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## TECHNICAL NOTE

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## NOTES ON THE FOUNDATIONS OF THE THEORY OF SMALL DISPLACEMENTS OF ORTHOTROPIC SHELLS

By F. B. Hildebrand, E. Reissner, and G. B. Thomas

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## SUMMARY

A survey has been made of various systems of equations which have been given in the literature for the analysis of small deflections of thin elastic shells. In this survey the results, previously known for isotropic shells, have been reformulated for shells which are orthotropic to the extent that the normal to the middle surface of the shell may be a preferred elastic axis.

A new system of equations has been derived for the analysis of shells, which includes the effects of transverse shear and normal stresses. The assumed orthotropy of the shell facilitates the identification of the separate effects of the ordinarily neglected transverse stresses.

## INTRODUCTION

In this report various methods are described of obtaining systems of equations which may be considered as forming the basis of a theory of small displacements of elastic plates and shells. In all cases one begins with the governing equations in the three-dimensional theory of elasticity and an attempt is made to reduce this system of equations, involving three independent space variables, to a new system involving only two space variables. These two variables are most conveniently taken as coordinates on the middle surface of the plate or shell.

Numerous reductions of this sort have been carried out in the past by different workers in the field of elasticity. Care will be taken to point out in the analysis the connection between the developments herein presented and the results of earlier studies.

One of the points of interest is the study of the effect of transverse shear deformation on the bending of shells. In the case of the flat plate it has been shown, by methods which differ somewhat from those used here (references 1 and 2), that inclusion of this effect resolves in a natural way well-known difficulties with regard to the boundary conditions which may be prescribed along the edges of a plate. It is to be expected that corresponding difficulties in the analysis of shells may be removed in an analogous way, and the present report shows that this is indeed the case. The practical significance of this analysis with reference to stress-concentration problems has been established in the earlier work on plates.

Attention is herein restricted to static problems involving small deformations, excluding for the present the study of vibrations, elastic stability, and finite deformations. It will be apparent, however, that much of the present analysis can be extended to the consideration of these problems.

In order that the separate effects of the transverse stresses may be segregated, the plates and shells are assumed to consist of material which may be orthotropic to the extent that the normal to the middle surface is a preferred elastic axis.

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## SYMBOLS

| | |
|---|---|
| $\xi_1$, $\xi_2$ | coordinates in middle surface of shell |
| $\zeta$ | coordinate normal to middle surface |
| $\overline{t}_1$, $\overline{t}_2$, $\overline{n}$ | unit vectors in directions of $\xi_1$, $\xi_2$, $\zeta$ |
| $A_1$, $A_2$, $\alpha_1$, $\alpha_2$ | parameters in linear element (equations (2) and (3)) |
| $R_1$, $R_2$ | principal radii of curvature |
| $\overline{U}$ | displacement vector $\left(\overline{U} = U_1\overline{t}_1 + U_2\overline{t}_2 + W\overline{n}\right)$ |
| $\overline{F}$ | body-force vector $\left(\overline{F} = F_1\overline{t}_1 + F_2\overline{t}_2 + F_\zeta\overline{n}\right)$ |
| $\epsilon_1$, $\epsilon_2$, $\epsilon_\zeta$ | components of direct strain |
| $\gamma_{12}$, $\gamma_{1\zeta}$, $\gamma_{2\zeta}$ | components of shearing strain |
| $\sigma_1$, $\sigma_2$, $\sigma_\zeta$ | components of direct stress |

| | |
|---|---|
| $\tau_{12}$, $\tau_{1\zeta}$, $\tau_{2\zeta}$ | components of shearing stress |
| $\nu$, $\nu_\zeta$ | Poisson's ratios |
| $E$, $E_\zeta$ | Young's moduli |
| $G$, $G_\zeta$ | shear moduli |
| $\bar{\nu}$ | parameter defined by $\bar{\nu} = \sqrt{\frac{E}{E_\zeta}}\,\nu_\zeta$ |
| $\nu^*$ | parameter defined by $\nu^* = \frac{\nu_\zeta E}{(1 - \nu)E_\zeta}$ |
| $\bar{E}$ | parameter defined by $\bar{E} = \frac{1 - \nu}{1 - \nu - 2\bar{\nu}^2}\,E$ |
| $\pi$ | potential energy |
| $P$ | strain energy per unit volume |
| $\bar{p}$ | effective external force per unit area; applied to middle surface $(\bar{p} = p_1\bar{t}_1 + p_2\bar{t}_2 + q\bar{n})$ |
| $\bar{m}$ | effective external moment per unit area; applied to middle surface $(\bar{m} = m_2\bar{t}_1 - m_1\bar{t}_2)$ |
| $\bar{N}_1$, $\bar{N}_2$ | stress-resultant vectors $(\bar{N}_1 = N_{11}\bar{t}_1 + N_{12}\bar{t}_2 + Q_1\bar{n};\ \bar{N}_2 = N_{21}\bar{t}_1 + N_{22}\bar{t}_2 + Q_2\bar{n})$ |
| $\bar{M}_1$, $\bar{M}_2$ | stress-couple vectors $(\bar{M}_1 = M_{12}\bar{t}_1 - M_{11}\bar{t}_2;\ \bar{M}_2 = M_{22}\bar{t}_1 - M_{21}\bar{t}_2)$ |
| $P_{11}$, $P_{12}$, $P_{21}$, $P_{22}$<br>$S_1$, $S_2$, $T_1$, $T_2$ | higher-order stress resultants defined by equations (73) |
| $A$, $B$ | auxiliary stress resultants defined by equations (79) |
| $u_1$, $u_1'$, $u_1''$<br>$u_2$, $u_2'$, $u_2''$<br>$w$, $w'$, $w''$ | displacement functions defined by equations (69) |
| $\epsilon^0$, $\epsilon'$, $\epsilon''$<br>$\beta^0$, $\beta'$, $\beta''$<br>$\mu^0$, $\mu'$, $\mu''$ | strain functions defined by equations (84), with appropriate subscripts |

$\gamma^0$, $\kappa$, $\tau$ strain functions defined by equations (46) and (48b), with appropriate subscripts

$a^0$, $a'$, $a''$ auxiliary parameters defined by equations (86), with appropriate subscripts

## THREE-DIMENSIONAL STRESS AND STRAIN IN CURVILINEAR COORDINATES

In this section there are collected for convenient reference certain known basic formulas pertaining to the analysis of stress and strain in terms of orthogonal curvilinear coordinates.

### THE COORDINATE SYSTEM

The position of any point in a plate or shell may be specified by three coordinates $\xi_1$, $\xi_2$, $\zeta$, where $\xi_1$ and $\xi_2$ specify position on the middle surface, while $\zeta$ measures the distance, along the outward normal, from the middle surface to the point. In order that the coordinate curves be orthogonal, it is required that the $\xi_1$- and $\xi_2$-curves be the lines of curvature on the middle surface $\zeta = 0$. The unit normal vector at a point of the middle surface is denoted by $\overline{n}$ and the unit tangent vectors to the $\xi_1$- and $\xi_2$-curves are denoted by $\overline{t}_1$ and $\overline{t}_2$, respectively. The coordinates $\xi_1$ and $\xi_2$ are to be chosen in such a way that the system is right-handed, in the sense that $\overline{t}_1$ is rotated into $\overline{t}_2$ by a right-handed rotation about $\overline{n}$.

If there is written for the position vector to a point in space

$$\overline{R}\left(\xi_1, \xi_2, \zeta\right) = \overline{r}\left(\xi_1, \xi_2\right) + \zeta\overline{n}\left(\xi_1, \xi_2\right) \tag{1}$$

where $\overline{r}$ is the position vector to a point on the middle surface, the linear element is of the form

$$ds^2 = d\overline{R}\cdot d\overline{R} = A_1^2\, d\xi_1^2 + A_2^2\, d\xi_2^2 + d\zeta^2 \tag{2}$$

The coefficients in the linear element are given by

$$\left.\begin{aligned} A_1 &= \alpha_1\left(1 + \frac{\zeta}{R_1}\right) \\ A_2 &= \alpha_2\left(1 + \frac{\zeta}{R_2}\right) \end{aligned}\right\} \tag{3}$$

where $R_1$ and $R_2$ are the principal radii of curvature of the middle surface and

$$\left.\begin{aligned} \alpha_1^2 &= \frac{\partial\bar{r}}{\partial\xi_1}\cdot\frac{\partial\bar{r}}{\partial\xi_1} & \alpha_2^2 &= \frac{\partial\bar{r}}{\partial\xi_2}\cdot\frac{\partial\bar{r}}{\partial\xi_2} \\ \frac{1}{R_1} &= \frac{1}{\alpha_1^2}\frac{\partial\bar{n}}{\partial\xi_1}\cdot\frac{\partial\bar{r}}{\partial\xi_1} & \frac{1}{R_2} &= \frac{1}{\alpha_2^2}\frac{\partial\bar{n}}{\partial\xi_2}\cdot\frac{\partial\bar{r}}{\partial\xi_2} \end{aligned}\right\} \tag{4}$$

There is also the relation

$$\bar{n} = \bar{t}_1 \times \bar{t}_2 = \frac{1}{\alpha_1\alpha_2}\frac{\partial\bar{r}}{\partial\xi_1} \times \frac{\partial\bar{r}}{\partial\xi_2} \tag{5}$$

The further relations

$$\left.\begin{aligned} \frac{1}{R_2}\frac{\partial\alpha_1}{\partial\xi_2} &= \frac{\partial}{\partial\xi_2}\left(\frac{\alpha_1}{R_1}\right) \\ \frac{1}{R_1}\frac{\partial\alpha_2}{\partial\xi_1} &= \frac{\partial}{\partial\xi_1}\left(\frac{\alpha_2}{R_2}\right) \end{aligned}\right\} \tag{6}$$

which are special cases of formulas obtained by Mainardi and Codazzi, are of frequent use. It is noted, for later reference, that equations (6) imply the truth of the equations

$$\left.\begin{aligned} \frac{\partial}{\partial\xi_2}\left[\alpha_1\left(1 + \frac{\zeta}{R_1}\right)\right] &= \frac{\partial\alpha_1}{\partial\xi_2}\left(1 + \frac{\zeta}{R_2}\right) \\ \frac{\partial}{\partial\xi_1}\left[\alpha_2\left(1 + \frac{\zeta}{R_2}\right)\right] &= \frac{\partial\alpha_2}{\partial\xi_1}\left(1 + \frac{\zeta}{R_1}\right) \end{aligned}\right\} \tag{7}$$

## COMPONENTS OF STRAIN

If the displacement vector $\bar{U}$ is written in the form

$$\bar{U} = U_1\bar{t}_1 + U_2\bar{t}_2 + W\bar{n} \tag{8}$$

the six components of ~~infinitesimal~~ strain are expressed in terms of the components of $\overline{U}$ by the equations

use eqs.(7)

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{\alpha_1(1+\zeta/R_1)}\left(\frac{\partial U_1}{\partial \xi_1} + \frac{U_2}{\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \alpha_1 \frac{W}{R_1}\right) \\
\epsilon_2 &= \frac{1}{\alpha_2(1+\zeta/R_2)}\left(\frac{\partial U_2}{\partial \xi_2} + \frac{U_1}{\alpha_1}\frac{\partial \alpha_2}{\partial \xi_1} + \alpha_2 \frac{W}{R_2}\right) \\
\epsilon_\zeta &= \frac{\partial W}{\partial \zeta}
\end{aligned}\right\} \tag{9a}$$

$$\left.\begin{aligned}
\gamma_{12} &= \frac{\alpha_2(1+\zeta/R_2)}{\alpha_1(1+\zeta/R_1)}\frac{\partial}{\partial \xi_1}\left[\frac{U_2}{\alpha_2(1+\zeta/R_2)}\right] + \frac{\alpha_1(1+\zeta/R_1)}{\alpha_2(1+\zeta/R_2)}\frac{\partial}{\partial \xi_2}\left[\frac{U_1}{\alpha_1(1+\zeta/R_1)}\right] \\
\gamma_{1\zeta} &= \frac{1}{\alpha_1(1+\zeta/R_1)}\frac{\partial W}{\partial \xi_1} + \left(1+\frac{\zeta}{R_1}\right)\frac{\partial}{\partial \zeta}\left(\frac{U_1}{1+\zeta/R_1}\right) \\
\gamma_{2\zeta} &= \frac{1}{\alpha_2(1+\zeta/R_2)}\frac{\partial W}{\partial \xi_2} + \left(1+\frac{\zeta}{R_2}\right)\frac{\partial}{\partial \zeta}\left(\frac{U_2}{1+\zeta/R_2}\right)
\end{aligned}\right\} \tag{9b}$$

## DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

If the body force per unit volume is denoted by the vector

$$\overline{F} = F_1\overline{t}_1 + F_2\overline{t}_2 + F_\zeta\overline{n} \tag{10}$$

the six components of stress $\sigma_1$, $\sigma_2$, $\sigma_\zeta$, $\tau_{12}$, $\tau_{1\zeta}$, and $\tau_{2\zeta}$ must satisfy the three equilibrium equations

$$\left.\begin{aligned}
&\frac{\partial}{\partial\xi_1}\left[\alpha_2\left(1 + \zeta/R_2\right)\sigma_1\right] + \frac{\partial}{\partial\xi_2}\left[\alpha_1\left(1 + \zeta/R_1\right)\tau_{12}\right] \\
&\quad + \alpha_1\alpha_2 \frac{\partial}{\partial\zeta}\left[\left(1 + \zeta/R_1\right)\left(1 + \zeta/R_2\right)\tau_{1\zeta}\right] \\
&\quad + \tau_{12}\frac{\partial}{\partial\xi_2}\left[\alpha_1\left(1 + \zeta/R_1\right)\right] + \tau_{1\zeta}\frac{\alpha_1\alpha_2}{R_1}\left(1 + \zeta/R_2\right) \\
&\quad - \sigma_2\frac{\partial}{\partial\xi_1}\left[\alpha_2\left(1 + \zeta/R_2\right)\right] + \alpha_1\alpha_2\left(1 + \zeta/R_1\right)\left(1 + \zeta/R_2\right)F_1 = 0 \\[2ex]
&\frac{\partial}{\partial\xi_2}\left[\alpha_1\left(1 + \zeta/R_1\right)\sigma_2\right] + \frac{\partial}{\partial\xi_1}\left[\alpha_2\left(1 + \zeta/R_2\right)\tau_{12}\right] \\
&\quad + \alpha_1\alpha_2 \frac{\partial}{\partial\zeta}\left[\left(1 + \zeta/R_1\right)\left(1 + \zeta/R_2\right)\tau_{2\zeta}\right] \\
&\quad + \tau_{12}\frac{\partial}{\partial\xi_1}\left[\alpha_2\left(1 + \zeta/R_2\right)\right] + \tau_{2\zeta}\frac{\alpha_1\alpha_2}{R_2}\left(1 + \zeta/R_1\right) \\
&\quad - \sigma_1\frac{\partial}{\partial\xi_2}\left[\alpha_1\left(1 + \zeta/R_1\right)\right] + \alpha_1\alpha_2\left(1 + \zeta/R_1\right)\left(1 + \zeta/R_2\right)F_2 = 0 \\[2ex]
&\alpha_1\alpha_2\frac{\partial}{\partial\zeta}\left[\left(1 + \zeta/R_1\right)\left(1 + \zeta/R_2\right)\sigma_\zeta\right] + \frac{\partial}{\partial\xi_1}\left[\alpha_2\left(1 + \zeta/R_2\right)\tau_{1\zeta}\right] \\
&\quad + \frac{\partial}{\partial\xi_2}\left[\alpha_1\left(1 + \zeta/R_1\right)\tau_{2\zeta}\right] - \frac{\alpha_1\alpha_2}{R_1}\left(1 + \zeta/R_2\right)\sigma_1 \\
&\quad - \frac{\alpha_1\alpha_2}{R_2}\left(1 + \zeta/R_1\right)\sigma_2 + \alpha_1\alpha_2\left(1 + \zeta/R_1\right)\left(1 + \zeta/R_2\right)F_\zeta = 0
\end{aligned}\right\} \quad (11)$$

## STRESS-STRAIN RELATIONS

If it is required that the normal to the middle surface be an axis of elastic symmetry and if isotropy is assumed in elements of the middle surface, the stress-strain equations may be written in the form

$$\left.\begin{aligned} \epsilon_1 &= \frac{\sigma_1 - \nu\sigma_2}{E} - \frac{\nu_\zeta \sigma_\zeta}{E_\zeta} \\ \epsilon_2 &= \frac{\sigma_2 - \nu\sigma_1}{E} - \frac{\nu_\zeta \sigma_\zeta}{E_\zeta} \\ \epsilon_\zeta &= \frac{\sigma_\zeta}{E_\zeta} - \frac{\nu_\zeta(\sigma_1 + \sigma_2)}{E_\zeta} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \gamma_{12} &= \frac{\tau_{12}}{G} \\ \gamma_{1\zeta} &= \frac{\tau_{1\zeta}}{G_\zeta} \\ \gamma_{2\zeta} &= \frac{\tau_{2\zeta}}{G_\zeta} \end{aligned}\right\} \tag{13}$$

If equations (12) are solved for $\sigma_1$ and $\sigma_2$ the results can be expressed in the form

$$\left.\begin{aligned} \sigma_1 &= \frac{E}{1 - \nu^2}\left(\epsilon_1 + \nu\epsilon_2\right) + \frac{\nu_\zeta E}{(1 - \nu)E_\zeta}\sigma_\zeta \\ \sigma_2 &= \frac{E}{1 - \nu^2}\left(\epsilon_2 + \nu\epsilon_1\right) + \frac{\nu_\zeta E}{(1 - \nu)E_\zeta}\sigma_\zeta \end{aligned}\right\} \tag{14}$$

where

$$\sigma_\zeta = \frac{E_\zeta}{1 - 2\frac{\left(\nu_\zeta\right)^2 E}{(1 - \nu)E_\zeta}}\left[\epsilon_\zeta + \frac{\nu_\zeta E}{(1 - \nu)E_\zeta}\left(\epsilon_1 + \epsilon_2\right)\right] \tag{14a}$$

If there are introduced the symbols

$$\left.\begin{aligned} \nu^* &= \frac{\nu_\zeta}{1 - \nu}\frac{E}{E_\zeta} \\ E^* &= \frac{(1 - \nu^2)E_\zeta}{1 - 2\nu_\zeta\nu^*} \end{aligned}\right\} \tag{15}$$

equations (14) and (14a) may also be written in the form

$$\left.\begin{aligned} \sigma_1 &= \frac{E}{1 - \nu^2}\left(\epsilon_1 + \nu\epsilon_2\right) + \nu^*\sigma_\zeta \\ \sigma_2 &= \frac{E}{1 - \nu^2}\left(\epsilon_2 + \nu\epsilon_1\right) + \nu^*\sigma_\zeta \\ \sigma_\zeta &= \frac{E^*}{1 - \nu^2}\left[\epsilon_\zeta + \nu^*\left(\epsilon_1 + \epsilon_2\right)\right] \end{aligned}\right\} \tag{16}$$

An equivalent form of equations (14) and (14a), which is more convenient for some purposes, is obtained in terms of the parameters

$$\left.\begin{aligned} \bar{\nu} &= \sqrt{\frac{E}{E_\zeta}}\,\nu_\zeta \\ \bar{E} &= \frac{1 - \nu}{1 - \nu - 2\bar{\nu}^2}\,E \end{aligned}\right\} \tag{17}$$

With this notation equations (14) and (14a) lead to explicit expressions in terms of the strains in the form

$$\left.\begin{aligned}\sigma_1 &= \frac{\bar{\bar{E}}}{1-\nu^2}\left[\left(1-\bar{\nu}^2\right)\epsilon_1 + \left(\nu+\bar{\nu}^2\right)\epsilon_2 + \nu_\zeta(1+\nu)\epsilon_\zeta\right] \\ \sigma_2 &= \frac{\bar{\bar{E}}}{1-\nu^2}\left[\left(1-\bar{\nu}^2\right)\epsilon_2 + \left(\nu+\bar{\nu}^2\right)\epsilon_1 + \nu_\zeta(1+\nu)\epsilon_\zeta\right]\end{aligned}\right\} \tag{18}$$

$$\sigma_\zeta = \bar{\bar{E}}\left[\frac{\nu_\zeta}{1-\nu}\left(\epsilon_1+\epsilon_2\right) + \frac{E_\zeta}{E}\,\epsilon_\zeta\right] \tag{19}$$

It is noted that in the limiting orthotropic case for which

$$\nu_\zeta = 0 \tag{20}$$

there follows $\bar{\nu} = 0$, $\bar{\bar{E}} = E$, and equations (18) and (19) become

$$\left.\begin{aligned}\sigma_1 &= \frac{E}{1-\nu^2}\left(\epsilon_1+\nu\epsilon_2\right) \\ \sigma_2 &= \frac{E}{1-\nu^2}\left(\epsilon_2+\nu\epsilon_1\right) \\ \sigma_\zeta &= E\epsilon_\zeta\end{aligned}\right\} \tag{21}$$

Further, in the isotropic case for which

$$\left.\begin{aligned}\nu_\zeta &= \nu \\ E_\zeta &= E\end{aligned}\right\} \tag{22}$$

there follows

$$\left.\begin{aligned} \bar{E}_{\text{isotropic}} &= \frac{1-\nu}{(1+\nu)(1-2\nu)} E \\ \bar{\nu}_{\text{isotropic}} &= \nu \end{aligned}\right\} \tag{23}$$

and equations (18) and (19) become

$$\left.\begin{aligned} \sigma_1 &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_1 + \nu\left(\epsilon_2 + \epsilon_\zeta\right)\right] \\ \sigma_2 &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_2 + \nu\left(\epsilon_1 + \epsilon_\zeta\right)\right] \\ \sigma_\zeta &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_\zeta + \nu\left(\epsilon_1 + \epsilon_2\right)\right] \end{aligned}\right\} \tag{24}$$

## THE MINIMUM PRINCIPLE FOR THE DISPLACEMENTS; PRINCIPLE OF MINIMUM POTENTIAL ENERGY

If no body forces are acting and if all boundary conditions are stress conditions, the principle sets forth that among all possible states of strain the state which actually exists is that one for which the potential energy $\pi$ takes on its minimum value. (See, for example, reference 3, page 281.) The potential energy $\pi_l$ of the load system is given by the surface integral

$$\pi_l = -\iint \bar{p}_s \cdot \bar{U} \, dS \tag{25a}$$

where $\bar{p}_s$ is the surface-stress vector, while the energy $\pi_{\text{strain}}$ is given by the volume integral

$$\pi_{\text{strain}} = \iiint P\left(\epsilon_1, \epsilon_2, \epsilon_\zeta, \gamma_{12}, \gamma_{1\zeta}, \gamma_{2\zeta}\right) dV \tag{25b}$$

where P is the strain energy per unit volume which is to be expressed in terms of the displacements $U_1$, $U_2$, and W by use of equations (9). The function P is determined by the relations

$$\left.\begin{aligned} &\frac{\partial P}{\partial \epsilon_1} = \sigma_1 \qquad \frac{\partial P}{\partial \epsilon_2} = \sigma_2 \qquad \frac{\partial P}{\partial \epsilon_\zeta} = \sigma_\zeta \\ &\frac{\partial P}{\partial \gamma_{12}} = \tau_{12} \qquad \frac{\partial P}{\partial \gamma_{1\zeta}} = \tau_{1\zeta} \qquad \frac{\partial P}{\partial \gamma_{2\zeta}} = \tau_{2\zeta} \end{aligned}\right\} \tag{26}$$

and, for the orthotropic material considered here, takes the form

$$\begin{aligned} P = \frac{1}{2}\Bigg\{&\frac{E}{1 - \nu^2}\left(\epsilon_1{}^2 + \epsilon_2{}^2 + 2\nu\epsilon_1\epsilon_2\right) + G\gamma_{12}{}^2 \\ &+ \frac{E^*}{1 - \nu^2}\left[\epsilon_\zeta + \nu^*\left(\epsilon_1 + \epsilon_2\right)\right]^2 + G_\zeta\left(\gamma_{1\zeta}{}^2 + \gamma_{2\zeta}{}^2\right)\Bigg\} \end{aligned} \tag{27a}$$

Equation (27a) can also be written in the form

$$\begin{aligned} P = \frac{1}{2}\Bigg\{&\frac{\overline{E}}{1 - \nu^2}\Bigg[\left(1 - \overline{\nu}^2\right)\left(\epsilon_1{}^2 + \epsilon_2{}^2\right) + \frac{E_\zeta}{\overline{E}}\left(1 - \nu^2\right)\epsilon_\zeta{}^2 + 2\left(\nu + \overline{\nu}^2\right)\epsilon_1\epsilon_2 \\ &+ 2\nu_\zeta(1 + \nu)\left(\epsilon_1 + \epsilon_2\right)\epsilon_\zeta\Bigg] + G\gamma_{12}{}^2 + G_\zeta\left(\gamma_{1\zeta}{}^2 + \gamma_{2\zeta}{}^2\right)\Bigg\} \end{aligned} \tag{27b}$$

The requirement that P be a positive definite quantity leads to the following restriction on the elastic constants

$$E_\zeta(1 - \nu) - 2E\nu_\zeta{}^2 > 0 \tag{28a}$$

and hence the restrictions

$$\left.\begin{aligned} \bar{\nu} &< \sqrt{\frac{1-\nu}{2}} \\ \nu^* &< \frac{1}{2\nu_\zeta} \end{aligned}\right\} \tag{28b}$$

on the parameters $\bar{\nu}$ and $\nu^*$ or, equivalently, to the requirement that $\bar{E}$ and $E^*$ be positive. With the definitions of equations (25a) and (25b), the minimal condition is of the form

$$\delta\pi = \delta\left(\pi_l + \pi_{strain}\right) = 0 \tag{29}$$

## THE MINIMUM PRINCIPLE FOR THE STRESSES; CASTIGLIANO'S THEOREM OF LEAST WORK

If linear stress-strain relations and small strains are assumed, in accordance with equations (9), and if the surface stresses are prescribed over the faces of the shell or plate, while the displacements are prescribed over the edge surfaces, the principle sets forth (see, for example, reference 3, page 286) that among all statically correct states of stress the true state is determined by the condition that the complementary energy $\pi_c$ be a minimum

$$\delta\pi_c = 0 \tag{30}$$

The complementary energy consists of two parts: (1) The volume integral of the strain energy, expressed in terms of the stresses, and (2) the work of the boundary stresses over that part of the surface over which the displacements are prescribed,

$$\pi_c = \iiint P\left(\sigma_1, \sigma_2, \sigma_\zeta, \tau_{12}, \tau_{1\zeta}, \tau_{2\zeta}\right) dV - \iint_{edge} \bar{p}\cdot\bar{U}\, dS \tag{31}$$

