NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1558

NOTES ON THE LAGRANGIAN MULTIPLIER METHOD IN ELASTIC-STABILITY ANALYSIS

By Bernard Budiansky, Pai C. Hu, and Robert W. Connor

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Langley Field, Va.

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Page 28: The last line of the page should read: $(j = 1,2,3,\ldots q)$

Page 39, equations for $\phi_m(y)$ and $\psi_m(y)$, middle of page: The x's in these equations (four places) should be changed to y's.

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SUMMARY

New applications of the Lagrangian multiplier method to stability analysis are described by means of elementary examples. The use of the method in analyzing the stability of (a) clamped plates in shear and (b) plate-stiffener combinations is demonstrated. A detailed analysis for finding upper and lower limits to critical stresses of clamped rectangular plates is presented in an appendix.

INTRODUCTION

The use of the Lagrangian multiplier method to calculate upper and lower limits to the critical compressive stress of a clamped plate was presented in reference 1. The procedures of reference 1 have been directly used to analyze the stability of clamped plates under compression in two directions (reference 2, footnote) and may be used, with little modification, to find vibration frequencies of clamped plates. The purpose of the present paper is to describe additional applications of the Lagrangian multiplier method to the elastic-stability analysis of (a) clamped plates in shear and (b) plate-stiffener combinations. Elementary examples are used to bring out the essential features involved in applying the method to these types of problems. A detailed analysis for finding upper and lower limits to critical shear stresses of finite clamped plates is given in an appendix.

SYMBOLS

a	length of plate
b	width of plate
λ	half-wave length
β	a/b or λ/b

t	plate thickness
E	Young's modulus of elasticity
μ	Poisson's ratio
D	plate stiffness in bending $\dfrac{Et^3}{12(1-\mu^2)}$
$(EI)_{stiff}$	effective flexural rigidity of stiffener attached to plate
A	stiffener cross-sectional area
$\gamma = \dfrac{EI}{bD}$	
$\delta = \dfrac{A}{bt}$	
σ	critical compressive stress
τ	critical shear stress
k	critical compressive stress coefficient in the formula $\sigma = k\left(\dfrac{\pi^2 D}{b^2 t}\right)$
k_s	critical shear stress coefficient in the formula $\tau = k_s\left(\dfrac{\pi^2 D}{b^2 t}\right)$
x	plate coordinate parallel to length
y	plate coordinate parallel to width
w	deflection normal to plane of the plate
a_n, b_n, d_n, Δ; $a_{mn}, b_{mn}, c_{mn}, d_{mn}$	Fourier coefficients
$\alpha, \mu, \zeta, \lambda_j, \eta_i, \zeta_j, \mu_i$; $\xi, \eta, \lambda'_j, \eta'_i, \zeta'_j, \mu'_i$	Lagrangian multipliers
V	internal bending energy

T	external work of applied stress
m,n,i,j,p,q	integers
δ_{mn}	Kronecker delta: 1 if $m = n$; 0 if $m \neq n$

$$A_n = \left(1 + 4n^2\beta^2\right)^2\left(1 + \delta_{0n}\right)$$

$$B_n = \frac{A_n}{A_n^{\,2} - D_n^{\,2}}$$

$$C_n = \frac{D_n}{A_n^{\,2} - D_n^{\,2}}$$

$$D_n = 4k_s\beta^3 n$$

$$E_n = \frac{1}{\left[\left(1 + n^2\beta^2\right)^2 - \beta^2 k\right]\left(1 + \delta_{0n}\right)}$$

$$H = \frac{1}{2\left(\gamma - \beta^2\delta k\right)}$$

$$A_{mn} = 2\left(m^2 + n^2\beta^2\right)^2\left(1 + \delta_{m0} + \delta_{0n}\right)$$

$$B_{mn} = \frac{A_{mn}}{A_{mn}^{\,2} - D_{mn}^{\,2}}$$

$$C_{mn} = \frac{D_{mn}}{A_{mn}^{\,2} - D_{mn}^{\,2}}$$

$$D_{mn} = k_s\beta^3 mn$$

$$A'_{mn} = \left(m^2 + 4n^2\beta^2\right)^2\left(1 + \delta_{0n}\right)$$

$$B'_{mn} = \frac{A'_{mn}}{\left(A'_{mn}\right)^2 - \left(D'_{mn}\right)^2}$$

$$C'_{mn} = \frac{D'_{mn}}{\left(A'_{mn}\right)^2 - \left(D'_{mn}\right)^2}$$

$$D'_{mn} = 4k_s\beta^3 mn$$

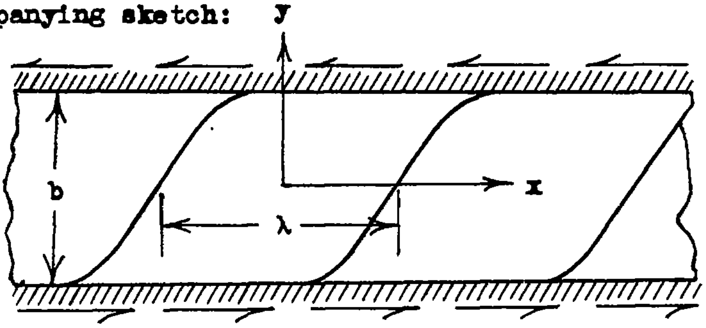
Subscripts:

pl plate

stiff stiffener

BUCKLING IN SHEAR OF AN INFINITELY LONG CLAMPED PLATE

Consider an infinitely long clamped plate loaded in shear as shown in the accompanying sketch:



The problem of finding the critical shear stress, for which there exists an exact solution by the differential equation approach (reference 3), will be solved by the Lagrangian multiplier method.

Boundary conditions.- It is evident that only the boundary conditions along the long edges affect the buckling stress. The boundary conditions of the problem are then:

Zero deflection, long edges

$$w\left(x, \frac{b}{2}\right) = w\left(x, -\frac{b}{2}\right) = 0 \tag{1}$$

Zero slope, long edges

$$\frac{\partial w}{\partial y}\left(x, \frac{b}{2}\right) = \frac{\partial w}{\partial y}\left(x, -\frac{b}{2}\right) = 0 \tag{2}$$

Fourier expansion.– The deflection surface of the buckled plate is known to be harmonic along longitudinal sections; the deflection surface may therefore be represented in the form

$$w = f_1(y) \sin \frac{\pi x}{\lambda} + f_2(y) \cos \frac{\pi x}{\lambda} \qquad (3)$$

where λ is the half-wave length. The necessity remains of choosing suitable Fourier expressions for $f_1(y)$ and $f_2(y)$ for use in the Lagrangian multiplier method. It is desired that the series chosen satisfy the conditions that:

(a) In the region $\left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, $\left(-\frac{b}{2}, \frac{b}{2}\right)$, the deflection w be symmetrical (see preceding sketch) about the origin

(b) The potential energy expression for the buckled plate, calculated on the basis of the specified expression for w, consist of integrals of products of functions which form orthogonal sets.

An expression satisfying these requirements is

$$w = \sin \frac{\pi x}{\lambda} \sum_{n=1}^{\infty} a_n \sin \frac{2n\pi y}{b} + \cos \frac{\pi x}{\lambda} \sum_{n=0}^{\infty} d_n \cos \frac{2n\pi y}{b} \qquad (4)$$

The fulfillment of condition (b) is verified in the following section, in which the energy expressions are calculated. The desirability of the condition will become evident when the final stability criterion is derived.

Energy expressions.– Substituting the value for w from equation (4) into the formulas for bending energy and work

$$V = \frac{D}{2} \int\int \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1 - \mu)\left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y}\right)^2\right]\right\} dx\, dy \qquad (5)$$

$$T = -\tau t \int\int \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \, dx\, dy \qquad (6)$$

gives, in the region $\left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, $\left(-\frac{b}{2}, \frac{b}{2}\right)$,

$$V = \frac{\pi^4 Db}{8\lambda^3} \sum_{n=0}^{\infty} \left(1 + 4\beta^2 n^2\right)^2 \left[a_n^2\left(1 - \delta_{0n}\right) + d_n^2\left(1 + \delta_{0n}\right)\right]$$

$$T = \pi^2 \tau t \sum_{n=1}^{\infty} n a_n d_n$$

Then,

$$\frac{(V - T)}{\left(\frac{\pi^4 Db}{8\lambda^3}\right)} = \sum_{n=0}^{\infty} \left\{\left(1 + 4n^2\beta^2\right)^2 \left[a_n^2\left(1 - \delta_{0n}\right) + d_n^2\left(1 + \delta_{0n}\right)\right] - 8k_s \beta^3 n a_n d_n\right\} \tag{7}$$

where

$$\beta = \frac{\lambda}{b}$$

$$\tau = k_s\left(\frac{\pi^2 D}{b^2 t}\right)$$

Constraining relationships.- Neither of the clamped-edge boundary conditions (1) and (2) are satisfied term by term by the expansion (4); constraining conditions must therefore be imposed on the Fourier coefficients a_n and d_n. These constraining relationships are, for zero deflection,

$$\sum_{n=0}^{\infty} (-1)^n d_n = 0 \tag{8}$$

and for zero slope,

$$\sum_{n=1}^{\infty} n(-1)^n a_n = 0 \tag{9}$$

Stability criterion.- The energy method requires that $V - T$ be minimized with respect to the a's and d's. Since the a's and d's are, however, bound by equations (8) and (9), the minimization is performed by the Lagrangian multiplier method by minimizing

$$F = \frac{(V - T)}{\left(\frac{\pi^4 Db}{8\lambda^3}\right)} - \mu \sum_{n=0}^{\infty} (-1)^n d_n - \zeta \sum_{n=1}^{\infty} n(-1)^n a_n \tag{10}$$

