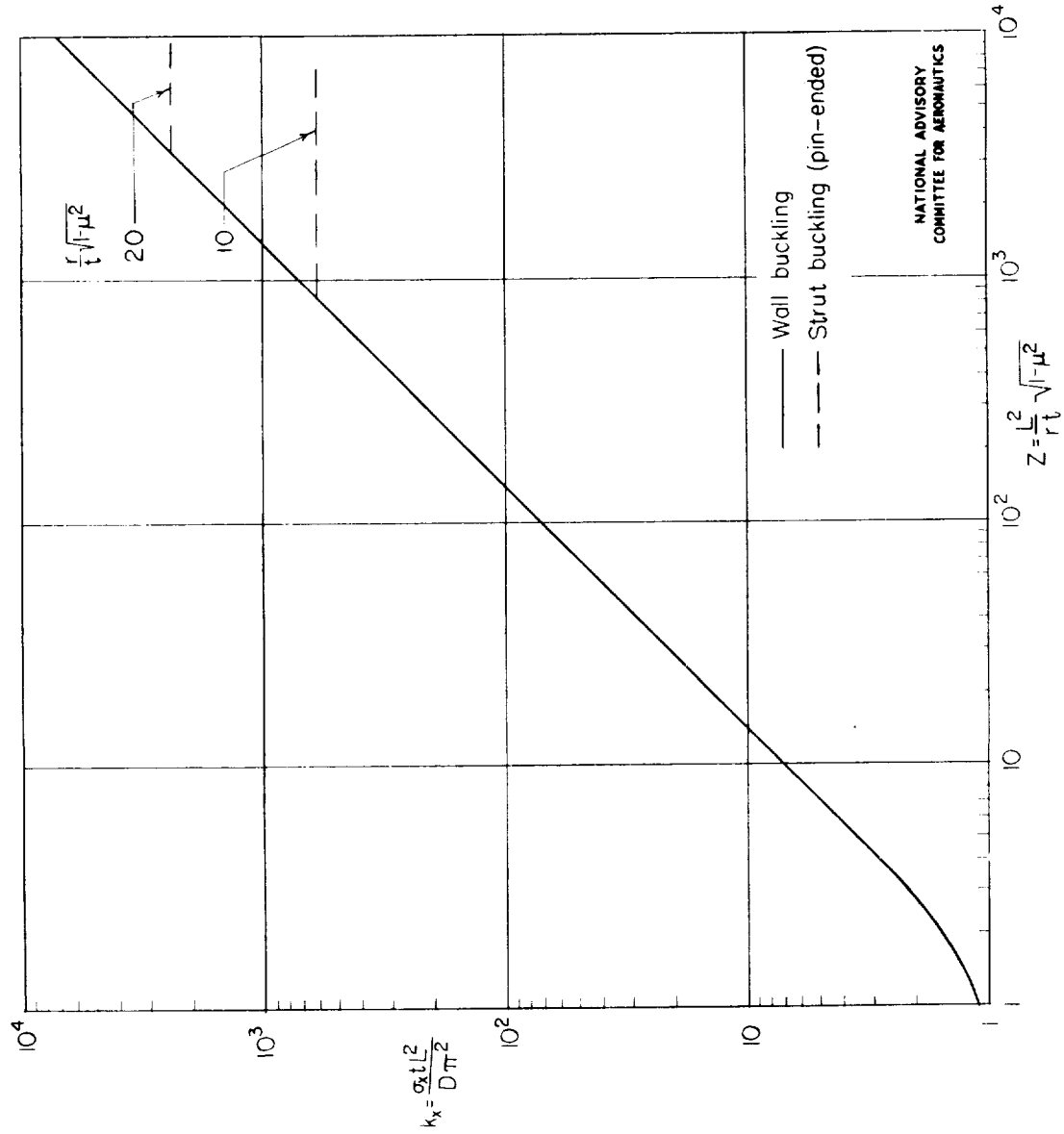

Figure 3.- Critical axial-stress coefficients for cylinders with simply supported edges.