The function P is determined by the relations

$$\left.\begin{aligned}
\frac{\partial P}{\partial \sigma_1} &= \epsilon_1 & \frac{\partial P}{\partial \sigma_2} &= \epsilon_2 & \frac{\partial P}{\partial \sigma_\zeta} &= \epsilon_\zeta \\
\frac{\partial P}{\partial \tau_{12}} &= \gamma_{12} & \frac{\partial P}{\partial \tau_{1\zeta}} &= \gamma_{1\zeta} & \frac{\partial P}{\partial \tau_{2\zeta}} &= \gamma_{2\zeta}
\end{aligned}\right\} \tag{32}$$

and, for the material under consideration here, is of the form

$$P = \frac{1}{2}\left\{\frac{1}{E}\left[\sigma_1{}^2 + \sigma_2{}^2 + \frac{E}{E_\zeta}\sigma_\zeta{}^2 - 2\nu\sigma_1\sigma_2 - 2\frac{\nu_\zeta E}{E_\zeta}(\sigma_1 + \sigma_2)\sigma_\zeta\right] + \frac{1}{G}\tau_{12}{}^2 + \frac{1}{G_\zeta}\left(\tau_{1\zeta}{}^2 + \tau_{2\zeta}{}^2\right)\right\} \tag{32a}$$

## STRESS RESULTANTS AND COUPLES

Equations (9), (11), (13), (18), and (19) comprise 15 equations involving the 15 unknown quantities $U_1$, $U_2$, $W$; $\epsilon_1$, $\epsilon_2$, $\epsilon_\zeta$, $\gamma_{12}$, $\gamma_{1\zeta}$, $\gamma_{2\zeta}$; $\sigma_1$, $\sigma_2$, $\sigma_\zeta$, $\tau_{12}$, $\tau_{1\zeta}$, $\tau_{2\zeta}$. These quantities here depend upon the three independent space variables $\xi_1$, $\xi_2$, and $\zeta$.

For the purpose of obtaining a two-dimensional theory of plates and shells it is customary to eliminate the $\zeta$-coordinate in the expressions involving stresses by the introduction of stress resultants and couples. Ten such quantities are conventionally defined, by the following equations:

$$\left.\begin{aligned}
N_{11} &= \int_{-h/2}^{h/2} \sigma_1\left(1 + \zeta/R_2\right) d\zeta & N_{22} &= \int_{-h/2}^{h/2} \sigma_2\left(1 + \zeta/R_1\right) d\zeta \\
N_{12} &= \int_{-h/2}^{h/2} \tau_{12}\left(1 + \zeta/R_2\right) d\zeta & N_{21} &= \int_{-h/2}^{h/2} \tau_{12}\left(1 + \zeta/R_1\right) d\zeta \\
Q_1 &= \int_{-h/2}^{h/2} \tau_{1\zeta}\left(1 + \zeta/R_2\right) d\zeta & Q_2 &= \int_{-h/2}^{h/2} \tau_{2\zeta}\left(1 + \zeta/R_1\right) d\zeta \\
M_{11} &= \int_{-h/2}^{h/2} \sigma_1\left(1 + \zeta/R_2\right)\zeta\, d\zeta & M_{22} &= \int_{-h/2}^{h/2} \sigma_2\left(1 + \zeta/R_1\right)\zeta\, d\zeta \\
M_{12} &= \int_{-h/2}^{h/2} \tau_{12}\left(1 + \zeta/R_2\right)\zeta\, d\zeta & M_{21} &= \int_{-h/2}^{h/2} \tau_{12}\left(1 + \zeta/R_1\right)\zeta\, d\zeta
\end{aligned}\right\} \quad (33)$$

The significance of the 10 resultants and couples so defined is suggested by the laws of mechanics, irrespective of the material of the shell and irrespective of the state of deformation of the elements of the shell. These resultants and couples can be considered as the components of the vectors

$$\left.\begin{aligned}
\overline{N}_1 &= N_{11}\overline{t}_1 + N_{12}\overline{t}_2 + Q_1\overline{n} & \overline{N}_2 &= N_{21}\overline{t}_1 + N_{22}\overline{t}_2 + Q_2\overline{n} \\
\overline{M}_1 &= M_{12}\overline{t}_1 - M_{11}\overline{t}_2 & \overline{M}_2 &= M_{22}\overline{t}_1 - M_{21}\overline{t}_2
\end{aligned}\right\} \quad (34)$$

It is evident that these vectors represent resultant force and moment, per unit of length along the parametric curves, acting on sections $\xi_1$ = Constant and $\xi_2$ = Constant. The couples are positive when they produce positive stresses on the part of the shell or plate on the positive side ($\zeta > 0$) of the middle surface. The absence of a third component in $\overline{M}_1$ and $\overline{M}_2$ is due to the fact that a differential element of area in a cross section has finite height $h$ and infinitesimal width.

To obtain differential equations of equilibrium for the resultants, the equilibrium equations (11) may be averaged over the thickness $h$ of the plate or shell. Making use of equations (7) where necessary, the results are obtained in the form

$$\left.\begin{aligned}
&\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2} + N_{12}\frac{\partial \alpha_1}{\partial \xi_2} - N_{22}\frac{\partial \alpha_2}{\partial \xi_1} + Q_1\frac{\alpha_1\alpha_2}{R_1} + \alpha_1\alpha_2 p_1 = 0\\
&\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + N_{21}\frac{\partial \alpha_2}{\partial \xi_1} - N_{11}\frac{\partial \alpha_1}{\partial \xi_2} + Q_2\frac{\alpha_1\alpha_2}{R_2} + \alpha_1\alpha_2 p_2 = 0\\
&\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - \left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right)\alpha_1\alpha_2 + \alpha_1\alpha_2 q = 0
\end{aligned}\right\} \qquad (35)$$

where

$$\left.\begin{aligned}
p_1 &= \left[\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\tau_{1\zeta}\right]_{-h/2}^{h/2} + \int_{-h/2}^{h/2}\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)F_1\,d\zeta\\
p_2 &= \left[\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\tau_{2\zeta}\right]_{-h/2}^{h/2} + \int_{-h/2}^{h/2}\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)F_2\,d\zeta\\
q &= \left[\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\sigma_{\zeta}\right]_{-h/2}^{h/2} + \int_{-h/2}^{h/2}\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)F_{\zeta}\,d\zeta
\end{aligned}\right\} \qquad (36)$$

The quantities $p_1$, $p_2$, and $q$ can be considered as the components of a vector

$$\overline{p} = p_1\overline{t}_1 + p_2\overline{t}_2 + q\overline{n} \tag{37}$$

representing effective external force per unit area applied to the middle surface of the plate or shell.

Two differential equations of equilibrium for the couples are obtained by multiplying both sides of the first two of equations (11) by $\zeta$ and integrating over the thickness. It may be noted that if the third equation were treated in a similar way new quantities not defined by equations (33) would be introduced. The two equations described are obtained in the form

$$\left.\begin{aligned}\frac{\partial\alpha_2 M_{11}}{\partial\xi_1} + \frac{\partial\alpha_1 M_{21}}{\partial\xi_2} + M_{12}\frac{\partial\alpha_1}{\partial\xi_2} - M_{22}\frac{\partial\alpha_2}{\partial\xi_1} - Q_1\alpha_1\alpha_2 - \alpha_1\alpha_2 m_1 &= 0\\ \frac{\partial\alpha_2 M_{12}}{\partial\xi_1} + \frac{\partial\alpha_1 M_{22}}{\partial\xi_2} + M_{21}\frac{\partial\alpha_2}{\partial\xi_1} - M_{11}\frac{\partial\alpha_1}{\partial\xi_2} - Q_2\alpha_1\alpha_2 + \alpha_1\alpha_2 m_2 &= 0\end{aligned}\right\} \tag{38}$$

where

$$\left.\begin{aligned}m_1 &= -\left[\zeta\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\tau_{1\zeta}\right]_{-h/2}^{h/2} - \int_{-h/2}^{h/2}\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\zeta F_1\,d\zeta\\ m_2 &= \left[\zeta\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\tau_{2\zeta}\right]_{-h/2}^{h/2} + \int_{-h/2}^{h/2}\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\zeta F_2\,d\zeta\end{aligned}\right\} \tag{39}$$

The quantities $m_1$ and $m_2$ can be considered as the components of a vector

$$\overline{m} = m_2\overline{t}_1 - m_1\overline{t}_2 \tag{40}$$

representing effective external moment per unit area applied to the middle surface of the plate or shell.

Equations (35) and (38) comprise 5 equations involving the 10 resultants and couples defined by equations (33). Two different points of view may now be adopted.

In the first place, the 10 quantities defined by equations (33) may be considered as a complete and adequate macroscopic description of the statics of the plate or shell. In this event, it becomes necessary to determine which quantities serve to provide a corresponding macroscopic description of the state of deformation of the plate or shell which is logically equivalent to the foregoing description of the state of stress. By this is meant that in the description of the state of deformation there should appear no quantities whose values are affected by stress distributions for which the 10 resultants and couples have zero values.

In the second place, quantities may be arbitrarily chosen to represent the state of deformation in the plate or shell. In this event, it may happen that the 10 resultants and moments defined by equations (33) are not sufficient to describe the corresponding state of stress, and additional quantities ("higher moments" of the stresses) may occur, together with corresponding additional equilibrium equations complementing equations (35) and (38).

It is evident that in various theories of plates and shells, developed along such lines, the final formulations may differ in consequence of differences in basic assumptions.

It is felt that neither of the foregoing two points of view has heretofore been adequately taken into account. In this report a definite formulation of the theory with reference to the first point of view is not obtained. However, results are arrived at which are believed to be definite with reference to the second aspect of the general problem.

## CONVENTIONAL ASSUMPTIONS

It may be expected that a state of small bending and stretching of a plate or shell is described in the first approximation by the following formulas for the displacement components occurring in equations (8) and (9):

$$\left.\begin{aligned} U_1(\xi_1, \xi_2, \zeta) &= u_1(\xi_1, \xi_2) + \zeta u_1'(\xi_1, \xi_2) \\ U_2(\xi_1, \xi_2, \zeta) &= u_2(\xi_1, \xi_2) + \zeta u_2'(\xi_1, \xi_2) \\ W(\xi_1, \xi_2, \zeta) &= w(\xi_1, \xi_2) \end{aligned}\right\} \tag{41}$$

The components of strain are determined in terms of the five displacement variables $u_1$, $u_1'$, $u_2$, $u_2'$, and $w$ by introducing equation (41) into equation (9). Equations (13), (18), and (19) then serve to express the components of stress in terms of the displacement variables. If these results are introduced into equations (33), the resultant 10 equations, in addition to the 5 equilibrium equations, comprise 15 equations involving the 10 resultants and couples and the 5 displacement variables.

The success of the customary procedure depends on the use of the following argument. An order-of-magnitude consideration of the equilibrium equations for the stresses shows that, unless the surface loads are highly concentrated, the transverse normal stress $\sigma_\zeta$ is in general of smaller order of magnitude than the stresses $\sigma_1$ and $\sigma_2$. In consequence of this fact it is conventional to neglect the term involving $\sigma_\zeta$ in equations (14).

The third of equations (9a) shows that the assumption of equation (41) implies the assumption of $\epsilon_\zeta = 0$. However, a theory which includes the two hypotheses $\sigma_\zeta = 0$ and $\epsilon_\zeta = 0$ would, in particular, fail to lead to correct results in the special case of a flat plate subjected to a state of homogeneous bending and stretching, for which problem the exact solution is easily established. This difficulty is usually avoided by neglecting $\sigma_\zeta$ in the stress-strain relation of equation (14a) and by then determining $\epsilon_\zeta$ from the resultant equation. To remove the resultant inconsistency, it would then be necessary to correct the original expression for $W$ by the addition of terms which are linear and quadratic in $\zeta$. If no boundary layers of width of the order of the thickness $h$ are present, these additional terms are found to be small in comparison with the leading term $w$. Thus, to obtain a first approximation theory the additional terms may be omitted in introducing $W$ into the expressions for the strains $\epsilon_1$, $\epsilon_2$, and $\gamma_{12}$.

If the assumptions of equations (41) are introduced into the last two of equations (9b) there follows

$$\left.\begin{aligned} \gamma_{1\zeta} &= \frac{1}{1 + \zeta/R_1}\left(u_1' + \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1}\right) \\ \gamma_{2\zeta} &= \frac{1}{1 + \zeta/R_2}\left(u_2' + \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2} - \frac{u_2}{R_2}\right) \end{aligned}\right\} \tag{42}$$

It has been customary to assume further that the transverse shear stresses $\tau_{1\zeta}$ and $\tau_{2\zeta}$ are also of smaller magnitude than $\sigma_1$, $\sigma_2$, and $\tau_{12}$. If these stresses are neglected in the stress-strain relations, two of equations (13) become $\gamma_{1\zeta} = \gamma_{2\zeta} = 0$. With these relations, equations (42) determine the bending terms $u_1'$ and $u_2'$ as functions of $u_1$, $u_2$, and $w$ as follows:

$$\left.\begin{aligned} u_1' &= -\frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} + \frac{u_1}{R_1} \\ u_2' &= -\frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2} + \frac{u_2}{R_2} \end{aligned}\right\} \tag{43}$$

It may be mentioned that the assumptions $\epsilon_\zeta = \gamma_{1\zeta} = \gamma_{2\zeta} = 0$ at all points are equivalent to requiring that straight lines which were originally normal to the undeformed middle surface remain straight lines after deformation, remain normal to the deformed middle surface, and suffer no extension.