with respect to the a's and d's. The Lagrangian multipliers are μ and ζ. The complete set of equations for minimizing $V - T$ with the a's and d's bound by the conditions given in equations (8) and (9) then becomes

$$\frac{\partial F}{\partial a_n} = 0 \qquad (n = 1,2,3,\ldots) \tag{11}$$

$$\frac{\partial F}{\partial d_n} = 0 \qquad (n = 0,1,2,\ldots) \tag{12}$$

$$\sum_{n=0}^{\infty} (-1)^n d_n = 0 \tag{8}$$

$$\sum_{n=1}^{\infty} n(-1)^n a_n = 0 \tag{9}$$

Substituting equation (7) into equation (10) and equation (10) into equations (11) and (12) gives

$$2\left(1 + 4\beta^2n^2\right)^2 a_n - \left(8k_s\beta^3 n\right)d_n = n(-1)^n \zeta \qquad (n = 1,2,3,\ldots) \qquad (13)$$

$$-\left(8k_s\beta^3 n\right)a_n + 2\left(1 + 4\beta^2n^2\right)^2\left(1 + \delta_{0n}\right)d_n = (-1)^n \mu \qquad (14)$$

$$(n = 0,1,2,\ldots)$$

Solving equations (13) and (14) simultaneously for a_n and d_n gives

$$a_n = \frac{1}{2}\left[B_n n(-1)^n \zeta + C_n(-1)^n \mu\right] \qquad (15)$$

$$d_n = \frac{1}{2}\left[C_n n(-1)^n \zeta + B_n(-1)^n \mu\right] \qquad (16)$$

where

$$B_n = \frac{A_n}{A_n^2 - D_n^2}$$

$$C_n = \frac{D_n}{A_n^2 - D_n^2}$$

in which

$$A_n = \left(1 + 4n^2\beta^2\right)^2\left(1 + \delta_{0n}\right)$$

$$D_n = 4k_s\beta^3 n$$

Substituting equations (15) and (16) into equations (9) and (8), respectively, yields

$$\left.\begin{aligned} \mu \sum_{n=0}^{\infty} B_n + \zeta \sum_{n=1}^{\infty} nC_n = 0 \\ \mu \sum_{n=1}^{\infty} nC_n + \zeta \sum_{n=1}^{\infty} n^2B_n = 0 \end{aligned}\right\} \qquad (17)$$

The condition that a nonvanishing solution exists for μ and ζ gives as the final stability criterion

$$\left(\sum_{n=0}^{\infty} B_n\right)\left(\sum_{n=1}^{\infty} n^2B_n\right) - \left(\sum_{n=1}^{\infty} nC_n\right)^2 = 0 \qquad (18)$$

For a given value of β, the value of k_s that satisfies equation (18) can be found by trial substitution and interpolation. The correct value of β is that which gives the lowest value of k_s. For $\beta = 0.80$, equation (18) yields $k_s = 8.989$ which agrees with the solution obtained by Southwell by the differential-equation approach (reference 3).

Discussion of method; further applications.- In the usual application of the Rayleigh-Ritz method, an infinite set of equations involving infinitely many deflection coefficients is obtained when the energy expression $V - T$ is minimized; the exact stability criterion is then an infinite determinant obtained from these equations. The simplicity of the solution just obtained, however, is due to the fact that it was practicable to transfer consideration of infinitely many Fourier coefficients to consideration of only two Lagrangian multipliers. An essential feature of the solution that permits this simplification is the fact that the substitution of the expansion chosen for w (equation (4)) into the expressions for V and T (equations (5) and (6)) leads to integrals of products of functions which form orthogonal sets. It then becomes a simple matter to solve the minimizing equations (13) and (14) for the Fourier coefficients in terms of the Lagrangian multiplier (equations (15) and (16)), substitute back into the constraining relationships (8) and (9), and derive the stability criterion from the condition that there be a nonvanishing solution for the Lagrangian multipliers.

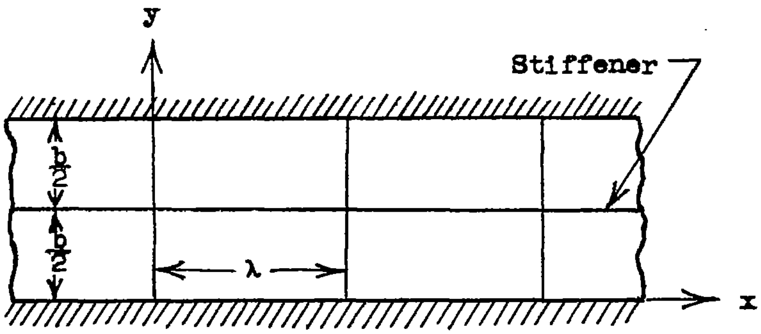
Stability problems involving finite rectangular plates may require double rather than single Fourier expansions for use in the Lagrangian multiplier method. However, a proper choice of series, with due attention to orthogonality considerations, would still make it possible to solve for infinitely many Fourier coefficients in terms of Lagrangian multipliers. Appendix A presents a solution by the Lagrangian multiplier method for the problem of the buckling in shear of a rectangular plate clamped on all sides. Although it was possible to obtain an exact solution for the infinitely long plate, the solution for finite plates is approximate; however, as in the compressive buckling problem discussed in reference 1, it is possible to obtain approximate solutions in two different ways, which permit the computation of an upper limit as well as a lower limit to the true critical stress. The true critical stress can thus be bracketed to within any desired degree of accuracy by taking sufficiently close upper- and lower-limit approximations.

The Lagrangian multiplier method may, with the use of appropriate deflection functions, find applications to other problems. A general discussion of Fourier series and their use in stability analysis is contained in appendix B.

BUCKLING OF PLATE-STIFFENER COMBINATIONS

The application of the Rayleigh-Ritz energy method to buckling problems involving plates with stiffeners usually results in energy expressions that are complicated functions of the deflection coefficients. That is, even if the terms of the assumed deflection function have the orthogonality properties previously discussed, energy terms due to stiffener deformations will usually involve quadratic cross products of all combinations of the deflection coefficients. Occasionally, for some special problems (see reference 4) relatively simple stability criteria can still be derived by algebraic manipulations; however, in general, it is to be expected that an exact stability criterion for a stiffened plate, derived by the Rayleigh-Ritz method, will consist of an infinite determinant that is obtained from explicit consideration of infinitely many deflection coefficients.

The Lagrangian multiplier method can be used to simplify the analysis considerably. As in the unstiffened plate buckling problems previously discussed, an appropriate application of the Lagrangian multiplier method makes it possible to solve for Fourier coefficients in terms of Lagrangian multipliers, so that explicit consideration of a finite number of Lagrangian multipliers takes into account infinitely many Fourier coefficients. The elements of the method of application of the Lagrangian multiplier method to stiffened plates will be presented by giving the analysis of the stability under longitudinal compression of an infinitely long clamped plate having a longitudinal stiffener along its center line as illustrated in the following sketch:



In this example, the stiffener is assumed to have no torsional rigidity. (See reference 5 for solution of this problem by the differential-equation approach.)

Boundary and continuity conditions.— The boundary conditions along the clamped edges are zero deflection,

$$w_{pl}(x,0) = w_{pl}(x,b) = 0 \tag{19}$$

and zero slope,

$$\frac{\partial w_{pl}}{\partial y}(x,0) = \frac{\partial w_{pl}}{\partial y}(x,b) = 0 \tag{20}$$

The condition that there be continuity between the plate and stiffener is given by

$$w_{stiff} - w_{pl} = 0 \tag{21}$$

Fourier expansions.— The buckled deflection surface is known to be sinusoidal in the long direction. However, the deflections in the short direction may be either symmetrical or antisymmetrical, depending on which mode corresponds to a lower buckling stress. For the present, the symmetrical mode will be considered. Then, let

$$w_{pl} = \sin\frac{\pi x}{\lambda} \sum_{n=0,2,4,\ldots}^{\infty} b_n \cos\frac{n\pi y}{b} \tag{22}$$

and

$$w_{stiff} = \Delta \sin \frac{\pi x}{\lambda} \tag{23}$$

where λ is the half-wave length of the buckled surface.

Energy expressions.– In addition to the plate-bending energy (equation 5), the stiffener-bending energy

$$V_{stiff} = \frac{(EI)_{stiff}}{2} \int \left(\frac{d^2 w_{stiff}}{dx^2}\right)^2 dx \tag{24}$$

the work of the plate stresses

$$T_{pl} = \frac{\sigma t}{2} \int\int \left(\frac{\partial w_{pl}}{\partial x}\right)^2 dx\, dy \tag{25}$$

and the work of the stiffener stresses

$$T_{stiff} = \frac{\sigma A}{2} \int \left(\frac{dw_{stiff}}{dx}\right)^2 dx \tag{26}$$

must be taken into account. Substituting equation (22) into equations (5) and (25) and equation (23) into equations (24) and (26) gives, in the region $(0,\lambda),(0,b)$,

$$\frac{V_{pl} + V_{stiff} - T_{pl} - T_{stiff}}{\frac{\pi^4 Db}{8\lambda^3}} = \sum_{n=0,2,4,\ldots}^{\infty} \left[\left(1 + n^2\beta^2\right)^2 - \beta^2 k\right]\left(1 + \delta_{0n}\right)b_n^{\,2} + 2\left(\gamma - \beta^2 \delta k\right)\Delta^2 \tag{27}$$

where

$$\beta = \frac{\lambda}{b}$$

$$\sigma = k\left(\frac{\pi^2 D}{b^2 t}\right)$$

$$\gamma = \frac{(EI)_{stiff}}{bD}$$

$$\delta = \frac{A}{bt}$$

Constraining relationships.- While the boundary conditions of zero slope (equation (20)) are satisfied by each term of equation (22), the boundary conditions of zero deflection (equation (19)) are satisfied only by making