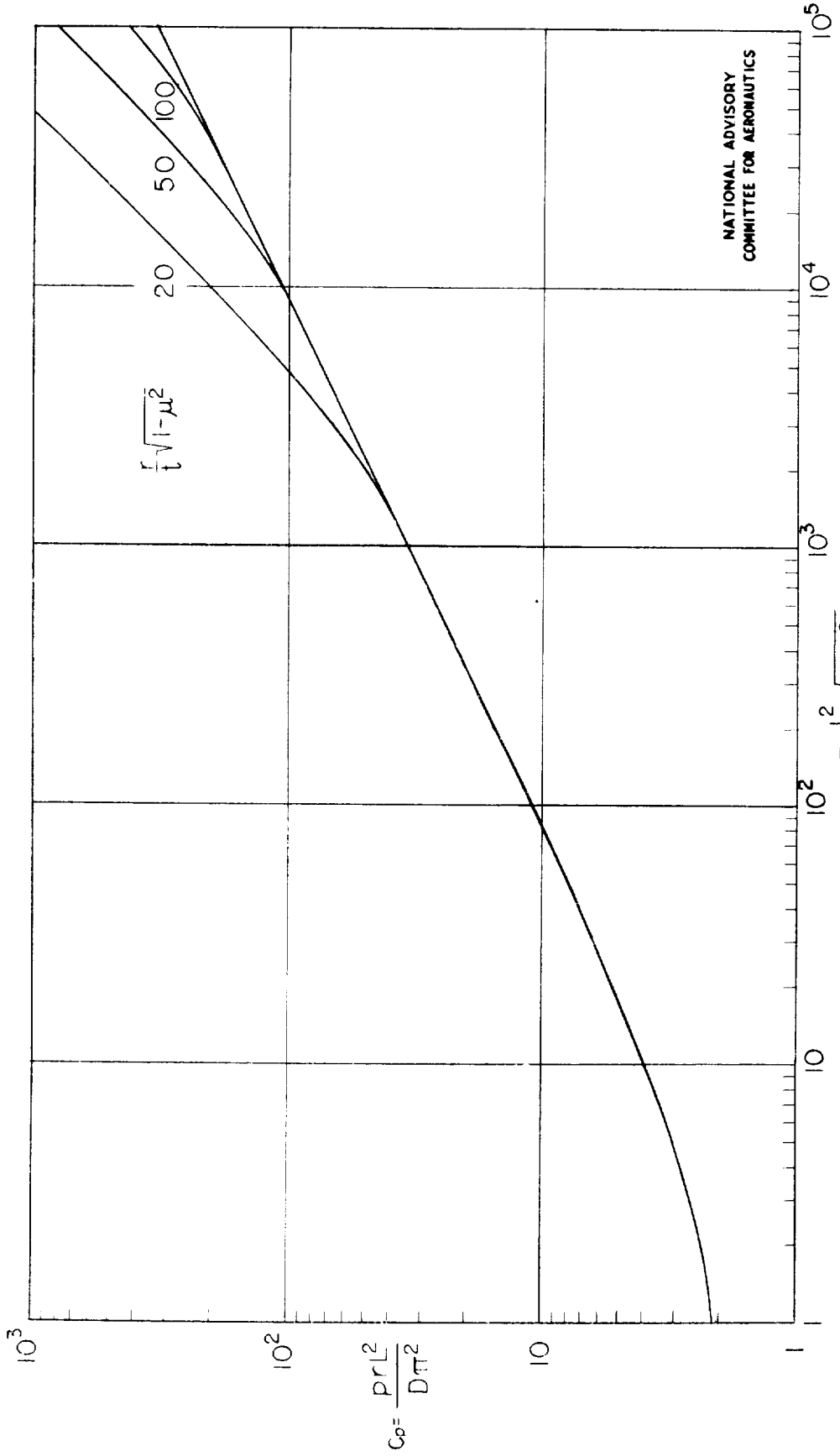

Figure 4.- Theoretical solution for hydrostatic pressure under which simply supported cylinders buckle.