With these simplifications, the expressions for $Q_1$ and $Q_2$ in equations (33) can no longer be retained and there results a set of 13 equations involving 13 unknown quantities, $u_1'$ and $u_2'$ being eliminated by means of equations (43).

If it is assumed that $\sigma_\zeta$ is negligibly small, there are obtained from equations (13), (14), and (33) the following equations relating the stress resultants and couples to the components of strain:

$$
\left.
\begin{aligned}
N_{11} &= \frac{E}{1 - \nu^2} \int_{-h/2}^{h/2} \left(\epsilon_1 + \nu\epsilon_2\right)\left(1 + \frac{\zeta}{R_2}\right) d\zeta \\
N_{22} &= \frac{E}{1 - \nu^2} \int_{-h/2}^{h/2} \left(\epsilon_2 + \nu\epsilon_1\right)\left(1 + \frac{\zeta}{R_1}\right) d\zeta \\
N_{12} &= G \int_{-h/2}^{h/2} \gamma_{12}\left(1 + \frac{\zeta}{R_2}\right) d\zeta \\
N_{21} &= G \int_{-h/2}^{h/2} \gamma_{12}\left(1 + \frac{\zeta}{R_1}\right) d\zeta
\end{aligned}
\right\} \qquad (44a)
$$

$$
\left.
\begin{aligned}
M_{11} &= \frac{E}{1 - \nu^2} \int_{-h/2}^{h/2} \left(\epsilon_1 + \nu\epsilon_2\right)\left(1 + \frac{\zeta}{R_2}\right)\zeta \, d\zeta \\
M_{22} &= \frac{E}{1 - \nu^2} \int_{-h/2}^{h/2} \left(\epsilon_2 + \nu\epsilon_1\right)\left(1 + \frac{\zeta}{R_1}\right)\zeta \, d\zeta \\
M_{12} &= G \int_{-h/2}^{h/2} \gamma_{12}\left(1 + \frac{\zeta}{R_2}\right)\zeta \, d\zeta \\
M_{21} &= G \int_{-h/2}^{h/2} \gamma_{12}\left(1 + \frac{\zeta}{R_1}\right)\zeta \, d\zeta
\end{aligned}
\right\} \qquad (44b)
$$

Equations (44) are listed as a basis of an outline of certain procedures which have been used to obtain relations between the stress resultants and couples and the displacement variables $u_1$, $u_2$, and $w$.

## LOVE'S FIRST APPROXIMATION

If the ratio $\zeta/R$ is neglected in comparison with unity in equations (44), as well as in equations (9) which define the strain components, the following stress-strain relations are obtained:

$$\left.\begin{aligned} N_{11} &= \frac{Eh}{1-\nu^2}\left(\epsilon_1{}^0 + \nu\epsilon_2{}^0\right) \\ N_{22} &= \frac{Eh}{1-\nu^2}\left(\epsilon_2{}^0 + \nu\epsilon_1{}^0\right) \\ N_{12} &= N_{21} = Gh\gamma_{12}{}^0 \end{aligned}\right\} \tag{45a}$$

$$\left.\begin{aligned} M_{11} &= \frac{Eh^3}{12\left(1-\nu^2\right)}\left(\kappa_1 + \nu\kappa_2\right) \\ M_{22} &= \frac{Eh^3}{12\left(1-\nu^2\right)}\left(\kappa_2 + \nu\kappa_1\right) \\ M_{12} &= M_{21} = G\,\frac{h^3}{12}\,\tau \end{aligned}\right\} \tag{45b}$$

where

$$
\left.
\begin{aligned}
\epsilon_1{}^0 &= \frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1} \\
\epsilon_2{}^0 &= \frac{1}{\alpha_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} + \frac{w}{R_2} \\
\gamma_{12}{}^0 &= \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{\alpha_1}\right) \\
\kappa_1 &= \frac{1}{\alpha_1}\frac{\partial u_1'}{\partial \xi_1} + \frac{u_2'}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} \\
\kappa_2 &= \frac{1}{\alpha_2}\frac{\partial u_2'}{\partial \xi_2} + \frac{u_1'}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1} \\
\tau &= \frac{\alpha_2}{\alpha_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2'}{\alpha_2}\right) + \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1'}{\alpha_1}\right)
\end{aligned}
\right\} \tag{46}
$$

and where $u_1'$ and $u_2'$ are defined by equations (43).

The system of equations (45), (46), (43), (35), and (38) was first given by Love (reference 4, page 531; see also reference 5) and has been used as the basis of many studies of specific problems with regard to flat plates, cylindrical shells, and shells of revolution. It is generally believed that this formulation of the problem contains all the essential facts necessary for the treatment of thin shells, as long as special conditions do not require inclusion of the effect of transverse shear and normal stresses.

It is well known that within the framework of this approximate theory fewer boundary conditions can be satisfied than is expected. It has been shown in earlier papers (references 1 and 2) that for flat plates this difficulty is resolved if transverse shear deformation is taken into account in an appropriate way. In this report it will be shown that the same is true with regard to the analysis of shells.

## MODIFICATION OF LOVE'S FIRST APPROXIMATION

A number of writers (references 6, 7, 8, and 9) have modified the foregoing first approximation by not neglecting the ratio $\zeta/R$ in comparison with unity in equations (9) and (44) but still retaining the assumptions of equations (41) and (43). In particular, if terms involving powers of h through the third are retained, and terms involving higher powers of h are neglected, in equations (44), equations (45) are replaced by the following forms (reference 9):

$$\left.\begin{aligned}
N_{11} &= \frac{Eh}{1-\nu^2}\left[\left(\epsilon_1{}^0 + \nu\epsilon_2{}^0\right) - \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\kappa_1 - \frac{\epsilon_1{}^0}{R_1}\right)\right] \\
N_{22} &= \frac{Eh}{1-\nu^2}\left[\left(\epsilon_2{}^0 + \nu\epsilon_1{}^0\right) - \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\kappa_2 - \frac{\epsilon_2{}^0}{R_2}\right)\right] \\
N_{12} &= Gh\left[\gamma_{12}{}^0 - \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\kappa_{12} - \frac{\gamma_{12}{}^0}{R_1}\right)\right] \\
N_{21} &= Gh\left[\gamma_{12}{}^0 - \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\kappa_{12} - \frac{\gamma_{12}{}^0}{R_2}\right)\right] \\
M_{11} &= \frac{Eh^3}{12\left(1-\nu^2\right)}\left[\left(\kappa_1 + \nu\kappa_2\right) - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\epsilon_1{}^0\right] \\
M_{22} &= \frac{Eh^3}{12\left(1-\nu^2\right)}\left[\left(\kappa_2 + \nu\kappa_1\right) - \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\epsilon_2{}^0\right] \\
M_{12} &= G\,\frac{h^3}{12}\left[\tau - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\beta_1{}^0\right] \\
M_{21} &= G\,\frac{h^3}{12}\left[\tau - \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\beta_2{}^0\right]
\end{aligned}\right\} \tag{47}$$

In addition to the quantities defined in equations (46), the quantities $\beta_1{}^0$ and $\beta_2{}^0$ are defined by the equations

$$\left.\begin{aligned}
\beta_1{}^0 &= \frac{1}{\alpha_1}\frac{\partial u_2}{\partial \xi_1} - \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} \\
\beta_2{}^0 &= \frac{1}{\alpha_2}\frac{\partial u_1}{\partial \xi_2} - \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_2}{\partial \xi_1}
\end{aligned}\right\} \tag{48a}$$

and $\kappa_{12}$ is defined by the equation

$$2\kappa_{12} = \tau + \frac{\beta_2^{\,0}}{R_1} + \frac{\beta_1^{\,0}}{R_2} \tag{48b}$$

It is noted that the equation

$$\beta_1^{\,0} + \beta_2^{\,0} = \gamma_{12}^{\,0} \tag{49}$$

is satisfied by virtue of equations (48a) and the third of equations (46).

It is seen that equations (47) differ from equations (45) in that certain terms of order $h^3$ are added to the original expressions. However, it may be said that the additional terms which appear in equations (47) cannot be expected to be of the same form as terms of the same order of magnitude which would be introduced if the simplifying assumptions $\epsilon_\zeta = \gamma_{1\zeta} = \gamma_{2\zeta} = \sigma_\zeta = 0$ were replaced by more flexible assumptions.

It may be noted that all the additional terms in equations (47) disappear when the principal radii of curvature of the middle surface are equal, that is, in the cases of flat plates and spherical shells.

## LOVE'S SECOND APPROXIMATION

A second approximation, given by Love (reference 4, page 533), introduces three types of corrections to his first formulation. Love states that such modifications are unnecessary unless the flexural strains $\zeta\kappa_1$, $\zeta\kappa_2$, and $\zeta\tau$ are large in comparison with the extensional strains $\epsilon_1^{\,0}$, $\epsilon_2^{\,0}$, and $\gamma_{12}^{\,0}$.

As a first modification, the transverse displacement $W$ is expressed in a more flexible form,

$$W\left(\xi_1, \xi_2, \zeta\right) = w\left(\xi_1, \xi_2\right) + \tilde{w}\left(\xi_1, \xi_2, \zeta\right)$$

according to which the direct strains, given by equations (9a), become

$$\left.\begin{aligned} \epsilon_1 &= \frac{1}{1 + \zeta/R_1}\left(\epsilon_1{}^0 + \zeta\kappa_1 + \frac{\tilde{w}}{R_1}\right) \\ \epsilon_2 &= \frac{1}{1 + \zeta/R_2}\left(\epsilon_2{}^0 + \zeta\kappa_2 + \frac{\tilde{w}}{R_2}\right) \\ \epsilon_\zeta &= \frac{\partial\tilde{w}}{\partial\zeta} \end{aligned}\right\} \tag{49a}$$

The second modification consists in not completely neglecting the ratio $\zeta/R$ with respect to unity but writing $\frac{1}{1 + \zeta/R_1} \approx 1 - \zeta/R_1$. However, in expanding the first two of equations (49a) Love assumes that the quantities $\frac{\zeta}{R}\,\epsilon^0$ and $\frac{\zeta}{R}\,\frac{\tilde{w}}{R}$ can be neglected. In this way, equations (49a) are approximated in the form

$$\left.\begin{aligned} \epsilon_1 &= \epsilon_1{}^0 + \left(\zeta - \frac{\zeta^2}{R_1}\right)\kappa_1 + \frac{\tilde{w}}{R_1} \\ \epsilon_2 &= \epsilon_2{}^0 + \left(\zeta - \frac{\zeta^2}{R_2}\right)\kappa_2 + \frac{\tilde{w}}{R_2} \\ \epsilon_\zeta &= \frac{\partial\tilde{w}}{\partial\zeta} \end{aligned}\right\} \tag{49b}$$

The term $\tilde{w}$ is considered as a small correction term. To obtain a first approximation to its value, the transverse normal stress $\sigma_\zeta$ is neglected in equation (19) and $\epsilon_1 + \epsilon_2$ is replaced by its first approximation $\left(\epsilon_1{}^0 + \epsilon_2{}^0\right) + \zeta\left(\kappa_1 + \kappa_2\right)$. The result then becomes

$$\epsilon_\zeta = \frac{\partial\tilde{w}}{\partial\zeta} = -\frac{\nu_\zeta E}{(1 - \nu)E_\zeta}\left[\left(\epsilon_1{}^0 + \epsilon_2{}^0\right) + \zeta\left(\kappa_1 + \kappa_2\right)\right] \tag{50}$$

The correction $\tilde{w}$ is then determined by integration, with the convention $\tilde{w}\left(\xi_1, \xi_2, 0\right) = 0$ in the form

$$\tilde{w} = -\frac{\nu_\zeta E}{(1-\nu)E_\zeta}\left[\zeta\left(\epsilon_1{}^0 + \epsilon_2{}^0\right) + \frac{1}{2}\zeta^2\left(\kappa_1 + \kappa_2\right)\right] \tag{51}$$

If this result is introduced into the first two of equations (49b) and quantities of the form $\frac{\zeta}{R}\epsilon^0$ are again neglected, there follows

$$\left.\begin{aligned}\epsilon_1 &= \epsilon_1{}^0 + \left(\zeta - \frac{\zeta^2}{R_1}\right)\kappa_1 - \frac{1}{2}\frac{\nu_\zeta E}{(1-\nu)E_\zeta}\zeta^2\left(\kappa_1 + \kappa_2\right)\\ \epsilon_2 &= \epsilon_2{}^0 + \left(\zeta - \frac{\zeta^2}{R_2}\right)\kappa_2 - \frac{1}{2}\frac{\nu_\zeta E}{(1-\nu)E_\zeta}\zeta^2\left(\kappa_1 + \kappa_2\right)\end{aligned}\right\} \tag{52}$$

The third modification consists in not neglecting $\sigma_\zeta$ in equations (14). In order to obtain a first approximation to the value of $\sigma_\zeta$, the third of the equilibrium equations (11) is used. It is supposed that no body force $F_\zeta$ is present and that the transverse shear stresses are negligible. Also, to this approximation the ratios $\zeta/R$ can be neglected, and the equation becomes

$$\frac{\partial\sigma_\zeta}{\partial\zeta} = \frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2}$$

If $\sigma_1$ and $\sigma_2$ are replaced by their first approximations and the extensional strains are neglected, then there follows

$$\frac{\partial\sigma_\zeta}{\partial\zeta} = \frac{E}{1-\nu^2}\left[\frac{\zeta}{R_1}\left(\kappa_1 + \nu\kappa_2\right) + \frac{\zeta}{R_2}\left(\kappa_2 + \nu\kappa_1\right)\right]$$

In the case of vanishing normal surface loads, $\sigma_\zeta\left(\xi_1, \xi_2, \pm\frac{h}{2}\right) = 0$, Love obtains by integration

$$\sigma_\zeta = -\frac{1}{2}\frac{E}{1-\nu^2}\left(\frac{h^2}{4} - \zeta^2\right)\left(\frac{\kappa_1 + \nu\kappa_2}{R_1} + \frac{\kappa_2 + \nu\kappa_1}{R_2}\right) \tag{53}$$

If the expressions for $\sigma_1$ and $\sigma_2$ obtained by introducing equations (52) and (53) into equations (14) are in turn introduced into the first two of equations (33) and if again quantities of the form $\frac{\zeta}{R}\epsilon^0$ are neglected, there follows finally

$$N_{11} = \frac{Eh}{1-\nu^2}\left\{\left(\epsilon_1{}^0 + \nu\epsilon_2{}^0\right) - \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\kappa_1 - \frac{h^2}{12}\frac{\nu_\zeta E}{(1-\nu)E_\zeta}\left[\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\left(\kappa_1 + \kappa_2\right) + \frac{\kappa_1 + \nu\kappa_2}{R_1} + \frac{\kappa_2 + \nu\kappa_1}{R_2}\right]\right\} \tag{54}$$

together with an analogous expression for $N_{22}$.

