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TECHNICAL NOTE

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LAMINAR FLOW OF A SLIGHTLY VISCOUS INCOMPRESSIBLE FLUID THAT ISSUES FROM A SLIT AND PASSES OVER A FLAT PLATE

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Washington
June 1948

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LAMINAR FLOW OF A SLIGHTLY VISCOUS INCOMPRESSIBLE FLUID THAT ISSUES FROM A SLIT AND PASSES OVER A FLAT PLATE

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SUMMARY

The laminar flow of a slightly viscous incompressible fluid that issues from a slit and passes over a flat plate is investigated in a region far enough from the slit for the boundary-layer equations to be valid. By assuming similar velocity distributions along the plate, the partial differential equation for the boundary layer is reduced to a third-order nonlinear ordinary differential equation. This equation is integrated by numerical means for the required boundary conditions. The solution gives the velocities at points in the fluid and the surface friction at points on the plate.

Some of the specific results obtained are that the velocities parallel to the plate vary inversely as the 1/2 power of the distance from the slit, that the velocities perpendicular to the plate vary inversely as the 3/4 power of the distance from the slit, that the width of the disturbed region increases as the 3/4 power of the distance from the slit, that the surface friction is independent of the viscosity of the fluid and varies inversely as the 5/4 power of the distance from the slit, that the rate at which momentum parallel to the plate passes through a plane normal to the plate varies inversely as the 1/4 power of the distance from the slit to the normal plane, and that the quantity of fluid passing through a plane normal to the plate per unit time varies directly as the 1/4 power of the distance from the slit to the normal plane.

INTRODUCTION

The laminar flow of an infinite uniform stream of incompressible fluid of small viscosity over a flat plate at zero angle of attack was first investigated analytically by Prandtl (reference 1). By a proper choice of variables the partial differential equation of the Prandtl boundary layer was reduced to a nonlinear ordinary differential equation of the third order. An accurate solution of the equation that gave the skin-friction coefficient for the plate and the velocities in the field of flow was first obtained by Blasius (reference 2) and later by others. (For example, see references 3 to 6.) The numerical values obtained by the different investigators, all of whom used either series or numerical methods to integrate the nonlinear differential equation, were about the same.

The two-dimensional laminar flow of a slightly viscous incompressible fluid from a slit into an infinite region of still fluid was investigated analytically by Schlichting (reference 7) and by Bickley (reference 8). By introducing new variables into the Prandtl boundary-layer equation and by using the fact that the rate of flow of momentum through a cross section of the jet is independent of the distance from the slit, the partial differential equation of the boundary layer was reduced to a third-order nonlinear ordinary differential equation. Schlichting integrated the equation by the method of series and Bickley later integrated the equation in a closed form. The solutions gave the velocities at points in the field of flow. In addition to the solutions for the flow over a flat plate and for the flow from a slit, reference 9 also gives the few other known solutions of the boundary-layer equations. The solutions are given for the flows near the stagnation points of plane bodies and of bodies of revolution, for the flow along a wall in a converging channel, and for the flow in a round jet from which the fluid issues from a small hole in a wall. Any additional solution of the boundary-layer equations is then, aside from any practical application, of some importance in boundary-layer theory.

The purpose of the present work is to investigate by means of the Prandtl boundary-layer equations the laminar flow of a slightly viscous incompressible fluid that issues from a slit and passes over a flat plate. The arrangement (fig. 1) may also be interpreted as the laminar flow into a very large container of still fluid from a slit at the intersection of two of the walls. By use of the Prandtl boundary-layer equations together with the momentum theorem, a substitution involving other variables was found for the variables that appear in the Prandtl boundary-layer equation. The substitutions reduced the boundary-layer equation from a partial differential equation to a nonlinear ordinary differential equation of the third order. The equation was integrated numerically for the required boundary conditions with the aid of the general purpose computing system of the Bell Telephone Laboratories in the Langley Bell computing section.

The reduction of the boundary-layer partial differential equation to an ordinary differential equation in the cases of the flat plate, the jet, and the combined flat plate and jet flow is made by finding suitable substitutions for the variables that appear in the Prandtl boundary-layer equations. By making these substitutions the velocity distributions in planes normal to the direction of the main flow are assumed to be similar. The success of the substitutions in reducing the partial differential equation of the boundary layer to an ordinary differential equation in these special cases means that the assumption of similar profiles is compatible with the equations describing the motion and that the similar profiles exist wherever the assumptions made in deriving the equations that describe the motion are valid.

The fact that the profiles are the similar profiles predicted by the laminar-boundary-layer theory has been verified experimentally both for the flow of a uniform infinite stream over a flat plate (reference 10) and for the flow in a round jet (reference 11). The experimental investigation of the flow from the jet verified the analysis (reference 9) for all regions except in the immediate neighborhood of the orifice. The application of the Prandtl boundary-layer equations to the present case, an application that implies assumptions similar to those made in analyzing the flow of a uniform stream over a flat plate and in analyzing the flow from a round orifice, is therefore expected to lead to results that are valid for regions not too close to the slit.

SYMBOLS

x	nondimensional distance parallel to surface of plate $(\bar{x}/\bar{a})$
$\bar{x}$	distance parallel to surface of plate
y	nondimensional distance normal to surface of plate $(\bar{y}/\bar{a})$
$\bar{y}$	distance normal to surface of plate
u	nondimensional velocity component in x-direction $\left(\bar{u}/\bar{U}_a\right)$
$\bar{u}$	velocity component in x-direction
v	nondimensional velocity component in y-direction $\left(\bar{v}/\bar{U}_a\right)$
$\bar{v}$	velocity component in y-direction
ρ	density
μ	coefficient of viscosity
ν	kinematic viscosity $\left(\mu/\rho\right)$
$\bar{U}_a$	reference velocity
$\bar{a}$	reference length
R_a	reference Reynolds number $\left(\bar{U}_a\bar{a}/\nu\right)$
f	function of η
$\eta = \frac{y}{\beta}$	
β	function of x

ϕ function of x

$$K = \frac{1}{R_a} \frac{\left(\frac{df}{d\eta}\right)_0}{\int_0^\infty f^2 d\eta} = \left(\frac{d^2G}{d\xi^2}\right)_0$$

p static pressure

C constant

C_1, C_2 constants of integration

M momentum in x-direction

M_a reference momentum in x-direction, one side of plate $\left(\rho \bar{U}_a{}^2 \bar{a}\right)$

$$F = \int_0^\eta f \, d\eta$$

$$\xi = \frac{\alpha\eta}{3}$$

ξ_L lower limit of ξ in $\int_{\xi_L}^\infty \left(\frac{dG}{d\xi}\right)^2 d\xi$

$$G = \frac{F}{\gamma}$$

$$\alpha = 9^{1/3} R_a{}^{2/3}$$

$$\gamma = \frac{9^{1/3}}{R_a{}^{1/3}}$$

τ_0 surface friction $\left(\mu\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)_0\right)$

Q_∞ quantity of flow, one side of plate

u_{max} maximum velocity at a section

y_{max} distance from surface of plate to point of maximum velocity u_{max}

Subscripts:

0 conditions at surface of plate

∞ conditions at infinity

ANALYSIS

After assuming that the flow is incompressible, that no static-pressure gradient exists in the direction of x-axis (fig. 1), and that the usual assumptions of the boundary-layer theory (reference 9) apply, the equations describing the motion are: The boundary-layer equation of motion with $\frac{\partial p}{\partial \overline{x}} = 0$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{1}$$

the equation of continuity for incompressible flow

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{2}$$

and the momentum theorem applied to the x-component of the flow

$$\frac{d \int_0^\infty \rho \bar{u}^2 d\bar{y}}{d\bar{x}} = -\mu \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_0 \tag{3}$$

In order to make the equations nondimensional, the velocities are divided by a reference velocity and the lengths by a reference length. The nondimensional velocities and lengths are

$$u = \frac{\bar{u}}{\bar{U}_a}$$

$$v = \frac{\bar{v}}{\bar{U}_a}$$

$$x = \frac{\bar{x}}{\bar{a}}$$

and

$$y = \frac{\bar{y}}{\bar{a}}$$

Now equation (3) becomes

$$\frac{d\int_0^\infty u^2\bar{U}_a{}^2\bar{a}\ dy}{\bar{a}\ dx} = -\nu\frac{\bar{U}_a}{\bar{a}}\left(\frac{\partial u}{\partial y}\right)_0$$

or

$$\frac{d\int_0^\infty u^2 dy}{dx} = -\frac{1}{R_a}\left(\frac{\partial u}{\partial y}\right)_0 \tag{4}$$

where

$$R_a = \frac{\bar{a}\bar{U}_a}{\nu}$$

The velocity distributions in planes perpendicular to the x-axis are now assumed to be similar to one another. Velocity distributions are said to be similar if all the velocity distributions are given by one curve when u/u_{max} is plotted against y/y_{max}. In order to make the assumption of similar profiles, the independent variables y and x are replaced by new independent variables η and x by means of the following substitutions (reference 12):

$$u = \frac{f}{\phi}$$

where $f = f(\eta)$ and $\phi = \phi(x)$, and

$$\eta = \frac{y}{\beta}$$

where $\beta = \beta(x)$. The expressions for u/u_{max} and for y/y_{max} for a fixed value of x then are

$$\frac{u}{u_{max}} = \frac{f}{(f)_{u=u_{max}}}$$

and

$$\frac{y}{y_{max}} = \frac{\eta}{(\eta)_{u=u_{max}}}$$

All the velocity distributions in planes perpendicular to the x-axis are therefore given by one curve of u/u_{max} plotted against y/y_{max} and consequently the foregoing substitutions for y and x mean that similar profiles have been assumed. Whether the assumption of similar profiles is compatible with equations (1) to (3) which describe the motion is decided by whether these equations reduce to ordinary differential equations.