$$\sum_{n=0,2,4,\ldots}^{\infty} b_n = 0 \tag{28}$$

The continuity condition (equation (21)) will be satisfied by means of the constraining relationship

$$\Delta - \sum_{n=0,2,4,\ldots}^{\infty} b_n \cos\frac{n\pi}{2} = 0$$

or

$$\Delta - \sum_{n=0,2,4,\ldots}^{\infty} (-1)^{n/2} b_n = 0 \tag{29}$$

Stability criterion.- The energy expression (27) must be minimized, with Δ and the b's bound by the constraining relationships (28) and (29). Let

$$F = \frac{V_{pl} + V_{stiff} - T_{pl} - T_{stiff}}{\left(\frac{\pi^4 Db}{8\lambda^3}\right)} - \xi \sum_{n=0,2,4,\ldots}^{\infty} b_n$$

$$- \eta\left[\Delta - \sum_{n=0,2,4,\ldots}^{\infty} (-1)^{n/2}\, b_n\right]$$

where ξ and η are Lagrangian multipliers. Then,

$$\frac{\partial F}{\partial b_n} = 0 = 2\left[\left(1 + n^2\beta^2\right)^2 - \beta^2 k\right]\left(1 + \delta_{0n}\right)b_n - \xi + \eta(-1)^n$$

$$\frac{\partial F}{\partial \Delta} = 0 = 4\left(\gamma - \beta^2 \delta k\right)\Delta - \mu$$

Solving for b_n and Δ gives

$$b_n = \frac{1}{2}E_n\left[\xi - \eta(-1)^n\right] \tag{30}$$

$$\Delta = \frac{1}{2}H\mu \tag{31}$$

where

$$E_n = \frac{1}{\left[\left(1 + n^2\beta^2\right)^2 - \beta^2 k\right]\left(1 + \delta_{0n}\right)}$$

and

$$H = \frac{1}{2\left(\gamma - \beta^2 \delta k\right)}$$

Substituting equations (30) and (31) into equations (28) and (29) gives

$$\left.\begin{aligned}
\xi \sum_{n=0,2,4,\ldots}^{\infty} E_n - \eta \sum_{n=0,2,4,\ldots}^{\infty} (-1)^n E_n &= 0 \\
-\xi \sum_{n=0,2,4,\ldots}^{\infty} (-1)^n E_n + \eta \left(H + \sum_{n=0,2,4,\ldots}^{\infty} E_n \right) &= 0
\end{aligned}\right\} \tag{32}$$

Setting the determinant of the coefficients of ξ and η in equations (32) equal to zero gives as the stability criterion for symmetrical buckling

$$\left[\sum_{n=0,2,4,\ldots}^{\infty} E_n \right] \cdot \left[H + \sum_{n=0,2,4,\ldots}^{\infty} E_n \right] - \left[\sum_{n=0,2,4,\ldots}^{\infty} (-1)^n E_n \right]^2 = 0 \tag{33}$$

For a given value of β, this criterion can be used to find k; the correct value of β is that which gives the lowest value of k.

The stability criterion for antisymmetrical buckling is obtained simply by letting

$$w = \sin \frac{\pi x}{\lambda} \sum_{n=1,3,5,\ldots}^{\infty} b_n \cos \frac{n \pi y}{b}$$

and using the Lagrangian multiplier method to introduce the zero deflection constraining relationship

$$\sum_{n=1,3,5,\ldots}^{\infty} b_n = 0$$

(Since the stiffener has no torsional rigidity and lies along a node, it does not affect the buckling stress.) The stability criterion for antisymmetrical buckling becomes

$$\sum_{n=1,3,5,\ldots}^{\infty} E_n = 0 \qquad (34)$$

If equation (34) gives a minimum value of k that is lower than that obtained from equation (33), the stiffener will remain straight and the plate will buckle antisymmetrically.

Further applications.- It is evident that the method given may be used to take into account any number of stiffeners with arbitrary spacing. Furthermore, the effect of stiffener torsional restraint can also be included by adding to the energy expression the twisting energy of the stiffener and introducing conditions of rotational continuity between plate and stiffener. The method can be extended to analyze stiffened plate problems in which double Fourier series must be used (for example, a finite rectangular stiffened plate, clamped along all edges); in such problems, approximate upper- and lower-limit solutions, rather than exact solutions, may be expected. Stability problems involving plate-stiffener combinations in shear, or in combined compression and shear, might also be conveniently solved by the present method.

CONCLUDING REMARKS

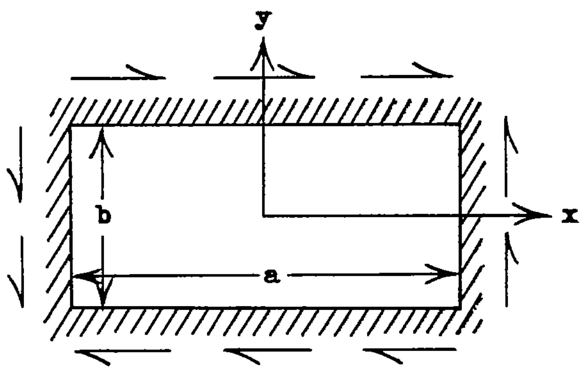
Elementary examples have been given to demonstrate the application of the Lagrangian multiplier method to the elastic-stability analysis of (a) flat rectangular clamped plates in shear and (b) plate-stiffener combinations. Exact solutions were obtained for the examples considered; for other problems, such as the shear buckling of a finite clamped plate (appendix A), approximate solutions may be obtained in two different ways providing upper and lower limits to the true value of the buckling stresses.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., September 12, 1947

APPENDIX A

BUCKLING IN SHEAR OF A CLAMPED RECTANGULAR PLATE

A thin rectangular plate clamped along all four sides and loaded in shear along the edges is illustrated in the following sketch:



The problem is to determine the critical value of the shear stress under which a plate of given aspect ratio begins to buckle.

Approximate analyses of this problem have been given by Smith (reference 6) and Iguchi (reference 7). Smith uses the Rayleigh-Ritz method and hence obtains an upper-limit solution, whereas Iguchi uses a method that leaves the type of solution unspecified. Both upper and lower limits to the true buckling stress may be obtained by the present Lagrangian multiplier solution.

The buckling configuration may be either symmetrical or antisymmetrical about the plate midpoint, depending on which buckling mode corresponds to a lower critical stress; the two buckling patterns will be considered separately.

Symmetrical Buckling

Boundary conditions.- The boundary conditions of the problem (see preceding sketch) are:

Zero deflection, all edges:

$$w\left(x, \frac{b}{2}\right) = w\left(x, -\frac{b}{2}\right) = 0 \qquad \text{(A1a)}$$

$$w\left(\frac{a}{2}, y\right) = w\left(-\frac{a}{2}, y\right) = 0 \qquad \text{(A1b)}$$

Zero slope, all edges:

$$\frac{\partial w}{\partial y}\left(x, \frac{b}{2}\right) = \frac{\partial w}{\partial y}\left(x, -\frac{b}{2}\right) = 0 \qquad \text{(A2a)}$$

$$\frac{\partial w}{\partial x}\left(\frac{a}{2}, y\right) = \frac{\partial w}{\partial x}\left(-\frac{a}{2}, y\right) = 0 \qquad \text{(A2b)}$$

Fourier expansions.– In appendix B, conditions to be considered in choosing Fourier series for use in stability analysis are discussed. On the basis of this discussion the following expansion was chosen to represent the symmetrical buckled surface:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{2m\pi x}{a} \sin\frac{2n\pi y}{b}$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn} \cos\frac{2m\pi x}{a} \cos\frac{2n\pi y}{b} \qquad \text{(A3)}$$

Energy expressions.– Substituting the expansion for w (equation (A3)) into equations (5) and (6) gives

$$V = 2D\pi^4 ab \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2\Big[a_{mn}^2\left(1 - \delta_{m0} - \delta_{0n}\right)$$

$$+ d_{mn}^2\left(1 + \delta_{m0} + \delta_{0n}\right)\Big]$$

$$T = 2\tau t\pi^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mn a_{mn} d_{mn}$$

Then

$$V - T = \frac{2D\pi^4 b}{a^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{1}{2} A_{mn} \left[a_{mn}^2 \left(1 - \delta_{m0} - \delta_{0n}\right) + d_{mn}^2 \right] - k_s \beta^3 mn a_{mn} d_{mn} \right\} \tag{A4}$$

where

$$A_{mn} = 2\left(m^2 + n^2\beta^2\right)^2 \left(1 + \delta_{m0} + \delta_{0n}\right)$$

Note that $V - T$ is independent of d_{00} since $A_{00} = 0$.