A comparison of equations (54) with the first of equations (47) shows that equation (54) includes the terms present in the former solution except for the term $\frac{\epsilon_1{}^0}{R_1}$ which was neglected in the present analysis. More important, however, is the fact that new terms, of the same order as the correction terms present in equation (47), are introduced in equation (54) in consequence of the partial inclusion of the effect of the transverse normal stress $\sigma_\zeta$. Furthermore, these terms do not vanish when $R_1 = R_2$. Still no account has been taken of the possible effect of the transverse shear stresses $\tau_{1\zeta}$ and $\tau_{2\zeta}$.

The expressions obtained, in the present procedure, for the remaining quantities listed in equations (47), are equivalent to the results of neglecting terms of the form $\frac{\epsilon^0}{R}$, $\frac{\gamma^0}{R}$, and $\frac{\beta^0}{R}$ in comparison with the flexural terms in equations (47). That is, no additional terms due to the effect of $\sigma_\zeta$ are introduced into the remaining quantities by this procedure.

It may be mentioned that the results of the developments of this section reduce to those presented in Love's treatise in the isotropic case $\nu_\zeta = \nu$, $E_\zeta = E$, which is the only case treated there.

## BASSET'S THEORY

In this section there is outlined an analysis given by Basset (reference 10) which, in the opinion of the present authors, has not received as much attention as it deserves. The reason for this may perhaps be found in the fact that Basset's work is difficult to read and that the notation employed is somewhat complicated and, from modern standards, somewhat unsystematic.

Basset begins his analysis with the assumption that the stress, strain, and displacement components in a shell can be expanded in series of powers of $\zeta$. Thus there may be written, for example,

$$U_1\left(\xi_1, \xi_2, \zeta\right) = u_1\left(\xi_1, \xi_2\right) + \zeta u_1'\left(\xi_1, \xi_2\right) + \frac{1}{2}\zeta^2 u_1''\left(\xi_1, \xi_2\right) + \ldots \tag{55}$$

where

$$\left.\begin{aligned} u_1\left(\xi_1, \xi_2\right) &= U_1\left(\xi_1, \xi_2, 0\right) \\ u_1'\left(\xi_1, \xi_2\right) &= \left.\frac{\partial U_1}{\partial \zeta}\right]_{\zeta=0} \end{aligned}\right\} \tag{56}$$

and so forth.

The derivation is based on the use of the principle of minimum potential energy (equations (25) to (28)). The strain-energy function P, as given by equation (27), is expanded in powers of $\zeta$ so that the first terms are of the form

$$P = \frac{1}{2}\left\{\frac{1 - \bar{\nu}^2}{1 - \nu^2}\bar{E}\left[\left(\epsilon_1 + \zeta\epsilon_1' + \frac{1}{2}\zeta^2\epsilon_1'' + \ldots\right)^2 + \left(\epsilon_2 + \zeta\epsilon_2' + \frac{1}{2}\zeta^2\epsilon_2'' + \ldots\right)^2\right] + \ldots\right\} \tag{57}$$

where $\epsilon_1$ is written for $\epsilon_1(\xi_1, \xi_2, 0)$; $\epsilon_1'$, for $\dfrac{\partial \epsilon_1(\xi_1, \xi_2, 0)}{\partial \zeta}$; and so forth. Only such terms are retained as will lead to terms of order $h^3$ or lower after the $\zeta$-integration of equation (25b) is carried out, terms involving the basic quantities $u_1$, $u_2$, and $w$ being considered as of zero order in $h$.

To express the primed quantities in terms of unprimed quantities, use is made of the strain-displacement relations of equations (9b), the last two of which can be written in the form

$$\left.\begin{aligned} \frac{\partial U_1}{\partial \zeta} &= \gamma_{1\zeta} - \frac{1}{\alpha_1(1 + \zeta/R_1)} \frac{\partial W}{\partial \xi_1} + \frac{U_1}{R_1 + \zeta} \\ \frac{\partial U_2}{\partial \zeta} &= \gamma_{2\zeta} - \frac{1}{\alpha_2(1 + \zeta/R_2)} \frac{\partial W}{\partial \xi_2} + \frac{U_2}{R_2 + \zeta} \end{aligned}\right\} \tag{58}$$

Thus, setting $\zeta = 0$, there follows

$$\left.\begin{aligned} u_1' &= \gamma_{1\zeta} - \frac{1}{\alpha_1} \frac{\partial w}{\partial \xi_1} + \frac{u_1}{R_1} \\ u_2' &= \gamma_{2\zeta} - \frac{1}{\alpha_2} \frac{\partial w}{\partial \xi_2} + \frac{u_2}{R_2} \end{aligned}\right\} \tag{59}$$

In this way the quantities $u_1'$ and $u_2'$ are expressed in terms of the displacement functions $u_1$, $u_2$, and $w$ on the middle surface and in terms of the transverse shearing strains on the middle surface. The third of equations (9a) gives the result

$$w' = \epsilon_\zeta \tag{60a}$$

This result can be expressed in a different form if equation (19) is used to express $\epsilon_\zeta$ in terms of $\epsilon_1$, $\epsilon_2$, and $\sigma_\zeta$ as follows:

$$w' = \frac{E}{E_\zeta}\left[\frac{1}{E}\,\sigma_\zeta - \frac{\nu_\zeta}{1 - \nu}\left(\epsilon_1 + \epsilon_2\right)\right] \tag{60b}$$

By further $\zeta$-differentiation, all other primed quantities can be expressed in terms of the values of $u_1$, $u_2$, $w$, and $\left(\epsilon_1 + \epsilon_2\right)$ on the middle surface; such expressions also involve the values of $\gamma_{1\zeta}$, $\gamma_{2\zeta}$, and $\sigma_\zeta$ and their $\zeta$-derivatives on the middle surface. The remaining three stress-strain relations of equations (9) then permit the determination of the coefficients in the power-series expansions of the strains $\epsilon_1$, $\epsilon_2$, and $\gamma_{12}$ in terms of the same quantities.

It is then assumed that in the expansion

$$\sigma_\zeta(\xi_1, \xi_2, \zeta) = \sigma_\zeta(\xi_1, \xi_2) + \zeta\sigma_\zeta'(\xi_1, \xi_2) + \frac{1}{2}\zeta^2\sigma_\zeta''(\xi_1, \xi_2) + \ldots$$

the leading term is at least of order $h^2$ and the coefficient of $\zeta$ is at least of order $h$. Next it is pointed out that the leading terms in the expressions for the couples $M_{11}$, $M_{22}$, $M_{12}$, and $M_{21}$ are at least of order $h^3$. On the basis of this fact it is stated that, if the external moments $m_1$ and $m_2$ are absent, the equilibrium equations (38) imply that the transverse shear resultants $Q_1$ and $Q_2$ must be at least of order $h^3$ and hence that the terms of lowest order in the transverse shearing strains $\gamma_{1\zeta}$ and $\gamma_{2\zeta}$ must be at least quadratic functions of $h$ and $\zeta$ since such functions when integrated over a section give rise to quantities of order $h^3$.

The fact may be pointed out here that this argument may break down in those cases where an appreciable change in the magnitude of a couple may occur over a distance of the order of magnitude of the thickness $h$ of the plate or shell or over a distance of order $\sqrt{ah}$, where $a$ is a representative dimension of the shell. For if such a change takes place over a distance $l$ in the $\xi_1$-direction it follows that $\frac{\partial M}{\partial \xi_1}$ is of the order $\frac{1}{l}M$ in the region considered. It is well known that for plates $l$ may be of order $h$, while for cylindrical and spherical shells the distance may be of order $\sqrt{ah}$, where $a$ is the radius of the circles of curvature, in problems of usual occurrence.

If such cases are excluded, it is found that the contributions of $\gamma_{1\zeta}$ and $\gamma_{2\zeta}$ to the strain-energy function of equation (57) give rise to terms of order $h^5$ or higher and hence the transverse shear effect may be neglected if only terms of order $h^3$ are to be retained.

With the assumptions noted, the variational equation (28) leads to expressions for the stress resultants and couples which include all third-order corrections to Love's first approximation, equations (45), which are consistent with the assumed order-of-magnitude relations involving $\gamma_{1\zeta}$, $\gamma_{2\zeta}$, and $\sigma_\zeta$. In particular, the corrections introduced by equations (47) are included, as well as the additional corrections introduced by Love's second approximation, and the remaining third-order corrections which are absent in the latter approximation are introduced.

Basset's analysis was carried out only for isotropic circular cylindrical and spherical shells but could equally well be adapted to the more general theory.

## TREFFTZ' THEORY

An alternate system of equations for small deflections of isotropic shells has been derived by E. Trefftz (reference 11) by means of the minimum principle for the stresses. Trefftz begins by writing

$$\left.\begin{aligned} \sigma_1 &= \sigma_1{}^0 + \zeta\sigma_1{}' \\ \sigma_2 &= \sigma_2{}^0 + \zeta\sigma_2{}' \\ \tau_{12} &= \tau_{12}{}^0 + \zeta\tau_{12}{}' \end{aligned}\right\} \qquad (61)$$

where the functions $\sigma^0$ and $\sigma'$ are expressed in terms of resultants and couples by means of equations (33). Equations (61) are introduced into expressions (30) and (32) for the complementary energy, in which all terms containing $\sigma_\zeta$, $\tau_{1\zeta}$, and $\tau_{2\zeta}$ are omitted.

The resultant expression for the complementary energy expressed in terms of the N's and M's is minimized subject to the restrictions imposed by the five equilibrium equations (35) and (38). The equilibrium equations are taken into account by means of the Lagrangian multiplier method, the multipliers being identified with appropriate displacement components.

When terms of order $\zeta/R$ are neglected, Trefftz' results agree with Love's first approximation as given by equations (45) and (46). When terms with $\zeta/R$ are retained a new system of equations is obtained.

With regard to this set of equations, however, the following point may be made. For certain exact solutions for circular rings it is found that, as $h/R$ increases, the linear displacement expressions corresponding to equations (41) are much more nearly correct than the linear stress expressions corresponding to equations (61). It is noted, however, that this distinction disappears for the special case of the flat plate.

The effect of transverse shear and normal stress in shell theory might be taken into account by calculating by means of the three-dimensional equations of equilibrium (11) the values of the stresses $\sigma_\zeta$, $\tau_{1\zeta}$, and $\tau_{2\zeta}$ which correspond to the values of $\sigma_1$, $\sigma_2$, and $\tau_{12}$ as given by equations (61).

It appears that this procedure would lead to rather complicated expressions for $\sigma_\zeta$, $\tau_{1\zeta}$, and $\tau_{2\zeta}$ when $R_1$ and $R_2$ are finite. A class of nonhomogeneous shells for which this complication can be avoided has been studied in reference 12.

In earlier papers (references 1 and 2) one of the present authors has treated the problem of the flat plate in this manner and has thereby established the importance of the role which is played by the deformations caused by the stresses $\sigma_\zeta$, $\tau_{1\zeta}$, and $\tau_{2\zeta}$ in certain problems of plate theory. It may be remarked that the use of the particular minimum principle employed in that reference is not essential to the analysis of the effects of transverse stresses. Similar results can be obtained in a number of ways, for example, as is shown in a later section of this report, by use of the minimum principle for the displacements.

It may be added that the system of equations obtained by the application of the minimum principle for the stresses in the manner just outlined, while furnishing approximations for the stresses themselves, determines certain weighted averages (taken over the thickness) of the displacements, rather than the displacements themselves. This is shown in reference 2 with regard to flat plates. It will be shown in the present report that just the reverse is true when the minimum principle for the displacements is used to obtain a system of two-dimensional equations for the theory of shells.

