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MATRIX METHODS FOR CALCULATING CANTILEVER-BEAM DEFLECTIONS

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SUMMARY

The method of numerical integration for calculation of beam deflection is presented in matrix form to give it the advantages which are inherent in an influence-coefficient method. Both the scalar method of numerical integration and the influence-coefficient method may be improved by introducing weighting matrices. Only distributed loading is considered. Examples are presented to show that the use of weighting matrices reduces the calculation time required to obtain a desired degree of accuracy.

INTRODUCTION

Two methods for calculating small deflections of non-uniform beams are in general use: a method of scalar numerical integration and a method employing influence coefficients. Either method may be expressed in matrix form to obtain maximum efficiency in the use of calculating machines.

The method of scalar numerical integration for obtaining deflections gives all quantities that may be needed in stress analysis; that is, shears, bending moments, section torques, curvatures, twists, slopes, and deflections. (See, for examples, references 1 and 2.) In problems such as vibration and aeroelasticity where only the deflections are of direct importance, matrix methods are preferable since they provide a direct linear relation between loads and deflections and avoid the intermediate calculations required in the scalar method. The method of influence coefficients gives deflections due to concentrated loads through direct linear relations but requires first the calculation of the influence coefficients and is not suited without modifications to accurate determination of deflections due to distributed loadings.

A matrix method of numerical integration incorporating weighting matrices and especially formulated for obtaining deflections due to distributed bending or torsional loadings is a feature presented herein. This method is based upon an equivalence between distributed and concentrated loading obtained by assuming the loading curve to be a series of parabolic arcs. The weighting matrices for bending contain weighting numbers introduced in reference 1 and applied in reference 2. The weighting matrices for torsion have not been previously published. Weighting matrices permit the use of lower-order matrices than would be needed in commonly used procedures to obtain a desired accuracy.

As an introduction to the weighting methods, the commonly used methods of numerical integration and influence coefficients for distributed loadings, based on step loading distributions, are formulated in matrix notation. The weighting methods described include, in addition to numerical integration, procedures for using weighting matrices in conjunction with influence coefficients.

In all, four matrix methods of different degrees of accuracy are considered, and each method gives deflection as a linear function of distributed loading through an array of coefficients which might be called "influence coefficients for distributed loading" or "transformation coefficients" to distinguish them clearly from standard influence coefficients.

Results of the application of three of the methods to a uniform cantilever beam are compared with exact solutions.

SYMBOLS

a	segmental area beneath torsional load curve
$\bar{a}$	segmental area beneath twist curve (increment of rotation)
E	Young's modulus of elasticity
G	shear modulus of elasticity
I	moment of inertia of cross-sectional area
I_m	mass moment of inertia, per unit of length, about the elastic axis
J	torsion constant
K	influence function
k_{ij}	influence coefficient
L	length of beam
m	bending moment
P	concentrated lateral load

p	distributed lateral load
Q	concentrated torsional load
S	static moment of mass, per unit of length, about the elastic axis
T	section torque
q	distributed torsional load
V	beam shear
x	spanwise coordinate
y	bending deflection (translation)
α	curvature
$\bar{\alpha}$	concentrated curvature (bend)
β	slope of bending deflection curve
ϵ	Simpson's numbers
ϕ	torsional deflection (rotation)
θ	twist (rate of rotation per unit length)
λ	length of a segment
ω	circular frequency of natural vibration
ρ	mass per unit of length
ξ	alternate spanwise coordinate

NUMERICAL INTEGRATION

If full advantage is to be taken of the new procedures to be presented, all relations between variables must be expressed in matrix form. In order to provide a simple illustration of the formulation of the matrix equations, the well-known process of numerical integration will be considered in some detail. A single typical scalar equation is written to illustrate the linear relationship being considered. The complete set of linear relations are then written in expanded matrix form. The expanded matrix equations are written for a beam span divided into four equal segments. The form of the matrices will be sufficiently general, however, to indicate the correct manner of extension to a larger number of segments. The matrix equations are also written in contracted symbolic form and may then be considered applicable to a beam with any number of segments.

Torsion

A cantilever beam with distributed torsional load is considered first as shown in figure 1(a). The loading curve is considered to be replaced, for purposes of numerical integration, by a step function as shown in figure 1(b). The step function is, in turn, replaced by a set of concentrated torques as shown in figure 1(c). The value of the concentrated torque is obtained by multiplying the ordinate to the load diagram by the width of the step. As a scalar example Q_3 is given by

$$Q_3 = \lambda q_3 \tag{1}$$

In matrix form the concentrated torques are given by

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \frac{\lambda}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \tag{2}$$

The validity of this matrix equation is immediately seen from an application of the row-by-column rule for matrix multiplication. (See reference 3.) In contracted form this matrix equation may be written as

$$\left[Q\right] = \frac{\lambda}{2} \left[A\right] \left[q\right] \tag{3}$$

In this equation $\left[Q\right]$ and $\left[q\right]$ are column matrices, or column vectors and $\left[A\right]$ is the square matrix of coefficients.

The section torques may now be computed. The section torque distribution is a step diagram as shown in figure 1(d). As a scalar example T_3 is given by

$$T_3 = Q_1 + Q_2 + Q_3 \tag{4}$$

The matrix equation for section torques becomes

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} \tag{5}$$

In this equation the matrix of coefficients may be regarded as an integrating matrix or a summation matrix. Throughout the paper there will appear, in various equations, two types of integrating matrices. The two integrating matrices differ from each other in having values of zero or unity on the principal diagonal. The two matrices are defined as follows:

$$\begin{bmatrix} \Sigma_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} \Sigma_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{7}$$

Equation (5) may now be written as follows:

$$\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} \Sigma_1 \end{bmatrix} \begin{bmatrix} Q \end{bmatrix} \tag{8}$$

The section torques may be expressed in terms of the applied load by substituting from equation (3) in equation (8); thus,

$$\left[T\right] = \frac{\lambda}{2}\left[\Sigma_1\right]\left[A\right]\left[q\right] \tag{9}$$

The next step in the numerical integration is to divide the section torques by the values of GJ at the corresponding stations to obtain values of twist θ (change of rotation per unit of length). (See figures 1(d), 1(e), 1(f), and 1(g).) A typical equation for the third segment is given by

$$\theta_3 = \frac{T_3}{GJ_3} \tag{10}$$

In matrix form the twists are given by

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{bmatrix} = \begin{bmatrix}1/GJ_1 & 0 & 0 & 0 & 0\\ 0 & 1/GJ_2 & 0 & 0 & 0\\ 0 & 0 & 1/GJ_3 & 0 & 0\\ 0 & 0 & 0 & 1/GJ_4 & 0\\ 0 & 0 & 0 & 0 & 1/GJ_5\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\\ T_5\end{bmatrix} \tag{11}$$

In contracted form this equation becomes,

$$\begin{aligned}\left[\theta\right] &= \left[1/GJ\right]\left[T\right]\\ &= \left[GJ\right]^{-1}\left[T\right]\end{aligned} \tag{12}$$

The inversion of a diagonal matrix is discussed in reference 3. The twists may be obtained in terms of the applied load by substituting from equation (9) in equation (12); thus,

$$\left[\theta\right] = \frac{\lambda}{2}\left[GJ\right]^{-1}\left[\Sigma_1\right]\left[A\right]\left[q\right] \tag{13}$$

The twist distribution, as shown in figure 1(g), is a step diagram. This diagram may be integrated to obtain rotations. In scalar form ϕ_3 is given by

$$\phi_3 = \lambda\theta_3 + \lambda\theta_4 + \lambda\theta_5 \tag{14}$$

In matrix form the rotations are given by

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \tag{15}$$

In contracted form equation (15) becomes

$$\left[\phi\right] = \lambda \left[\Sigma_1\right]' \left[\theta\right] \tag{16}$$

The matrix $\left[\Sigma_1\right]'$ is the transpose of $\left[\Sigma_1\right]$. The transpose of a matrix is obtained by rotating the matrix about its principal diagonal. (See reference 3.) Substituting from equation (13) in equation (16) gives the final formula for rotations as follows:

$$\left[\phi\right] = \frac{\lambda^2}{2} \left[\Sigma_1\right]' \left[GJ\right]^{-1} \left[\Sigma_1\right] \left[A\right] \left[q\right] \tag{17}$$

The square coefficient matrices in equation (17) may be multiplied together to obtain a single matrix for convenience in deflection calculations. The resulting matrix $\left[C_t\right]$ is dependent only upon the structural properties of the beam and is defined as follows:

$$\left[C_t\right] = \frac{\lambda^2}{2} \left[\Sigma_1\right]' \left[GJ\right]^{-1} \left[\Sigma_1\right] \left[A\right] \tag{18}$$

Substituting from equation (18) in equation (17) gives

$$\left[\phi\right] = \left[C_t\right] \left[q\right] \tag{19}$$

The elements of the matrix $\left[C_t\right]$ are coefficients which express a linear relationship between the rotations and the ordinates to the distributed load curve. Since the numerical method that has been used for obtaining equivalent concentrated loads is not highly accurate, a large number of stations along the span must be used to obtain accurate deflections.

Bending

The calculation of bending deflections by numerical integration involves the same procedures as those used in torsion. The matrix equations are therefore written generally in contracted form. The diagrams of figures 2(a), 2(b), and 2(c) show the concentrated loads P_i to be related to the distributed load ordinates p_i by the equation

$$\left[P\right] = \frac{\lambda}{2}\left[A\right]\left[p\right] \tag{20}$$

Direct summation of the concentrated loads gives the shears according to the following equation:

$$\left[V\right] = \left[\Sigma_1\right]\left[P\right] \tag{21}$$

Substituting from equation (20) in equation (21) gives

$$\left[V\right] = \frac{\lambda}{2}\left[\Sigma_1\right]\left[A\right]\left[p\right] \tag{22}$$

The diagrams of figures 2(d) and 2(e) indicate that the moment-curve ordinates may be obtained by direct summation of the areas beneath the shear curve. The moments are thus related to the shears by the equation

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} \tag{23}$$

This equation is written in expanded form to illustrate the use of the summation matrix $\left[\Sigma_0\right]$. In contracted form equation (23) becomes

$$\left[m\right] = \lambda \left[\Sigma_0\right] \left[v\right] \tag{24}$$

Substitution of values from equation (22) in equation (24) gives

$$\left[m\right] = \frac{\lambda^2}{2} \left[\Sigma_0\right] \left[\Sigma_1\right] \left[A\right] \left[p\right] \tag{25}$$

The moment values must now be divided by the corresponding EI values which are illustrated in figure 2(f). The resulting diagram defines the curvature of the beam. These curvatures may be considered, for purposes of numerical integration, to be the loading on a conjugate beam (reference 4) as shown in figure 2(g). In matrix form the curvatures are given by

$$\begin{aligned} \left[\alpha\right] &= \left[1/EI\right] \left[m\right] \\ &= \left[EI\right]^{-1} \left[m\right] \end{aligned} \tag{26}$$

The matrix $\left[EI\right]^{-1}$ is a diagonal matrix. Substituting from equation (25) in equation (26) gives

$$\left[\alpha\right] = \frac{\lambda^2}{2} \left[EI\right]^{-1} \left[\Sigma_0\right] \left[\Sigma_1\right] \left[A\right] \left[p\right] \tag{27}$$

The loading on the conjugate beam may now be converted to equivalent concentrated loads in the same manner used for the original beam loading; thus,

$$\left[\bar{\alpha}\right] = \frac{\lambda}{2} \left[A\right] \left[\alpha\right] \tag{28}$$

The concentrated loads $\bar{\alpha}_1$, shown in figure 2(h), may also be regarded as concentrated curvatures. These concentrated curvatures may be visualized as bends in a broken line. Substituting from equation (27) in equation (28) gives

$$\left[\bar{\alpha}\right] = \frac{\lambda^3}{4} \left[A\right] \left[EI\right]^{-1} \left[\Sigma_0\right] \left[\Sigma_1\right] \left[A\right] \left[p\right] \tag{29}$$

The slopes β_i, shown in figure 2(i), are computed by direct integration according to the following equation:

$$\left[\beta\right] = \left[\Sigma_0\right]' \left[\bar{\alpha}\right] \tag{30}$$

Substitution from equation (29) in equation (30) gives

$$\left[\beta\right] = \frac{\lambda^3}{4} \left[\Sigma_0\right]' \left[A\right] \left[EI\right]^{-1} \left[\Sigma_a\right] \left[\Sigma_1\right] \left[A\right] \left[p\right] \tag{31}$$

The deflections, as shown in figure 2(j), are obtained by an integration of the slopes according to the following equation:

$$\left[y\right] = \lambda \left[\Sigma_1\right]' \left[\beta\right] \tag{32}$$

Substitution of values from equation (31) in equation (32) gives

$$\left[y\right] = \frac{\lambda^4}{4} \left[\Sigma_1\right]' \left[\Sigma_0\right]' \left[A\right] \left[EI\right]^{-1} \left[\Sigma_a\right] \left[\Sigma_1\right] \left[A\right] \left[p\right] \tag{33}$$

Equation (33) may be written in the following form for convenience in deflection calculations, as explained in the section on torsion:

$$\left[y\right] = \left[C_b\right] \left[p\right] \tag{34}$$

where

$$\left[C_b\right] = \frac{\lambda^4}{4} \left[\Sigma_1\right]' \left[\Sigma_0\right]' \left[A\right] \left[EI\right]^{-1} \left[\Sigma_a\right] \left[\Sigma_1\right] \left[A\right] \tag{35}$$

Just as in the torsion case, the matrix $\left[C_b\right]$ defines the relation between ordinates to a distributed load curve and the ordinates to a deflection curve.

INFLUENCE COEFFICIENTS

When influence coefficients are available, they may be used to determine deformations directly without the necessity for numerical integration. The influence coefficients may be computed by the integration method of the section dealing with numerical integration or they may be obtained experimentally. The relationships between load and deflection are given by very simple matrix equations and are shown only in contracted form.