Determination of Relations for ϕ, β, and u

The functions ϕ and β are to be determined by using equations (1) to (3). If the values

$$u = \frac{f}{\phi}$$

and

$$\frac{\partial u}{\partial y} = \frac{1}{\phi\beta}\frac{df}{d\eta}$$

are substituted in equation (4), the result is

$$\frac{d\dfrac{\beta}{\phi^2}\displaystyle\int_0^\infty f^2 d\eta}{dx} = -\frac{1}{R_a}\frac{1}{\phi\beta}\left(\frac{df}{d\eta}\right)_0$$

or, since $\displaystyle\int_0^\infty f^2 d\eta$ is independent of x

$$\frac{1}{\beta}\frac{d\beta}{dx} - \frac{2}{\phi}\frac{d\phi}{dx} = -\frac{1}{R_a}\frac{\left(\dfrac{df}{d\eta}\right)_0}{\displaystyle\int_0^\infty f^2 d\eta}\frac{\phi}{\beta^2}$$

Now let

$$\frac{1}{R_a}\frac{\left(\dfrac{df}{d\eta}\right)_0}{\displaystyle\int_0^\infty f^2 d\eta} = K$$

then

$$\frac{1}{\beta}\frac{d\beta}{dx} - \frac{2}{\phi}\frac{d\phi}{dx} = -\frac{K\phi}{\beta^2} \tag{5}$$

Equation (5) can be solved by using equation (1) to provide a relation, which does not involve x, between ϕ and β. After equation (1) is made nondimensional, the result is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R_a}\frac{\partial^2 u}{\partial y^2} \tag{6}$$

Equation (6) is to be rewritten in terms of f, ϕ, β, and η. From the relation

$$u = \frac{f}{\phi}$$

it follows that

$$\frac{\partial u}{\partial y} = \frac{1}{\phi\beta} \frac{df}{d\eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\phi\beta^2} \frac{d^2 f}{d\eta^2}$$

and

$$\frac{\partial u}{\partial x} = -\left(\frac{\eta \frac{df}{d\eta} \frac{d\beta}{dx}}{\beta\phi} + \frac{f}{\phi^2} \frac{d\phi}{dx}\right)$$

From equation (2)

$$v = \int_0^y \frac{\partial v}{\partial y} dy = -\int_0^y \frac{\partial u}{\partial x} dy = -\int_0^\eta \frac{\partial u}{\partial x}\beta \, d\eta$$

where

$$dy = \beta \, d\eta$$

or

$$v = \frac{\frac{d\beta}{dx}}{\phi} \int_0^\eta \eta \frac{df}{d\eta} d\eta + \frac{\beta \frac{d\phi}{dx}}{\phi^2} \int_0^\eta f \, d\eta$$

By partial integration

$$\int_0^\eta \eta \frac{df}{d\eta} d\eta = f\eta - \int_0^\eta f \, d\eta$$

then, the result for v is

$$v = \frac{f\eta}{\phi} \frac{d\beta}{dx} + \int_0^\eta f \, d\eta \left(\frac{\beta}{\phi^2} \frac{d\phi}{dx} - \frac{1}{\phi} \frac{d\beta}{dx}\right)$$

If values for u, v, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial^2 u}{\partial y^2}$ are substituted in equation (6), the result is

$$-\frac{f}{\phi}\left(\frac{\eta \frac{df}{d\eta}}{\beta\phi} \frac{d\beta}{dx} + \frac{f}{\phi^2} \frac{d\phi}{dx}\right) + \left[\frac{f\eta}{\phi} \frac{d\beta}{dx} + \int_0^\eta f \, d\eta \left(\frac{\beta}{\phi^2} \frac{d\phi}{dx} - \frac{1}{\phi} \frac{d\beta}{dx}\right)\right] \frac{1}{\phi\beta} \frac{df}{d\eta} = \frac{1}{R_a} \frac{1}{\phi\beta^2} \frac{d^2f}{d\eta^2} \qquad (7)$$

The left-hand and right-hand sides of equation (7) are multiplied by ϕ^2, and the factor ϕ/β^2 which then appears on the right-hand side of the equation is replaced by the value of ϕ/β^2 from equation (5). After further simplification, equation (7) can be written as

$$\frac{\frac{1}{\phi} \frac{d\phi}{dx}}{\frac{1}{\beta} \frac{d\beta}{dx}} = \frac{\frac{df}{d\eta} \int_0^\eta f \, d\eta - \frac{1}{R_a K} \frac{d^2f}{d\eta^2}}{-f^2 + \frac{df}{d\eta} \int_0^\eta f \, d\eta - \frac{2}{R_a K} \frac{d^2f}{d\eta^2}} \qquad (8)$$

The variables x and η are independent. Consequently, a change in x does not change the right-hand side of equation (8) and therefore does not change the left-hand side either. A change in η likewise does not change the left-hand side of equation (8) and thus does not change the right-hand side either. Both the left-hand and right-hand sides of equation (8) are therefore equal to a constant C. Then

$$\frac{\dfrac{1}{\phi}\dfrac{d\phi}{dx}}{\dfrac{1}{\beta}\dfrac{d\beta}{dx}} = C$$

or

$$\frac{d\phi}{\phi} = C\,\frac{d\beta}{\beta}$$

or, by integrating

$$\phi = C_1\beta^C \tag{9}$$

When equation (9) is substituted in equation (5), the result is

$$\beta^{1-C}\,\frac{d\beta}{dx} = -\,\frac{KC_1}{1 - 2C}$$

or, after integrating $\left(\text{for}\quad C \neq \frac{1}{2}\quad \text{and}\quad C \neq 2\right)$

$$\beta = \left[\left(\frac{KC_1}{2C - 1}x + C_2\right)(2 - C)\right]^{\frac{1}{2-C}}$$

and

$$\phi = C_1\left[\left(\frac{KC_1}{2C - 1}x + C_2\right)(2 - C)\right]^{\frac{C}{2-C}}$$

Then

$$u = \frac{1}{C_1\left[\left(\dfrac{KC}{2C - 1}x + C_2\right)(2 - C)\right]^{\frac{C}{2-C}}}\, f\left\{\frac{y}{\left[\left(\dfrac{KC_1}{2C - 1}x + C_2\right)(2 - C)\right]^{\frac{1}{2-C}}}\right\}$$

The expression for u contains two arbitrary constants C_1 and C_2. The constant C_1 is a scale factor; the constant C_2 is a translation factor that determines the location of the slit along the x-axis. Therefore, the constant C_1 can be made unity and the constant C_2 can be made zero without affecting the generality of the solution in any way other than fixing the origin of the slit at $x = 0$. The expressions for β, ϕ, and u therefore become

$$\beta = \left[\frac{Kx}{2C - 1}(2 - C)\right]^{\frac{1}{2-C}}$$

$$\phi = \left[\frac{Kx}{2C - 1}(2 - C)\right]^{\frac{C}{2-C}}$$

and

$$u = \frac{1}{\left[\frac{Kx}{2C - 1}(2 - C)\right]^{\frac{C}{2-C}}} f\left\{\frac{y}{\left[\frac{Kx}{2C - 1}(2 - C)\right]^{\frac{1}{2-C}}}\right\}$$

The constant C can be evaluated by use of the expression for the momentum in the x-direction:

$$M = \rho \int_0^\infty \bar{u}^2 d\bar{y}$$

or

$$\frac{M}{\rho \bar{U}_a{}^2 \bar{a}} = \int_0^\infty u^2 dy = \frac{\beta}{\phi^2} \int_0^\infty f^2 d\eta$$

or

$$\frac{M}{\rho\bar{U}_a{}^2\bar{a}} = \left[\frac{Kx}{2C - 1}(2 - C)\right]^{\frac{1-2C}{2-C}} \int_0^\infty f^2 d\eta \qquad (10)$$

The arbitrary length $\bar{a}$ is interpreted as the distance along the x-axis from the origin to the point where the similar profile has been established. At this distance $\bar{a}$ from the origin the rate of momentum flow is equated to $\rho\bar{U}_a{}^2\bar{a}$; thus, the reference velocity $\bar{U}_a$ is defined. When $\bar{x} = \bar{a}$ or $x = 1$, $M = M_a$ and, consequently, equation (10) at $\bar{x} = \bar{a}$ becomes

$$1 = \left[\frac{K}{2C - 1}(2 - C)\right]^{\frac{1-2C}{2-C}} \int_0^\infty f^2 d\eta \qquad (11)$$

The expression $\int_0^\infty f^2 d\eta$ is a constant.