## THEORY OF SYNGE AND CHIEN

A theory of the finite deflections of isotropic shells has been developed in a series of recent papers by J. L. Synge (reference 13) and W. C. Chien (references 13 and 14). The basis of this theory, which is of great generality, appears in the case of small deflections to reduce to the following considerations.

All stresses and displacements are expressed as power series in $\zeta$ and the equilibrium equations and stress-strain relations are written accordingly as equalities between power series in $\zeta$. By equating the coefficients of respective powers of $\zeta$ in each of these equations, an infinite system of simultaneous differential equations in the independent variables $\xi_1$ and $\xi_2$ is obtained.

Next a representative length $a$ on the middle surface of the shell is introduced and each coefficient of each power series in $\zeta$ is considered to be developed in powers of the ratio $\eta = h/a$. Various choices of the lowest exponent (which may be negative) of $\eta$ are made, such choices leading to different formulations of shell theory.

Finally, only the leading terms of the respective developments are retained.

The fact must be stressed that this interpretation of the Synge–Chien procedure is tentative and that a more detailed explanation of this work in terms of the notions of the present report is believed to be desirable. It may be remarked that the formulation of boundary conditions appropriate to the various theories evolved by this procedure does not appear to be incorporated in the work of Synge and Chien.

## INCORPORATION OF TRANSVERSE SHEAR AND NORMAL STRESS EFFECTS WITHOUT USE OF A MINIMUM PRINCIPLE

In this section there is outlined a possible extension of the classical general methods which takes into account a first approximation to the effects of transverse shear and normal stresses.

Again there are assumed for the displacements the expressions

$$\left.\begin{aligned} U_1(\xi_1, \xi_2, \zeta) &= u_1(\xi_1, \xi_2) + \zeta u_1'(\xi_1, \xi_2) \\ U_2(\xi_1, \xi_2, \zeta) &= u_2(\xi_1, \xi_2) + \zeta u_2'(\xi_1, \xi_2) \\ W(\xi_1, \xi_2, \zeta) &= w(\xi_1, \xi_2) \end{aligned}\right\} \qquad (62)$$

It is not, however, assumed that the transverse shear strains are negligible. Thus equations (43) no longer hold. In their place now there are the definitions of $Q_1$ and $Q_2$ as given in equations (33). To a first approximation these equations take the form

$$\left.\begin{aligned} Q_1 &= G_\zeta \int_{-h/2}^{h/2} \gamma_{1\zeta}\, d\zeta \\ Q_2 &= G_\zeta \int_{-h/2}^{h/2} \gamma_{2\zeta}\, d\zeta \end{aligned}\right\} \tag{63}$$

where the factors $1 + \zeta/R$ are replaced by unity. With the same approximations, expressions for $\gamma_{1\zeta}$ and $\gamma_{2\zeta}$ are obtained from equations (9b) in the form

$$\left.\begin{aligned} \gamma_{1\zeta} &= \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1} + u_1' \\ \gamma_{2\zeta} &= \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2} - \frac{u_2}{R_2} + u_2' \end{aligned}\right\} \tag{64}$$

Thus, in place of equations (33) now there are the relations

$$\left.\begin{aligned} u_1' + \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1} &= \frac{Q_1}{hG_\zeta} \\ u_2' + \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2} - \frac{u_2}{R_2} &= \frac{Q_2}{hG_\zeta} \end{aligned}\right\} \tag{65}$$

The next step consists in introducing the values of $\epsilon_1$, $\epsilon_2$, and $\tau_{12}$ which correspond to equations (41) into the first of equations (13) and into equations (14), to determine corresponding expressions for $\sigma_1$, $\sigma_2$, and $\gamma_{12}$ and in then substituting the results into equations (33), to determine corresponding expressions for the stress resultants and couples. In doing this, again the factors $1 + \zeta/R$ are replaced by unity in calculating terms which are in addition to those which appear in equations (47). If the right-hand members of equations (47) are denoted by $N_{11}^0$, $N_{22}^0$, and so forth and the parameter

$$\nu^* = \frac{\nu_\zeta E}{(1 - \nu)E_\zeta} \tag{66}$$

which appears in equations (15) and (16) is used, the resultant equations take the form

$$\left.\begin{aligned}
N_{11} &= N_{11}{}^0 + \nu^* \int_{-h/2}^{h/2} \sigma_\zeta \, d\zeta \\
N_{22} &= N_{22}{}^0 + \nu^* \int_{-h/2}^{h/2} \sigma_\zeta \, d\zeta \\
N_{12} &= N_{12}{}^0 \\
N_{21} &= N_{21}{}^0
\end{aligned}\right\} \tag{67a}$$

$$\left.\begin{aligned}
M_{11} &= M_{11}{}^0 + \nu^* \int_{-h/2}^{h/2} \sigma_\zeta \zeta \, d\zeta \\
M_{22} &= M_{22}{}^0 + \nu^* \int_{-h/2}^{h/2} \sigma_\zeta \zeta \, d\zeta \\
M_{12} &= M_{12}{}^0 \\
M_{21} &= M_{21}{}^0
\end{aligned}\right\} \tag{67b}$$

Equations (67) and (63), together with the 5 equilibrium equations (35) and (38), comprise 15 equations, involving a total of 17 unknown quantities, namely: 10 stress resultants and couples, the 2 integrals involving $\sigma_\zeta$ in equations (67), and the 5 displacement variables of equations (62).

Two additional equations may be obtained by requiring that the results of integrating equation (19), and the product of $\zeta$ and the two members of equation (19), be satisfied. Since the third assumption of equations (62) implies the vanishing of the transverse direct strain $\epsilon_\zeta$, these equations take the form

$$\left.\begin{aligned} \int_{-h/2}^{h/2} \sigma_\zeta \, d\zeta &= \nu^* \overline{E} \frac{E_\zeta}{E} \int_{-h/2}^{h/2} \left(\epsilon_1 + \epsilon_2\right) d\zeta \\ \int_{-h/2}^{h/2} \sigma_\zeta \zeta \, d\zeta &= \nu^* \overline{E} \frac{E_\zeta}{E} \int_{-h/2}^{h/2} \left(\epsilon_1 + \epsilon_2\right) \zeta \, d\zeta \end{aligned}\right\} \tag{68}$$

Equations (67), (68), and (63) serve to determine the conventional 10 stress resultants and couples, as well as the 2 auxiliary resultants, in terms of the 5 displacement functions. The introduction of these results into the five equilibrium equations (35) and (38) then leads to a set of five differential equations in the five displacement functions. The solution of the given set of equations is then apparently determinate if these five functions are prescribed along the boundary of the plate or shell. Alternatively, it may be expected that along an edge $\xi_1 = \text{Constant}$ the five quantities $N_{11}$, $N_{12}$, $M_{11}$, $M_{12}$, and $Q_1$ may be independently prescribed, as statical considerations would require. If this is indeed the case the classical difficulties first investigated by Kirchoff would be resolved.

A new difficulty may, however, be noted. In the special case of pure bending of a flat plate, by couples distributed uniformly along the boundary, the known exact solution is derived from equations (67b) only if the corrective terms involving $\sigma_\zeta$ are absent, that is, if $M_{11} = M_{11}{}^0$ and $M_{22} = M_{22}{}^0$. But if equations (68) are to be satisfied it is readily verified that the corrective terms will be absent only in the special case when the prescribed uniform edge couples satisfy the relation $M_{11} + M_{22} = 0$. The presence of this difficulty may be explained as follows. The assumption $W = w\left(\xi_1, \xi_2\right)$ implies the assumption $\epsilon_\zeta = 0$, and hence (if the relevant stress-strain relation is retained) there follows also

$$\sigma_\zeta = \nu_\zeta\left(\sigma_1 + \sigma_2\right)$$

Thus a transverse normal stress effect is automatically introduced, in this formulation, in connection with effects due to the stress $\sigma_1 + \sigma_2$. This coupling is avoided in the classical theory by disregarding the third stress-strain relation and taking $\sigma_\zeta = 0$ identically, even though this in general contradicts the assumption $\epsilon_\zeta = 0$ which is also part of the classical theory. A similar procedure might be adopted here, wherein equations (68) would be disregarded and an expression for $\sigma_\zeta$ such as that used by Love (equation (53)) in his second approximation would be used. The validity of such a procedure would, however, be open to question.

It is clear that the difficulty is not present in the special orthotropic case when $\nu_\zeta$ is negligibly small.

A second possible procedure consists in replacing the assumption $\frac{\partial W}{\partial \zeta} = 0$ by a more general type of assumption under which $\epsilon_\zeta$ and $\sigma_\zeta$ are not so restricted. In this connection, however, it may be remarked that if the assumptions of equations (62) were replaced by more flexible assumptions, in which more than five displacement functions were involved, the present procedure would not supply in a rational way the additional relations needed for the determination of the additional functions.

To clarify the entire situation and, in particular, to investigate in a rational way the manner in which boundary conditions may be imposed, the formulation of the problem is considered next from the point of view of the principle of minimum potential energy.

## APPLICATION OF THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY

In place of the assumptions of equations (62), here the following more flexible forms are taken:

$$\left.\begin{aligned}
U_1\left(\xi_1, \xi_2, \zeta\right) &= u_1\left(\xi_1, \xi_2\right) + \zeta u_1'\left(\xi_1, \xi_2\right) + \frac{1}{2}\zeta^2 u_1''\left(\xi_1, \xi_2\right) \\
U_2\left(\xi_1, \xi_2, \zeta\right) &= u_2\left(\xi_1, \xi_2\right) + \zeta u_2'\left(\xi_1, \xi_2\right) + \frac{1}{2}\zeta^2 u_2''\left(\xi_1, \xi_2\right) \\
W\left(\xi_1, \xi_2, \zeta\right) &= w\left(\xi_1, \xi_2\right) + \zeta w'\left(\xi_1, \xi_2\right) + \frac{1}{2}\zeta^2 w''\left(\xi_1, \xi_2\right)
\end{aligned}\right\} \quad (69)$$

and a system of equations determining the nine displacement functions present is derived by an application of the principle of minimum potential energy (equations (25) to (28)).

A shell is considered whose boundary edges lie in coordinate surfaces $\xi_1$ = Constant and $\xi_2$ = Constant, and there is first calculated the potential energy $\pi_l$ associated with the load system. According to equations (25a),

$$\begin{aligned}\pi_l = &-\iint\Bigg[\left(p_{1+}U_{1+} + p_{2+}U_{2+} + q_+W_+\right)\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) \\ &+ \left(p_{1-}U_{1-} + p_{2-}U_{2-} + q_-W_-\right)\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right)\Bigg]\alpha_1\alpha_2\, d\xi_1\, d\xi_2 \\ &- \oint\left[\int_{-h/2}^{h/2}\left(\sigma_n U_n + \tau_{nt}U_t + \tau_{n\zeta}W\right)\left(1 + \frac{\zeta}{R_t}\right) d\zeta\right]\alpha_t\, d\xi_t \\ \equiv\ &\pi_{ls} + \pi_{le}\end{aligned} \tag{70}$$

The first (double) integral $\pi_{ls}$ represents the energy associated with the surface loads $p_{1+}$, $p_{2+}$, and $q_+$ acting on the surface $\zeta = \frac{h}{2}$ and the surface loads $p_{1-}$, $p_{2-}$, and $q_-$ acting on the surface $\zeta = -\frac{h}{2}$. The abbreviations

$$U_{1+} = U_1\left(\xi_1,\ \xi_2,\ +\frac{h}{2}\right) \qquad U_{1-} = U_1\left(\xi_1,\ \xi_2,\ -\frac{h}{2}\right)$$

and so forth have been introduced. The second integral $\pi_{le}$ in equation (70) represents the energy associated with the edge stresses. Here the subscripts n and t refer to the normal and tangential directions on the boundary faces; thus, on a face $\xi_1$ = Constant, n = 1, t = 2.

If the assumptions of equations (69) are introduced into the expression for $\pi_{ls}$ the coefficients of the displacement functions are found to involve quantities which may be denoted as follows:

$$\left.\begin{aligned}
p_1 &= p_{1+}\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) + p_{1-}\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right) \\
p_2 &= p_{2+}\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) + p_{2-}\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right) \\
q &= q_{+}\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) + q_{-}\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right)
\end{aligned}\right\} \tag{71a}$$

$$\left.\begin{aligned}
m_1 &= -\frac{h}{2}\left[p_{1+}\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) - p_{1-}\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right)\right] \\
m_2 &= \frac{h}{2}\left[p_{2+}\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) - p_{2-}\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right)\right] \\
n &= \frac{h}{2}\left[q_{+}\left(1 + \frac{h}{2R_1}\right)\left(1 + \frac{h}{2R_2}\right) - q_{-}\left(1 - \frac{h}{2R_1}\right)\left(1 - \frac{h}{2R_2}\right)\right]
\end{aligned}\right\} \tag{71b}$$

With these abbreviations, there follows

$$\begin{aligned}
\pi_{ls} = -\iint\Big(&p_1u_1 + p_2u_2 + qw + m_1u_1' + m_2u_2' + nw' \\
&+ \frac{1}{8}h^2p_1u_1'' + \frac{1}{8}h^2p_2u_2'' + \frac{1}{8}h^2qw''\Big)\alpha_1\alpha_2\,d\xi_1\,d\xi_2
\end{aligned} \tag{72}$$

where the integration is taken over the area of the middle surface of the plate or shell.