Torsion

Influence coefficients define a linear relationship between a finite number of concentrated loads and a finite number of ordinates to the deflection curve. The matrix of torsional influence coefficients will be indicated by $\left[K_t\right]$. This matrix is used to express the relation between rotations and concentrated torque loads as follows:

$$\left[\phi\right] = \left[K_t\right]\left[Q\right] \tag{36}$$

The common method of determining equivalent concentrated loads from a distributed load is to replace the distributed-load curve by a step curve as used previously. The equivalent of $[Q]$ from equation (3) may therefore be substituted in equation (36) to obtain

$$\left[\phi\right] = \frac{\lambda}{2}\left[K_t\right]\left[A\right]\left[q\right] \tag{37}$$

If the influence coefficients are to be computed by the common method of numerical integration, a simple matrix formula may be used for this purpose. A comparison of equations (17) and (37) shows that the influence coefficients are given by the following formula;

$$\left[K_t\right] = \lambda\left[\Sigma_1\right]'\left[GJ\right]^{-1}\left[\Sigma_1\right] \tag{38}$$

Equation (37), when compared with equation (19), provides a second definition of the matrix $\left[C_t\right]$, as follows:

$$\left[C_t\right] = \frac{\lambda}{2}\left[K_t\right]\left[A\right] \tag{39}$$

Bending

The matrix of bending influence coefficients will be indicated by $\left[K_b\right]$. The relation between bending deflections and concentrated bending loads is given by the following equation:

$$\left[y\right] = \left[K_b\right] \left[P\right] \tag{40}$$

The column vector of concentrated loads is replaced by a vector of distributed-load ordinates by substituting from equation (20) in equation (40); thus,

$$\left[y\right] = \frac{\lambda}{2} \left[K_b\right] \left[A\right]\left[p\right] \tag{41}$$

A comparison of equations (33) and (41) indicates that the influence coefficients could be computed from the following formula:

$$\left[K_b\right] = \frac{\lambda^3}{2} \left[\Sigma_1\right]' \left[\Sigma_o\right]' \left[A\right] \left[EI\right]^{-1} \left[\Sigma_o\right] \left[\Sigma_1\right] \tag{42}$$

Equation (41) when compared with equation (34) provides a second definition of the matrix $\left[C_b\right]$, as follows:

$$\left[C_b\right] = \frac{\lambda}{2} \left[K_b\right] \left[A\right] \tag{43}$$

NUMERICAL INTEGRATION WITH WEIGHTING MATRICES

The matrix methods of numerical integration described in the foregoing sections correspond to the commonly used scalar methods. In those methods, the manner of converting distributed loadings into concentrated loadings is rather arbitrary. Such conversions usually lead to appreciable errors in the deflections when only a small number of stations is used.

The following sections show that the arbitrariness of the common methods can be largely removed by regarding the loading curves as series of parabolic arcs. With such an approach, the matrix $\left[A\right]$ in the equations of the foregoing sections is replaced by matrices which are designated "weighting matrices."

Torsion

Consider the distributed loading curve of figure 3(a). If, for example, the part of the curve between ordinates q_2 and q_4 is assumed to be a second-degree parabola defined by ordinates q_2, q_3, and q_4, the area a_2 between ordinates q_2 and q_3 is given by

$$a_2 = \frac{\lambda}{12}\left(5q_2 + 8q_3 - q_4\right) \tag{44}$$

The derivation of this formula is shown in appendix A. Proceeding in like manner for the other areas permits each area to be written in terms of three ordinates, as follows:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ 0 \end{bmatrix} = \frac{\lambda}{12} \begin{bmatrix} 5 & 8 & -1 & 0 & 0 \\ 0 & 5 & 8 & -1 & 0 \\ 0 & 0 & 5 & 8 & -1 \\ 0 & 0 & -1 & 8 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \tag{45}$$

In contracted form, equation (45) becomes

$$\left[a\right] = \frac{\lambda}{12}\left[W_1\right]\left[q\right] \tag{46}$$

and $\left[W_1\right]$ is referred to as a weighting matrix. The areas $\left[a\right]$ correspond to increments in section torque as indicated in figure 3(b). The section torques T at the five designated stations are related to the increments in section torque by the equation

$$\left[T\right] = \left[\Sigma_o\right]\left[a\right] \tag{47}$$

Substitution of values from equation (46) in equation (47) gives

$$\left[T\right] = \frac{\lambda}{12}\left[\Sigma_o\right]\left[W_1\right]\left[q\right] \tag{48}$$

In the section on torsion dealing with numerical integration, a diagonal matrix of values of $1/GJ_i$ was introduced as shown in equation (11) The individual values of GJ_i were assumed to be the average values within a bay. This assumption corresponds to a replacement of the GJ diagram by a step diagram as shown in figure 1(f). It is now necessary to make a slight modification of the definition of the diagonal elements of the matrix $\left[GJ\right]$. The elements of this matrix must now be defined as the values of GJ_i at the stations where the rotations are to be determined. These values are illustrated in figure 3(c). With this revised definition of the matrix $\left[GJ\right]$ the distribution of twist (fig. 3(d)) is given by the equation

$$\left[\theta\right] = \left[GJ\right]^{-1}\left[T\right] \tag{49}$$

or, upon substitution of values from equation (48) in equation (49), by

$$\left[\theta\right] = \frac{\lambda}{12}\left[GJ\right]^{-1}\left[\Sigma_o\right]\left[W_1\right]\left[q\right] \tag{50}$$

In figure 3(d), the areas $\bar{a}$ correspond to increments of rotation as indicated in figure 3(e). These areas are computed on the assumption that the twist curve can be represented by a series of parabolic arcs. For example, the area $\bar{a}_2$ is found by the formula

$$\bar{a}_2 = \frac{\lambda}{12}\left(-\theta_1 + 8\theta_2 + 5\theta_3\right) \tag{51}$$

Equation (51) is similar to equation (44), but the order of the coefficients is reversed. This reversal is not necessary but is used because it provides a convenience in the matrix equations which will be derived and discussed subsequently. The expanded form of the matrix equation relating increments in rotation to twists is

$$\begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_4 \\ 0 \end{bmatrix} = \frac{\lambda}{12}\begin{bmatrix} 5 & 8 & -1 & 0 & 0 \\ -1 & 8 & 5 & 0 & 0 \\ 0 & -1 & 8 & 5 & 0 \\ 0 & 0 & -1 & 8 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \tag{52}$$

The elements in the matrices of equation (52) can be rearranged in such a manner as to provide a convenience in the matrix equations that are to be derived. All elements of the first two matrices must be moved downward one position to give the equation in the following form:

$$\begin{bmatrix} 0 \\ \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_4 \end{bmatrix} = \frac{\lambda}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & -1 & 0 & 0 \\ -1 & 8 & 5 & 0 & 0 \\ 0 & -1 & 8 & 5 & 0 \\ 0 & 0 & -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \tag{53}$$

Although the elements of the column matrix on the left-hand side of the equation are out of their natural order, application of the row-by-column rule for multiplication reveals that all of the linear relationships defined by equation (52) are preserved in equation (53). In contracted form, equation (53) is written as

$$\left[\bar{a}\right] = \frac{\lambda}{12} \left[W_1\right]'' \left[\theta\right] \tag{54}$$

in which $\left[W_1\right]''$ is a weighting matrix. The double prime is used to indicate a double transposition, or double rotation, of the matrix. The matrix is first rotated about its principal diagonal in the usual manner of transposition. The transposed matrix is then given a rotation about the secondary diagonal. The possibility of relating the vector $\left[\bar{a}\right]$ to the vector $\left[\theta\right]$ by means of the matrix $\left[W_1\right]$, which has already been introduced, arises because of the precautions that have been previously taken in making particular arrangements of the elements within the various matrices.

Substituting from equation (50) in equation (54) gives

$$\left[\bar{a}\right] = \frac{\lambda^2}{144} \left[W_1\right]'' \left[GJ\right]^{-1} \left[\Sigma_o\right] \left[W_1\right] \left[q\right] \tag{55}$$

The rotations ϕ are obtained by a summation of the increments of rotation according to the formula

$$\left[\phi\right] = \left[\Sigma_o\right]' \left[\bar{a}\right] \tag{56}$$

From equation (55) and equation (56) the final formula for rotations is obtained, as follows:

$$\left[\phi\right] = \frac{\lambda^2}{144} \left[\Sigma_o\right]' \left[W_1\right]'' \left[GJ\right]^{-1} \left[\Sigma_o\right] \left[W_1\right] \left[q\right] \tag{57}$$

A comparison of equation (57) with equation (19) indicates that the matrix of coefficients $\left[C_t\right]$ may be computed by the following formula:

$$\left[C_t\right] = \frac{\lambda^2}{144} \left[\Sigma_o\right]' \left[W_1\right]'' \left[GJ\right]^{-1} \left[\Sigma_o\right] \left[W_1\right] \tag{58}$$

Since this formula for $\left[C_t\right]$ incorporates the weighting matrices, this definition is expected to yield more accurate solutions than the definition given by equation (18), which is based upon numerical integration without weighting numbers. In equation (58) the product $\left[\Sigma_o\right] \left[W_1\right]$ may be represented by a single matrix $\left[M\right]$:

$$\left[M\right] = \left[\Sigma_o\right] \left[W_1\right] \tag{59}$$

The product $\left[\Sigma_o\right]' \left[W_1\right]''$ can be obtained by a double transposition of $\left[M\right]$:

$$\left[M\right]'' = \left[\Sigma_o\right]' \left[W_1\right]'' \tag{60}$$

The validity of equation (60) is proven in appendix B. Substitution of values from equations (59) and (60) in equation (58) gives the following formula for $\left[C_t\right]$:

$$\left[C_t\right] = \frac{\lambda^2}{144} \left[M\right]'' \left[GJ\right]^{-1} \left[M\right] \tag{61}$$

The matrices $\left[M\right]$ and $\left[M\right]''$ are standard universal matrices which are not dependent upon the properties of the beam or its loading. These matrices have therefore been computed and are given in appendix C for systems of 5, 7, 9, and 11 stations. The form of the linear relationship between the rotations and the ordinates to the distributed load curve remains as given by equation (19).

Bending

The conversion of a distributed bending load of the type shown in figure 4(a) into a set of equivalent concentrated loads of the kind illustrated in figure 4(c) may be accomplished in a manner analogous to the conversion of a distributed torsional load into increments of torque. The conversion to be used has been presented and illustrated in scalar form in reference 2. As in the section on torsion, the distributed loading curve is regarded as a series of second-degree parabolic arcs. The principle used in the conversion is indicated in figure 4(b), which shows the load applied to a set of simply supported sub-beams that react on the cantilever beam at the five designated stations. The reactions of the sub-beams on the cantilever beam are the concentrated loads P. The statical equivalence of the concentrated loading and the distributed loading is restricted to bending moments at the five designated stations, but the effects at these stations are the only effects of direct interest. By appropriate integration, the following typical formulas for the equivalent concentrated loads are obtained:

At end station 1

$$P_1 = \frac{\lambda}{24}\left(7p_1 + 6p_2 - p_3\right) \tag{62}$$

and at the intermediate station 2

$$P_2 = \frac{\lambda}{12}\left(p_1 + 10p_2 + p_3\right) \tag{63}$$

In expanded matrix form, the complete expression of the equivalent concentrated loads in terms of the ordinates to the distributed loading curve is

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \frac{\lambda}{24} \begin{bmatrix} 7 & 6 & -1 & 0 & 0 \\ 2 & 20 & 2 & 0 & 0 \\ 0 & 2 & 20 & 2 & 0 \\ 0 & 0 & 2 & 20 & 2 \\ 0 & 0 & -1 & 6 & 7 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} \tag{64}$$

Written in contracted form, equation (64) becomes

$$\left[P\right] = \frac{\lambda}{24}\left[W_2\right]\left[p\right] \tag{65}$$

and $\left[W_2\right]$ is referred to as a weighting matrix.

The average shears in the bays of the beam, shown graphically in figure 4(d), are found by a summation of the concentrated loads according to the formula

$$\left[v\right] = \left[\Sigma_1\right]\left[P\right] \tag{66}$$

Substitution of values from equation (65) in equation (66) gives

$$\left[v\right] = \frac{\lambda}{24}\left[\Sigma_1\right]\left[W_2\right]\left[p\right] \tag{67}$$

The bending moments at the five stations (fig. 4(e)) are obtained by integration of the shears with the formula

$$\left[m\right] = \lambda\left[\Sigma_o\right]\left[v\right] \tag{68}$$

and, upon substitution of values from equation (67), equation (68) becomes

$$\left[m\right] = \frac{\lambda^2}{24}\left[\Sigma_o\right]\left[\Sigma_1\right]\left[W_2\right]\left[p\right] \tag{69}$$

At this point, the reader is reminded that the bending-moment diagram for the concentrated loads is a broken line whereas the bending-moment diagram for the distributed loading is a smooth curve. Both diagrams, however, have exactly the same set of ordinates at the five designated stations if the distributed loading curve is truly a second-degree curve.