Constraining relationships.– In order to satisfy the boundary conditions of zero deflection (equations (A1a) and (A1b)) it is necessary to impose the constraining relationships

$$\sum_{m=0}^{\infty} (-1)^m d_{mj} = 0 \qquad (j = 0,1,2,\ldots) \tag{A5a}$$

$$\sum_{n=0}^{\infty} (-1)^n d_{in} = 0 \qquad (i = 0,1,2,\ldots) \tag{A5b}$$

Similarly, in order to satisfy the zero slope conditions (equations (A2a) and (A2b)), it must be true that

$$\sum_{m=1}^{\infty} m(-1)^m a_{mj} = 0 \qquad (j = 1,2,3,\ldots) \tag{A6a}$$

$$\sum_{n=1}^{\infty} n(-1)^n a_{in} = 0 \qquad (i = 1,2,3,\ldots) \tag{A6b}$$

Equations (A5a) and (A5b) contain the term d_{00}, which is missing from the expression for $V - T$. The term d_{00} may be eliminated from the constraint relationships by subtracting the first of equations (A5b), the equation for $i = 0$, from the first of equations (A5a), the equation for $j = 0$. The final set of necessary constraining relationships then becomes

$$\left.\begin{aligned} &\sum_{m=1}^{\infty} (-1)^m d_{m0} - \sum_{n=1}^{\infty} (-1)^n d_{0n} = 0 \\ &\sum_{m=0}^{\infty} (-1)^m d_{mj} = 0 \qquad (j = 1,2,3,\ldots) \\ &\sum_{n=0}^{\infty} (-1)^n d_{in} = 0 \qquad (i = 1,2,3,\ldots) \end{aligned}\right\} \qquad (A7a)$$

$$\left.\begin{aligned} &\sum_{m=1}^{\infty} m(-1)^m a_{mj} = 0 \qquad (j = 1,2,3,\ldots) \\ &\sum_{n=1}^{\infty} n(-1)^n a_{in} = 0 \qquad (i = 1,2,3,\ldots) \end{aligned}\right\} \qquad (A7b)$$

Lower-limit solution.- As described in reference 1, a lower limit to the buckling stress can be found by minimizing the energy expression $V - T$ (equation (A4)) with respect to all the coefficients a_{mn} and d_{mn} but satisfying only some of the constraining relationships (A7a) and (A7b), say, as far as $i = p$ and $j = q$.

If $V - T$ is divided through by the constant term $\frac{2D\pi^4 b}{a^3}$, the function to be minimized becomes

$$G = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{1}{2} A_{mn} \left[a_{mn}^2 \left(1 - \delta_{m0} - \delta_{0n}\right) + d_{mn}^2 \right] - k_s \beta^3 mn a_{mn} d_{mn} \right\}$$

$$- \alpha \left[\sum_{m=1}^{\infty} (-1)^m d_{m0} - \sum_{n=1}^{\infty} (-1)^n d_{0n} \right]$$

$$- \sum_{j=1}^{q} \lambda_j \sum_{m=0}^{\infty} (-1)^m d_{mj} - \sum_{i=1}^{p} \mu_i \sum_{n=0}^{\infty} (-1)^n d_{in}$$

$$- \sum_{j=1}^{q} \zeta_j \sum_{m=1}^{\infty} m(-1)^m a_{mj} - \sum_{i=1}^{p} \eta_i \sum_{n=1}^{\infty} n(-1)^n a_{in} \tag{A8}$$

The quantities α, λ_j, μ_i, ζ_j, and η_i are Lagrangian multipliers. The minimizing equations become

$$\left.\begin{aligned} &\frac{\partial G}{\partial a_{mn}} = 0 \qquad (m,n = 1,2,3,\ldots) \\ &\frac{\partial G}{\partial d_{mn}} = 0 \qquad (m,n = 0,1,2,\ldots) \\ &\text{Equations (A7a) and (A7b) taken up to } i = p \text{ and } j = q \end{aligned}\right\} \tag{A9}$$

By evaluation,

$$\frac{\partial G}{\partial a_{mn}} = A_{mn} a_{mn} - k_s \beta^3 mn d_{mn} - \zeta_n m(-1)^m - \eta_m n(-1)^n = 0 \tag{A10}$$

$$\frac{\partial G}{\partial d_{mn}} = A_{mn} d_{mn} - k_s \beta^3 mn a_{mn} - \lambda_n (-1)^m - \mu_m (-1)^n = 0 \qquad (m,n \neq 0) \tag{A11}$$

$$\frac{\partial G}{\partial d_{m0}} = A_{m0} d_{m0} - \alpha(-1)^m - \mu_m = 0 \tag{A12}$$

$$\frac{\partial G}{\partial d_{0n}} = A_{0n}d_{0n} + \alpha(-1)^n - \lambda_n = 0 \tag{A13}$$

In equations (A10) to (A13), ζ_n and λ_n do not appear for values of $n > q$, nor do η_m and μ_m appear for values of $m > p$. When both $m > p$ and $n > q$, ζ_n, λ_n, η_m, and μ_m vanish from equations (A10) and (A11). Then one of two conditions is possible, either

$$A_{mn}^2 - k_s^2\beta^6m^2n^2 = 0$$

or

$$a_{mn} = d_{mn} = 0$$

The first alternative, however, for given values of $m > p$ and $n > q$ will ordinarily lead to a very high value of the buckling stress coefficient k_s. For the lowest buckling load, therefore, when $m > p$ and $n > q$,

$$a_{mn} = d_{mn} = 0$$

For the remaining a's and d's, solving equations (A10) and (A11) gives

$$a_{mn} = B_{mn}\left[m(-1)^m \zeta_n + n(-1)^n \eta_m\right] + C_{mn}\left[(-1)^m \lambda_n + (-1)^n \mu_m\right] \tag{A14a}$$

$$d_{mn} = B_{mn}\left[(-1)^m \lambda_n + (-1)^n \mu_m\right] + C_{mn}\left[m(-1)^m \zeta_n + n(-1)^n \eta_m\right] \tag{A14b}$$

where

$$B_{mn} = \frac{A_{mn}}{A_{mn}^2 - D_{mn}^2}$$

$$C_{mn} = \frac{D_{mn}}{A_{mn}^2 - D_{mn}^2}$$

$$D_{mn} = k\beta^3 mn$$

From equations (A12) and (A13),

$$d_{m0} = \frac{1}{A_{m0}}\left[\alpha(-1)^m + \mu_m\right] \qquad \text{(A15)}$$

$$d_{0n} = \frac{1}{A_{0n}}\left[-\alpha(-1)^n + \lambda_n\right] \qquad \text{(A16)}$$

It is to be emphasized that in equations (A14a) to (A16), for values of $m > p$

$$\eta_m = \mu_m = 0$$

and for values of $n > q$

$$\zeta_n = \lambda_n = 0$$

Substituting the values of a's and d's given by equations (A14a) to (A16) back into the constraining relationships (A7a) and (A7b) taken up to $j = q$ and $i = p$ gives

$$\left.\begin{aligned}
&\alpha\left(\sum_{m=1}^{\infty} B_{m0} + \sum_{n=1}^{\infty} B_{0n}\right) + \sum_{m=1}^{p} (-1)^{m} B_{m0}\mu_m - \sum_{n=1}^{q} (-1)^{n} B_{0n}\lambda_n = 0 \\
&\lambda_j \sum_{m=0}^{\infty} B_{mj} - \alpha(-1)^{j} B_{0j} + \sum_{m=1}^{p} (-1)^{m+j} B_{mj}\mu_m + \zeta_j \sum_{m=1}^{\infty} mC_{mj} + \sum_{m=1}^{p} j(-1)^{m+j} C_{mj}\eta_m = 0 \qquad (j = 1,2,3,..q) \\
&\sum_{n=1}^{q} (-1)^{i+n} B_{in}\lambda_n + \alpha(-1)^{i} B_{i0} + \mu_i \sum_{n=0}^{\infty} B_{in} + \sum_{n=1}^{q} i(-1)^{i+n} C_{in}\zeta_n + \eta_i \sum_{n=1}^{\infty} nC_{in} = 0 \qquad (i = 1,2,3,..p) \\
&\zeta_j \sum_{m=1}^{\infty} m^2 B_{mj} + \sum_{m=1}^{p} mj(-1)^{m+j} B_{mj}\eta_m + \lambda_j \sum_{m=1}^{\infty} mC_{mj} + \sum_{m=1}^{p} m(-1)^{m+j} C_{mj}\mu_m = 0 \qquad (j = 1,2,3,..q) \\
&\sum_{n=1}^{q} in(-1)^{i+n} B_{in}\zeta_n + \eta_i \sum_{n=1}^{\infty} n^2 B_{in} + \sum_{n=1}^{q} n(-1)^{i+n} C_{in}\lambda_n + \mu_i \sum_{n=1}^{\infty} nC_{in} = 0 \qquad (i = 1,2,3,..p)
\end{aligned}\right\} \text{(A17)}$$

In order for this set of $2(p + q) + 1$ equations to be compatible, the determinant of the coefficients of the Lagrangian multipliers must vanish. This requirement leads to a determinantal equation from which the critical value of the buckling coefficient may be found by trial. An example of an eleventh-order determinant, with $p = 3$ and $q = 2$, is shown in table 1.

Upper-limit solution.- The theory of the upper-limit solution in the Lagrangian multiplier method (reference 1) requires that some a's and d's arbitrarily be set equal to zero, that expression (A4) be minimized with respect to the remaining a's and d's, and that all the constraining relationships (A7a) and (A7b) be satisfied.

As a result of the necessity for satisfying all the constraint relationships, a redundancy exists among equations (A7a) (see reference 1); this redundancy can be removed by discarding the first of equations (A7a). That another redundancy exists in equations (A7b) may be shown as follows: If

$$\sum_{m=1}^{\infty} m(-1)^m a_{mj} = 0$$

is multiplied by $j(-1)^j$ and summed over j, the result is

$$\sum_{j=1}^{\infty} j(-1)^j \sum_{m=1}^{\infty} m(-1)^m a_{mj} = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} jm(-1)^{j+m} a_{mj} = 0 \qquad \text{(A18)}$$

and if

$$\sum_{n=1}^{\infty} n(-1)^n a_{in} = 0$$

is multiplied by $i(-1)^i$ and summed over i, the result is

$$\sum_{i=1}^{\infty} i(-1)^i \sum_{n=1}^{\infty} n(-1)^n a_{in} = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} in(-1)^{i+n} a_{in} = 0 \qquad \text{(A19)}$$

Equations (A18) and (A19) are identical; hence a redundancy exists, which may be removed by discarding one of equations (A7b), for example, the equation for $i = 1$.