Let

$$\int_0^\infty f^2 d\eta = 1$$

The function f to be found must satisfy this condition. Then equation (11) becomes

$$1 = \left[\frac{K}{2C - 1}(2 - C)\right]^{\frac{1-2C}{2-C}}$$

Therefore

$$C = \frac{2K + 1}{K + 2}$$

Then

$$\beta = x^{\frac{K+2}{3}} \tag{12}$$

$$\phi = x^{\frac{2K+1}{3}} \tag{13}$$

and

$$u = \frac{1}{x^{\frac{2K+1}{3}}} f\left(\frac{y}{x^{\frac{K+2}{3}}}\right) \tag{14}$$

where now

$$K = \frac{\left(\frac{df}{d\eta}\right)_0}{R_a} \tag{15}$$

and

$$x \geqq 1$$

It can now be shown that the requirement that $C \neq 2$ and that $C \neq \frac{1}{2}$, which was stipulated in determining β as a function of x by using equations (5) and (9), is satisfied. The expression that was previously obtained for C is

$$C = \frac{2K + 1}{K + 2}$$

From physical considerations $0 < \left(\frac{df}{d\eta}\right)_0 < \infty$ and $0 < R_a < \infty$. Therefore, $0 < K < \infty$. Thus, because $K > 0$, the expression for C gives

$$C > \frac{1}{2}$$

Because $K < \infty$,

$$C < 2$$

Therefore,

$$C \neq 2$$

and

$$C \neq \frac{1}{2}$$

In order to show that the flow appears to come from a slit, the edge of the region in which the velocity u is other than zero can be arbitrarily defined by choosing a fixed value of η that corresponds to a small fixed value of u. Thus, let the edge of the jet region be given by

$$\eta = \eta_{edge} = \text{Constant}$$

Then

$$\eta_{edge} = \frac{y_{edge}}{x^{\frac{K+2}{3}}} = \text{Constant}$$

or

$$y_{edge} = x^{\frac{K+2}{3}} \times \text{Constant}$$

The flow therefore does appear to come from a slit located at $(x = 0, y = 0)$. The slit can be placed at a position of x that is not zero by choosing a value for C_2 that is not zero. When the flat plate is removed, $\left(\frac{df}{d\eta}\right)_0 = 0$ or $K = 0$; and then from equation (14)

$$u = \frac{1}{x^{1/3}} f\left(\frac{y}{x^{2/3}}\right) \tag{16}$$

which is similar to the relation for $\bar{u}$ in reference 8.

Derivation of the Differential Equation for G

The first step in determining the differential equation for G is to substitute equations (12) and (13) in equation (7). The equation of motion, therefore, becomes

$$-\frac{2K + 1}{3}f^2 + \frac{df}{d\eta}\int_0^\eta f\ d\eta\ \frac{K - 1}{3} = \frac{1}{R_a}\frac{d^2f}{d\eta^2} \tag{17}$$

The assumption of similar profiles has reduced the partial differential equation of the boundary layer to an ordinary differential equation; thus, an indication is given that the assumption of similar profiles is consistent with equations (1) to (3) which describe the motion. Now, let

$$\int_0^\eta f\ d\eta = F$$

where $F = F(\eta)$. Then, because

$$f = \frac{dF}{d\eta}$$

$$\frac{df}{d\eta} = \frac{d^2F}{d\eta^2}$$

and

$$\frac{d^2f}{d\eta^2} = \frac{d^3F}{d\eta^3}$$

equation (17) becomes

$$\frac{1}{R_a}\frac{d^3F}{d\eta^3} + \frac{1 - K}{3} F \frac{d^2F}{d\eta^2} + \frac{2K + 1}{3}\left(\frac{dF}{d\eta}\right)^2 = 0 \qquad (18)$$

Now, let

$$\xi = \frac{\alpha\eta}{3}$$

$$\gamma = \frac{\alpha}{R_a}$$

and

$$F = \gamma G$$

where $G = G(\xi)$. Thus,

$$\frac{dF}{d\eta} = \frac{\alpha\gamma}{3}\frac{dG}{d\xi}$$

$$\frac{d^2F}{d\eta^2} = \frac{\alpha^2\gamma}{9}\frac{d^2G}{d\xi^2}$$

and

$$\frac{d^3F}{d\eta^3} = \frac{\alpha^3\gamma}{27}\frac{d^3G}{d\xi^3}$$

Equation (18) then becomes

$$\frac{d^3G}{d\xi^3} + (1 - K)G\frac{d^2G}{d\xi^2} + (1 + 2K)\left(\frac{dG}{d\xi}\right)^2 = 0 \qquad (19)$$

where

$$K = \frac{\left(\frac{df}{d\eta}\right)_0}{R_a} = \frac{\alpha^2\gamma}{9R_a}\left(\frac{d^2G}{d\xi^2}\right)_0 = \frac{\alpha^3}{9R_a^{\,2}}\left(\frac{d^2G}{d\xi^2}\right)_0$$

Now, let

$$\alpha = 9^{1/3}R_a^{\,2/3}$$

Then, equation (19) becomes

$$\frac{d^3G}{d\xi^3} + \left[1 - \left(\frac{d^2G}{d\xi^2}\right)_0\right] G\,\frac{d^2G}{d\xi^2} + \left[1 + 2\left(\frac{d^2G}{d\xi^2}\right)_0\right]\left(\frac{dG}{d\xi}\right)^2 = 0 \qquad (20)$$

The conditions to be satisfied by G are, at $\xi = 0$,

$$G = 0$$

$$\left(\frac{dG}{d\xi}\right)_0 = 0$$

and, at $\xi = \infty$,

$$G = \text{Constant}$$

Integration of the Differential Equation for G

The differential equation for G, equation (20), is integrated numerically by use of the method of reference 13. The numerical integration was started by computing the values of G, $\frac{dG}{d\xi}$, $\frac{d^2G}{d\xi^2}$, and $\frac{d^3G}{d\xi^3}$ for the first five intervals of ξ by means of a Taylor's series. The Taylor's series that is valid for small values of ξ is

$$G = \frac{\xi^2}{2!}\left(\frac{d^2G}{d\xi^2}\right)_0 + \frac{\xi^5}{5!}\left(\frac{d^5G}{d\xi^5}\right)_0 + \frac{\xi^8}{8!}\left(\frac{d^8G}{d\xi^8}\right)_0 + \frac{\xi^{11}}{11!}\left(\frac{d^{11}G}{d\xi^{11}}\right)_0 + \frac{\xi^{14}}{14!}\left(\frac{d^{14}G}{d\xi^{14}}\right)_0 + \frac{\xi^{17}}{17!}\left(\frac{d^{17}G}{d\xi^{17}}\right)_0$$

$$\frac{dG}{d\xi} = \xi\left(\frac{d^2G}{d\xi^2}\right)_0 + \frac{\xi^4}{4!}\left(\frac{d^5G}{d\xi^5}\right)_0 + \frac{\xi^7}{7!}\left(\frac{d^8G}{d\xi^8}\right)_0 + \frac{\xi^{10}}{10!}\left(\frac{d^{11}G}{d\xi^{11}}\right)_0 + \frac{\xi^{13}}{13!}\left(\frac{d^{14}G}{d\xi^{14}}\right)_0 + \frac{\xi^{16}}{16!}\left(\frac{d^{17}G}{d\xi^{17}}\right)_0$$