The distribution of curvatures α, shown in figure 4(g), is obtained by dividing the true bending moments by the appropriate values of EI at all stations along the beam. This distribution gives a smooth curve, to which the ordinates at the five designated stations are given by the equation

$$\left[\alpha\right] = \left[EI\right]^{-1}\left[m\right] \tag{70}$$

Substitution of $\left[m\right]$ from equation (69) in equation (70) gives

$$\left[\alpha\right] = \frac{\lambda^2}{24}\left[EI\right]^{-1}\left[\Sigma_o\right]\left[\Sigma_1\right]\left[W_2\right]\left[p\right] \tag{71}$$

According to the conjugate beam theory, the diagram of curvatures is considered to be a loading on the conjugate beam. By the use of formulas typified by equations (62) and (63), the distributed loading on the conjugate beam is then converted into equivalent concentrated loads $\bar{\alpha}$ on the conjugate beam as shown in figure 4(h). In contracted matrix form, these equivalent concentrated loads are given as

$$\left[\bar{\alpha}\right] = \frac{\lambda}{24}\left[W_2\right]\left[\alpha\right] \tag{72}$$

Substituting from equation (71) in equation (72) gives

$$\left[\bar{\alpha}\right] = \frac{\lambda^3}{576}\left[W_2\right]\left[EI\right]^{-1}\left[\Sigma_0\right]\left[\Sigma_1\right]\left[W_2\right]\left[p\right] \tag{73}$$

The average shears in the four bays of the conjugate beam, diagramatically shown in figure 4(i), give the average slopes of the actual beam. They are obtained by a summation of the concentrated loads on the conjugate beam with the formula

$$\left[\beta\right] = \left[\Sigma_1\right]'\left[\bar{\alpha}\right] \tag{74}$$

Substitution from equation (73) in equation (74) gives

$$\left[\beta\right] = \frac{\lambda^3}{576}\left[\Sigma_1\right]'\left[W_2\right]\left[EI\right]^{-1}\left[\Sigma_0\right]\left[\Sigma_1\right]\left[W_2\right]\left[p\right] \tag{75}$$

The deflections of the actual beam are equal to the bending moments in the conjugate beam. Ordinates to the deflection curve of the actual beam, shown in figure 4(j), are found by integration of the slopes of the actual beam according to the formula

$$\left[y\right] = \lambda\left[\Sigma_0\right]'\left[\beta\right] \tag{76}$$

The final formula for deflections is obtained by substituting from equation (75) in equation (76) as follows:

$$\left[y\right] = \frac{\lambda^4}{576}\left[\Sigma_0\right]'\left[\Sigma_1\right]'\left[W_2\right]\left[EI\right]^{-1}\left[\Sigma_0\right]\left[\Sigma_1\right]\left[W_2\right]\left[p\right] \tag{77}$$

A comparison of equation (77) with equation (34) indicates that the matrix $\left[C_b\right]$ may be defined as follows:

$$\left[C_b\right] = \frac{\lambda^4}{576}\left[\Sigma_o\right]'\left[\Sigma_1\right]'\left[W_2\right]\left[EI\right]^{-1}\left[\Sigma_o\right]\left[\Sigma_1\right]\left[W_2\right] \tag{78}$$

Since this formula for $\left[C_b\right]$ includes weighting matrices, it is expected to yield more accurate deflections than the formula given by equation (35) obtained by numerical integration without weighting matrices.

The formula for $\left[C_b\right]$ may be simplified by introducing the following definitions:

$$\left[N\right] = \left[\Sigma_o\right]\left[\Sigma_1\right]\left[W_2\right] \tag{79}$$

$$\left[N\right]'' = \left[\Sigma_o\right]'\left[\Sigma_1\right]'\left[W_2\right] \tag{80}$$

The validity of equation (80) is proven in appendix B. Substituting from equations (79) and (80) in equation (78) gives the following formula for $\left[C_b\right]$:

$$\left[C_b\right] = \frac{\lambda^4}{576}\left[N\right]''\left[EI\right]^{-1}\left[N\right] \tag{81}$$

The matrices $\left[N\right]$ and $\left[N\right]''$ are standard matrices which are independent of the properties of the beam or its loading. They have been computed and are given in appendix C for systems of 5, 7, 9, and 11 stations. The form of the linear relationship between the deflections and the ordinates to the distributed-load curve remains as given by equation (34).

INFLUENCE COEFFICIENTS WITH WEIGHTING MATRICES

The use of influence coefficients measured on the structure may be desirable in the final stages of design as a check on preliminary calculations or in order to account for the effects of items such as large discontinuities in structure and the restraint of warping of cross-sections. Weighting numbers can be introduced that will increase the accuracy of deflection calculations for distributed loading when influence coefficients are used. Weighting matrices involving the appropriate weighting numbers are introduced in the following sections.

Torsion

In the method of numerical integration with weighting matrices, no consideration of concentrated torsional loads was necessary. It is therefore necessary to develop a concept of an equivalent concentrated torsional load for use with influence coefficients. This concept is developed in detail in appendix A and the result is stated here. If the distributed torsional-loading curve of figure 3(a), for example, is truly a second-degree curve and if GJ is constant over the length of the beam, the equivalent concentrated torsional loads are given exactly by the formula

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} = \frac{\lambda}{24} \begin{bmatrix} 7 & 6 & -1 & 0 & 0 \\ 2 & 20 & 2 & 0 & 0 \\ 0 & 2 & 20 & 2 & 0 \\ 0 & 0 & 2 & 20 & 2 \\ 0 & 0 & -1 & 6 & 7 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \tag{82}$$

In contracted form, equation (82) becomes

$$\begin{bmatrix} Q \end{bmatrix} = \frac{\lambda}{24} \begin{bmatrix} W_2 \end{bmatrix} \begin{bmatrix} q \end{bmatrix} \tag{83}$$

If the equivalent concentrated torsional loads $\begin{bmatrix} Q \end{bmatrix}$ are used in conjunction with influence coefficients, the deflection obtained will be exact for a uniform beam. The rotations are obtained by substitution of values from equation (83) in equation (36) as follows:

$$\begin{bmatrix} \phi \end{bmatrix} = \frac{\lambda}{24} \begin{bmatrix} K_t \end{bmatrix} \begin{bmatrix} W_2 \end{bmatrix} \begin{bmatrix} q \end{bmatrix} \tag{84}$$

Equation (84) when compared with equation (19) provides another definition of the matrix $\begin{bmatrix} C_t \end{bmatrix}$, as follows:

$$\begin{bmatrix} C_t \end{bmatrix} = \frac{\lambda}{24} \begin{bmatrix} K_t \end{bmatrix} \begin{bmatrix} W_2 \end{bmatrix} \tag{85}$$

This definition of $\left[C_t\right]$ involves influence coefficients and weighting numbers and should prove to be more accurate than the formula given by equation (39) which involved influence coefficients without weighting numbers.

Bending

When bending deflections due to a distributed load are computed with the use of influence coefficients, the method of equivalent concentrated loads gives more accurate results than the common method involving the step-function approximation. The equivalent concentrated loads as given by equation (65), however, are appropriate for the purpose of computing bending moments but not for the purpose of computing deflections when used with the influence coefficients. Certain advantages may consequently be obtained if a new approach is made to the deflection problem in order to develop weighting matrices for use in conjunction with influence coefficients.

When an analytical influence function $K(x,\xi)$ is known for the deflection at station x due to a unit load at station ξ, the deflection due to a loading $p(\xi)$ may be expressed by means of a definite integral in the following form (see reference 5, p. 266):

$$y(x) = \int_0^L K(x,\xi)p(\xi)d\xi \tag{86}$$

Equation (86) must be replaced by a system of linear algebraic equations by letting x and ξ take on a finite set of values x_i and ξ_j corresponding to equally spaced stations. The evaluation of the definite integral can be performed accurately by using Simpson's rule. The value of y at station x_i may be indicated by y_i and the value of p at station ξ_j by p_j. The value of $K(x,\xi)$ for $x = x_i$ and $\xi = \xi_j$ may be indicated as an influence coefficient k_{ij}. Simpson's numbers may be indicated by ϵ_j. A single scalar equation of the system is written for deflection at station 3 as an example

$$y_3 = k_{31}\epsilon_1 p_1 + k_{32}\epsilon_2 p_2 + k_{33}\epsilon_3 p_3 + k_{34}\epsilon_4 p_4 + k_{35}\epsilon_5 p_5 \tag{87}$$

From this equation the influence coefficients may be seen to be formed into a square matrix and the weighting numbers to be formed into a diagonal matrix in order to express the system of equations in matrix form. These matrices appear as follows:

$$\begin{bmatrix} k_{11} & k_{12} & \cdot & \cdot & \cdot \\ k_{21} & k_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & k_{55} \end{bmatrix} = \begin{bmatrix} K_b \end{bmatrix} \tag{88}$$

$$\begin{bmatrix} \epsilon_1 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_4 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_5 \end{bmatrix} = \frac{\lambda}{3} \begin{bmatrix} W_3 \end{bmatrix} \tag{89}$$

Simpson's numbers are well known (see reference 5, p. 5) and may be used in equation (89) to obtain

$$\begin{bmatrix} W_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{90}$$

The system of linear algebraic equations which is equivalent to equation (86) may now be expressed in the following matrix form:

$$\begin{bmatrix} y \end{bmatrix} = \frac{\lambda}{3} \begin{bmatrix} K_b \end{bmatrix} \begin{bmatrix} W_3 \end{bmatrix} \begin{bmatrix} p \end{bmatrix} \tag{91}$$

Equation (91) when compared with equation (34) provides another definition of the matrix $\begin{bmatrix} C_b \end{bmatrix}$ as follows:

$$\begin{bmatrix} C_b \end{bmatrix} = \frac{\lambda}{3} \begin{bmatrix} K_b \end{bmatrix} \begin{bmatrix} W_3 \end{bmatrix} \tag{92}$$

Since this formula for $\left[C_b\right]$ includes weighting numbers it should provide more accurate deflections than the formula in equation (43).

APPLICATIONS TO NATURAL VIBRATION PROBLEMS

Equations (19) and (34) may be formally stated in a single matrix equation, as follows:

$$\begin{bmatrix} C_t & 0 \\ 0 & C_b \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \phi \\ y \end{bmatrix} \qquad (93)$$

The submatrices $\left[C_t\right]$ and $\left[C_b\right]$ in equation (93) are the square matrices relating distributed loadings to deflections and have been defined previously in four ways with varying degrees of accuracy.

The loads may be applied statically or dynamically. Natural vibration is an example of dynamic loading, the dynamic loads being the inertia loads which accompany the accelerations of the system during vibration. The cantilever beam under consideration vibrates naturally in any of an infinite number of modes with each of which is associated a frequency of vibration. If the deflections are small and in the elastic range, the vibration in each mode is a simple harmonic motion in which all parts of the beam oscillate about the position of static equilibrium in phase with each other and with the same frequency. The dynamic or inertia loads for any mode may be expressed in terms of the deflections and mass properties of the beam and the frequency as follows:

$$\begin{bmatrix} q \\ p \end{bmatrix} = \omega^2 \begin{bmatrix} I_m & S \\ S & \rho \end{bmatrix} \begin{bmatrix} \phi \\ y \end{bmatrix} \qquad (94)$$

In equation (94), ω is the frequency of vibration, $\left[\rho\right]$ is a diagonal submatrix of values of mass per unit length, $\left[S\right]$ is a diagonal submatrix of static moments of mass per unit length about the elastic axis of the beam, and $\left[I_m\right]$ is a diagonal submatrix of moments of inertia of mass per unit length about the elastic axis of the beam. The elements of the matrix $\left[S\right]$ are sometimes called the coupling terms, and when $\left[S\right]$ is different from zero, each natural mode contains both torsion and bending deflections.

Combining equation (94) and equation (93) gives the complete equation for natural vibrations, as follows:

$$\begin{bmatrix} C_t & 0 \\ 0 & C_b \end{bmatrix} \begin{bmatrix} I_m & S \\ S & \rho \end{bmatrix} \begin{bmatrix} \phi \\ y \end{bmatrix} = \frac{1}{\omega^2} \begin{bmatrix} \phi \\ y \end{bmatrix} \tag{95}$$

Equation (95) may be consolidated for practical use by carrying out the multiplication of the two square matrices on the left side. The resulting equation may be stated in contracted form as follows:

$$\begin{bmatrix} V \end{bmatrix} \begin{bmatrix} \phi \\ y \end{bmatrix} = \frac{1}{\omega^2} \begin{bmatrix} \phi \\ y \end{bmatrix} \tag{96}$$

The matrix $\begin{bmatrix} V \end{bmatrix}$ depends only on the geometric, mass, and elastic properties of the cantilever beam and therefore remains constant for all modes of vibration. The solution of the vibration problem consists in finding sets of numerical values of the vector $\begin{bmatrix} \phi \\ y \end{bmatrix}$ and associated frequency ω which satisfy equation (96). Iteration is a convenient method for obtaining the solution. Attention is drawn to the fact that, although the actual cantilever beam has an infinite number of natural modes of vibration, equation (96) can be used to calculate only a finite number of modes approximating with varying degrees of precision the exact modes in the initial range of the frequency spectrum. Equation (96) determines $n - 1$ modes, n being the number of designated stations on the beam. The precision with which equation (96) determines the exact modes is greatest for the first mode, becoming less as the order of the mode increases.

COMPARISON OF METHODS

In order to provide a simple comparison of the relative merits of the various methods of analysis that have been described, a uniform cantilever beam has been analyzed for both static and vibrational deflections. Exact values have been obtained by solving the well-known differential equations which govern the deflections. In the static analyses the loading is assumed to be of triangular shape with a value of zero at the tip and a maximum value at the root of the beam. In the vibration analyses the mass axis is assumed to coincide with the elastic axis of the beam so that torsional and bending vibrations may be considered separately.

The four methods of deflection analysis that have been described are as follows:

(1) Numerical integration

(2) Influence coefficients

(3) Numerical integration with weighting matrices

(4) Influence coefficients with weighting matrices

Since the first method is well known and is usually the least accurate of the four methods, it has been omitted from consideration in this section. Calculations of deflections have been made by the other three methods. The first two natural frequencies and the static tip deflection have been computed for 3, 5, and 7 stations and are recorded in table 1. The exact values obtained by solving the differential equations are also shown in table 1. The torsional deflections corresponding to static loading and vibrational motion are governed by the following two differential equations, respectively:

$$GJ \frac{d^2\phi}{dx^2} = -q \tag{97}$$

$$GJ \frac{d^2\phi}{dx^2} + \rho r^2 \omega^2 \phi = 0 \tag{98}$$

In these equations ρ is the mass per unit length of the beam, r is the radius of gyration of the mass, and ω is a natural frequency. The bending deflections corresponding to static loading and vibrational motion are governed by the following two differential equations, respectively:

$$EI \frac{d^4y}{dx^4} = p \tag{99}$$

$$EI \frac{d^4y}{dx^4} - \rho\omega^2 y = 0 \tag{100}$$

The second method of calculation involves the use of influence coefficients. The influence coefficients for a uniform beam can be easily computed from simple formulas. These formulas constitute the influence function, or Green's function, for the problem. The influence function is a solution of the homogeneous differential equation for static deflections. It gives the deflections due to a unit concentrated load. Since the function is discontinuous at the point of application of the load, it must be defined separately for the regions on either side of the load. If the deflection is to be determined at station x for a unit load at station ξ (see fig. 5), the influence function for torsion is defined by the following two formulas:

$$K_t(x, \xi) = \frac{\xi}{GJ} \qquad (x > \xi) \qquad (101)$$

$$K_t(x, \xi) = \frac{x}{GJ} \qquad (x < \xi) \qquad (102)$$

The influence function for bending is defined by the following two equations:

$$K_b(x, \xi) = \frac{\xi^2}{6EI}(3x - \xi) \qquad (x > \xi) \qquad (103)$$

$$K_b(x, \xi) = \frac{x^2}{6EI}(3\xi - x) \qquad (x < \xi) \qquad (104)$$

Using the exact values, as shown in table 1, permits computation of the percentage error resulting from the use of each of the calculation methods for each number of stations. Graphs of this percentage error are shown in figures 6 to 11. In most cases the absolute value of the percentage error decreases with an increase in the number of stations. In all cases the use of weighting matrices with either numerical integration or influence coefficients brings about an appreciable reduction in the percentage error.