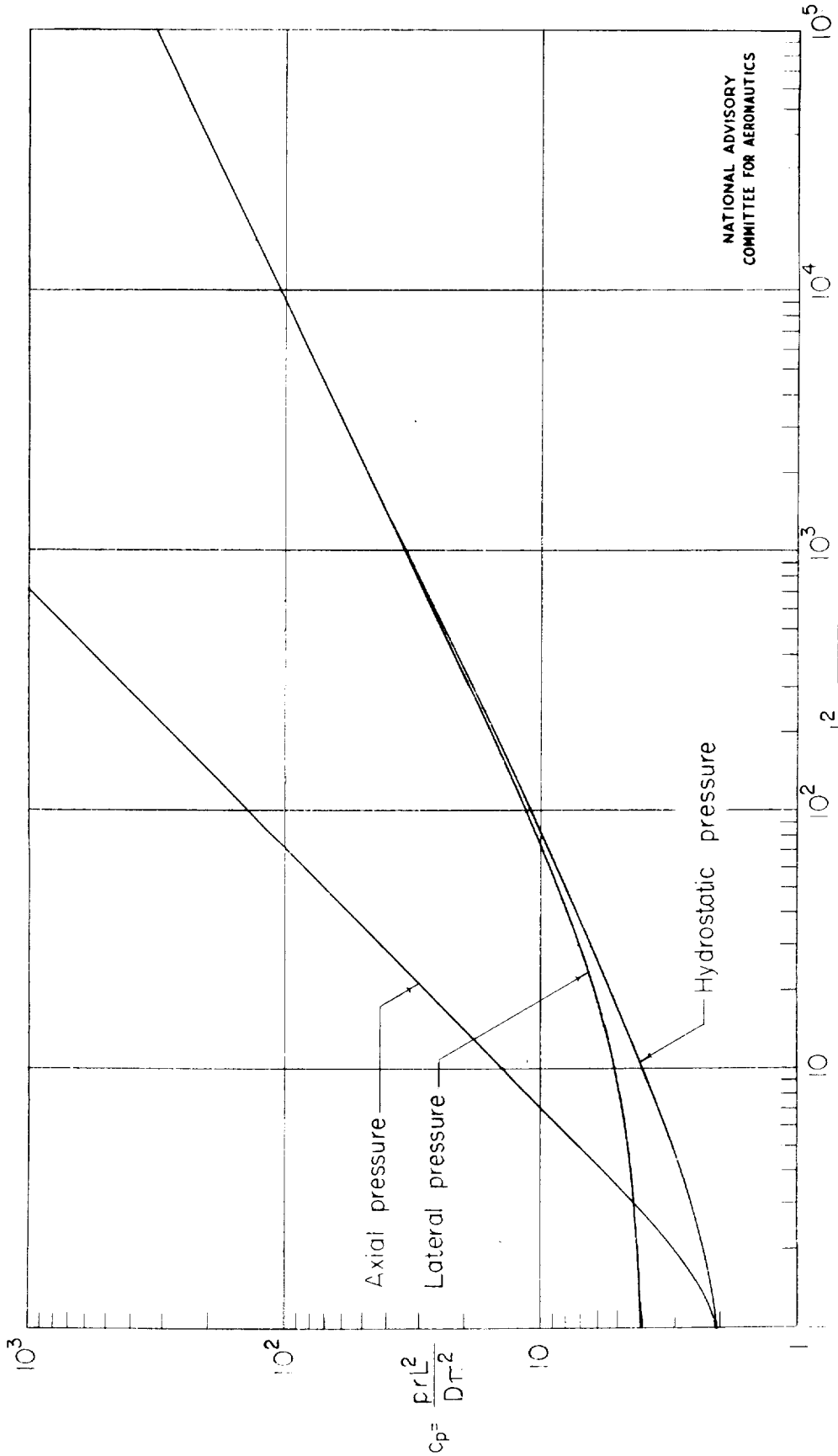

Figure 5.- Comparison of solution for buckling of simply supported cylinders under hydrostatic pressure with solutions for buckling under axial pressure alone and lateral pressure alone.