$$\frac{d^2G}{d\xi^2} = \left(\frac{d^2G}{d\xi^2}\right)_0 + \frac{\xi^3}{3!}\left(\frac{d^5G}{d\xi^5}\right)_0 + \frac{\xi^6}{6!}\left(\frac{d^8G}{d\xi^8}\right)_0 + \frac{\xi^9}{9!}\left(\frac{d^{11}G}{d\xi^{11}}\right)_0 + \frac{\xi^{12}}{12!}\left(\frac{d^{14}G}{d\xi^{14}}\right)_0 + \frac{\xi^{15}}{15!}\left(\frac{d^{17}G}{d\xi^{17}}\right)_0$$

and

$$\frac{d^3G}{d\xi^3} = \frac{\xi^2}{2!}\left(\frac{d^5G}{d\xi^5}\right)_0 + \frac{\xi^5}{5!}\left(\frac{d^8G}{d\xi^8}\right)_0 + \frac{\xi^8}{8!}\left(\frac{d^{11}G}{d\xi^{11}}\right)_0 + \frac{\xi^{11}}{11!}\left(\frac{d^{14}G}{d\xi^{14}}\right)_0 + \frac{\xi^{14}}{14!}\left(\frac{d^{17}G}{d\xi^{17}}\right)_0$$

where

$$\left(\frac{d^5G}{d\xi^5}\right)_0 = -3\left(\frac{d^2G}{d\xi^2}\right)_0^2\left[1 + \left(\frac{d^2G}{d\xi^2}\right)_0\right]$$

$$\left(\frac{d^8G}{d\xi^8}\right)_0 = 9\left(\frac{d^2G}{d\xi^2}\right)_0^3\left[1 + \left(\frac{d^2G}{d\xi^2}\right)_0\right]\left[7 + 3\left(\frac{d^2G}{d\xi^2}\right)_0\right]$$

$$\left(\frac{d^{11}G}{d\xi^{11}}\right)_0 = -\left\{\left(\frac{d^2G}{d\xi^2}\right)_0 \left(\frac{d^8G}{d\xi^8}\right)_0 \left[45 + 3\left(\frac{d^2G}{d\xi^2}\right)_0\right] + \left(\frac{d^5G}{d\xi^5}\right)_0^2 \left[126 + 84\left(\frac{d^2G}{d\xi^2}\right)_0\right]\right\}$$

$$\left(\frac{d^{14}G}{d\xi^{14}}\right)_0 = -\left\{\left(\frac{d^2G}{d\xi^2}\right)_0 \left(\frac{d^{11}G}{d\xi^{11}}\right)_0 \left[78 - 12\left(\frac{d^2G}{d\xi^2}\right)_0\right] + \left(\frac{d^5G}{d\xi^5}\right)_0 \left(\frac{d^8G}{d\xi^8}\right)_0 \left[1287 + 693\left(\frac{d^2G}{d\xi^2}\right)_0\right]\right\}$$

and

$$\left(\frac{d^{17}G}{d\xi^{17}}\right)_0 = -\left\{\left(\frac{d^2G}{d\xi^2}\right)_0 \left(\frac{d^{14}G}{d\xi^{14}}\right)_0 \left[120 - 36\left(\frac{d^2G}{d\xi^2}\right)_0\right] + \left(\frac{d^5G}{d\xi^5}\right)_0 \left(\frac{d^{11}G}{d\xi^{11}}\right)_0 \left[4368 + 1638\left(\frac{d^2G}{d\xi^2}\right)_0\right] + \left(\frac{d^8G}{d\xi^8}\right)_0^2 \left[6435 + 3861\left(\frac{d^2G}{d\xi^2}\right)_0\right]\right\}$$

The value of $\left(\frac{d^2G}{d\xi^2}\right)_0 = K$, which appears in the Taylor's series as the only unknown, can be determined from equation (20). When equation (20) is multiplied by G, the equation becomes

$$G\frac{d^3G}{d\xi^3} + \left[1 - \left(\frac{d^2G}{d\xi^2}\right)_0\right]G^2\frac{d^2G}{d\xi^2} + \left[1 + 2\left(\frac{d^2G}{d\xi^2}\right)_0\right]G\left(\frac{dG}{d\xi}\right)^2 = 0$$

From the nature of the problem each term of the equation is continuous in the interval from 0 to ∞; the equation may therefore be integrated directly term by term. Then

$$\int_0^\infty G \frac{d^3G}{d\xi^3} d\xi + \left[1 - \left(\frac{d^2G}{d\xi^2}\right)_0\right] \int_0^\infty G^2 \frac{d^2G}{d\xi^2} d\xi + \left[1 + 2\left(\frac{d^2G}{d\xi^2}\right)_0\right] \int_0^\infty G\left(\frac{dG}{d\xi}\right)^2 d\xi = 0 \quad (21)$$

Integrating $\int_0^\infty G \frac{d^3G}{d\xi^3} d\xi$ by parts gives

$$\int_0^\infty G \frac{d^3G}{d\xi^3} d\xi = G \frac{d^2G}{d\xi^2}\Bigg]_0^\infty - \int_0^\infty \frac{d^2G}{d\xi^2} \frac{dG}{d\xi} d\xi = -\int_0^\infty \frac{d^2G}{d\xi^2} \frac{dG}{d\xi} d\xi$$

Integrating $\int_0^\infty \frac{d^2G}{d\xi^2} \frac{dG}{d\xi} d\xi$ by parts gives

$$\int_0^\infty \frac{d^2G}{d\xi^2} \frac{dG}{d\xi} d\xi = \left(\frac{dG}{d\xi}\right)^2\Bigg]_0^\infty - \int_0^\infty \frac{dG}{d\xi} \frac{d^2G}{d\xi^2} d\xi = -\int_0^\infty \frac{d^2G}{d\xi^2} \frac{dG}{d\xi} d\xi$$

Therefore

$$\int_0^\infty \frac{d^2G}{d\xi^2} \frac{dG}{d\xi} d\xi = 0$$

and thus

$$\int_0^\infty G \frac{d^3G}{d\xi^3} d\xi = 0$$

Now, integrating $\int_0^\infty G^2 \frac{d^2G}{d\xi^2}\, d\xi$ by parts gives

$$\int_0^\infty G^2 \frac{d^2G}{d\xi^2}\, d\xi = G^2 \frac{dG}{d\xi}\Bigg]_0^\infty - \int_0^\infty \frac{dG}{d\xi}\, 2G\, \frac{dG}{d\xi}\, d\xi = -\int_0^\infty 2G \left(\frac{dG}{d\xi}\right)^2 d\xi$$

Then equation (21) becomes

$$-2\left[1 - \left(\frac{d^2G}{d\xi^2}\right)_0\right] \int_0^\infty G \left(\frac{dG}{d\xi}\right)^2 d\xi + \left[1 + 2\left(\frac{d^2G}{d\xi^2}\right)_0\right] \int_0^\infty G \left(\frac{dG}{d\xi}\right)^2 d\xi = 0$$

or

$$\left[4\left(\frac{d^2G}{d\xi^2}\right)_0 - 1\right] \int_0^\infty G \left(\frac{dG}{d\xi}\right)^2 d\xi = 0 \qquad (22)$$

The quantity G is always positive, except on the plate where it is zero; therefore, equation (22) cannot be true unless $\left(\frac{d^2G}{d\xi^2}\right)_0 = \frac{1}{4}$. Consequently, $\left(\frac{d^2G}{d\xi^2}\right)_0 = \frac{1}{4}$ is taken as the starting value of $\frac{d^2G}{d\xi^2}$ in the numerical integration of equation (20).

The equation was integrated by use of the "fourth approximation" of reference 13 for an increment in ξ, $\Delta\xi = 0.10$; the numerical integration began at $\xi = 0.50$. In order to determine whether the increment in ξ, $\Delta\xi = 0.10$, was sufficiently small, the equation was also integrated for an increment in ξ, $\Delta\xi = 0.05$; the numerical integration began at $\xi = 0.50$. Some of the results from the integration with $\Delta\xi = 0.05$, the more accurate integration, are given in table I. A comparison of some of the values of G and its derivatives obtained by integrating with $\Delta\xi = 0.05$ and with $\Delta\xi = 0.10$ is given in table II; the values at $\xi = 0.60$ computed by the Taylor's series are also included for comparison with those computed by Falkner's method (reference 13).

The results of the two integrations agree to five significant figures up to $\xi = 8.50$ at which value $\frac{dG}{d\xi}$ is less than 1 percent of its maximum value. From $\xi = 8.50$ to $\xi = 13.10$, at which value the computation for $\Delta\xi = 0.10$ was stopped, the agreement between the computations for the two different increments in ξ becomes poorer. The values of G, however, are the same to five significant figures for both computations. At $\xi = 13.00$, G has already attained its asymptotic value, a value that is necessary for the computation of v at $\xi = \infty$. (See table I.)