From practical considerations of economy in calculation effort, these graphs show that, for an allowable percentage error, the use of weighting matrices permits the deflection analysis to be made with a smaller number of stations. In order to illustrate this point an examination of the graphs has been made to determine the number of stations required to obtain satisfactory accuracy defined to allow the following percentage errors:

Static tip deflection 1
First vibrational frequency 1
Second vibrational frequency 2

The number of stations required to give satisfactory accuracy, according to the above definition, has been read from the graphs and recorded in table 2. From an inspection of this table it is seen that, in five out of six deflection analyses, the use of influence coefficients alone would require more than seven stations to obtain satisfactory accuracy. When weighting numbers are used with influence coefficients, satisfactory accuracy is obtained in five out of six of the deflection analyses with less than seven stations. When weighting numbers are used with numerical integration, satisfactory accuracy is obtained in all six deflection analyses with less than seven stations.

The example that has been used for illustration purposes is a uniform beam and the relative percentage errors illustrated in figures 6 to 11 cannot be considered as strictly of general applicability. This indicates clearly the need for future research in studies of nonuniform beams. Future research must also deal with the development of more accurate weighting numbers and practical methods of analysis with concentrated loads.

CONCLUDING REMARKS

The advantage of an influence coefficient method of deflection analysis is that it provides a direct linear relation between the loading and the deflection in explicit form. The same advantage may be obtained in a numerical integration process, employing beam stiffness properties, if the analysis is expressed in matrix form. The linear relationships for distributed loading have been developed.

The accuracy of both the numerical integration and influence coefficient methods can be improved by the introduction of weighting matrices. Consequently, for a desired degree of accuracy, smaller matrices may be used. This procedure results in an appreciable saving in calculation time since the computing work varies as the square of the order of the matrices.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., December 7, 1948

APPENDIX A

DERIVATIONS BASED ON THE PARABOLA

Formula for Areas

In the method of numerical integration with weighting matrices, increments of section torque and increments of rotation correspond, respectively, to increments of area under the curve of distributed torsional loading and increments of area under the curve of twists. In the calculations, these curves are approximated by a series of second-degree parabolic arcs defined by groups of three ordinates. The formula for increments of area under a second-degree parabola are derived as follows: In figure 12, the ordinates f_1, f_2, and f_3, separated by the distance λ, represent a typical group of three ordinates to a loading curve or a curve of twists. With the coordinate system shown in figure 12, the equation of the second-degree curve defined by the three ordinates may be written in the form of Lagrange's interpolation formula (reference 6) as follows:

$$f = \frac{f_1}{2} \frac{\lambda - x}{\lambda} \frac{2\lambda - x}{\lambda} + f_2 \frac{x}{\lambda} \frac{2\lambda - x}{\lambda} - \frac{f_3}{2} \frac{x}{\lambda} \frac{\lambda - x}{\lambda} \tag{A1}$$

The area a_1 between ordinates f_1 and f_2 is found by integrating the function f between the limits $x = 0$ and $x = \lambda$, and the result is

$$a_1 = \int_0^\lambda f\,dx = \frac{\lambda}{12}\left(5f_1 + 8f_2 - f_3\right) \tag{A2}$$

Formulas for other areas may be obtained by increasing the subscripts in equation (A2). For example, the formula for area a_2 in figure 12 would have the form

$$a_2 = \frac{\lambda}{12}\left(5f_2 + 8f_3 - f_4\right) \tag{A3}$$

The area a_2 may also be computed by using equation (A1) and integrating from λ to 2λ. The resulting formula is

$$a_2 = \frac{\lambda}{12}\left(-f_1 + 8f_2 + 5f_3\right) \tag{A4}$$

A formula of the type given by equations (A2) or (A3) must be used at the left end of a beam. The type of formula shown by equation (A4) must be used at the right end of a beam. Either type of formula may be used for intermediate segments of the span.

Formulas for Equivalent Concentrated Torques

The concept of equivalent concentrated torques, necessary for the use of weighting matrices with influence coefficients, is based on the following condition of equivalence: In the beam shown in figure 13, the set of rotations, at the five designated stations, due to the concentrated loading must be identical with the set of rotations due to the distributed loading. If this condition is fulfilled, the concentrated torques Q of figure 13(b) must produce a set of increments of rotation in the bays of length λ equal to those produced by the distributed torsional loading of figure 13(a).

Consider first the increment of rotation between stations 1 and 2 produced by the distributed loading. The section torque at a distance x from the tip is given by

$$T_x = \int_0^x q \, dx \tag{A5}$$

The twist at the distance x is given by

$$\theta_x = \frac{T_x}{GJ_x} = \frac{1}{GJ_x}\int_0^x q \, dx \tag{A6}$$

The increment of rotation between stations 1 and 2 is then given by

$$\Delta\phi_{12} = \int_0^\lambda \theta_x \, dx = \int_0^\lambda \frac{1}{GJ_x}\int_0^x q \, dx \, dx \tag{A7}$$

Consider next the increment of rotation between stations 1 and 2 produced by the concentrated loading. The section torque between stations 1 and 2 is equal to Q_1. The twist at the distance x between stations 1 and 2 is given by

$$\theta_x = \frac{Q_1}{GJ_x} \tag{A8}$$

The increment of rotation between stations 1 and 2 is

$$\Delta\phi_{12} = \int_0^\lambda \theta_x \, dx = Q_1 \int_0^\lambda \frac{dx}{GJ_x} \tag{A9}$$

Since the two expressions of equations (A7) and (A9) for $\Delta\phi_{12}$ are to be equal, the following equation is obtained

$$Q_1 \int_0^\lambda \frac{dx}{GJ_x} = \int_0^\lambda \frac{1}{GJ_x} \int_0^x q \, dx \, dx \tag{A10}$$

Equation (A10) may be integrated if the values of q and GJ_x are given as functions of x. The present consideration will be limited to the case of constant GJ; equation (A10) can therefore be simplified to the form

$$Q_1 = \frac{1}{\lambda} \int_0^\lambda \int_0^x q \, dx \, dx \tag{A11}$$

Equation (A11) is found to apply also when GJ is a step function. The variation of q is assumed to be given by the second-degree parabola defined by the ordinates q_1, q_2, and q_3. The expression for q is then obtained from equation (A1) by substituting q for f, as follows:

$$q = \frac{q_1}{2} \frac{\lambda - x}{\lambda} \frac{2\lambda - x}{\lambda} + q_2 \frac{x}{\lambda} \frac{2\lambda - x}{\lambda} - \frac{q_3}{2} \frac{x}{\lambda} \frac{\lambda - x}{\lambda} \tag{A12}$$

With this expression for q, equation (A11) becomes, after integration,

$$Q_1 = \frac{\lambda}{24} \left(7q_1 + 6q_2 - q_3\right) \tag{A13}$$

The expression for Q_2 in figure 13(b) will now be derived. It is necessary to consider Q_2 as consisting of two parts; a part Q_{21} associated with the distributed loading between stations 1 and 2, and a part Q_{23} associated with the distributed loading between stations 2 and 3. The sum of Q_{21} and Q_{23} equals Q_2. In figure 13(b), if the sum of Q_1 and Q_{21} equals the total loading between stations 1 and 2 in figure 13(a), the rotations at the five designated stations of the beam due to the loading between stations 1 and 2 in figure 13(a) will be equal to the rotations due to the loads Q_1 and Q_{21}. This consideration provides the definition of Q_{21}, as follows:

$$Q_{21} = \int_0^{\lambda} q\ dx - Q_1 \tag{A14}$$

The integral in equation (A14) is the area under the q-curve between stations 1 and 2 and in accordance with equation (A2) is given by

$$\int_0^{\lambda} q\ dx = \frac{\lambda}{12}\left(5q_1 + 8q_2 - q_3\right) \tag{A15}$$

Substituting from equations (A13) and (A15) in equation (A14) gives

$$Q_{21} = \frac{\lambda}{24}\left(3q_1 + 10q_2 - q_3\right) \tag{A16}$$

The concentrated load Q_{23} bears the same relation to the distributed loading between stations 2 and 3 as the concentrated load Q_1 bears to the distributed loading between stations 1 and 2; this relation gives that part of the total increment of rotation between stations 2 and 3 which is due to the distributed loading between stations 2 and 3. The defining equation for Q_{23}, similar to equation (A11) is, as follows:

$$Q_{23} = \frac{1}{\lambda}\int_{\lambda}^{2\lambda}\int_{\lambda}^{x} q\ dx\ dx \tag{A17}$$

Substitution of values from equation (A12) in equation (A17) gives, after integration,

$$Q_{23} = \frac{\lambda}{24}\left(-q_1 + 10q_2 + 3q_3\right) \tag{A18}$$

The sum of equations (A16) and (A18) gives

$$Q_2 = \frac{\lambda}{12}\left(q_1 + 10q_2 + q_3\right) \tag{A19}$$

Formulas for the equivalent concentrated loads at the other stations may be derived by proceeding in the manner used to derive equations (A13) and (A19).

APPENDIX B

DOUBLE TRANSPOSITION OF MATRICES

A matrix $\left[A\right]$ is transposed by rotating it about its principal diagonal. The transposed matrix is indicated with a prime as $\left[A\right]'$. If the matrix $\left[A\right]'$ is now rotated about the secondary diagonal the new matrix may be indicated with a double prime as $\left[A\right]''$. The matrix $\left[A\right]''$ may be said to be doubly transposed. These transposition processes may be illustrated by the following three equations:

$$\left[A\right] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{B1}$$

$$\left[A\right]' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{B2}$$

$$\left[A\right]'' = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix} \tag{B3}$$

The object of the present appendix is to prove the validity of the formulas for $\left[M\right]''$ and $\left[N\right]''$ as given in the main body of the paper. For this purpose the several types of symmetry which matrices may have must first be considered. If a matrix is equal to its transpose, the matrix is said to be symmetrical. This condition might be referred to more specifically as a principal symmetry. A matrix that remains unchanged after rotation about its secondary diagonal might be said to have secondary symmetry. If a matrix remains unchanged after a double transposition, the matrix is described as being centrosymmetrical (reference 7). These three types of symmetry may be expressed by equations relating the matrices $\begin{bmatrix} A \end{bmatrix}$, $\begin{bmatrix} A \end{bmatrix}'$, and $\begin{bmatrix} A \end{bmatrix}''$ as follows:

Principal symmetry:

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}' \tag{B4}$$

Secondary symmetry:

$$\begin{bmatrix} A \end{bmatrix}' = \begin{bmatrix} A \end{bmatrix}'' \tag{B5}$$

Central Symmetry:

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} A \end{bmatrix}'' \tag{B6}$$

A relationship showing that any matrix $\begin{bmatrix} A \end{bmatrix}''$ can be obtained from the matrix $\begin{bmatrix} A \end{bmatrix}$ by simple matrix multiplications is now necessary. A matrix $\begin{bmatrix} J \end{bmatrix}$ must be introduced according to the following definition:

$$\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{B7}$$

This matrix has values of unity in its secondary diagonal and zeros elsewhere. If a matrix $\begin{bmatrix} A \end{bmatrix}$ is premultiplied by $\begin{bmatrix} J \end{bmatrix}$, the procedure merely interchanges the rows of $\begin{bmatrix} A \end{bmatrix}$; thus,

$$\begin{bmatrix} J \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} \tag{B8}$$

If a matrix $\begin{bmatrix} A \end{bmatrix}$ is postmultiplied by $\begin{bmatrix} J \end{bmatrix}$, the procedure merely interchanges the columns of $\begin{bmatrix} A \end{bmatrix}$; thus,

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{bmatrix} \tag{B9}$$

If a matrix $\begin{bmatrix} A \end{bmatrix}$ is premultiplied by $\begin{bmatrix} J \end{bmatrix}$ and postmultiplied by $\begin{bmatrix} J \end{bmatrix}$, the procedure brings about a double rotation of the matrix about horizontal and vertical central axes. The result is as follows:

$$\begin{bmatrix} J \end{bmatrix}\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix} \tag{B10}$$

If equation (B10) is compared with equation (B3), the expanded matrix on the right-hand side is found to be exactly the same in both equations and therefore

$$\begin{bmatrix} A \end{bmatrix}'' = \begin{bmatrix} J \end{bmatrix}\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} J \end{bmatrix} \tag{B11}$$

An unusual but important property of the matrix $\begin{bmatrix} J \end{bmatrix}$ must now be noted. The matrix is its own reciprocal, which may be expressed by the following equation:

$$\begin{bmatrix} J \end{bmatrix}^2 = \begin{bmatrix} J \end{bmatrix}\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} I \end{bmatrix} \tag{B12}$$

In this equation $\begin{bmatrix} I \end{bmatrix}$ is the identity matrix.