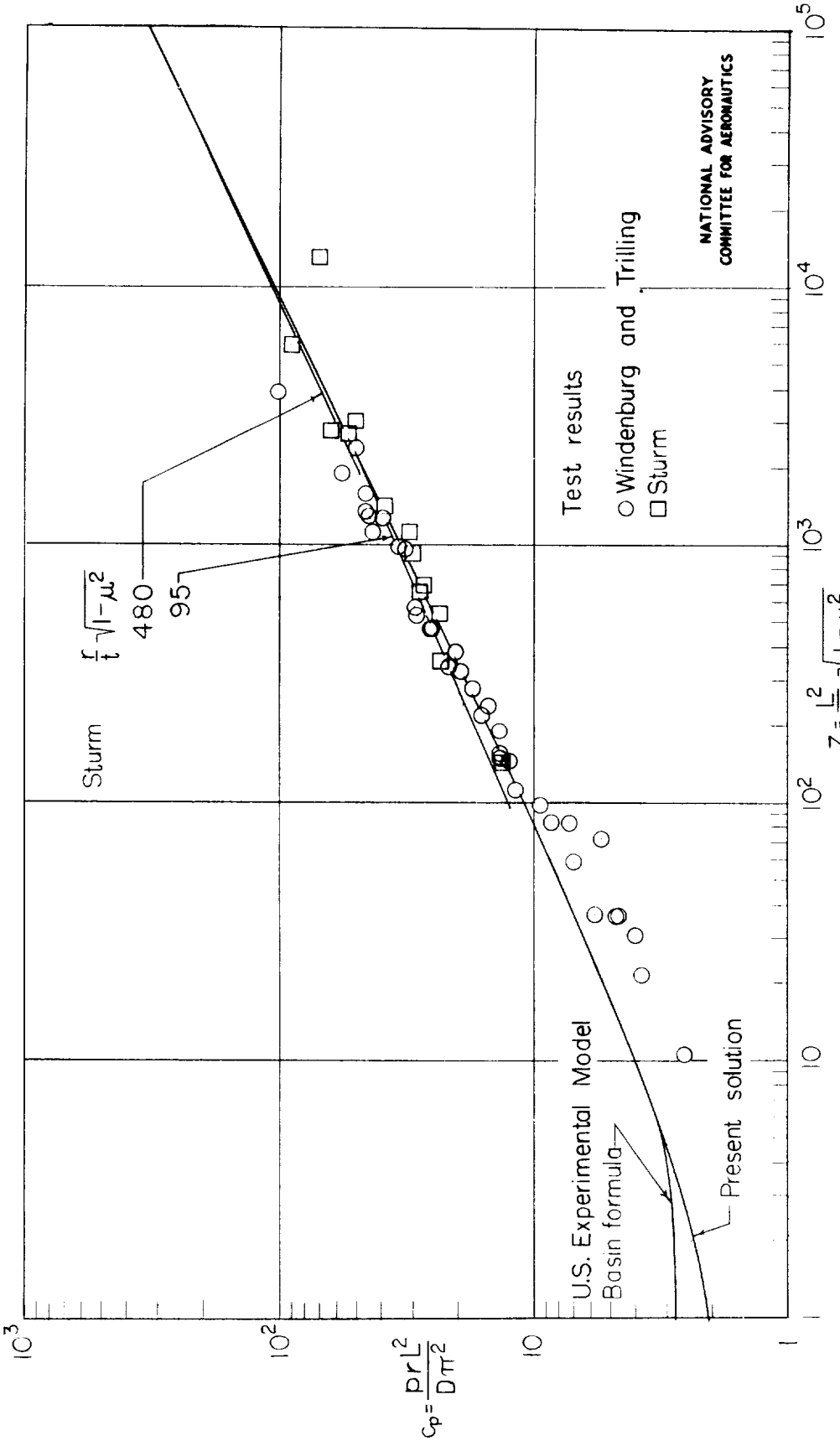

Figure 6.- Comparison of present solution for buckling of simply supported cylinders under hydrostatic pressure with other theoretical solutions and test results. (Sturm's solution is from reference 6 and Windenburg and Trilling's solution is from reference 14.)

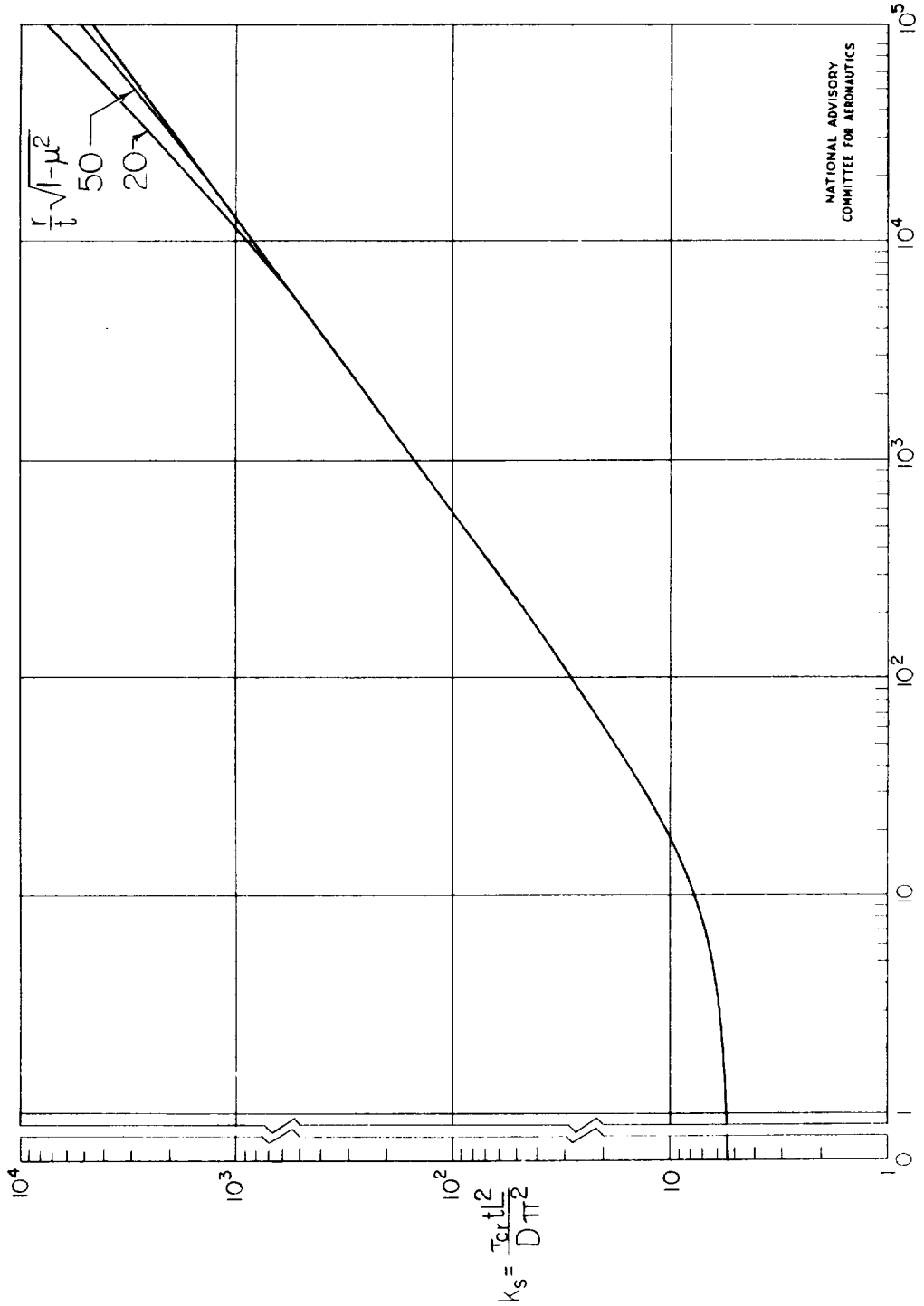

Figure 7.- Critical shear-stress coefficients for simply supported cylinders subjected to torsion.