As a further check on the accuracy of the numerical integration, the following condition was used:

$$\int_0^\infty f^2 d\eta = 1$$

which is the same as

$$\int_0^\infty \left(\frac{dG}{d\xi}\right)^2 d\xi = \frac{1}{3} \tag{23}$$

The values of $\int_0^\xi \left(\frac{dG}{d\xi}\right)^2 d\xi$ obtained by using Weddle's rule (reference 14) with $\Delta\xi = 0.05$ are given in table III. It is seen that the foregoing condition (equation (23)) is confirmed to five significant figures. In order to determine whether the integral $\int_0^\infty \left(\frac{dG}{d\xi}\right)^2 d\xi$ is approximated with sufficient accuracy by the integral $\int_0^{12.30} \left(\frac{dG}{d\xi}\right)^2 d\xi$, the assumption was made that for large values of ξ the curve of $\frac{dG}{d\xi}$ plotted against ξ could be approximated by the function

$$\frac{dG}{d\xi} = Ae^{-B\xi}$$

The constants A and B were evaluated in the range of ξ from $\xi = 9.45$ to $\xi = 14.95$. The constant B was observed to increase slowly. For

$$\frac{dG}{d\xi} = Ae^{-B\xi}$$

the integral from ξ_L to ∞ is

$$\int_{\xi_L}^{\infty} \left(\frac{dG}{d\xi}\right)^2 d\xi = \frac{\left(\frac{dG}{d\xi}\right)_{\xi_L}^{2}}{2B}$$

$$= \frac{(0.00009145710)^2}{2(0.9452)}$$

$$= 0.4425 \times 10^{-8}$$

where

$$\xi_L = 12.30$$

$$\frac{dG}{d\xi} = 0.00009145710$$

and

$$B = 0.9452$$

Because $\frac{dG}{d\xi}$ decreases as ξ_L increases and because B increases as ξ_L increases, the integral $\int_0^{\infty} \left(\frac{dG}{d\xi}\right)^2 d\xi$ is given accurately to five significant figures by the integral $\int_0^{12.30} \left(\frac{dG}{d\xi}\right)^2 d\xi$.

In order to determine whether the integral $\int_0^\infty \left(\frac{dG}{d\xi}\right) d\xi$ is approximated with sufficient accuracy by the integral $\int_0^{14.95} \left(\frac{dG}{d\xi}\right) d\xi$, the same function for $\frac{dG}{d\xi}$ for large values of ξ was assumed; that is, for

$$\frac{dG}{d\xi} = Ae^{-B\xi}$$

the integral from ξ_L to ∞ is

$$\int_{\xi_L}^{\infty} \frac{dG}{d\xi}\, d\xi = \frac{\left(\frac{dG}{d\xi}\right)_{\xi_L}}{B}$$

$$= \frac{0.000007433791}{0.950682}$$

$$= 0.00000782$$

where

$$\xi_L = 14.95$$

$$\frac{dG}{d\xi} = 0.000007433791$$

and

$$B = 0.950682$$

Because $\frac{dG}{d\xi}$ decreases as ξ_L increases and because B increases as ξ_L increases, adding $\int_{14.95}^{\infty} \frac{dG}{d\xi}\,d\xi$ to the value of G at $\xi = 14.95$ does not change its value to five significant figures.

The last value in table III for $\int_0^{\infty} \left(\frac{dG}{d\xi}\right)^2 d\xi$ and the asymptotic value of G are therefore both unchanged by extension of the curves to $\xi = \infty$. The results of the computation for $\Delta\xi = 0.05$ are consequently taken as correct to five significant figures for $0 \leqq \xi \leqq 8.50$ and the asymptotic value of G is taken as 1.2599. The curve of $\frac{dG}{d\xi}$ plotted against ξ is shown in figure 2.

Determination of Final Expressions

Now that the values of G, $\frac{dG}{d\xi}$, $\frac{d^2G}{d\xi^2}$, and $\frac{d^3G}{d\xi^3}$ have been tabulated, the completion of the solution of the problem requires that expressions for u, $\bar{u}$, v, $\bar{v}$, Q_∞, τ_0, M, and ξ be determined.

In order to obtain the expressions for u and $\bar{u}$, use

$$u = \frac{f}{\phi} = \frac{\frac{dF}{d\eta}}{x^{\frac{2K+1}{3}}} = \frac{\frac{\alpha\gamma}{3}\frac{dG}{d\xi}}{x^{1/2}} = \frac{9^{2/3}}{3}R_a{}^{1/3}\frac{\frac{dG}{d\xi}}{x^{1/2}} \tag{24}$$

The expression for R_a is

$$R_a = \frac{\bar{U}_a\bar{a}}{\nu} = \frac{\bar{a}}{\nu}\sqrt{\frac{M_a}{\rho\bar{a}}} = \sqrt{\frac{M_a\bar{a}}{\rho\nu^2}}$$

therefore, u may now be written as

$$u = \frac{9^{2/3}}{3}\left(\frac{M_a\bar{a}}{\rho\nu^2}\right)^{1/6}\frac{\frac{dG}{d\xi}}{x^{1/2}} \tag{25}$$

The expression for $\bar{u}$ is

$$\bar{u} = u\bar{U}_a = \frac{9^{2/3}}{3}\left(\frac{M_a\bar{a}}{\rho\nu^2}\right)^{1/6} \frac{\bar{a}^{1/2}\,\dfrac{dG}{d\xi}}{\bar{x}^{1/2}} \sqrt{\frac{M_a}{\rho\bar{a}}}$$

or

$$\bar{u} = \frac{9^{2/3}}{3}\left(\frac{M_a{}^2}{\rho^2\nu}\right)^{1/3} \frac{\bar{a}^{1/6}}{\bar{x}^{1/2}} \frac{dG}{d\xi} \tag{26}$$

The $\bar{u}$ component of the velocity therefore varies inversely as the 1/2 power of the distance from the slit.

The expression for v is

$$v = \frac{f\eta}{\phi}\frac{d\beta}{dx} + \int_0^\eta f\,d\eta\left(\frac{\beta}{\phi^2}\frac{d\phi}{dx} - \frac{1}{\phi}\frac{d\beta}{dx}\right)$$

where

$$f = \frac{\alpha\gamma}{3}\frac{dG}{d\xi}$$

$$\eta = \frac{y}{x^{\frac{K+2}{3}}}$$

$$\beta = x^{\frac{K+2}{3}}$$

$$\phi = x^{\frac{2K+1}{3}}$$

$$\int_0^\eta f \, d\eta = \gamma G$$

and

$$\xi = \frac{\alpha\eta}{3}$$

Then

$$v = \frac{\gamma}{x^{\frac{K+2}{3}}}\left(\frac{K+2}{3}\,\xi\,\frac{dG}{d\xi} + \frac{K-1}{3}\,G\right)$$

and, after the appropriate substitutions for γ and K have been made,

$$v = \frac{9^{1/3}}{R_a{}^{1/3}}\,\frac{1}{x^{3/4}}\left(\frac{3}{4}\,\xi\,\frac{dG}{d\xi} - \frac{G}{4}\right)$$

or

$$v = \frac{9^{1/3}}{4}\,\frac{1}{\left(\frac{M_a\bar{a}}{\rho\nu^2}\right)^{1/6}}\,\frac{1}{x^{3/4}}\left(3\xi\,\frac{dG}{d\xi} - G\right) \tag{27}$$

The expression for $\bar{v}$ is

$$\bar{v} = v\bar{U}_a = \frac{9^{1/3}}{4}\,\frac{1}{\left(\frac{M_a\bar{a}}{\rho\nu^2}\right)^{1/6}}\,\frac{\bar{a}^{3/4}}{\bar{x}^{3/4}}\left(3\xi\,\frac{dG}{d\xi} - G\right)\sqrt{\frac{M_a}{\rho\bar{a}}}$$

or

$$\bar{v} = \frac{9^{1/3}}{4}\left(\frac{M_a\nu}{\rho}\right)^{1/3}\frac{\bar{a}^{1/12}}{\bar{x}^{3/4}}\left(3\xi\,\frac{dG}{d\xi} - G\right) \tag{28}$$

The magnitude of the vertical velocity $\overline{v}$ thus varies inversely as the 3/4 power of the distance from the slit. Whether the vertical velocity is away from or toward the plate depends on the sign of the term $3\xi \frac{dG}{d\xi} - G$ in equation (28).