Development of the desired formulas is now possible. The torsion case will be considered first. According to equation (59) the matrix $\begin{bmatrix} M \end{bmatrix}$ is defined by the formula

$$\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} \Sigma_o \end{bmatrix} \begin{bmatrix} W_1 \end{bmatrix} \tag{B13}$$

According to equation (B11), the doubly-transposed matrix may be determined from the equation

$$\begin{bmatrix} M \end{bmatrix}'' = \begin{bmatrix} J \end{bmatrix} \begin{bmatrix} \Sigma_o \end{bmatrix} \begin{bmatrix} W_1 \end{bmatrix} \begin{bmatrix} J \end{bmatrix} \tag{B14}$$

The identity matrix may be inserted as a factor at any point on either side of an equation without changing the value of either side. Equation (B14) may therefore be written in the following form:

$$\begin{bmatrix} M \end{bmatrix}'' = \begin{bmatrix} J \end{bmatrix} \begin{bmatrix} \Sigma_o \end{bmatrix} \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} W_1 \end{bmatrix} \begin{bmatrix} J \end{bmatrix} \tag{B15}$$

Equation (B12) may now be combined with equation (B15) to obtain

$$\left[M\right]'' = \left[J\right]\left[\Sigma_o\right]\left[J\right]\left[J\right]\left[W_1\right]\left[J\right] \tag{B16}$$

Formula (B11) applied in equation (B16) gives

$$\left[M\right]'' = \left[\Sigma_o\right]''\left[W_1\right]'' \tag{B17}$$

An interesting feature of the process of double transposition can be noted at this point. Comparing equations (B13) and (B17) indicates that the double transposition of a product of matrices can be obtained by a double transposition of the individual matrices without changing their order.

It must now be noted that $\left[\Sigma_o\right]'$ is symmetrical about its secondary diagonal. Consequently,

$$\left[\Sigma_o\right]' = \left[\Sigma_o\right]'' \tag{B18}$$

Substitution of values from equation (B18) in equation (B17) gives the following formula:

$$\left[M\right]'' = \left[\Sigma_o\right]'\left[W_1\right]'' \tag{B19}$$

This formula was previously given in the main text as equation (60).

The bending case is now to be considered. According to equation (79) the matrix $\left[N\right]$ is defined by the formula

$$\left[N\right] = \left[\Sigma_o\right]\left[\Sigma_1\right]\left[W_2\right] \tag{B20}$$

Employing the rule that has just been developed permits both sides of this equation to be doubly transposed to give

$$\left[N\right]'' = \left[\Sigma_o\right]''\left[\Sigma_1\right]''\left[W_2\right]'' \tag{B21}$$

The matrices $\left[\Sigma_o\right]'$ and $\left[\Sigma_1\right]'$ both have secondary symmetry so that equation (B21) may be written as

$$\left[N\right]'' = \left[\Sigma_0\right]' \left[\Sigma_1\right]' \left[W_2\right]'' \tag{B22}$$

A consideration of the expanded form of the matrix $\left[W_2\right]$, as shown in equation (64), indicates that this matrix has central symmetry; hence

$$\left[W_2\right] = \left[W_2\right]'' \tag{B23}$$

Substitution of values of equation (B23) in equation (B22) gives the following formula:

$$\left[N\right]'' = \left[\Sigma_0\right]' \left[\Sigma_1\right]' \left[W_2\right] \tag{B24}$$

This formula was given in the main text as equation (80).

APPENDIX C

STANDARD MATRICES

The matrices $\left[M\right]$ and $\left[M\right]''$ serve to weight and integrate the torsional loading curve and the twist curve, respectively. The matrices $\left[N\right]$ and $\left[N\right]''$ serve to weight and perform a double integration of the load curve and the curvature curve in bending, respectively. These matrices are standard matrices which may be tabulated and used for various cantilever-beam analyses. The matrices must be computed for each different number of subdivisions of the span. The order of each matrix will be one greater than the number of segments into which the span is divided. Equation (59) gives the following formula for the matrix $\left[M\right]$:

$$\left[M\right] = \left[\Sigma_0\right] \left[W_1\right] \tag{C1}$$

The formula for the matrix $\left[N\right]$, equation (79), is

$$\left[N\right] = \left[\Sigma_0\right] \left[\Sigma_1\right] \left[W_2\right] \tag{C2}$$

The matrices $\left[M\right]''$ and $\left[N\right]''$ are obtained from $\left[M\right]$ and $\left[N\right]$ by a double transposition. Equations (C1) and (C2) have been used for computing the matrices $\left[M\right]$ and $\left[N\right]$.

The matrices of fifth order are as follows:

Fifth Order Matrices

$$[M] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & -1 & 0 & 0 \\ 5 & 13 & 7 & -1 & 0 \\ 5 & 13 & 12 & 7 & -1 \\ 5 & 13 & 11 & 15 & 4 \end{bmatrix} \qquad [M]'' = \begin{bmatrix} 4 & 15 & 11 & 13 & 5 \\ -1 & 7 & 12 & 13 & 5 \\ 0 & -1 & 7 & 13 & 5 \\ 0 & 0 & -1 & 8 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[N] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & -1 & 0 & 0 \\ 16 & 32 & 0 & 0 & 0 \\ 25 & 60 & 21 & 2 & 0 \\ 34 & 88 & 44 & 24 & 2 \end{bmatrix} \qquad [N]'' = \begin{bmatrix} 2 & 24 & 44 & 88 & 34 \\ 0 & 2 & 21 & 60 & 25 \\ 0 & 0 & 0 & 32 & 16 \\ 0 & 0 & -1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is seen in the foregoing examples that the matrices $[M]''$ and $[N]''$ are obtained from $[M]$ and $[N]$ by a double transposition. Since these examples illustrate the transposition process clearly, the higher order matrices will not be shown in transposed form. The higher order matrices are as follows:

Seventh Order Matrices

$$[M]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 8 & -1 & 0 & 0 & 0 & 0 \\ 5 & 13 & 7 & -1 & 0 & 0 & 0 \\ 5 & 13 & 12 & 7 & -1 & 0 & 0 \\ 5 & 13 & 12 & 12 & 7 & -1 & 0 \\ 5 & 13 & 12 & 12 & 12 & 7 & -1 \\ 5 & 13 & 12 & 12 & 11 & 15 & 4 \end{bmatrix}$$

$$[N]$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & -1 & 0 & 0 & 0 & 0 \\ 16 & 32 & 0 & 0 & 0 & 0 & 0 \\ 25 & 60 & 21 & 2 & 0 & 0 & 0 \\ 34 & 88 & 44 & 24 & 2 & 0 & 0 \\ 43 & 116 & 67 & 48 & 24 & 2 & 0 \\ 52 & 144 & 90 & 72 & 48 & 24 & 2 \end{bmatrix}$$

Ninth Order Matrices

$$[M]$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 13 & 7 & -1 & 0 & 0 & 0 & 0 & 0 \\
5 & 13 & 12 & 7 & -1 & 0 & 0 & 0 & 0 \\
5 & 13 & 12 & 12 & 7 & -1 & 0 & 0 & 0 \\
5 & 13 & 12 & 12 & 12 & 7 & -1 & 0 & 0 \\
5 & 13 & 12 & 12 & 12 & 12 & 7 & -1 & 0 \\
5 & 13 & 12 & 12 & 12 & 12 & 12 & 7 & -1 \\
5 & 13 & 12 & 12 & 12 & 12 & 11 & 15 & 4
\end{bmatrix}$$

$$[N]$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 6 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
25 & 60 & 21 & 2 & 0 & 0 & 0 & 0 & 0 \\
34 & 88 & 44 & 24 & 2 & 0 & 0 & 0 & 0 \\
43 & 116 & 67 & 48 & 24 & 2 & 0 & 0 & 0 \\
52 & 144 & 90 & 72 & 48 & 24 & 2 & 0 & 0 \\
61 & 172 & 113 & 96 & 72 & 48 & 24 & 2 & 0 \\
70 & 200 & 136 & 120 & 96 & 72 & 48 & 24 & 2
\end{bmatrix}$$

Eleventh Order Matrices

$$[M]$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 8 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 13 & 7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 13 & 12 & 7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 13 & 12 & 12 & 7 & -1 & 0 & 0 & 0 & 0 & 0 \\
5 & 13 & 12 & 12 & 12 & 7 & -1 & 0 & 0 & 0 & 0 \\
5 & 13 & 12 & 12 & 12 & 12 & 7 & -1 & 0 & 0 & 0 \\
5 & 13 & 12 & 12 & 12 & 12 & 12 & 7 & -1 & 0 & 0 \\
5 & 13 & 12 & 12 & 12 & 12 & 12 & 12 & 7 & -1 & 0 \\
5 & 13 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 7 & -1 \\
5 & 13 & 12 & 12 & 12 & 12 & 12 & 12 & 11 & 15 & 4
\end{bmatrix}$$

$$[N]$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 6 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
16 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
25 & 60 & 21 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
34 & 88 & 44 & 24 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
43 & 116 & 67 & 48 & 24 & 2 & 0 & 0 & 0 & 0 & 0 \\
52 & 144 & 90 & 72 & 48 & 24 & 2 & 0 & 0 & 0 & 0 \\
61 & 172 & 113 & 96 & 72 & 48 & 24 & 2 & 0 & 0 & 0 \\
70 & 200 & 136 & 120 & 96 & 72 & 48 & 24 & 2 & 0 & 0 \\
79 & 228 & 159 & 144 & 120 & 96 & 72 & 48 & 24 & 2 & 0 \\
88 & 256 & 182 & 168 & 144 & 120 & 96 & 72 & 48 & 24 & 2
\end{bmatrix}$$

REFERENCES

1. Newmark, N. M.: Numerical Procedure for Computing Deflections, Moments and Buckling Loads. Proc. A. S. C. E., vol. 68, no. 5, May 1942, pp. 691-718. (Also, Paper No. 2202, Trans. A. S. C. E., vol. 108, 1943, pp. 1161-1188.)

2. Houbolt, John C., and Anderson, Roger A.: Calculation of Uncoupled Modes and Frequencies in Bending or Torsion of Nonuniform Beams. NACA TN No. 1522, 1948.

3. Benscoter, S. U.: Matrix Analyses of Continuous Beams. Trans. A. S. C. E., vol. 112, 1947, pp. 1109-1172.

4. Westergaard, H. M.: Deflection of Beams by the Conjugate Beam Method. Jour. Western Soc. Eng., vol. 26, no. 11, Nov. 1921, pp. 369-396.

5. Von Kármán, Theodore, and Biot, Maurice A.: Mathematical Methods in Engineering. First ed., McGraw-Hill Book Co., Inc., 1940.

6. Sokolnikoff, Ivan S., and Sokolnikoff, Elizabeth S.: Higher Mathematics for Engineers and Physicists. McGraw-Hill Book Co., Inc., 1934, p. 423.

7. Aitken, A. C.: Determinants and Matrices. Interscience Publishers, Inc., New York, 1939, p. 124.

TABLE I

NATURAL FREQUENCIES AND STATIC TIP DEFLECTIONS

Number of stations	Torsion			Bending		
	First frequency, $\omega_1\sqrt{\frac{I_m L^2}{GJ}}$	Second frequency, $\omega_2\sqrt{\frac{I_m L^2}{GJ}}$	Static tip deflection, $\phi_1 \frac{GJ}{L^2 q_r}$ (a)	First frequency, $\omega_1\sqrt{\frac{\rho L^4}{EI}}$	Second frequency, $\omega_2\sqrt{\frac{\rho L^4}{EI}}$	Static tip deflection, $y_1 \frac{EI}{L^4 p_r}$ (b)
Influence-coefficient method						
3	1.530	3.11	0.125	3.16	13.70	0.0261
5	1.560	4.38	.156	3.42	20.54	.0316
7	1.567	4.57	.162	3.47	21.29	.0326
Weighted-influence-coefficient method						
3	1.575	5.39	0.167	3.56	15.63	0.0347
5	1.571	4.73	.167	3.52	22.80	.0334
7	1.571	4.72	.167	3.52	22.08	.0334
Weighted-integration method						
3	1.582	7.59	0.167	3.58	14.58	0.0347
5	1.573	4.81	.167	3.52	22.08	.0334
7	1.572	4.74	.167	3.52	22.26	.0334
Exact values						
	1.571	4.71	0.167	3.52	22.03	0.0333

[a] Load distribution triangular with intensity q_r at root.

[b] Load distribution triangular with intensity p_r at root.