With the elimination of the redundant conditions, the necessary constraint relationships become

$$\left.\begin{aligned} &\sum_{m=0}^{\infty}(-1)^m d_{mj} = 0 \qquad (j = 1,2,3,\ldots) \\ &\sum_{n=0}^{\infty}(-1)^n d_{in} = 0 \qquad (i = 1,2,3,\ldots) \end{aligned}\right\} \qquad \text{(A20a)}$$

$$\left.\begin{aligned} &\sum_{m=1}^{\infty} m(-1)^m a_{mj} = 0 \qquad (j = 1,2,3,\ldots) \\ &\sum_{n=1}^{\infty} n(-1)^n a_{in} = 0 \qquad (i = 2,3,4,\ldots) \end{aligned}\right\} \qquad \text{(A20b)}$$

The accuracy of the upper-limit result, as well as the ease of solution, depends in part on which Fourier coefficients are retained in the analysis. Several possible groupings of included terms were tried out in solutions for the special case of a square plate. The trials indicated that the optimum arrangement for practical applications was a finite rectangular array of cosine coefficients (d_{mn}) together with certain infinite rows and columns of sine coefficients (a_{mn}), as illustrated in table 2.

These limits on the existence of the coefficients can be expressed as follows:

$$d_{mn} = 0 \qquad (\text{when either } m > p \text{ or } n > q)$$

$$a_{mn} = 0 \qquad (\text{when both } m > p \text{ and } n > q)$$

When these limits are imposed, the constraint relationships (A20a) and (A20b) take the form

$$\left.\begin{aligned} &\sum_{m=0}^{p}(-1)^m d_{mj} = 0 \qquad (j = 1,2,3,..q) \\ &\sum_{n=0}^{q}(-1)^n d_{in} = 0 \qquad (i = 1,2,3,..p) \end{aligned}\right\} \qquad \text{(A21)}$$

$$\left.\begin{aligned}\sum_{m=1}^{\infty} m(-1)^m a_{mj} &= 0 \qquad (j = 1,2,3,\ldots q)\\ \sum_{m=1}^{p} m(-1)^m a_{mj} &= 0 \qquad (j = q+1,q+2,\ldots)\end{aligned}\right\} \quad \text{(A22a)}$$

$$\left.\begin{aligned}\sum_{n=1}^{\infty} n(-1)^n a_{in} &= 0 \qquad (i = 2,3,4,\ldots p)\\ \sum_{n=1}^{q} n(-1)^n a_{in} &= 0 \qquad (i = p+1,p+2,\ldots)\end{aligned}\right\} \quad \text{(A22b)}$$

The function to be minimized is

$$\begin{aligned}G = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{1}{2}A_{mn}\left[a_{mn}^{2}\left(1 - \delta_{m0} - \delta_{0n}\right) + d_{mn}^{2}\right] - k_s\beta^3 mn a_{mn} d_{mn}\right\}\\ &- \sum_{j=1}^{q}\lambda_j \sum_{m=0}^{p}(-1)^m d_{mj} - \sum_{i=1}^{p}\mu_i \sum_{n=0}^{q}(-1)^n d_{in}\\ &- \sum_{j=1}^{q}\zeta_j \sum_{m=1}^{\infty} m(-1)^m a_{mj} - \sum_{j=q+1}^{\infty}\zeta_j \sum_{m=1}^{p} m(-1)^m a_{mj}\\ &- \sum_{i=1}^{p}\eta_i \sum_{n=1}^{\infty} n(-1)^n a_{in} - \sum_{i=p+1}^{\infty}\eta_i \sum_{n=1}^{q} n(-1)^n a_{in}\end{aligned} \quad \text{(A23)}$$

Setting $\frac{\partial G}{\partial a_{mn}} = \frac{\partial G}{\partial d_{mn}} = 0$ then gives

$$d_{mn} = B_{mn}\left[(-1)^m \lambda_n + (-1)^n \mu_m\right] + C_{mn}\left[m(-1)^m \zeta_n + n(-1)^n \eta_m\right]$$

For $d_{mn} \neq 0$,

$$a_{mn} = B_{mn}\left[m(-1)^m \zeta_n + n(-1)^n \eta_m\right] + C_{mn}\left[(-1)^m \lambda_n + (-1)^n \mu_m\right]$$

For $d_{mn} = 0$,

$$a_{mn} = \frac{1}{A_{mn}}\left[m(-1)^m \zeta_n + n(-1)^n \eta_m\right]$$

Substituting these values back into constraint relationships (A21) through (A22b) gives

$$\lambda_j \sum_{m=0}^{p} B_{mj} + \sum_{m=0}^{p} (-1)^{m+j} B_{mj}\mu_m + \zeta_j \sum_{m=0}^{p} mC_{mj} + \sum_{m=0}^{p} (-1)^{m+j} jC_{mj}\eta_m = 0 \qquad (j = 1,2,3,..q) \tag{A24a}$$

$$\sum_{n=0}^{q} (-1)^{i+n} B_{in}\lambda_n + \mu_i \sum_{n=0}^{q} B_{in} + \sum_{n=0}^{q} (-1)^{i+n} iC_{in}\zeta_n + \eta_i \sum_{n=0}^{q} nC_{in} = 0 \qquad (i = 1,2,3,..p) \tag{A24b}$$

$$\zeta_j \sum_{m=1}^{p} m^2 B_{mj} + \zeta_j \sum_{m=p+1}^{\infty} \frac{m^2}{A_{mj}} + \sum_{m=1}^{p} (-1)^{m+j} mjB_{mj}\eta_m + \sum_{m=p+1}^{\infty} \frac{(-1)^{m+j} mj}{A_{mj}} \eta_m + \lambda_j \sum_{m=1}^{p} mC_{mj} + \sum_{m=1}^{p} (-1)^{m+j} mC_{mj}\mu_m = 0 \tag{A25a}$$

$$(j = 1, 2, 3 \ldots q)$$

$$\zeta_j \sum_{m=1}^{p} \frac{m^2}{A_{mj}} + \sum_{m=1}^{p} \frac{(-1)^{m+j}\, mj}{A_{mj}} \eta_m = 0 \qquad (j = q+1, q+2, \ldots) \tag{A25b}$$

$$\sum_{n=1}^{q} (-1)^{i+n}\, inB_{in}\zeta_n + \sum_{n=q+1}^{\infty} \frac{(-1)^{i+n}\, in}{A_{in}} \zeta_n - \eta_i \sum_{n=1}^{q} n^2 B_{in}$$

$$+ \eta_i \sum_{n=q+1}^{\infty} \frac{n^2}{A_{in}} + \sum_{n=1}^{q} (-1)^{i+n}\, nC_{in}\lambda_n + \mu_i \sum_{n=1}^{q} nC_{in} = 0 \tag{A26a}$$

$$(i = 2,3,4,..p)$$

$$\sum_{n=1}^{q} \frac{(-1)^{i+n}\, in}{A_{in}} \zeta_n + \eta_i \sum_{n=1}^{q} \frac{n^2}{A_{in}} = 0 \qquad (i = p+1, p+2, \ldots) \tag{A26b}$$

These equations involve all the multipliers ζ_j and η_i. They can be reduced, however, to a set of equations including only the multipliers up to $i = p$, $j = q$ as follows:

From equation (A25b), for $j = q+1, q+2, \ldots$

$$\zeta_j = \frac{-\sum_{m=1}^{p} \frac{(-1)^{m+j}\, mj}{A_{mj}} \eta_m}{\sum_{m=1}^{p} \frac{m^2}{A_{mj}}}$$

From equation (A26b), for $i = p+1, p+2, \ldots$

$$\eta_i = \frac{-\sum_{n=1}^{q} \frac{(-1)^{i+n}\, in}{A_{in}} \zeta_n}{\sum_{n=1}^{q} \frac{n^2}{A_{in}}}$$

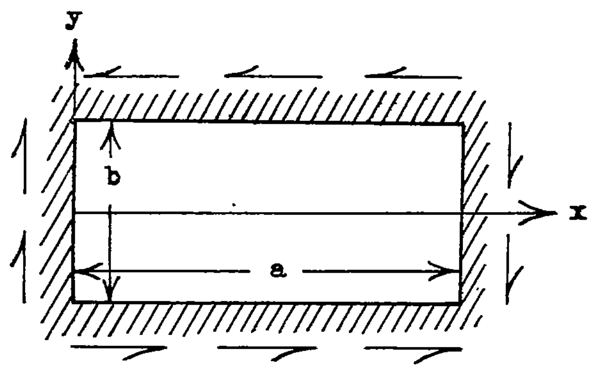
Substituting these expressions for ζ_j and η_i into equations (A26a) and (A25a) respectively gives, along with equations (A24a) and (A24b), the final stability equations:

$$
\left.
\begin{aligned}
&\eta_i \sum_{n=1}^{q} n^2 B_{in} + \sum_{n=1}^{q} n(-1)^{i+n} C_{in}\lambda_n + \sum_{n=1}^{q} ni(-1)^{i+n} B_{in}\zeta_n + \mu_i \sum_{n=1}^{q} nC_{in} + \eta_i \sum_{n=q+1}^{\infty} \frac{n^2}{A_{in}} \\
&\qquad - \sum_{m=2}^{p} \sum_{n=q+1}^{\infty} \frac{\dfrac{min^2(-1)^{m+1}}{A_{in}A_{mn}}\,\eta_m}{\displaystyle\sum_{m=1}^{p} \frac{m^2}{A_{mn}}} = 0 \qquad (i = 2,3,4,..p) \\
&\zeta_j \sum_{m=1}^{p} m^2 B_{mj} + \sum_{m=2}^{p} mj(-1)^{m+j} B_{mj}\eta_m + \lambda_j \sum_{m=1}^{p} mC_{mj} + \sum_{m=1}^{p} m(-1)^{m+j} C_{mj}\mu_m + \zeta_j \sum_{m=p+1}^{\infty} \frac{m^2}{A_{mj}} \\
&\qquad - \sum_{n=1}^{q} \sum_{m=p+1}^{\infty} \frac{\dfrac{m^2 nj(-1)^{n+j}}{A_{mj}A_{mn}}\,\zeta_n}{\displaystyle\sum_{n=1}^{q} \frac{n^2}{A_{mn}}} = 0 \qquad (j = 1,2,3,..q) \\
&\lambda_j \sum_{m=0}^{p} B_{mj} + \sum_{m=1}^{p} (-1)^{m+j} B_{mj}\mu_m + \zeta_j \sum_{m=1}^{p} mC_{mj} + \sum_{m=2}^{p} j(-1)^{m+j} C_{mj}\eta_m = 0 \qquad (j = 1,2,3,..q) \\
&\mu_i \sum_{n=0}^{q} B_{in} + \sum_{n=1}^{q} (-1)^{i+n} B_{in}\lambda_n + \sum_{n=1}^{q} i(-1)^{i+n} C_{in}\zeta_n + \eta_i\left(1 - \delta_{1i}\right) \sum_{n=1}^{q} nC_{in} = 0 \qquad (i = 1,2,3,..p)
\end{aligned}
\right\} \quad \text{(A27)}
$$

The stability criterion is a determinant based on equations (A27), as in the lower limit solution. An example of a ninth-order determinant, with $p = 3$ and $q = 2$, is shown in table 3.

Antisymmetrical Buckling

For the analysis of buckling antisymmetrical about the plate midpoint, the origin of coordinates was taken as shown in the following sketch:



Boundary conditions.— The boundary conditions are now expressed as:
Zero deflection, all edges

$$w\left(x, \frac{b}{2}\right) = w\left(x, -\frac{b}{2}\right) = 0$$

$$w(o, y) = w(a, y) = 0$$

Zero slope, all edges:

$$\frac{\partial w}{\partial y}\left(x, \frac{b}{2}\right) = \frac{\partial w}{\partial y}\left(x, -\frac{b}{2}\right) = 0$$

$$\frac{\partial w}{\partial x}(o, y) = \frac{\partial w}{\partial x}(a, y) = 0$$

Fourier expansion.- In accordance with the ideas summarized in appendix A, the chosen expansion is

$$w = \sum_{m=1,3}^{\infty} \sum_{n=0}^{\infty} b_{mn} \cos \frac{m\pi x}{a} \cos \frac{2n\pi y}{b}$$

$$+ \sum_{m=1,3}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{a} \sin \frac{2n\pi y}{b} \qquad \text{(A28)}$$

Energy expression.- When the expansion (A28) is substituted in the general formulas (5) and (6), the result for the total energy expression is

$$V - T = \frac{D\pi^4 b}{4a^3} \left\{ \sum_{m=1,3}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2} A'_{mn} \left[b_{mn}^2 + c_{mn}^2 \left(1 - \delta_{0n}\right) \right] - 4k_s \beta^3 mn b_{mn} c_{mn} \right\} \text{(A29)}$$

where

$$A'_{mn} = \left(m^2 + 4\beta^2 n^2\right)^2 \left(1 + \delta_{0n}\right)$$

Constraint relationships.- The necessity of satisfying the boundary conditions imposes the following constraining relationships on the coefficients:

$$\left.\begin{aligned} \sum_{n=0}^{\infty} (-1)^n b_{in} &= 0 \qquad (i = 1,3,5,\ldots) \\ \sum_{m=1,3}^{\infty} b_{mj} &= 0 \qquad (j = 0,1,2,\ldots) \end{aligned}\right\} \qquad \text{(A30a)}$$

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} n(-1)^n c_{in} = 0 \qquad (i = 1,3,5,\ldots) \\ &\sum_{m=1,3}^{\infty} mc_{mj} = 0 \qquad (j = 1,2,3,\ldots) \end{aligned}\right\} \qquad \text{(A30b)}$$

Note that there is no term b_{00} to be removed from these expressions.

Lower-limit solution.- The function to be minimized becomes

$$G' = \sum_{m=1,3}^{\infty}\sum_{n=0}^{\infty} \tfrac{1}{2}A'_{mn}\left[b_{mn}^2 + c_{mn}^2\left(1 - \delta_{0n}\right)\right]$$

$$- 4k_s\beta^3 mnb_{mn}c_{mn} - \sum_{i=1,3}^{p} \mu'_i \sum_{n=0}^{\infty} (-1)^n b_{in}$$

$$- \sum_{j=0}^{q} \lambda'_j \sum_{m=1,3}^{\infty} b_{mj} - \sum_{i=1,3}^{p} \eta'_i \sum_{n=1}^{\infty} n(-1)^n c_{in}$$

$$- \sum_{j=1}^{q} \zeta'_j \sum_{m=1,3}^{\infty} mc_{mj}$$

The stability equations analogous to equations (A17) for the symmetrical case take the form

$$\lambda'_j \sum_{m=1,3}^{\infty} B'_{mj} + \sum_{m=1,3}^{p} (-1)^j B'_{mj}\mu'_m + \zeta'_j\left(1 - \delta_{0j}\right) \sum_{m=1,3}^{\infty} mC'_{mj} + \sum_{m=1,3}^{p} j(-1)^j C'_{mj}\eta'_m = 0$$

$$(j = 0,1,2,..q)$$

$$\sum_{n=0}^{q} (-1)^n B'_{in}\lambda'_n + \mu'_i \sum_{n=0}^{\infty} B'_{in} + \sum_{n=1}^{q} i(-1)^n C'_{in}\zeta'_n + \eta'_i \sum_{n=1}^{\infty} nC'_{in} = 0 \qquad (i = 1,3,5,..p)$$

$$\zeta'_j \sum_{m=1,3}^{\infty} m^2 B'_{mj} + \sum_{m=1,3}^{p} mj(-1)^j B'_{mj}\eta'_m + \lambda'_j \sum_{m=1,3}^{\infty} mC'_{mj} + \sum_{m=1,3}^{p} m(-1)^j C'_{mj}\mu'_m = 0$$

$$(j = 1,2,3,..q)$$

$$\sum_{n=1}^{q} (-1)^n inB'_{in}\zeta'_n + \eta'_i \sum_{n=1}^{\infty} n^2 B'_{in} + \sum_{n=1}^{q} n(-1)^n C'_{in}\lambda'_n + \mu'_i \sum_{n=1}^{\infty} nC'_{in} = 0$$

$$(i = 1,3,5,..p)$$

(A31)

From these equations a determinant can be formed to give the critical value of the buckling coefficient.

Upper-limit solution.- Since all of constraint relations (A30a) and (A30b) are to be satisfied according to the theory of an upper-limit solution, it can be seen that one of equations (A30a) and one of equations (A30b) are redundant. These redundancies can be proved in the same manner as for the symmetrical case and are eliminated by discarding the equation for $i = 1$ of equations (A30a) and also the equation for $j = 1$ of equations (A30b). As was done in the upper-limit solution for the symmetrical case, a rectangular array of cosine coefficients and infinite strips of sine coefficients will be retained; thus, let

$$b_{mn} = 0 \qquad (\text{when either } m > p \text{ or } n > q)$$

and

$$c_{mn} = 0 \qquad (\text{when both } m > p \text{ and } n > q)$$

The final set of constraining relationships then becomes

$$\sum_{n=0}^{q} (-1)^n b_{in} = 0 \qquad (i = 3,5,7,..p)$$

$$\sum_{m=1,3}^{p} b_{mj} = 0 \qquad (j = 0,1,2,..q)$$

$$\sum_{n=1}^{\infty} n(-1)^n b_{in} = 0 \qquad (i = 1,3,5,..p)$$

$$\sum_{n=1}^{q} n(-1)^n b_{in} = 0 \qquad (i = p+2,p+4,...)$$

$$\sum_{m=1,3}^{\infty} mc_{mj} = 0 \qquad (j = 2,3,4,..q)$$

$$\sum_{m=1}^{p} mc_{mj} = 0 \qquad (j = q+1,q+2,...)$$

Performing operations similar to those for the symmetrical case gives as the final stability equations:

$$\eta'_i \sum_{n=1}^{q} n^2 B'_{in} + \sum_{n=2}^{q} in(-1)^n B'_{in}\zeta'_n + \sum_{n=1}^{q} n(-1)^n C'_{in}\lambda'_n + \mu'_i\left(1 - \delta_{1i}\right)\sum_{n=1}^{q} nC'_{in}$$

$$+ \eta_i \sum_{n=q+1}^{\infty} \frac{n^2}{A'_{in}} + \sum_{m=1,3}^{p}\ \sum_{n=q+1}^{\infty} \frac{\dfrac{n^2 mi}{A'_{in}A'_{mn}}\eta'_m}{\displaystyle\sum_{m=1,3}^{p} \frac{m^2}{A'_{mn}}} = 0 \qquad (i = 1,3,5,..p)$$

$$\zeta'_j \sum_{m=1,3}^{p} m^2 B'_{mj} + \sum_{m=1,3}^{p} mj(-1)^j B'_{mj}\eta'_m + \lambda'_j \sum_{m=1,3}^{p} mC'_{mj} + \sum_{m=3,5}^{p} m(-1)^j C'_{mj}\mu'_m$$

$$+ \zeta'_j \sum_{m=p+2}^{\infty} \frac{m^2}{A'_{mj}} - \sum_{n=2}^{q}\ \sum_{m=p+2,p+4}^{\infty} \frac{\dfrac{m^2 nj(-1)^{n+j}}{A'_{mj}A'_{mn}}\zeta'_n}{\displaystyle\sum_{n=1}^{q} \frac{n^2}{A'_{mn}}} = 0 \qquad (j = 2,3,4,..q)$$

$$\mu'_i \sum_{n=0}^{q} B'_{in} + \sum_{n=0}^{q} (-1)^n B'_{in}\lambda'_n + \sum_{n=2}^{q} i(-1)^n C'_{in}\zeta'_n + \eta'_i \sum_{n=1}^{q} nC'_{in} = 0 \qquad (i = 3,5,7,..p)$$

$$\lambda'_j \sum_{m=1,3}^{p} B'_{mj} + \sum_{m=3,5}^{p} (-1)^j B'_{mj}\mu'_m + \zeta'_j\left(1 - \delta_{0j} - \delta_{1j}\right) \sum_{m=1,3}^{p} mC'_{mj} + \sum_{m=1,3}^{p} j(-1)^j C'_{mj}\eta_m = 0$$

$$(j = 0,1,2,..q)$$

(A32)