When $\xi \longrightarrow \infty$, it can be shown that $\xi \frac{dG}{d\xi} \longrightarrow 0$ because $G \longrightarrow$ Constant as $\xi \longrightarrow \infty$. Therefore,

$$\overline{v}_\infty = -\frac{9^{1/3}}{4}\left(\frac{M_a \nu}{\rho}\right)^{1/3} \frac{\overline{a}^{1/12}}{\overline{x}^{3/4}} G_\infty$$

or

$$\overline{v}_\infty = -\frac{(1.2599)9^{1/3}}{4}\left(\frac{M_a \nu}{\rho}\right)^{1/3} \frac{\overline{a}^{1/12}}{\overline{x}^{3/4}} \tag{29}$$

The flow from the slit thus induces a velocity towards the plate at sufficiently large distances above or below the plate.

The quantity of fluid passing through an imaginary plane normal to the plate and extending to infinity in one direction is given as

$$Q_\infty = \int_0^\infty \overline{u}\, d\overline{y} = \overline{U}_a \overline{a} \int_0^\infty u\, dy$$

and, after the appropriate substitutions for $\overline{U}_a$, u, and y have been made,

$$Q_\infty = 9^{1/3} x^{1/4} \left(\frac{M_a \overline{a}}{\rho}\right)^{1/3} \nu^{1/3} \int_0^\infty \frac{dG}{d\xi}\, d\xi$$

or, finally,

$$Q_\infty = 9^{1/3} \overline{x}^{1/4} \left(\frac{M_a \nu}{\rho}\right)^{1/3} \overline{a}^{1/12} G_\infty \tag{30}$$

where

$$G_\infty = 1.2599$$

The quantity of fluid passing through a plane normal to the plate per unit time thus varies directly as the 1/4 power of the distance from the slit to the normal plane.

The expression for the surface friction τ_0 is

$$\tau_0 = \mu\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)_0$$

$$= \frac{\mu\bar{U}_a}{\bar{a}}\left(\frac{\partial u}{\partial y}\right)_0$$

$$= \frac{\mu\bar{U}_a}{\bar{a}}\frac{1}{\phi\beta}\left(\frac{df}{d\eta}\right)_0$$

$$= \frac{\mu\bar{U}_a}{\bar{a}}\frac{1}{x^{K+1}}\frac{\alpha^2\gamma}{9}\left(\frac{d^2G}{d\xi^2}\right)_0$$

$$= \frac{\mu\bar{U}_a}{\bar{a}}\frac{\bar{a}^{5/4}}{\bar{x}^{5/4}}R_a\left(\frac{d^2G}{d\xi^2}\right)_0$$

and, after the appropriate substitutions for $\bar{U}_a$, R_a, and $\left(\frac{d^2G}{d\xi^2}\right)_0$ have been made,

$$\tau_0 = \mu\sqrt{\frac{M_a}{\rho\bar{a}}}\frac{\bar{a}^{1/4}}{\bar{x}^{5/4}}\sqrt{\frac{M_a\bar{a}}{\rho\nu^2}}\frac{1}{4}$$

or

$$\tau_o = \mu \frac{M_a}{\nu\rho} \frac{\bar{a}^{1/4}}{\bar{x}^{5/4}} \frac{1}{4} = \frac{M_a}{4} \frac{\bar{a}^{1/4}}{\bar{x}^{5/4}} \tag{31}$$

The surface friction is thus independent of the viscosity of the fluid and varies inversely as the 5/4 power of the distance from the slit.

The shearing stress is independent of the viscosity of the fluid not only at the surface but also between any two adjacent layers in the flow. For the flow from a slit without the presence of the plate (reference 8), the shearing stress between any two adjacent layers of fluid is also noted to be independent of the viscosity of the fluid. The shearing stresses between adjacent layers are independent of the viscosity in both the problem presented herein and in the flow from the slit without the plate, because in both cases the velocity derivative $\frac{\partial \bar{u}}{\partial \bar{y}}$ is inversely proportional to the viscosity μ of the fluid.

The ratio of the rate of flow of momentum parallel to the plate at distance $\bar{x}$ from the slit to the rate of momentum flow at the reference distance $\bar{a}$ from the slit can be obtained from equation (10) as

$$\frac{M}{M_a} = \frac{M}{\rho \bar{U}_a{}^2 \bar{a}} = \frac{\bar{a}^{1/4}}{\bar{x}^{1/4}} \tag{32}$$

The rate at which momentum parallel to the plate passes through a plane normal to the plate therefore varies inversely as the 1/4 power of the distance from the slit to the normal plane.

The expression for ξ can be obtained as follows:

$$\xi = \frac{\alpha\eta}{3} = \frac{9^{1/3} R_a{}^{2/3}}{3} \frac{y}{x^{3/4}} = \frac{9^{1/3}}{3} \left(\frac{\bar{a}M_a}{\nu^2\rho}\right)^{1/3} \frac{y}{x^{3/4}}$$

or

$$\xi = \frac{9^{1/3}}{3} \left(\frac{\bar{a}M_a}{\nu^2\rho}\right)^{1/3} \frac{\bar{y}}{\bar{x}^{3/4}} \frac{1}{\bar{a}^{1/4}} = \frac{9^{1/3}}{3} \left(\frac{M_a}{\nu^2\rho}\right)^{1/3} \bar{a}^{1/12} \frac{\bar{y}}{\bar{x}^{3/4}} \tag{33}$$

The width of the disturbed region, given by $\bar{y}$ for $\xi = \xi_{edge}$, thus increases as the 3/4 power of the distance from the slit.

Validity of Solution

A restriction on the region in which the solution is valid can be obtained by noting that for the boundary-layer equations to be valid the value of $\frac{\bar{v}}{\bar{u}}$ must be small. By use of equations (26) and (28) the ratio $\frac{\bar{v}}{\bar{u}}$ becomes

$$\frac{\bar{v}}{\bar{u}} = \frac{\dfrac{9^{1/3}}{4}\left(\dfrac{\nu M_a}{\rho}\right)^{1/3} \dfrac{\bar{a}^{1/12}}{\bar{x}^{3/4}}\left(3\xi \dfrac{dG}{d\xi} - G\right)}{\dfrac{9^{2/3}}{3}\left(\dfrac{M_a{}^2}{\rho^2 \nu}\right)^{1/3} \dfrac{\bar{a}^{1/6}}{\bar{x}^{1/2}} \dfrac{dG}{d\xi}}$$

or

$$\frac{\bar{v}}{\bar{u}} = \frac{3}{4\left(9^{1/3}\right)}\left(\frac{\rho\nu^2}{M_a}\right)^{1/3} \frac{1}{\bar{x}^{1/4}\bar{a}^{1/12}}\left(3\xi - \frac{G}{\dfrac{dG}{d\xi}}\right)$$

Large values of $\frac{M_a}{\rho\nu^2}$ give small values of $\frac{\bar{v}}{\bar{u}}$, and so the region in which the solution is valid is enlarged when $\frac{M_a}{\rho\nu^2}$ is increased.

By use of the complete equations of motion instead of the boundary-layer equations the difference between the static pressure on the plate and the static pressure very far above the plate is given approximately at a given value of $\bar{x}$ by

$$\left(\frac{\Delta p}{\rho \bar{u}_{max}{}^2}\right)_{\bar{x}} < \left|\frac{10}{\bar{a}^{1/6}} \frac{1}{\bar{x}^{1/2}}\left(\frac{\rho\nu^2}{M_a}\right)^{2/3}\right|$$

and the pressure gradient in the direction of x is given approximately by

$$\left(\frac{1}{\rho \bar{u}_{max}^2} \frac{\partial p}{\partial x}\right)_{\bar{x}} < \left| \frac{5}{\bar{a}^{1/6}} \frac{1}{\bar{x}^{3/2}} \left(\frac{\rho \nu^2}{M_a}\right)^{2/3} \right|$$

When it is noted that ν is a small quantity, the approximation that the static pressure is constant throughout the entire field which was made in equations (1) and (3) is justified by the solution. All the terms neglected in the boundary-layer approximations can likewise be shown to be truly negligable on the basis of the obtained solution. All the approximations become more accurate as $\bar{x}$ increases and as $\frac{M_a}{\rho \nu^2}$ increases.

The constant $\bar{a}$ seems to appear in a rather arbitrary manner but, because $\frac{M_a}{M_{a_1}} = \left(\frac{\bar{a}_1}{\bar{a}}\right)^{1/4}$ where $\bar{a}_1$ is an arbitrary value of $\bar{a}$ and M_{a_1} is the momentum corresponding to the distance $\bar{a}_1$ from the slit, it can be shown that a change in $\bar{a}$ does not change $\bar{u}$, $\bar{v}$, ξ, or τ_0 provided that $\bar{x}$ and $\bar{y}$ are kept fixed.