TABLE II

STATIONS REQUIRED FOR SATISFACTORY ACCURACY

Quantity	Influence coefficients	Weighted influence coefficients	Weighted integration
Torsion			
ω_1	5	3	3
ω_2	>7	5	5
ϕ_1	>7	3	3
Bending			
ω_1	>7	4	4
ω_2	>7	7	5
y_1	>7	5	5

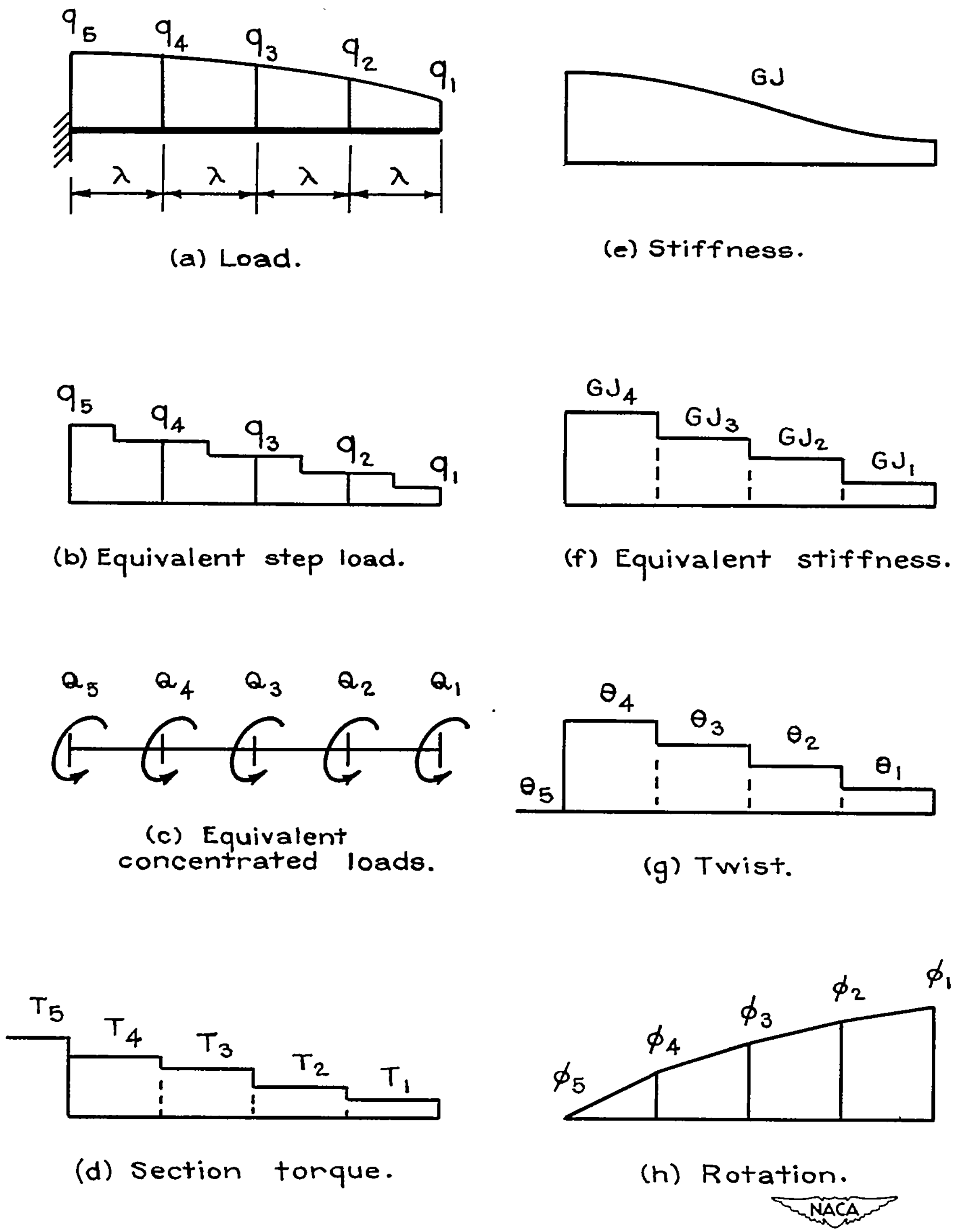


Figure 1.- Torsional deflections obtained by step functions.

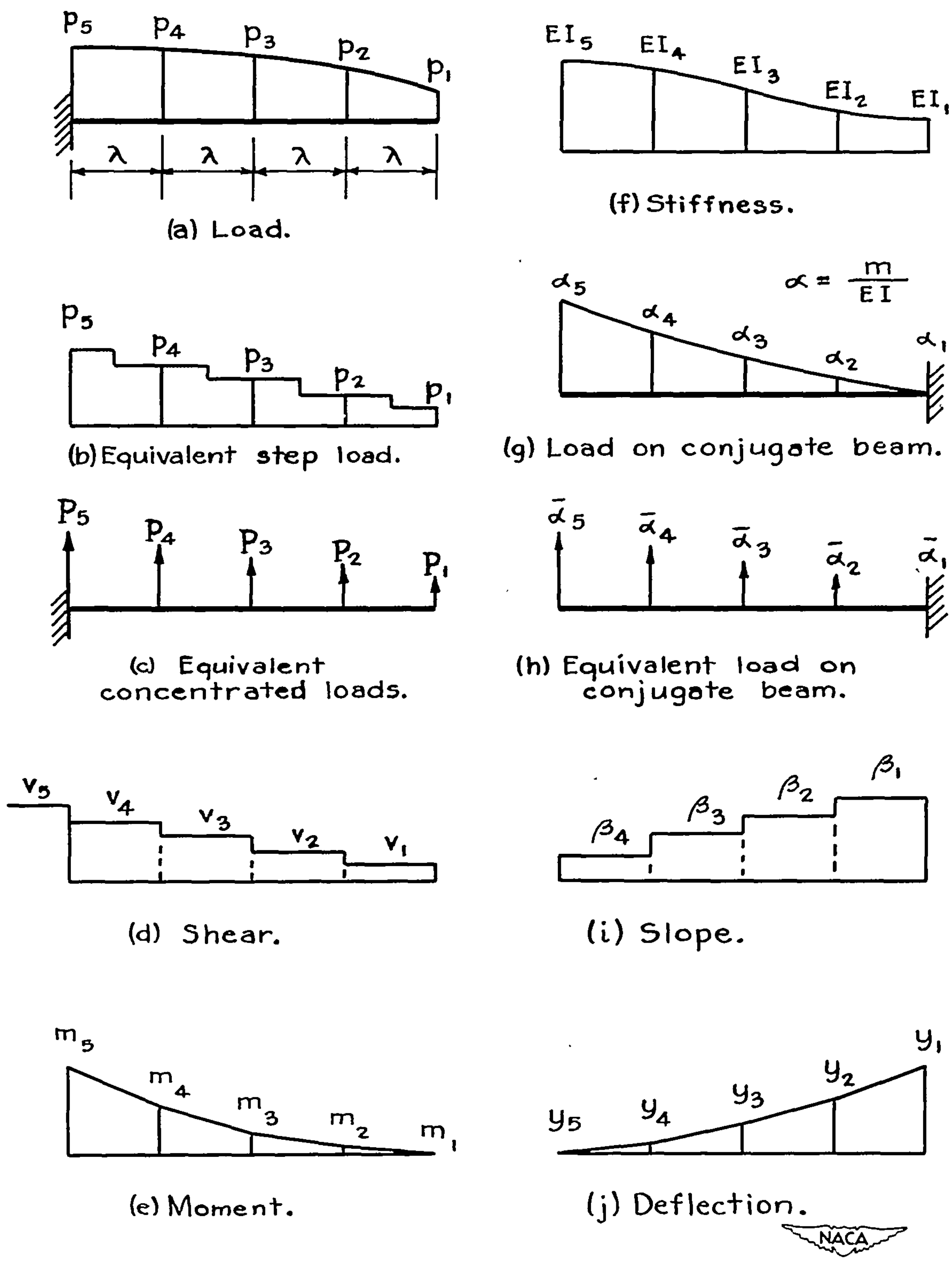


Figure 2.- Bending deflections obtained by step functions.

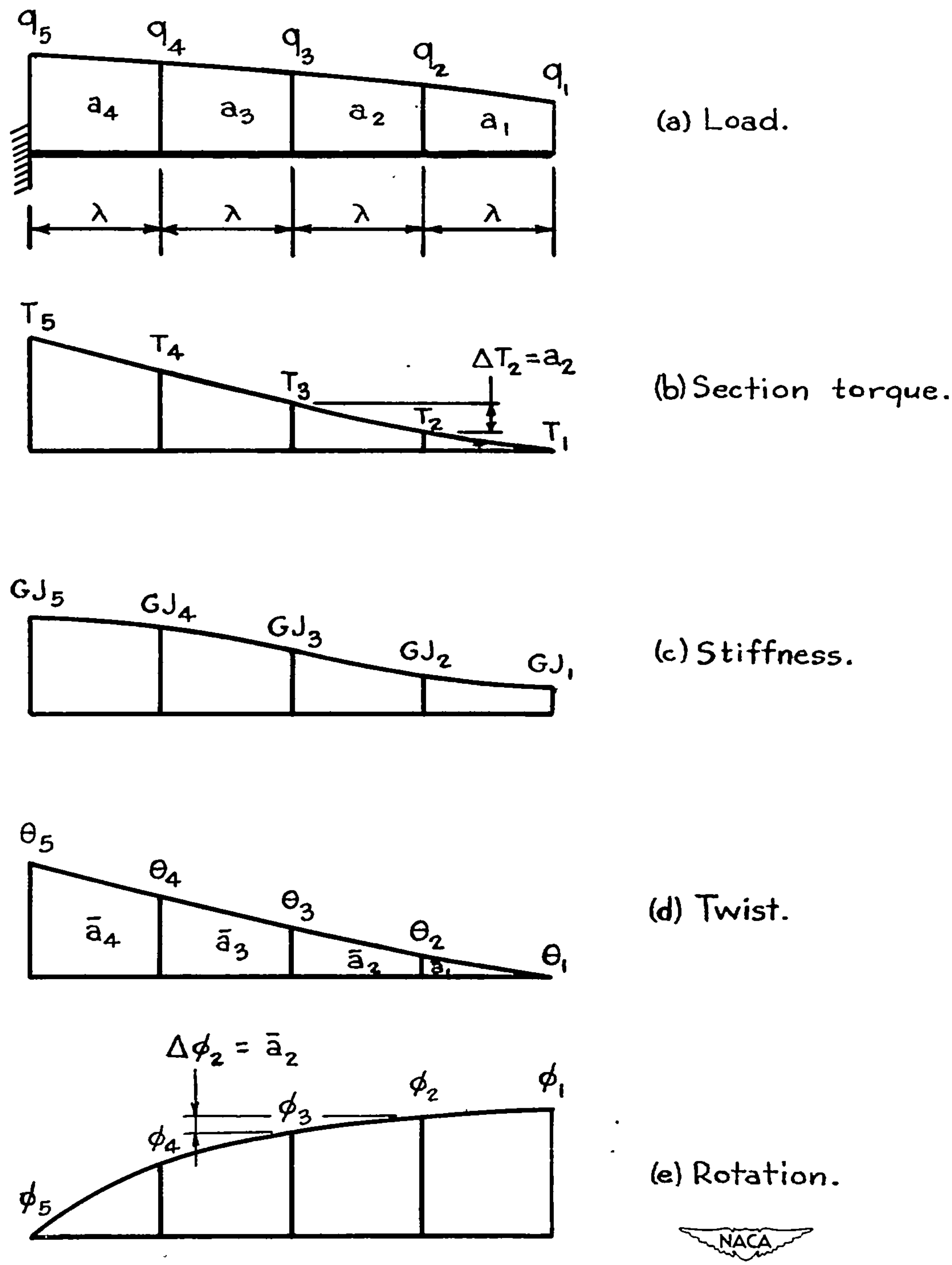


Figure 3.- Torsional deflections obtained by parabolic arcs.

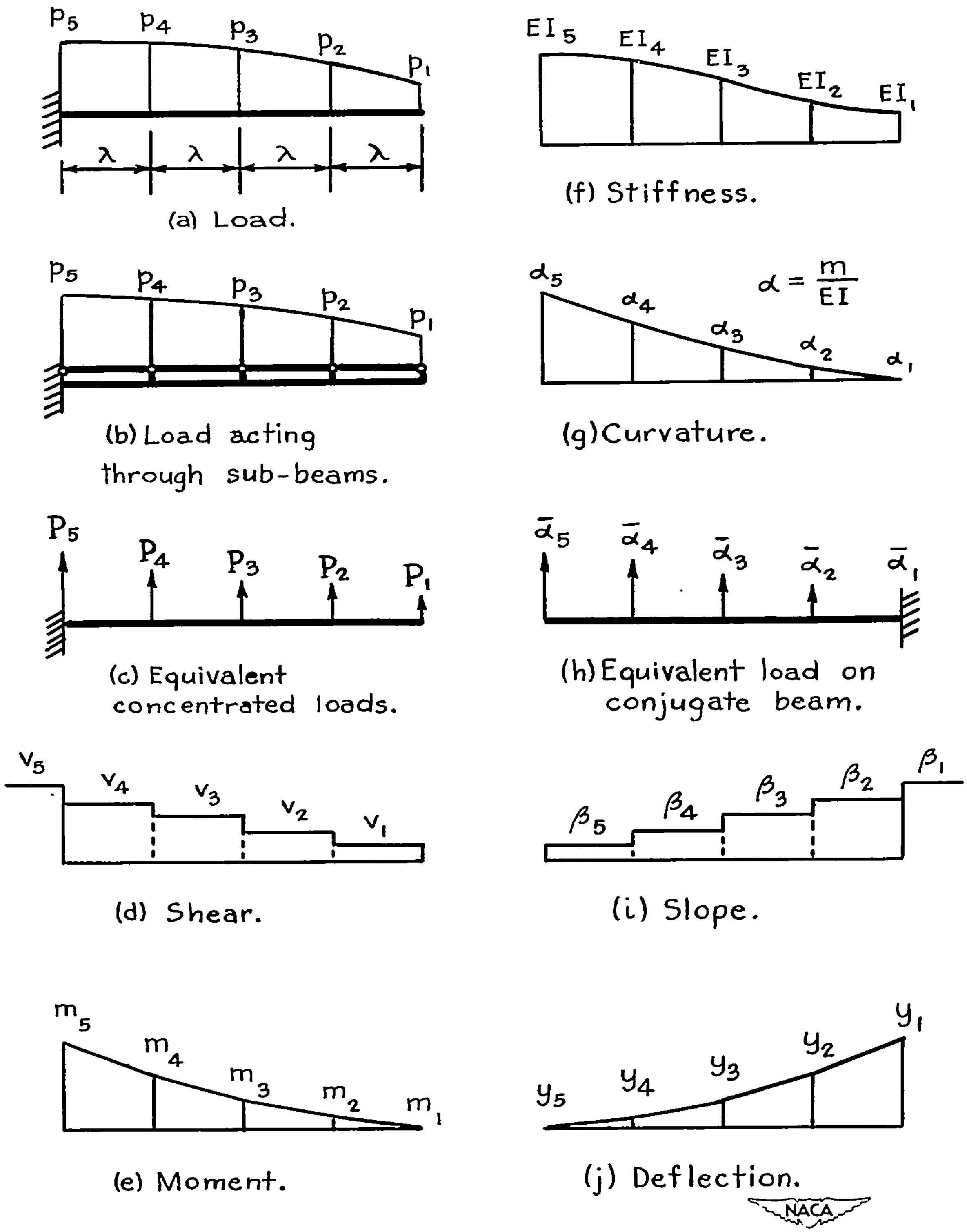
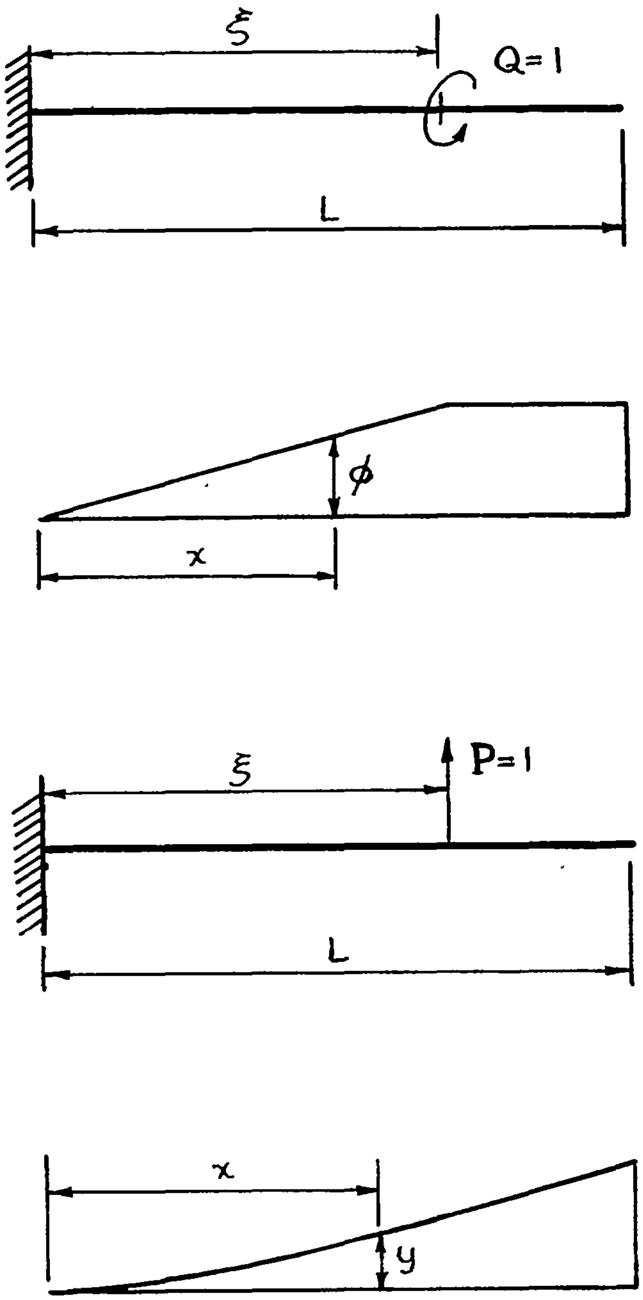