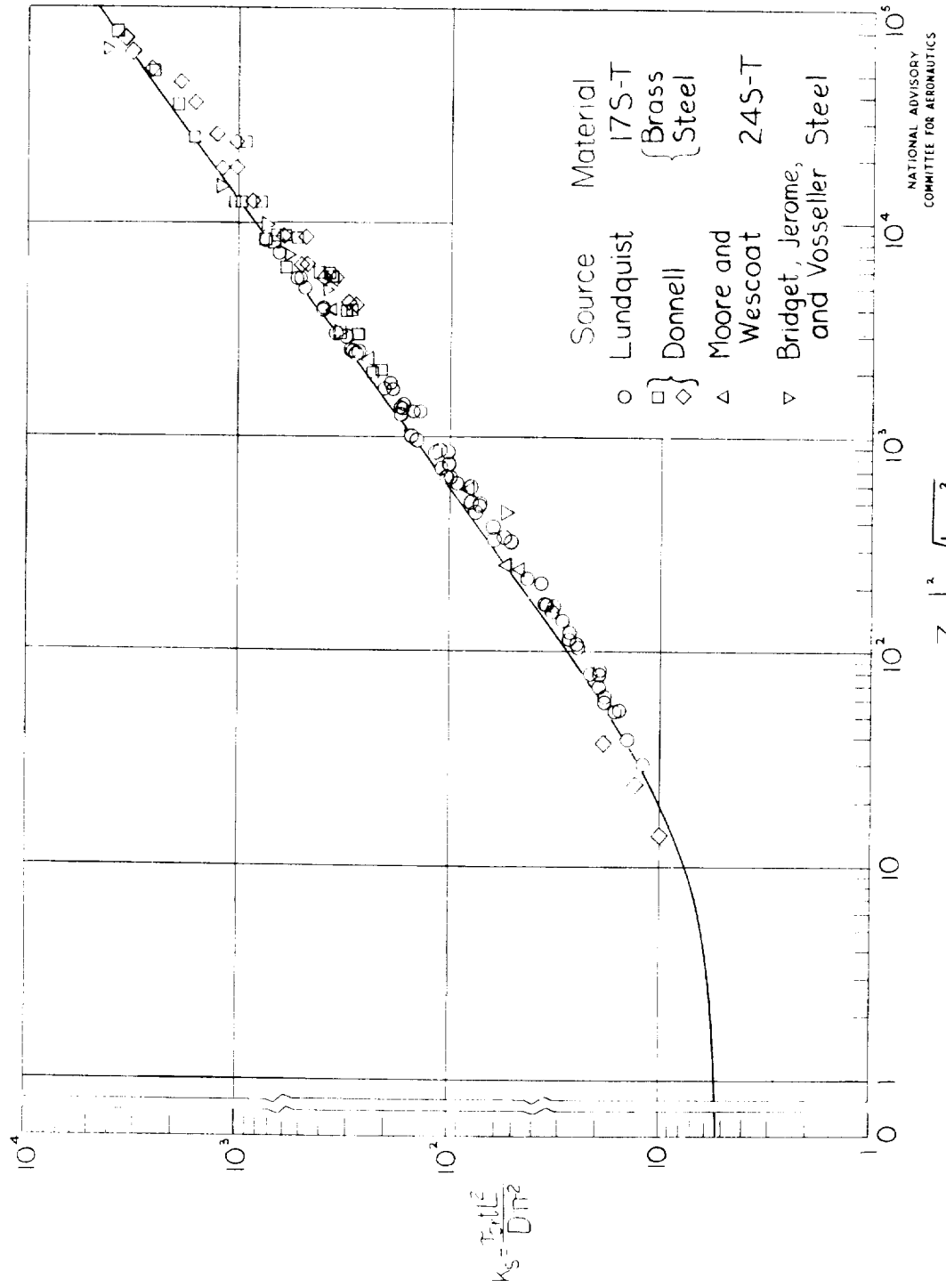

Figure 8.- Comparison of theoretical solution for critical shear stress of simply supported cylinders in torsion with experimental ultimate stresses. (Lundquist's solution is from reference 11, Donnell's solution is from reference 5, Moore and Wescoat's solution is from reference 17, and Bridget, Jerome, and Vosseller's solution is from reference 18.)