If the assumptions of equation (69) are introduced into the expression for $\pi_{le}$ and if the $\zeta$-integration is carried out, the coefficients of the displacement functions involve quantities which may be denoted as follows:

$$\left.\begin{aligned}
N_{nn} &= \int_{-h/2}^{h/2} \sigma_n\left(1 + \frac{\zeta}{R_t}\right) d\zeta & N_{nt} &= \int_{-h/2}^{h/2} \tau_{nt}\left(1 + \frac{\zeta}{R_t}\right) d\zeta & Q_n &= \int_{-h/2}^{h/2} \tau_{n\zeta}\left(1 + \frac{\zeta}{R_t}\right) d\zeta \\
M_{nn} &= \int_{-h/2}^{h/2} \sigma_n\left(1 + \frac{\zeta}{R_t}\right)\zeta\ d\zeta & M_{nt} &= \int_{-h/2}^{h/2} \tau_{nt}\left(1 + \frac{\zeta}{R_t}\right)\zeta\ d\zeta & S_n &= \int_{-h/2}^{h/2} \tau_{n\zeta}\left(1 + \frac{\zeta}{R_t}\right)\zeta\ d\zeta \\
P_{nn} &= \frac{1}{2}\int_{-h/2}^{h/2} \sigma_n\left(1 + \frac{\zeta}{R_t}\right)\zeta^2\ d\zeta & P_{nt} &= \frac{1}{2}\int_{-h/2}^{h/2} \tau_{nt}\left(1 + \frac{\zeta}{R_t}\right)\zeta^2\ d\zeta & T_n &= \frac{1}{2}\int_{-h/2}^{h/2} \tau_{n\zeta}\left(1 + \frac{\zeta}{R_t}\right)\zeta^2\ d\zeta
\end{aligned}\right\}\quad(73)$$

Of the 18 "resultants" so defined, corresponding to edges $\xi_1$ = Constant and $\xi_2$ = Constant, it is seen that 10 are the familiar stress resultants and couples of equations (33), while the remaining 8 quantities are new.

With the notation of equations (73), there follows

$$\pi_{le} = -\oint \left(\overline{N}_{nn}u_n + \overline{M}_{nn}u_n' + \overline{P}_{nn}u_n'' + \overline{N}_{nt}u_t + \overline{M}_{nt}u_t' + \overline{P}_{nt}u_t'' + \overline{Q}_n w + \overline{S}_n w' + \overline{T}_n w''\right)\alpha_t\ d\xi_t \qquad (74)$$

where the line integral is taken along the coordinate curves which form the complete boundary of the middle surface. The fact that edge values of the resultants are involved in equations (74) is indicated by the use of bars.

The internal energy $\pi_{strain}$ is given by the volume integral

$$\pi_{strain} = \iint \left[ \int_{-h/2}^{h/2} P\alpha_1\alpha_2\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) d\zeta \right] d\xi_1 \, d\xi_2 \tag{75}$$

where P represents the strain energy per unit volume and is defined by equation (27). Thus, the variational principle of equation (28) is of the form

$$\delta\pi_{strain} + \delta\pi_{ls} + \delta\pi_{le} = 0 \tag{76}$$

where

$$\delta\pi_{strain} = \iint \left[ \int_{-h/2}^{h/2} (\delta P)\alpha_1\alpha_2\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) d\zeta \right] d\xi_1 \, d\xi_2 \tag{77}$$

If use is made of equations (26), the expression for $\delta P$ can be put in the form

$$\delta P = \frac{\partial P}{\partial \epsilon_1} \delta\epsilon_1 + \frac{\partial P}{\partial \epsilon_2} \delta\epsilon_2 + \dots + \frac{\partial P}{\partial \gamma_{2\zeta}} \delta\gamma_{2\zeta}$$

$$= \sigma_1\delta\epsilon_1 + \sigma_2\delta\epsilon_2 + \sigma_\zeta\delta\epsilon_\zeta + \tau_{12}\delta\gamma_{12} + \tau_{1\zeta}\delta\gamma_{1\zeta} + \tau_{2\zeta}\delta\gamma_{2\zeta} \tag{78}$$

where the stress and strain components are to be considered as functions of the nine displacement functions appearing in equations (69).

First there is calculated the contribution of the first term $\sigma_1\delta\epsilon_1$ to the variation $\delta\pi_{strain}$. If the assumptions of equations (69) are introduced into the first of equations (9a), there is obtained

$$\epsilon_1 = \frac{1}{\alpha_1(1 + \zeta/R_1)}\left[\left(\frac{\partial u_1}{\partial \xi_1} + \zeta \frac{\partial u_1'}{\partial \xi_1} + \frac{1}{2}\zeta^2 \frac{\partial u_1''}{\partial \xi_1}\right) + \frac{1}{\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}\left(u_2 + \zeta u_2' + \frac{1}{2}\zeta^2 u_2''\right)\right.$$
$$\left. + \frac{\alpha_1}{R_1}\left(w + \zeta w' + \frac{1}{2}\zeta^2 w''\right)\right]$$

Thus, the result of replacing $\delta P$ by $\sigma_1 \delta\epsilon_1$ in the right-hand member of equation (77) is of the form

$$\iint \left\{\int_{-h/2}^{h/2} \sigma_1\left(1 + \frac{\zeta}{R_2}\right)\left[\alpha_2\left(\frac{\partial \delta u_1}{\partial \xi_1} + \zeta \frac{\partial \delta u_1'}{\partial \xi_1} + \frac{1}{2}\zeta^2 \frac{\partial \delta u_1''}{\partial \xi_1}\right)\right.\right.$$
$$+ \frac{\partial \alpha_1}{\partial \xi_2}\left(\delta u_2 + \zeta \delta u_2' + \frac{1}{2}\zeta^2 \delta u_2''\right) + \frac{\alpha_1 \alpha_2}{R_1}\left(\delta w + \zeta \delta w'\right.$$
$$\left.\left.\left. + \frac{1}{2}\zeta^2 \delta w''\right)\right] d\zeta\right\} d\xi_1\, d\xi_2$$

If the $\zeta$-integration is carried out and the notation of equations (73) is introduced, this expression takes the form

$$\iint \left[N_{11}\left(\alpha_2 \frac{\partial \delta u_1}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2}\delta u_2 + \frac{\alpha_1 \alpha_2}{R_1}\delta w\right) + M_{11}\left(\alpha_2 \frac{\partial \delta u_1'}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2}\delta u_2' + \frac{\alpha_1 \alpha_2}{R_1}\delta w'\right)\right.$$
$$\left. + P_1\left(\alpha_2 \frac{\partial \delta u_1''}{\partial \xi_1} + \frac{\partial \alpha_1}{\partial \xi_2}\delta u_2'' + \frac{\alpha_1 \alpha_2}{R_1}\delta w''\right)\right] d\xi_1\, d\xi_2$$

where the integration is taken over the middle surface of the plate or shell.

The contributions of the remaining terms in equation (78) to the value of $\delta\pi_{strain}$ can be calculated in a similar way. In calculating the contribution of the term $\tau_{12}\delta\gamma_{12}$ it is convenient to make use of the identities of equations (7).

The contribution of the term $\sigma_\zeta \delta\epsilon_\zeta$ involves two new "resultants," not defined by equations (73), for which the following notations are adopted:

$$\left.\begin{aligned} A &= \int_{-h/2}^{h/2} \sigma_\zeta\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right) d\zeta \\ B &= \int_{-h/2}^{h/2} \sigma_\zeta\left(1 + \frac{\zeta}{R_1}\right)\left(1 + \frac{\zeta}{R_2}\right)\zeta \, d\zeta \end{aligned}\right\} \qquad (79)$$

In the resultant expression for $\delta\pi_{strain}$ the terms involving derivatives of the independent variations may be integrated by parts, this procedure leading to an equivalent expression for $\delta\pi_{strain}$ consisting of the sum of a line integral taken around the boundary of the middle surface and a new double integral involving the independent variations in a linear way.

Details of the calculation are omitted, and only the form is presented that is assumed by equation (76) when the separate integrals are combined and the coefficients of the independent variations are collected:

$$
\begin{aligned}
\iint \Bigg\{ & \left[\frac{\partial \alpha_2 N_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{21}}{\partial \xi_2} + N_{12}\frac{\partial \alpha_1}{\partial \xi_2} - N_{22}\frac{\partial \alpha_2}{\partial \xi_1} + Q_1\frac{\alpha_1\alpha_2}{R_1} + \alpha_1\alpha_2 p_1\right]\delta u_1 \\
& + \left[\frac{\partial \alpha_2 N_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 N_{22}}{\partial \xi_2} + N_{21}\frac{\partial \alpha_2}{\partial \xi_1} - N_{11}\frac{\partial \alpha_1}{\partial \xi_2} + Q_2\frac{\alpha_1\alpha_2}{R_2} + \alpha_1\alpha_2 p_2\right]\delta u_2 \\
& + \left[\frac{\partial \alpha_2 Q_1}{\partial \xi_1} + \frac{\partial \alpha_1 Q_2}{\partial \xi_2} - \left(\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2}\right)\alpha_1\alpha_2 + \alpha_1\alpha_2 q\right]\delta w \\
& + \left[\frac{\partial \alpha_2 M_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{21}}{\partial \xi_2} + M_{12}\frac{\partial \alpha_2}{\partial \xi_2} - M_{22}\frac{\partial \alpha_2}{\partial \xi_1} - Q_1\alpha_1\alpha_2 - \alpha_1\alpha_2 m_1\right]\delta u_1' \\
& + \left[\frac{\partial \alpha_2 M_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 M_{22}}{\partial \xi_2} + M_{21}\frac{\partial \alpha_2}{\partial \xi_1} - M_{11}\frac{\partial \alpha_1}{\partial \xi_2} - Q_2\alpha_1\alpha_2 + \alpha_1\alpha_2 m_2\right]\delta u_2' \\
& + \left[\frac{\partial \alpha_2 S_1}{\partial \xi_1} + \frac{\partial \alpha_1 S_2}{\partial \xi_2} - \left(\frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right)\alpha_1\alpha_2 - A\alpha_1\alpha_2 + \alpha_1\alpha_2 n\right]\delta w' \\
& + \left[\frac{\partial \alpha_2 P_{11}}{\partial \xi_1} + \frac{\partial \alpha_1 P_{21}}{\partial \xi_2} + P_{12}\frac{\partial \alpha_1}{\partial \xi_2} - P_{22}\frac{\partial \alpha_2}{\partial \xi_1} - \left(S_1 + \frac{T_1}{R_1}\right)\alpha_1\alpha_2 \right. \\
& \left. + \frac{1}{8}\alpha_1\alpha_2 h^2 p_1\right]\delta u_1'' + \left[\frac{\partial \alpha_2 P_{12}}{\partial \xi_1} + \frac{\partial \alpha_1 P_{22}}{\partial \xi_2} + P_{21}\frac{\partial \alpha_2}{\partial \xi_1} - P_{11}\frac{\partial \alpha_1}{\partial \xi_2} \right. \\
& \left. - \left(S_2 + \frac{T_2}{R_2}\right)\alpha_1\alpha_2 + \frac{1}{8}\alpha_1\alpha_2 h^2 p_2\right]\delta u_2'' + \left[\frac{\partial \alpha_2 T_1}{\partial \xi_1} + \frac{\partial \alpha_1 T_2}{\partial \xi_2} \right. \\
& \left. - \left(\frac{P_{11}}{R_1} + \frac{P_{22}}{R_2}\right)\alpha_1\alpha_2 - B\alpha_1\alpha_2 + \frac{1}{8}\alpha_1\alpha_2 h^2 q\right]\delta w'' \Bigg\}\, d\xi_1\, d\xi_2 \\
& + \oint \Big\{\left(\overline{N}_{nn} - N_{nn}\right)\delta u_n + \left(\overline{N}_{nt} - N_{nt}\right)\delta u_t + \left(\overline{Q}_n - Q_n\right)\delta w \\
& + \left(\overline{M}_{nn} - M_{nn}\right)\delta u_n' + \left(\overline{M}_{nt} - M_{nt}\right)\delta u_t' + \left(\overline{S}_n - S_n\right)\delta w' \\
& + \left(\overline{P}_{nn} - P_{nn}\right)\delta u_n'' + \left(\overline{P}_{nt} - P_{nt}\right)\delta u_t'' + \left(\overline{T}_n - T_n\right)\delta w''\Big\}\alpha_t\, d\xi_t \\
& = 0 \qquad (80)
\end{aligned}
$$

Since the variations appearing in this equation are entirely arbitrary in the interior of the middle surface, the coefficients of the nine variations in the double integral must each vanish identically. In this way nine differential equations are obtained, the first five of which are the conventional equilibrium equations (35) and (38), except for the interpretation of the loading terms, and the last four of which are new. These additional equations correspond to the introduction of the four new displacement functions $w'$, $w''$, $u_1''$, and $u_2''$.