Figure 4.- Bending deflections obtained by parabolic arcs.



(a) Beam with unit torque.

(b) Rotation.

(c) Beam with unit force.

(d) Translation.

Figure 5.- Deflections of a cantilever beam with unit loads.

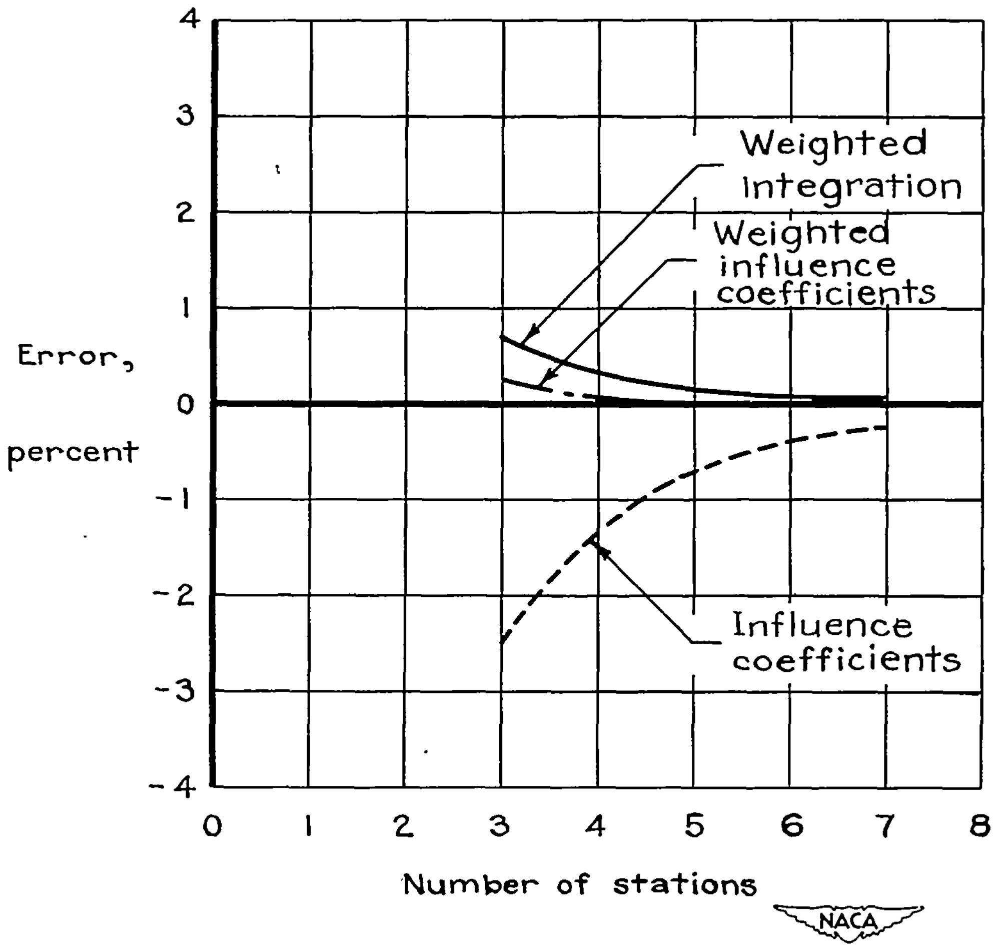


Figure 6.- Comparison of errors in first torsional frequency.

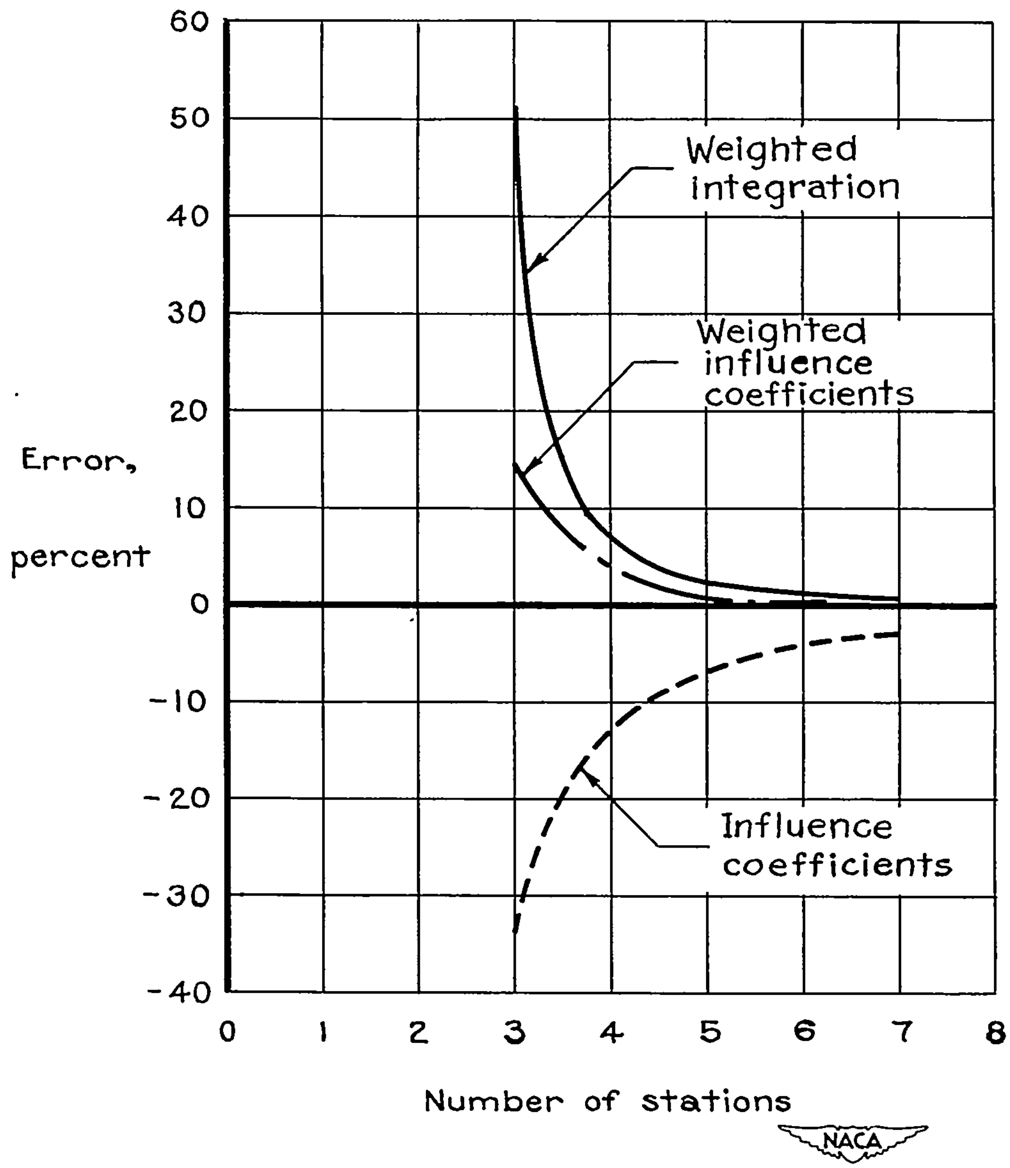


Figure 7.- Comparison of errors in second torsional frequency.

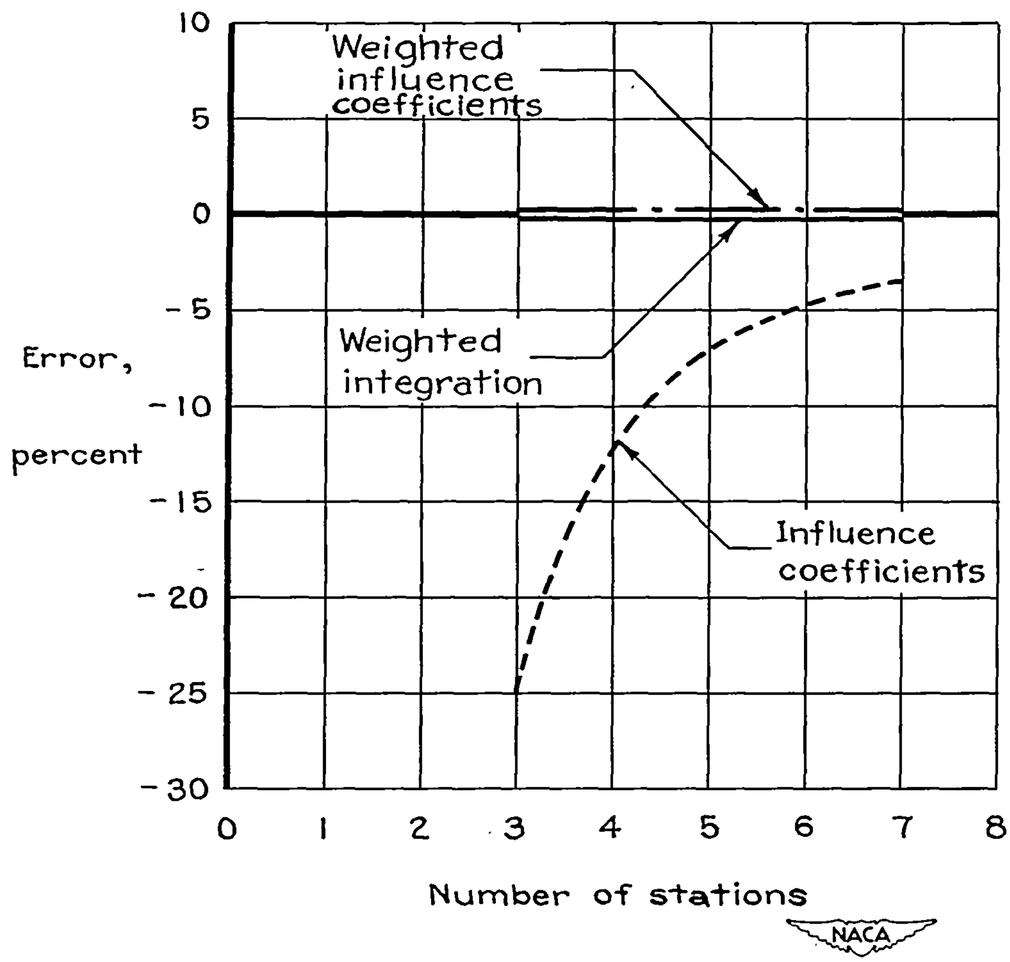


Figure 8.- Comparison of errors in static torsional tip deflection.