When the plate is removed, the flow becomes the same as the flow in a two-dimensional laminar jet. This problem has been treated in references 7 and 8. The removal of the plate in the present problem makes $\left(\frac{d^2G}{d\xi^2}\right)_0 = K = 0$. Equation (20) then becomes

$$\frac{d^3G}{d\xi^3} + G \frac{d^2G}{d\xi^2} + \left(\frac{dG}{d\xi}\right)^2 = 0$$

When the conditions that

$$G = 0$$

$$\frac{d^2G}{d\xi^2} = 0$$

at $\xi = 0$ and that

$$\int_0^\infty \left(\frac{dG}{d\xi}\right)^2 d\xi = \frac{1}{3}$$

are used, the solution, which in this case is obtained in closed form, is

$$G = \tanh \frac{\xi}{2} \tag{34}$$

Because the momentum M_a is the momentum for one side of the plate in contrast to the momentum used by Bickley M_B (reference 8) which was the total momentum flow from the slit and therefore equal to twice M_a, the result $\xi = 2\xi_B$ is obtained. When this relation between the value of ξ of the present analysis and the value of ξ_B in reference 8 is used together with equation (34), exactly the same relations are obtained for all the quantities of interest as were obtained by Bickley (reference 8).

CONCLUSIONS

The laminar flow of a slightly viscous incompressible fluid that issues from a slit and passes over a flat plate is investigated in a region far enough from the slit for the boundary-layer equations to be valid. The results obtained are that the velocities parallel to the plate vary inversely as the 1/2 power of the distance from the slit, that the velocities perpendicular to the plate vary inversely as the 3/4 power of the distance from the slit, that the width of the disturbed region increases as the 3/4 power of the distance from the slit, that the surface friction is independent of the viscosity of the fluid and varies inversely as the 5/4 power of the distance from the slit, that the rate at which momentum parallel to the plate passes through a plane normal to the plate varies inversely as the 1/4 power of the distance from the slit to the normal plane, and that the quantity of fluid passing through a plane normal to the plate per unit time varies directly as the 1/4 power of the distance from the slit to the normal plane.

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National Advisory Committee for Aeronautics
Langley Field, Va., March 9, 1948

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TABLE I

VALUES OF G AND ITS DERIVATIVES FROM NUMERICAL INTEGRATION

OF EQUATION (20) FOR $\Delta \xi = 0.05$

ξ	G	$\frac{dG}{d\xi}$	$\frac{d^2G}{d\xi^2}$	$\frac{d^3G}{d\xi^3}$	ξ	G	$\frac{dG}{d\xi}$	$\frac{d^2G}{d\xi^2}$	$\frac{d^3G}{d\xi^3}$
0	0	0	0.2500000	0	5.40	1.196679	0.05750777	−0.05023198	0.04012295
[a].10	.001249981	.02499902	.2499609	−.001171762	5.60	1.207228	.04823052	−.04270551	.03517719
[a].20	.004999375	.04998438	.2496877	−.004683869	5.80	1.216065	.04036134	−.03614174	.03051947
[a].30	.01124526	.07492092	.2489467	−.01051933	6.00	1.223454	.03371425	−.03047135	.02625524
[a].40	.01998002	.09975046	.2475078	−.01863413	6.20	1.229620	.02811885	−.02561013	.02243204
[a].50	.03118910	.1243918	.2451467	−.02894438	6.40	1.234759	.02342222	−.02146850	.01905842
.60	.04484869	.1487419	.2416503	−.04131449	6.60	1.239038	.01948938	−.01795784	.01611808
.70	.06092368	.1726774	.2368218	−.05554727	6.80	1.242596	.01620262	−.01499442	.01358022
.80	.07936563	.1960561	.2304877	−.07137659	7.00	1.245555	.01346028	−.01250146	.01140667
.90	.1001111	.2187200	.2225047	−.08846404	7.20	1.248012	.01117529	−.01041018	.009556691
1.00	.1230801	.2404986	.2127667	−.1063999	7.40	1.250051	.009273506	−.008659909	.007989999
1.20	.1752811	.2806800	.1878300	−.1428642	7.60	1.251743	.007692134	−.007197836	.006668627
1.40	.2349715	.3151565	.1558129	−.1764441	7.80	1.253147	.006378208	−.005978429	.005557865
1.60	.3008741	.3426000	.1177466	−.2026324	8.00	1.254311	.005287195	−.004962730	.004626673
1.80	.3714726	.3619759	.07551285	−.2175781	8.20	1.255274	.004381755	−.004117618	.003847754
2.00	.4450861	.3726944	.03162764	−.2189093	8.40	1.256073	.003630653	−.003415066	.003197406
2.20	.5199684	.3747034	−.01111474	−.2062695	8.60	1.256734	.003007807	−.002831453	.002655216
2.40	.5944190	.3685029	−.05005941	−.1813744	8.80	1.257283	.002491472	−.002346937	.002203762
2.60	.6668871	.3550772	−.08306875	−.1475716	9.00	1.257738	.002063540	−.001944892	.001828236
2.80	.7360571	.3357643	−.1087741	−.1090585	9.50	1.258560	.001287780	−.001214945	.001144323
3.00	.8009023	.3120908	−.1266588	−.07001997	10.00	1.259073	.0008033527	−.0007583967	.0007151896
3.20	.8607064	.2856064	−.1369813	−.03393091	10.50	1.259393	.0005010319	−.0004731907	.0004465731
3.40	.9150536	.2577468	−.1405886	−.003165544	11.00	1.259591	.0003124300	−.0002951561	.0002786856
3.60	.9637958	.2297388	−.1386845	.02107780	11.50	1.259715	.0001947988	−.0001840728	.0001738525
3.80	1.007005	.2025513	−.1326068	.03861124	12.00	1.259794	.0001214426	−.0001147835	.0001084306
4.00	1.044919	.1768875	−.1236529	.04997166	12.30	1.259826	.00009145710	−.00008645644	.00008167751
4.20	1.077893	.1532052	−.1129664	.05611652	12.50	1.259842	.00007570118	−.00007157115	.00006761765
4.40	1.106351	.1317539	−.1014795	.05816533	13.00	1.259873	.00004718050	−.00004462497	.00004216300
4.60	1.130750	.1126192	−.08989998	.05721618	13.50	1.259893	.00002939794	−.00002782315	.00002628934
4.80	1.151552	.09576637	−.07872794	.05423768	14.00	1.259903	.00001831081	−.00001734711	.00001639125
5.00	1.169201	.08107896	−.06828708	.05002028	14.50	1.259913	.00001139829	−.00001081543	.00001021968
5.20	1.184117	.06839029	−.05876165	.04516965	14.95	1.259916	.000007433791	−.000007069283	.000006679943

[a]Values obtained by Taylor's series.

TABLE II

COMPARISON OF RESULTS FOR $\Delta\xi = 0.05$, $\Delta\xi = 0.10$ AND VALUES

FOR $\xi = 0.60$ FROM THE TAYLOR'S SERIES

ξ	G	$\frac{dG}{d\xi}$	$\frac{d^2G}{d\xi^2}$	$\frac{d^3G}{d\xi^3}$
Numerical integration; $\Delta\xi = 0.05$				
0.60	0.04484869	0.1487419	0.2416503	−0.04131449
2.00	.4450861	.3726944	.03162764	−.2189093
4.00	1.044919	.1768875	−.1236529	.04997166
6.00	1.223454	.03371425	−.03047135	.02625524
8.50	1.256419	.003304650	−.003109714	.002913947
13.10	1.259877	.00004292233	−.00004060171	.00003836209
Numerical integration; $\Delta\xi = 0.10$				
0.60	0.04484870	0.1487419	0.2416502	−0.04131451
2.00	.4450859	.3726940	.03162748	−.2189089
4.00	1.044918	.1768876	−.1236527	.04997135
6.00	1.223450	.03371429	−.03047143	.02625523
8.50	1.256417	.003304591	−.003109728	.002913955
13.10	1.259877	.00004284263	−.00004060191	.00003836229
Taylor's series				
0.60	0.04484868	0.1487419	0.2416504	−0.04131449

TABLE III

VALUES OF INTEGRAL $\int_0^{\xi} \left(\frac{dG}{d\xi}\right)^2 d\xi$ OBTAINED BY USE

OF WEDDLE'S RULE FOR $\Delta\xi = 0.05$

ξ	$\int_0^{\xi} \left(\frac{dG}{d\xi}\right)^2 d\xi$	ξ	$\int_0^{\xi} \left(\frac{dG}{d\xi}\right)^2 d\xi$
0	0	6.30	0.3329781
.30	.0005619072	6.60	.3331301
.60	.004462288	6.90	.3332176
.90	.01476475	7.20	.3332678
1.20	.03369614	7.50	.3332965
1.50	.06194261	7.80	.3333129
1.80	.09818613	8.10	.3333222
2.10	.1392639	8.40	.3333275
2.40	.1810349	8.70	.3333305
2.70	.2195948	9.00	.3333322
3.00	.2522585	9.30	.3333332
3.30	.2779223	9.60	.3333338
3.60	.2968261	9.90	.3333341
3.90	.3100159	10.20	.3333343
4.20	.3188173	10.50	.3333344
4.50	.3244822	10.80	.3333345
4.80	.3280249	11.10	.3333345
5.10	.3301909	11.40	.3333345
5.40	.3314920	11.70	.3333345
5.70	.3322630	12.00	.3333345
6.00	.3327151	12.30	.3333345