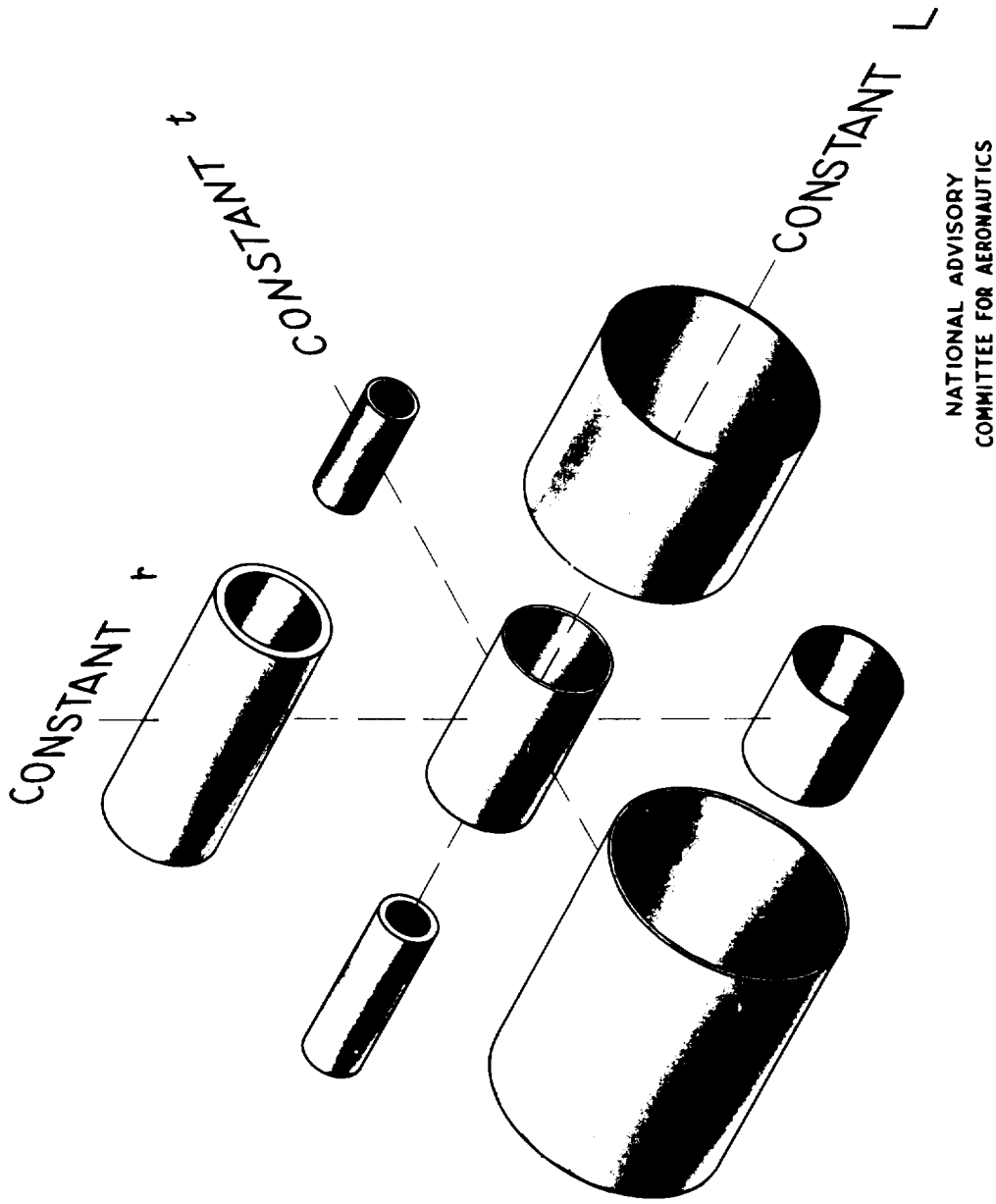

Figure 9.- Representative cylinders corresponding to the same value of Z (Z about 150).