Numerical Example

Upper and lower limits to the symmetrical buckling stress of a square plate have been computed; certain simplifications are possible in the operations for that case. From the symmetry about both diagonals present in the buckle pattern of a square plate, it can be shown that

$$a_{mn} = a_{nm}$$

$$d_{mn} = d_{nm}$$

where the a's and d's are the Fourier coefficients of expansion (A3). Then, in the stability equations (A17) and (A27),

$$\alpha = 0$$

and, for $i = j$,

$$\lambda_j = \mu_i$$

$$\zeta_j = \eta_i$$

The following upper and lower limits to the true symmetrical buckling stress coefficient were computed for the square plate, with $p = q = 3$ in equations (A17) and (A27):

Lower limit $k_s = 14.64$

Upper limit $k_s = 14.79$

Thus, the true value of k_s must differ by less than 1 percent from the mean of the upper and lower limits. The numerical results for the square plate given by Smith (reference 6) and Iguchi (reference 7) are 14.72 and 14.58, respectively.

APPENDIX B

FOURIER SERIES IN STABILITY ANALYSIS

The primary condition in the selection of a Fourier expansion to be used as a deflection function is that it be a complete set over the region in question. The choice of the best series for a given buckling mode, however, depends on certain other considerations which will be outlined herein, with emphasis on application of series in the Lagrangian multiplier method.

Series in one variable.– It can be shown that any arbitrary function $f(x)$ in the interval $(0,a)$ may be expressed as the sum of two other functions, one symmetrical and the other antisymmetrical about the midpoint $x = \frac{a}{2}$.

If

$$f(x) = S(x) + A(x)$$

where

$$S(x) = \frac{1}{2}f(x) + \frac{1}{2}f(a - x)$$

and

$$A(x) = \frac{1}{2}f(x) - \frac{1}{2}f(a - x)$$

it is true that $S(x) = S(a - x)$ is symmetrical and $A(x) = - A(a - x)$ is antisymmetrical.

When the function $f(x)$ is to be represented by Fourier series, both $S(x)$ and $A(x)$ can be given by a series of either sines or cosines, as is shown in figure 1. Thus

$$S(x) = \sum_{m} a_m S_m(x)$$

and

$$A(x) = \sum_{m} b_m A_m(x)$$

Since $f(x) = S(x) + A(x)$, and both $S(x)$ and $A(x)$ are capable of representation by either sines or cosines, there are thus (2×2) or 4 possible general series for $f(x)$ that can be constructed from the functions of figure 1.

<u>Series in two variables</u>.- Let it be required to represent $f(x,y)$ in the region $(0,a),(0,b)$ by a double Fourier series. This double series representation can be derived by writing a single Fourier series in x and letting the Fourier coefficients depend on y; thus,

$$f(x,y) = \sum_m \phi_m(y) S_m(x) + \sum_m \psi_m(y) A_m(x)$$

Each of the functions ϕ_m and ψ_m can in turn be given by Fourier series in y, in the region $(0,b)$ as follows

$$\phi_m(y) = \sum_n c_{mn} A_n(y) + \sum_n d_{mn} S_n(y)$$

$$\psi_m(y) = \sum_n a_{mn} A_n(y) + \sum_n b_{mn} S_n(y)$$

Then $f(x,y)$ becomes

$$f(x,y) = \sum_m \sum_n a_{mn} A_m(x) A_n(y) + \sum_m \sum_n b_{mn} A_m(x) S_n(y)$$

$$+ \sum_m \sum_n c_{mn} S_m(x) A_n(y) + \sum_m \sum_n d_{mn} S_m(x) S_n(y)$$

It can be proved that

$$f(x,y) = S(x,y) + A(x,y)$$

where

$$S(x,y) = \sum_m \sum_n a_{mn}A_m(x)A_n(y) + \sum_m \sum_n d_{mn}S_m(x)S_n(y)$$

is symmetrical about the midpoint of the region and

$$A(x,y) = \sum_m \sum_n b_{mn}A_m(x)S_n(y) + \sum_m \sum_n c_{mn}S_m(x)A_n(y)$$

is antisymmetrical about the same point.

Since the terms S_m, S_n, A_m, A_n in these expressions can be given by either sines or cosines from the group depicted in figure 1, there are thus $(2 \times 2)^2$ or 16 combinations of Fourier series by which a symmetrical or antisymmetrical buckling pattern may be represented. For a general pattern, neither symmetrical nor antisymmetrical, a choice of 16 × 16 or 256 possible series is available.

Choice of series.– The simplest application of the Rayleigh-Ritz energy method occurs when the series chosen for the buckled surface not only satisfies the boundary conditions term by term, but also leads to integrals of products of orthogonal functions in the evaluation of the energy expressions. This occurs, for example, when a double sine series is used in the analysis of the compressive buckling of a simply supported plate (reference 8).

In more difficult buckling problems, such as clamped-plate problems, the simplicity of this calculation can be approached by choosing series which do not satisfy the geometric boundary conditions term by term but which do have the desirable orthogonal properties; the Lagrangian multiplier method is then used to make the series as a whole satisfy the boundary conditions. Thus, in reference 1, a double cosine series was employed in the Lagrangian multiplier method of finding the compressive buckling load of a clamped plate. One important consideration to be kept in mind in choosing a particular series for use in this method is that, for reasons of rapid convergence, the use of cosines rather than sines for clamped-edge deflection surfaces is preferable.

For the case of symmetrical shear buckling of a rectangular clamped plate, the following series (equation (A3)) was chosen (origin at plate midpoint):

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{2m\pi x}{a} \sin \frac{2n\pi y}{b} + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{mn} \cos \frac{2m\pi x}{a} \cos \frac{2n\pi y}{b}$$

This expansion, having the general form

$$\sum_{m} \sum_{n} a_{mn} A_m(x) A_n(y) + \sum_{m} \sum_{n} d_{mn} S_m(x) S_n(y)$$

is a combination of symmetrical cosines and antisymmetrical sines. As is desired, the energy expressions involve the products of orthogonal functions; the boundary conditions of zero slope and deflection are applied to the sines and cosines, respectively, by means of Lagrangian multipliers. It was necessary to include sines as well as cosines in the series in order to achieve the desired orthogonal properties; however, the portion of the deflection function symbolized by $\sum_{m} \sum_{n} a_{mn} S_m(x) S_n(y)$ was intuitively believed to be the more important and therefore, for reasons of rapid convergence, was chosen to be in terms of cosines rather than sines. Similar considerations dictated the choice of equation (A28) as the deflection function for antisymmetrical buckling.

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TABLE 1.– LOWER-LIMIT STABILITY CRITERION FOR SYMMETRICAL BUCKLING, WITH $p = 3$ AND $q = 2$

α	λ_1	λ_2	μ_1	μ_2	μ_3	ζ_1	ζ_2	η_1	η_2	η_3
$\sum_{m=1}^{\infty} B_{m0} + \sum_{n=1}^{\infty} B_{0n}$	B_{01}	$-B_{02}$	$-B_{10}$	B_{20}	$-B_{30}$	0	0	0	0	0
B_{01}	$\sum_{m=0}^{\infty} B_{m1}$	0	B_{11}	$-B_{21}$	B_{31}	$\sum_{m=1}^{\infty} mC_{m1}$	0	C_{11}	$-C_{21}$	C_{31}
$-B_{02}$	0	$\sum_{m=0}^{\infty} B_{m2}$	$-B_{12}$	B_{22}	$-B_{32}$	0	$\sum_{m=1}^{\infty} mC_{m2}$	$-2C_{12}$	$2C_{22}$	$-2C_{32}$
$-B_{10}$	B_{11}	$-B_{12}$	$\sum_{n=0}^{\infty} B_{1n}$	0	0	C_{11}	$-C_{12}$	$\sum_{n=1}^{\infty} nC_{1n}$	0	0
B_{20}	$-B_{21}$	B_{22}	0	$\sum_{n=0}^{\infty} B_{2n}$	0	$-2C_{21}$	$2C_{22}$	0	$\sum_{n=1}^{\infty} nC_{2n}$	0
$-B_{30}$	B_{31}	$-B_{32}$	0	0	$\sum_{n=0}^{\infty} B_{3n}$	$3C_{31}$	$-3C_{32}$	0	0	$\sum_{n=1}^{\infty} nC_{3n}$
0	$\sum_{m=1}^{\infty} mC_{m1}$	0	C_{11}	$-2C_{21}$	$3C_{31}$	$\sum_{m=1}^{\infty} m^2B_{m1}$	0	B_{11}	$-2B_{21}$	$3B_{31}$
0	0	$\sum_{m=1}^{\infty} mC_{m2}$	$-C_{12}$	$2C_{22}$	$-3C_{32}$	0	$\sum_{m=1}^{\infty} m^2B_{m2}$	$-2B_{12}$	$4B_{22}$	$-6B_{32}$
0	C_{11}	$-2C_{12}$	$\sum_{n=1}^{\infty} nC_{1n}$	0	0	B_{11}	$-2B_{12}$	$\sum_{n=1}^{\infty} n^2B_{1n}$	0	0
0	$-C_{21}$	$2C_{22}$	0	$\sum_{n=1}^{\infty} nC_{2n}$	0	$-2B_{21}$	$4B_{22}$	0	$\sum_{n=1}^{\infty} n^2B_{2n}$	0
0	C_{31}	$-2C_{32}$	0	0	$\sum_{n=1}^{\infty} nC_{3n}$	$3B_{31}$	$-6B_{32}$	0	0	$\sum_{n=1}^{\infty} n^2B_{3n}$