NACA

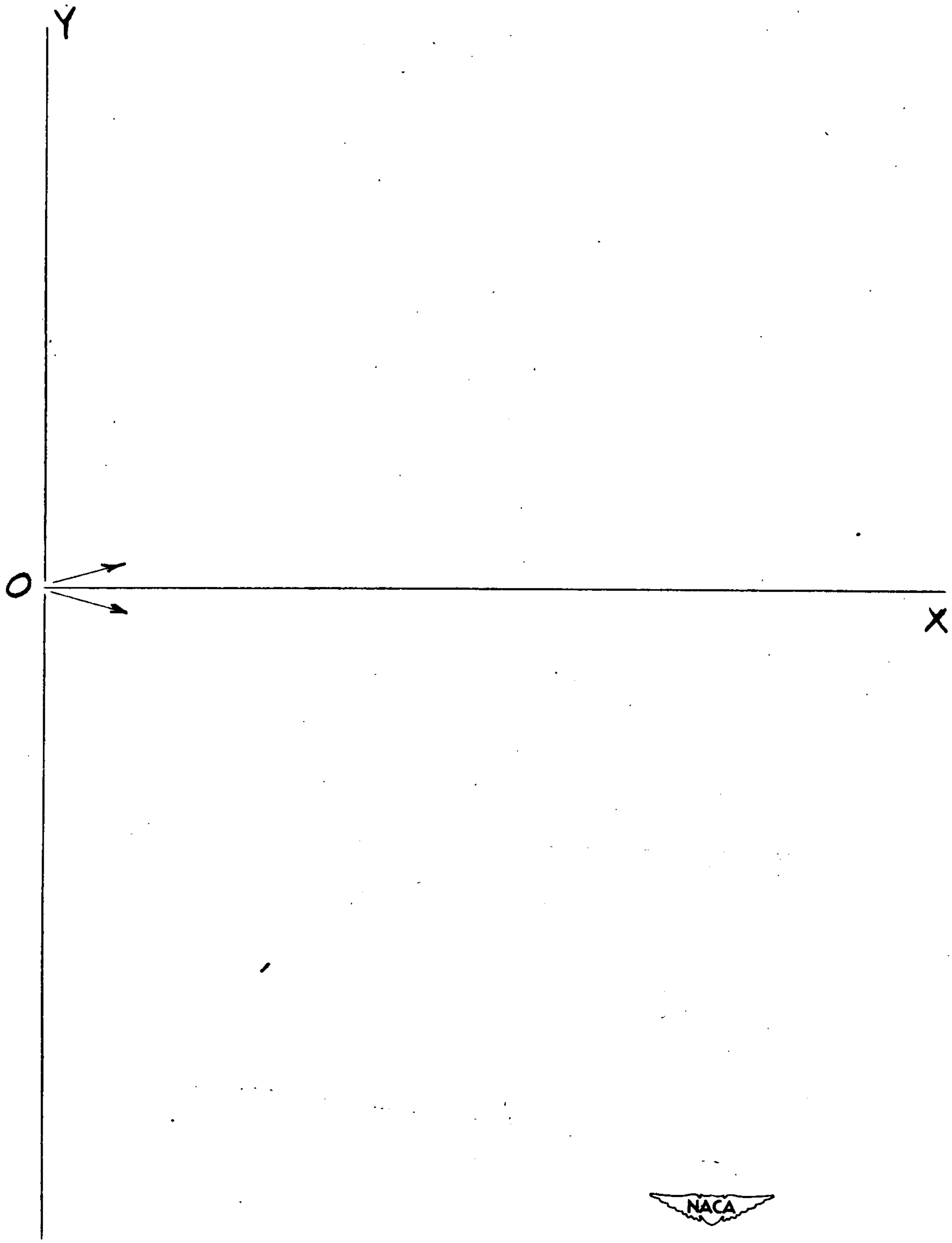

Figure 1.- Flow from a slit at 0 into the XY-plane with a plate along 0X.

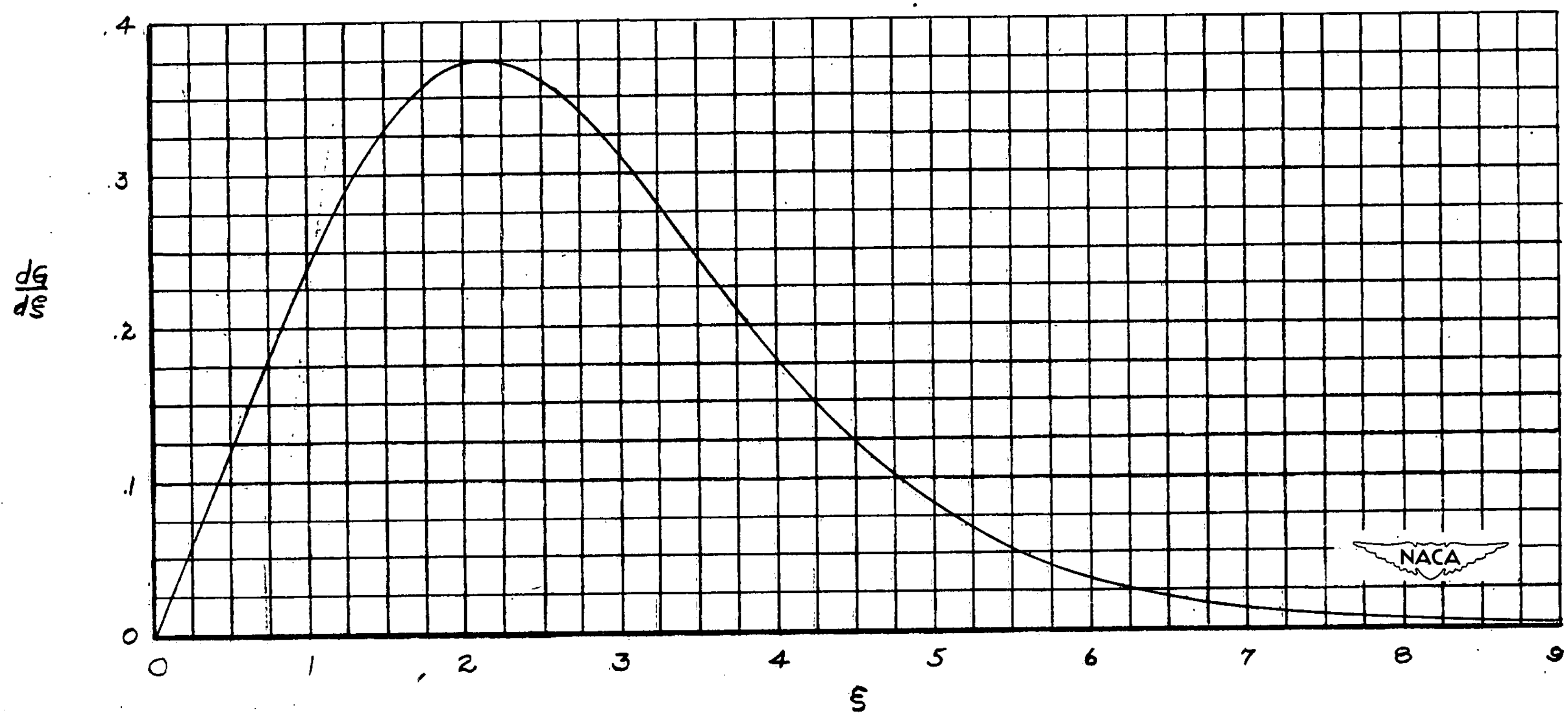

Figure 2.- Curve of $\frac{dG}{d\xi}$ plotted against ξ from integration of equation (20).

$$\bar{u} = \frac{9^{2/3}}{3}\left(\frac{M_a{}^2}{\rho\,{}^2\nu}\right)^{1/3} \frac{\bar{a}^{1/6}}{\bar{x}^{1/2}}\,\frac{dG}{d\xi};\quad \xi = \frac{9^{1/3}}{3}\left(\frac{M_a}{\nu^2\rho}\right)^{1/3} \bar{a}^{1/12}\,\frac{\bar{y}}{\bar{x}^{3/4}}.$$