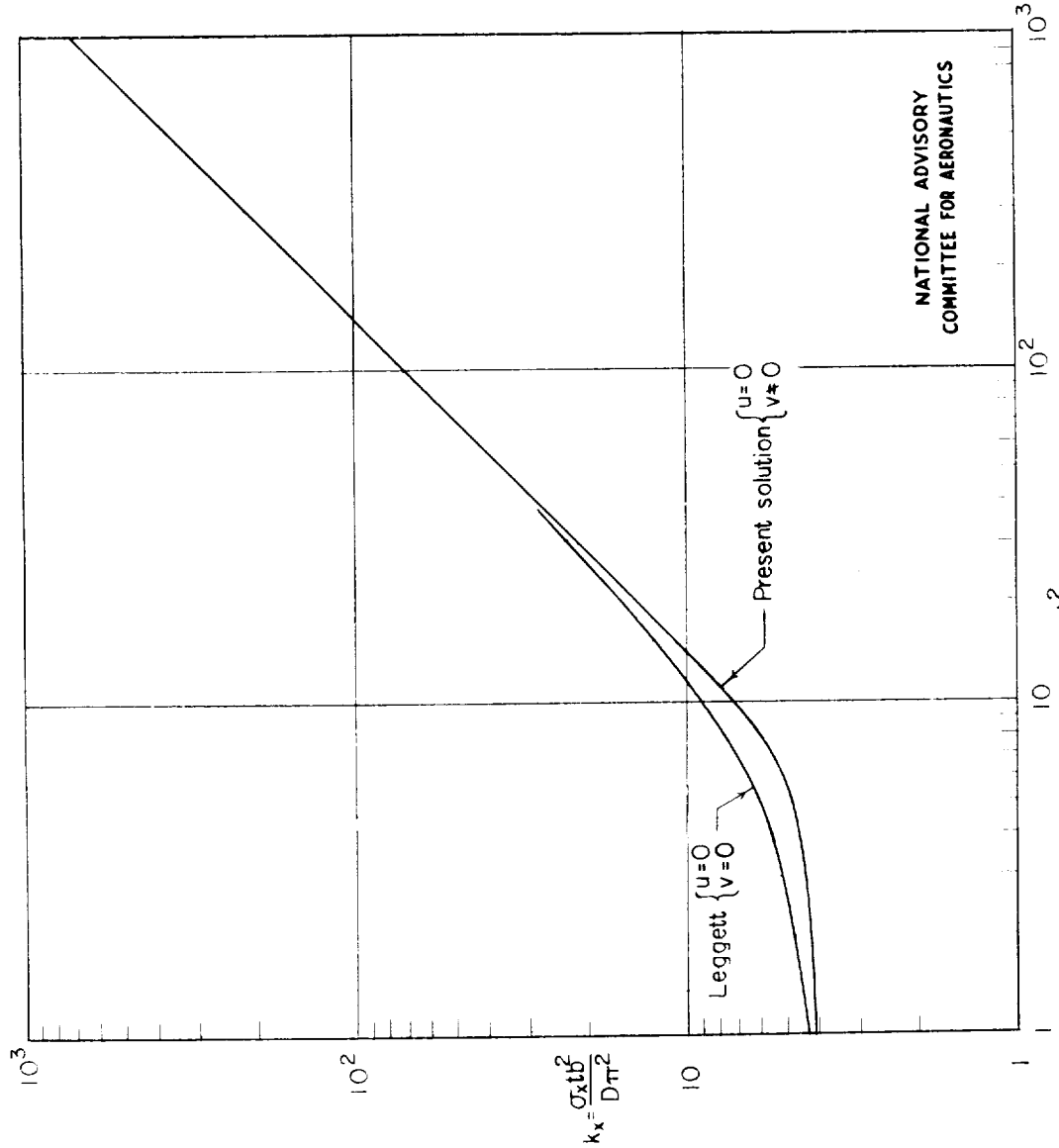

Figure 10.- Comparison of the present solution for the buckling under axial compression of a curved strip infinitively long in the axial direction, with solution found by Leggett (reference 9).