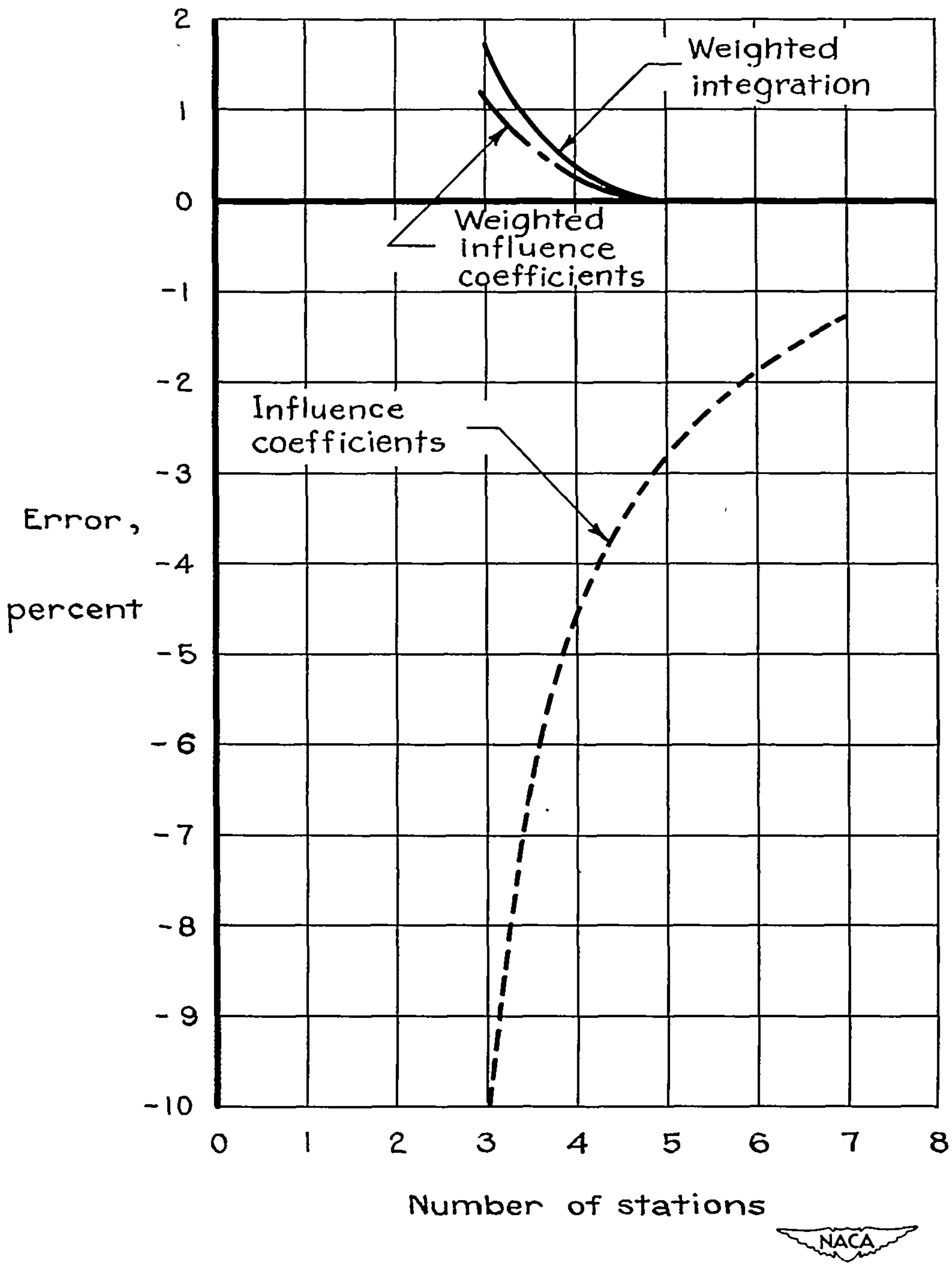


Figure 9.- Comparison of errors in first bending frequency.

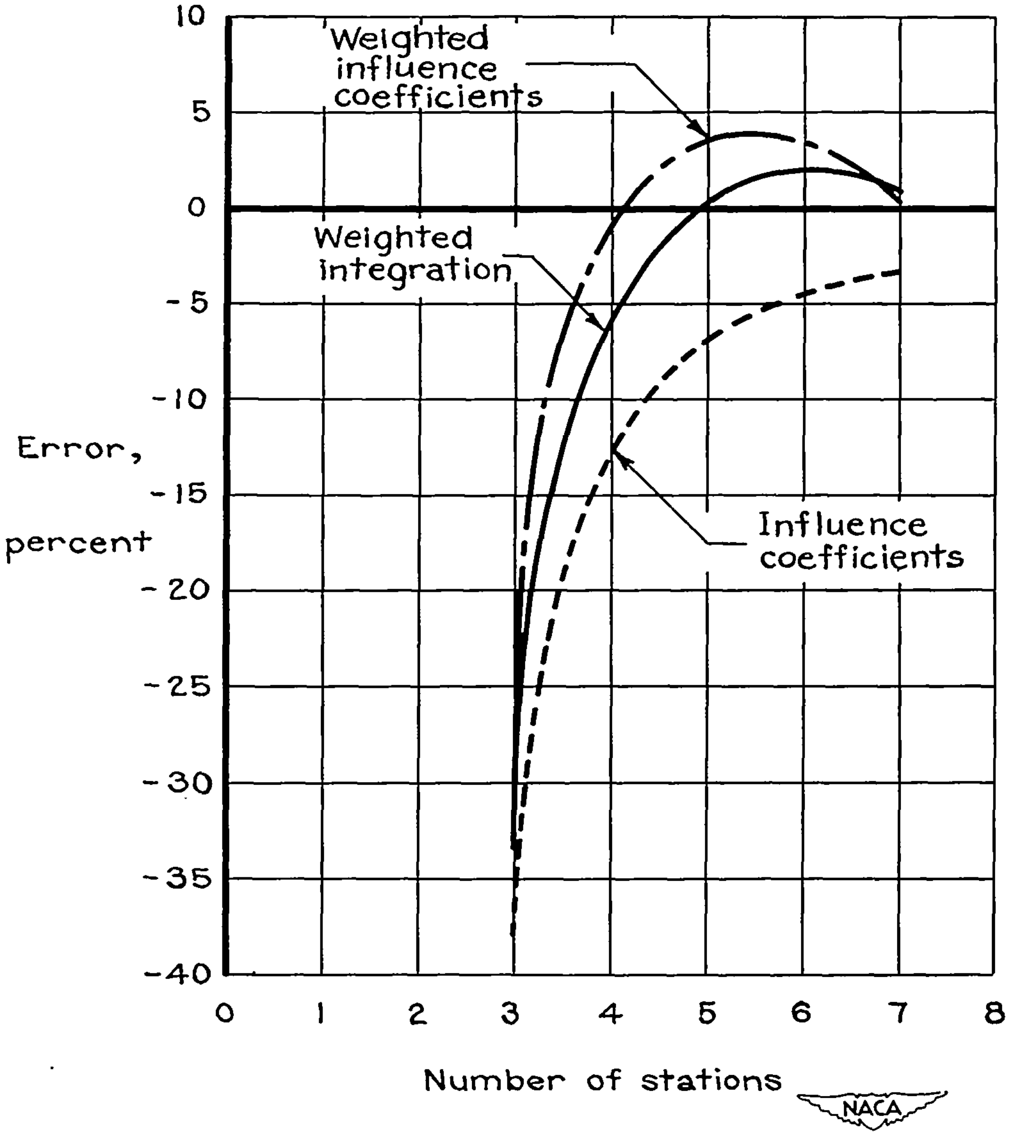


Figure 10.- Comparison of errors in second bending frequency.

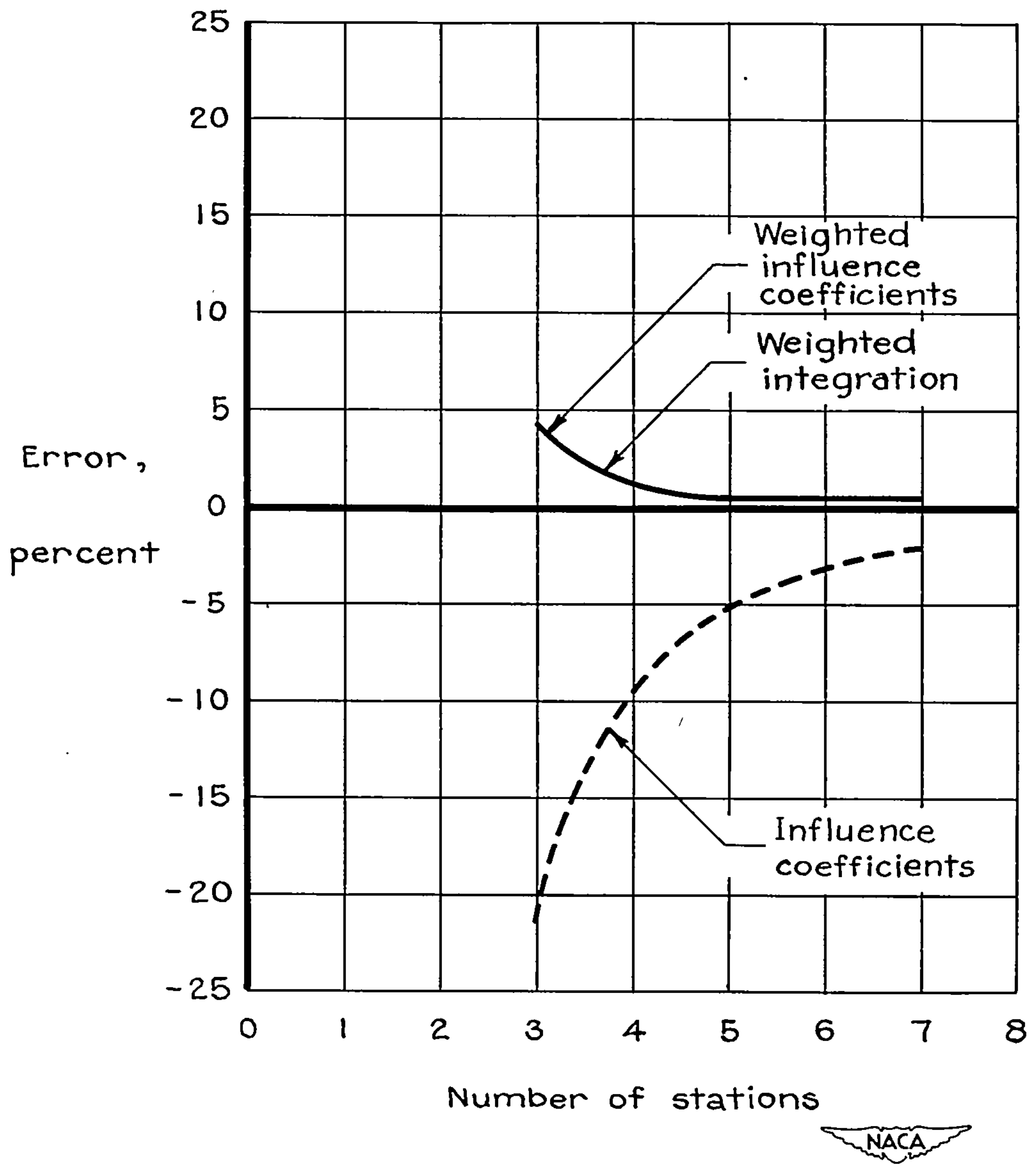


Figure 11.- Comparison of errors in static bending tip deflection.

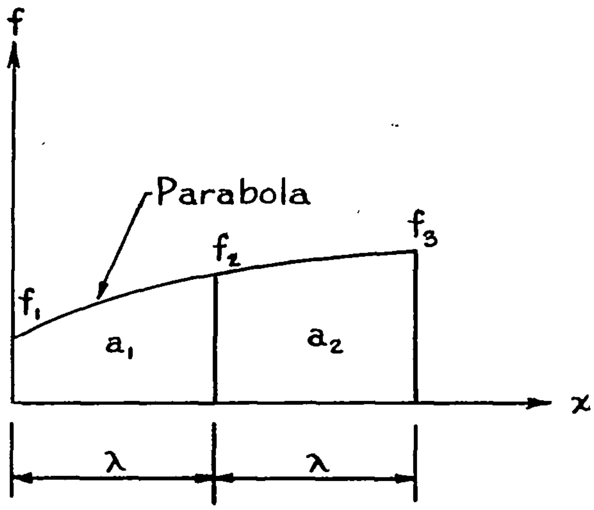


Figure 12.- Areas under a parabolic curve.

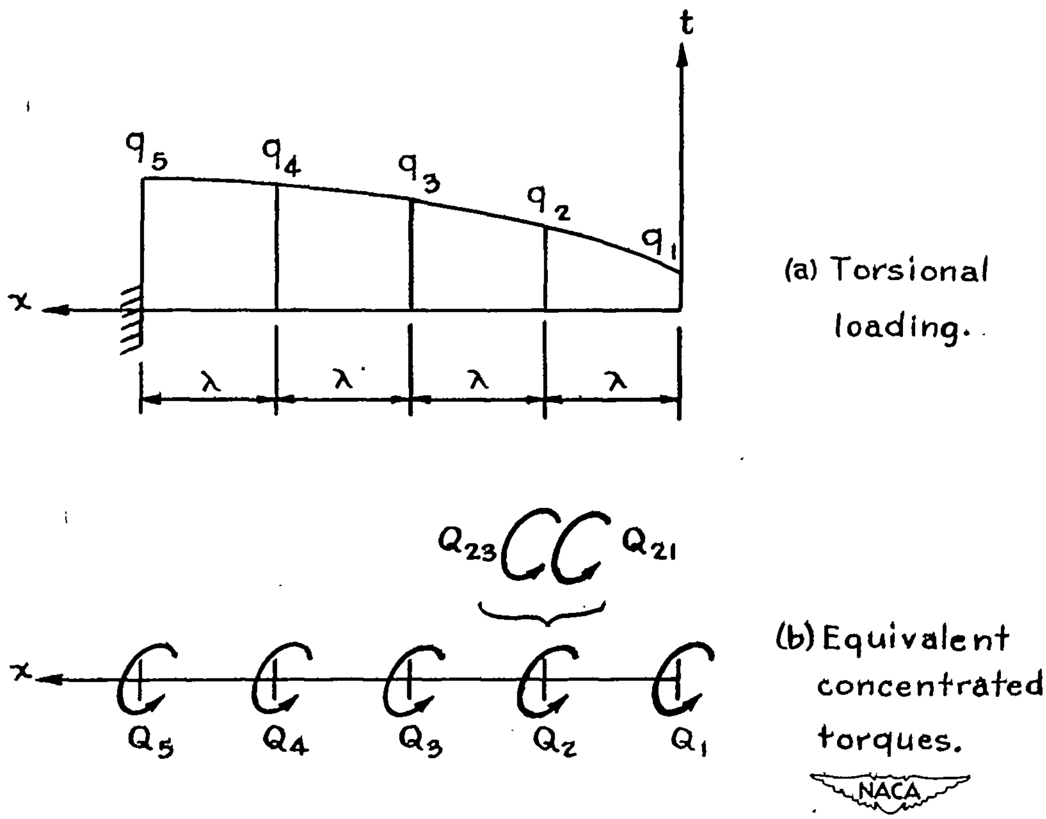


Figure 13.- Equivalent concentrated torsional loading.