$= 0$

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TABLE 2.– CHOSEN ARRAYS OF SINE AND COSINE TERMS FOR UPPER-LIMIT SOLUTION OF SYMMETRICAL BUCKLING, WITH $p = 3$ AND $q = 2$

n \ m	1	2	3	4	5	6	7	8	. . .
1	a_{11}	a_{21}	a_{31}	a_{41}	a_{51}	a_{61}	a_{71}	a_{81}	⟶
2	a_{12}	a_{22}	a_{32}	a_{42}	a_{52}	a_{62}	a_{72}	a_{82}	⟶
3	a_{13}	a_{23}	a_{33}						
4	a_{14}	a_{24}	a_{34}						
5	a_{15}	a_{25}	a_{35}						
6	a_{16}	a_{26}	a_{36}						
. . .	↓	↓	↓						

n \ m	0	1	2	3	4	5	6	7	. . .
0	d_{00}	d_{10}	d_{20}	d_{30}					
1	d_{01}	d_{11}	d_{21}	d_{31}					
2	d_{02}	d_{12}	d_{22}	d_{32}					
3									
4									
. . .									

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TABLE 3.– UPPER-LIMIT STABILITY CRITERION FOR SYMMETRICAL BUCKLING

$$\left[p = 3 \text{ AND } q = 2;\ K_m = \frac{1}{A_{m1}} + \frac{4}{A_{m2}} \text{ AND } C_n = \frac{1}{A_{1n}} + \frac{4}{A_{2n}} + \frac{9}{A_{3n}}\right]$$

η_2	η_3	ζ_1	ζ_2	λ_1	λ_2	μ_1	μ_2	μ_3
$\sum_{n=1}^{2} n^2B_{2n} - \sum_{n=3}^{\infty}\frac{n^2}{A_{2n}} - \sum_{n=3}^{\infty}\frac{4n^2}{A_{2n}^2C_n}$	$\sum_{n=3}^{\infty}\frac{6n^2}{A_{2n}A_{3n}C_n}$	$-2B_{21}$	$4B_{22}$	$-C_{21}$	$2C_{22}$	0	$\sum_{n=1}^{2} nC_{2n}$	0
$\sum_{n=3}^{\infty}\frac{6n^2}{A_{2n}A_{3n}C_n}$	$\sum_{n=1}^{2} n^2B_{3n} - \sum_{n=3}^{\infty}\frac{n^2}{A_{3n}} - \sum_{n=3}^{\infty}\frac{9n^2}{A_{3n}^2C_n}$	$3B_{31}$	$-6B_{32}$	C_{31}	$-2C_{32}$	0	0	$\sum_{n=1}^{2} nC_{3n}$
$-2B_{21}$	$3B_{31}$	$\sum_{m=1}^{3} m^2B_{m1} - \sum_{m=4}^{\infty}\frac{m^2}{A_{m1}} - \sum_{m=4}^{\infty}\frac{m^2}{A_{m1}^2K_m}$	$\sum_{m=4}^{\infty}\frac{2m^2}{A_{m1}A_{m2}K_m}$	$\sum_{m=1}^{3} mC_{m1}$	0	C_{11}	$-2C_{21}$	$3C_{31}$
$4B_{22}$	$-6B_{32}$	$\sum_{m=4}^{\infty}\frac{2m^2}{A_{m1}A_{m2}K_m}$	$\sum_{m=1}^{3} m^2B_{m2} - \sum_{m=4}^{\infty}\frac{m^2}{A_{m2}} - \sum_{m=4}^{\infty}\frac{4m^2}{A_{m2}^2K_m}$	0	$\sum_{m=1}^{3} mC_{m2}$	$-C_{12}$	$2C_{22}$	$-3C_{32}$
$-C_{21}$	C_{31}	$\sum_{m=1}^{3} mC_{m1}$	0	$\sum_{m=0}^{3} B_{m1}$	0	B_{11}	$-B_{21}$	B_{31}
$2C_{22}$	$-2C_{32}$	0	$\sum_{m=1}^{3} mC_{m2}$	0	$\sum_{m=0}^{3} B_{m2}$	B_{12}	B_{22}	$-B_{32}$
0	0	C_{11}	$-C_{12}$	B_{11}	$-B_{12}$	$\sum_{n=0}^{2} B_{1n}$	0	0
$\sum_{n=1}^{2} nC_{2n}$	0	$-2C_{21}$	$2C_{22}$	$-B_{21}$	B_{22}	0	$\sum_{n=0}^{2} B_{2n}$	0
0	$\sum_{n=1}^{2} nC_{3n}$	$3C_{31}$	$-3C_{32}$	B_{31}	$-B_{32}$	0	0	$\sum_{n=0}^{2} B_{3n}$

= 0

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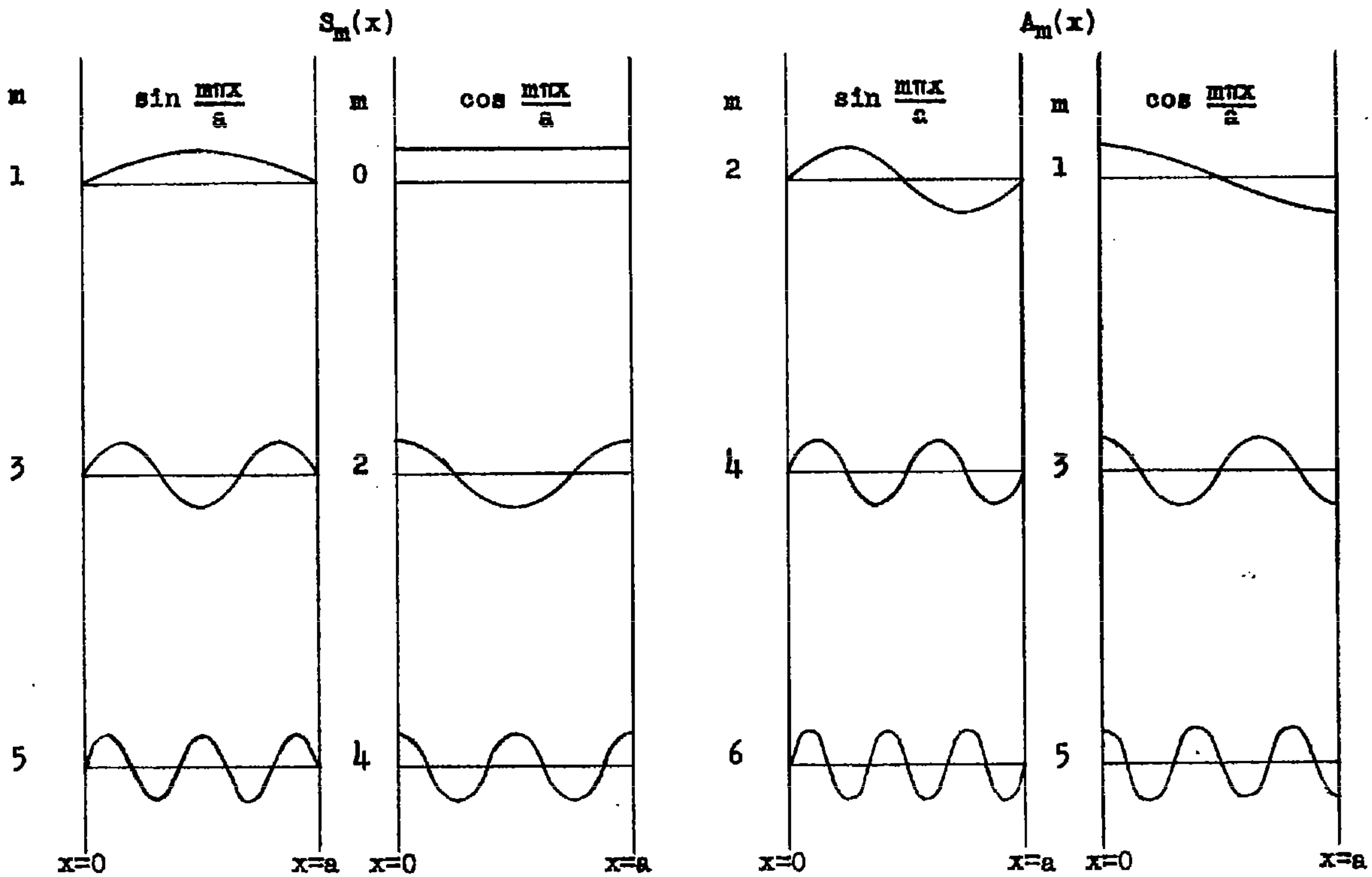

(a) Complete symmetrical sets.

(b) Complete antisymmetrical sets.

Figure 1.- Complete sets of trigonometric functions.

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