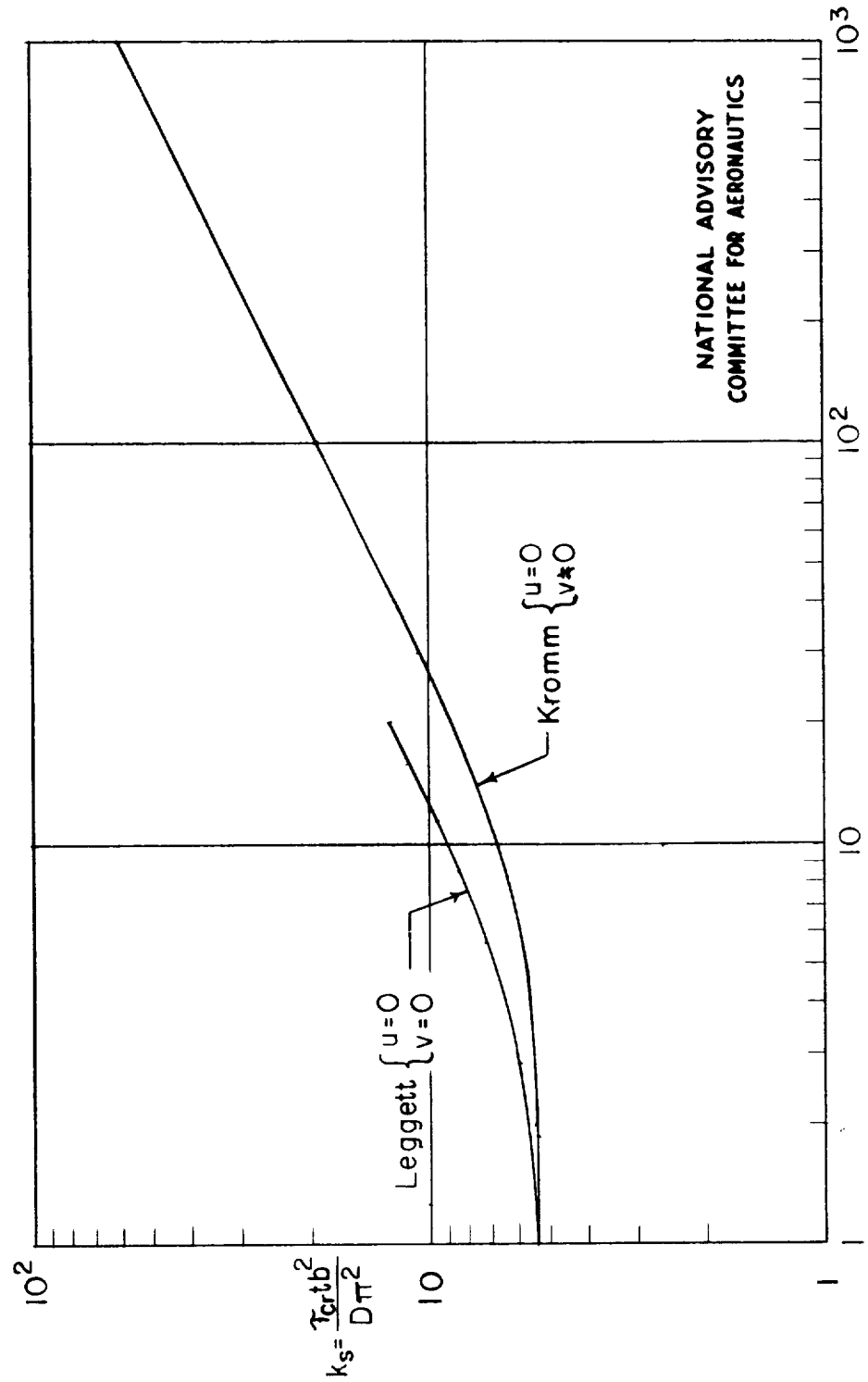

Figure 11.- Comparison of theoretical solutions for the buckling under shearing stresses of a curved strip infinitely long in the axial direction. (Leggett's solution is from reference 8 and Kromm's solution is from reference 7.)

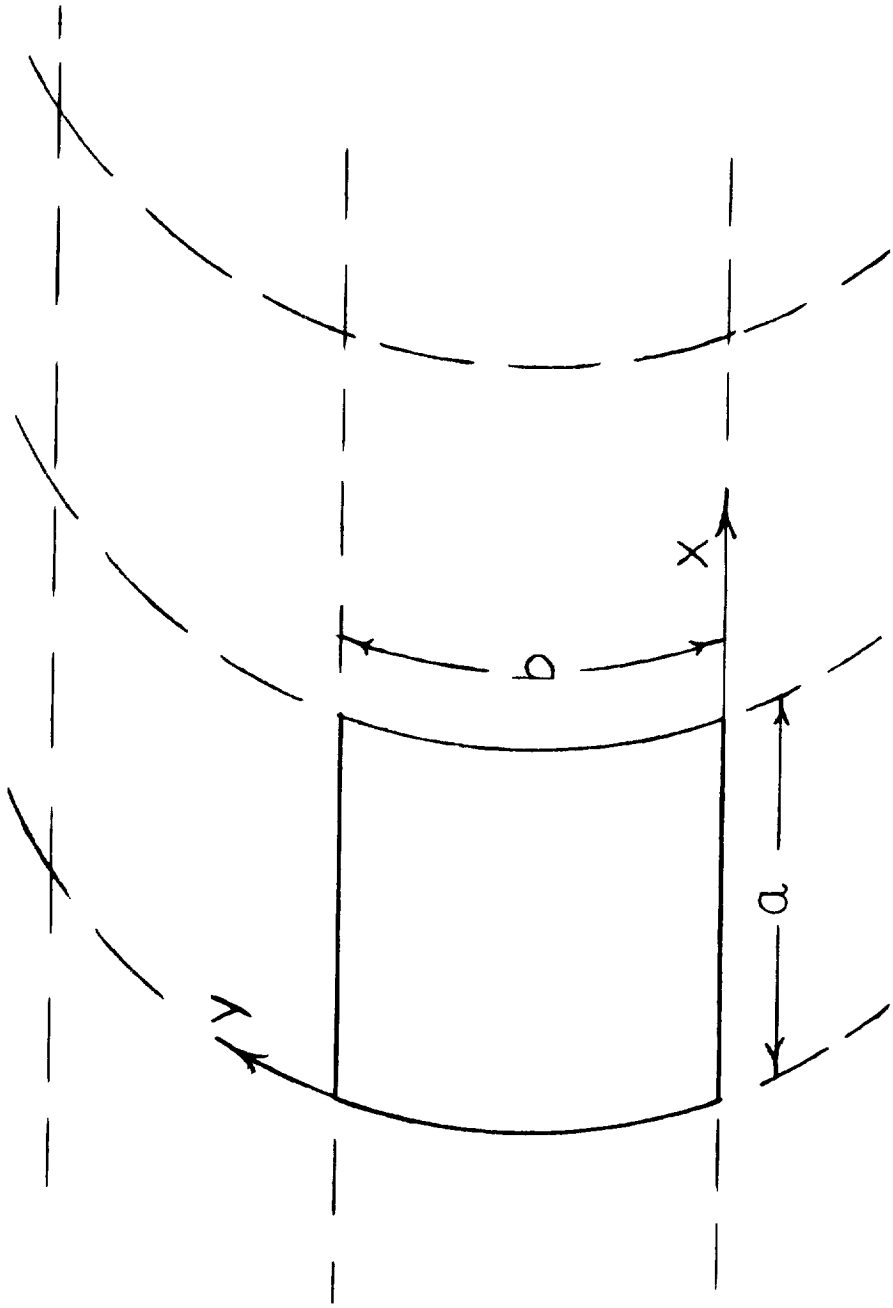

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Figure 12.- Two-dimensional array of identical curved panels.