The corresponding nine boundary conditions to be prescribed along the edges of the plate or shell are obtained as a consequence of the independent vanishing of the nine terms in the line integral of equations (80). Thus, along a boundary $\xi_1$ = Constant, the first term of the line integral becomes $\left(\overline{N}_{11} - N_{11}\right)\delta u_1$. The vanishing of this term is assured if the displacement $u_1 = U_1(\xi_1, \xi_2, 0)$ is prescribed along this boundary, since then $\delta u_1 = 0$. Alternatively, the term will vanish if the resultant $N_{11}$ is required to take on a prescribed value $\overline{N}_{11}$ along this boundary. Considering the other terms in the same way, it is seen that the following nine boundary conditions along a boundary $\xi_1 = C$ are consistent with the displacement assumptions of equations (69):

$$\left.\begin{array}{llll}
N_{11} = \overline{N}_{11} & \text{or} & u_1 = U_1(C, \xi_2, 0) & \text{prescribed} \\
N_{12} = \overline{N}_{12} & \text{or} & u_2 = U_2(C, \xi_2, 0) & \text{prescribed} \\
Q_1 = \overline{Q}_1 & \text{or} & w = W(C, \xi_2, 0) & \text{prescribed} \\
M_{11} = \overline{M}_{11} & \text{or} & u_1' = \dfrac{\partial U_1(C, \xi_2, 0)}{\partial \zeta} & \text{prescribed} \\
M_{12} = \overline{M}_{12} & \text{or} & u_2' = \dfrac{\partial U_2(C, \xi_2, 0)}{\partial \zeta} & \text{prescribed}
\end{array}\right\} \quad (81a)$$

$$\left.\begin{array}{llll}
S_1 = \overline{S}_1 & \text{or} & w' = \dfrac{\partial W(C, \xi_2, 0)}{\partial \zeta} & \text{prescribed} \\
T_1 = \overline{T}_1 & \text{or} & w'' = \dfrac{\partial^2 W(C, \xi_2, 0)}{\partial \zeta^2} & \text{prescribed} \\
P_{11} = \overline{P}_{11} & \text{or} & u_1'' = \dfrac{\partial^2 U_1(C, \xi_2, 0)}{\partial \zeta^2} & \text{prescribed} \\
P_{12} = \overline{P}_{12} & \text{or} & u_2'' = \dfrac{\partial^2 U_2(C, \xi_2, 0)}{\partial \zeta^2} & \text{prescribed}
\end{array}\right\} \quad (81b)$$

In particular, it is seen that if $w' = w'' = u_1'' = u_2'' = 0$ everywhere, and hence equations (69) are specialized to the classical assumptions of equations (41), the five boundary conditions which remain are those of equations (81a). Thus, for example, at a free edge of a plate the twisting couple $M_{12}$ and the shearing force $Q_1$ can be required to vanish independently. These results are in contrast with the corresponding situation in the classical theory, where Kirchoff first showed that for a flat plate the physically desirable conditions $M_{12} = Q_1 = 0$ at a free edge cannot both be satisfied but that they must be replaced by the single condition $Q_1 + \dfrac{\partial M_{12}}{\partial \xi_2} = 0$. Basset, Lamb, and others have since shown that in the classical theory the resultants $N_{12}$, $M_{12}$, and $Q_1$ cannot in general be separately prescribed along an edge of a shell. (See reference 4, pp. 536 and 537.)

The next step in the analysis consists in expressing the resultants of equations (73) and (79) in terms of the nine displacement functions. The expressions for the relevant stress components are obtained by first introducting equations (69) into equations (9) to determine the strain components and then substituting those results into the stress-strain relations of equations (13) and (18). The remainder of the analysis consists basically in introducing the expressions for the resultants into the equations obtained by setting the nine bracketed expressions in equation (80) equal to zero and so obtaining nine differential equations in the nine displacement functions. In practice, however, the latter part of the analysis may frequently be carried out more conveniently in terms of some other appropriate set of nine independent quantities.

Explicit expressions for the "auxiliary resultants" A and B of equations (79), which do not enter into the boundary conditions, may be listed as follows:

$$\left.\begin{aligned} A &= \nu_\zeta\left(N_{11} + N_{22} + \frac{M_{11}}{R_1} + \frac{M_{22}}{R_2}\right) + E_\zeta h\left[\left(1 + \frac{h^2}{12R_1R_2}\right)w' + \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)w''\right] \\ B &= \nu_\zeta\left(M_{11} + M_{22} + \frac{2P_{11}}{R_1} + \frac{2P_{22}}{R_2}\right) + E_\zeta \frac{h^3}{12}\left[\left(\frac{1}{R_1} + \frac{1}{R_2}\right)w' + \left(1 + \frac{3h^2}{20R_1R_2}\right)w''\right] \end{aligned}\right\} \quad (82)$$

In obtaining these expressions, use was made of the third of equations (12) to express $\sigma_\zeta$ in terms of $(\sigma_1 + \sigma_2)$ and $\epsilon_\zeta$.

By introducing equations (69) into equations (9) expressions are obtained for the strain components which may be written in the form

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{1 + \zeta/R_1}\left(\epsilon_1{}^0 + \zeta\epsilon_1{}' + \frac{1}{2}\zeta^2\epsilon_1{}''\right)\\
\epsilon_2 &= \frac{1}{1 + \zeta/R_2}\left(\epsilon_2{}^0 + \zeta\epsilon_2{}' + \frac{1}{2}\zeta^2\epsilon_2{}''\right)\\
\epsilon_\zeta &= w' + \zeta w''\\
\gamma_{12} &= \frac{1}{1 + \zeta/R_1}\left(\beta_1{}^0 + \zeta\beta_1{}' + \frac{1}{2}\zeta^2\beta_1{}''\right) + \frac{1}{1 + \zeta/R_2}\left(\beta_2{}^0 + \zeta\beta_2{}' + \frac{1}{2}\zeta^2\beta_2{}''\right)\\
\gamma_{1\zeta} &= \frac{1}{1 + \zeta/R_1}\left(\mu_1{}^0 + \zeta\mu_1{}' + \frac{1}{2}\zeta^2\mu_1{}''\right)\\
\gamma_{2\zeta} &= \frac{1}{1 + \zeta/R_2}\left(\mu_2{}^0 + \zeta\mu_2{}' + \frac{1}{2}\zeta^2\mu_2{}''\right)
\end{aligned}\right\} \tag{83}$$

where the relevant functions are defined by the equations

$$\left.\begin{aligned}
\epsilon_1{}^0 &= \frac{1}{\alpha_1}\frac{\partial u_1}{\partial \xi_1} + \frac{u_2}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w}{R_1}\\
\epsilon_1{}' &= \frac{1}{\alpha_1}\frac{\partial u_1{}'}{\partial \xi_1} + \frac{u_2{}'}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w'}{R_1}\\
\epsilon_1{}'' &= \frac{1}{\alpha_1}\frac{\partial u_1{}''}{\partial \xi_1} + \frac{u_2{}''}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} + \frac{w''}{R_1}
\end{aligned}\right\} \tag{84a}$$

$$\left.\begin{aligned}
\beta_1{}^0 &= \frac{1}{\alpha_1}\frac{\partial u_2}{\partial \xi_1} - \frac{u_1}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} \\
\beta_1{}' &= \frac{1}{\alpha_1}\frac{\partial u_2{}'}{\partial \xi_1} - \frac{u_1{}'}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2} \\
\beta_1{}'' &= \frac{1}{\alpha_1}\frac{\partial u_2{}''}{\partial \xi_1} - \frac{u_1{}''}{\alpha_1\alpha_2}\frac{\partial \alpha_1}{\partial \xi_2}
\end{aligned}\right\} \tag{84b}$$

$$\left.\begin{aligned}
\mu_1{}^0 &= \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} - \frac{u_1}{R_1} + u_1{}' \\
\mu_1{}' &= \frac{1}{\alpha_1}\frac{\partial w'}{\partial \xi_1} + u_1{}'' \\
\mu_1{}'' &= \frac{1}{\alpha_1}\frac{\partial w''}{\partial \xi_1} + \frac{u_1{}''}{R_1}
\end{aligned}\right\} \tag{84c}$$

and by the additional equations obtained by permuting the subscripts 1 and 2 in equations (84).

The calculation of the resultants involves integrals for which there are introduced the following notations

$$\left.\begin{aligned}
&\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1}\, d\zeta = h\left[1 + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^2}{12R_1}\, a_1{}^0\right] \\
&\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1}\, \zeta\, d\zeta = -\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^3}{12}\, a_1{}^0 \\
&\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1}\, \zeta^2\, d\zeta = \frac{h^3}{12}\left[1 + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{3h^2}{20R_1}\, a_1{}'\right] \\
&\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1}\, \zeta^3\, d\zeta = -\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^5}{80}\, a_1{}' \\
&\int_{-h/2}^{h/2} \frac{1 + \zeta/R_2}{1 + \zeta/R_1}\, \zeta^4\, d\zeta = \frac{h^5}{80}\left[1 + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{5h^2}{28R_1}\, a_1{}''\right]
\end{aligned}\right\} \qquad (85)$$

together with the analogous expressions obtained by permutation of subscripts. The parameters introduced in equations (85) possess the following expansions in powers of $h$:

$$\left.\begin{aligned}
a_1{}^0 &= 1 + \frac{3}{5}\left(\frac{h}{2R_1}\right)^2 + \frac{3}{7}\left(\frac{h}{2R_1}\right)^4 + \frac{3}{9}\left(\frac{h}{2R_1}\right)^6 + \ldots \\
a_1{}' &= 1 + \frac{5}{7}\left(\frac{h}{2R_1}\right)^2 + \frac{5}{9}\left(\frac{h}{2R_1}\right)^4 + \frac{5}{11}\left(\frac{h}{2R_1}\right)^6 + \ldots \\
a_1{}'' &= 1 + \frac{7}{9}\left(\frac{h}{2R_1}\right)^2 + \frac{7}{11}\left(\frac{h}{2R_1}\right)^4 + \frac{7}{13}\left(\frac{h}{2R_1}\right)^6 + \ldots
\end{aligned}\right\} \qquad (86)$$

With the notation of equations (84) and (86), the expressions for the resultants defined by equations (73) can be written as follows:

$$
\left.
\begin{aligned}
N_{11} &= \frac{\overline{E}h}{1 - \nu^2} \left\{ \left[ (1 - \overline{\nu}^2)\epsilon_1{}^0 + (\nu + \overline{\nu}^2)\epsilon_2{}^0 + \nu_\zeta(1 + \nu)w' \right] \right. \\
&\quad + \frac{h^2}{24}\left[ (1 - \overline{\nu}^2)\epsilon_1{}'' + (\nu + \overline{\nu}^2)\epsilon_2{}'' + 2\nu_\zeta(1 + \nu)\frac{w''}{R_2} \right] \\
&\quad \left. + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^2}{12}(1 - \overline{\nu}^2)\left[ a_1{}^0\left(\frac{\epsilon_1{}^0}{R_1} - \epsilon_1{}'\right) + \frac{3h^2}{40R_1}\, a_1{}'\epsilon_1{}'' \right] \right\} \\
N_{12} &= Gh\left\{ \left(\beta_1{}^0 + \beta_2{}^0\right) + \frac{h^2}{24}\left(\beta_1{}'' + \beta_2{}''\right) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^2}{12}\left[ a_1{}^0\left(\frac{\beta_1{}^0}{R_1} - \beta_1{}'\right) \right.\right. \\
&\quad \left.\left. + \frac{3h^2}{40R_1}\, a_1{}'\beta_1{}'' \right] \right\} \\
M_{11} &= \frac{\overline{E}h^3}{12(1 - \nu^2)} \left\{ \left[ (1 - \overline{\nu}^2)\epsilon_1{}' + (\nu + \overline{\nu}^2)\epsilon_2{}' + \nu_\zeta(1 + \nu)\left(w'' + \frac{w'}{R_2}\right) \right] \right. \\
&\quad \left. - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)(1 - \overline{\nu}^2)\left[ a_1{}^0\epsilon_1{}^0 - \frac{3h^2}{20}\, a_1{}'\left(\frac{\epsilon_1{}'}{R_1} - \frac{1}{2}\,\epsilon_1{}''\right) \right] \right\} \\
M_{12} &= \frac{Gh^3}{12}\left\{ \left(\beta_1{}' + \beta_2{}'\right) - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left[ a_1{}^0\beta_1{}^0 - \frac{3h^2}{20}\, a_1{}'\left(\frac{\beta_1{}'}{R_1} - \frac{1}{2}\,\beta_1{}''\right) \right] \right\}
\end{aligned}
\right\} \quad (87a)
$$

$$
Q_1 = G\zeta h\left\{ \mu_1{}^0 + \frac{h^2}{24}\,\mu_1{}'' + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^2}{12}\left[ a_1{}^0\left(\frac{\mu_1{}^0}{R_1} - \mu_1{}'\right) + \frac{3h^2}{40R_1}\, a_1{}'\mu_1{}'' \right] \right\} \quad (87b)
$$

$$
\left.
\begin{aligned}
S_1 &= \frac{G_\zeta h^3}{12}\left\{\mu_1{}' - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left[a_1{}^0\mu_1{}^0 - \frac{3h^2}{20}\, a_1{}'\left(\frac{\mu_1{}'}{R_1} - \frac{1}{2}\,\mu_1{}''\right)\right]\right\} \\
T_1 &= \frac{G_\zeta h^3}{24}\left\{\mu_1{}^0 + \frac{3h^2}{40}\,\mu_1{}'' + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{3h^2}{20}\left[a_1{}'\left(\frac{\mu_1{}^0}{R_1} - \mu_1{}'\right)\right.\right. \\
&\qquad \left.\left. + \frac{5h^2}{56R_1}\, a_1{}''\mu_1{}''\right]\right\} \\
P_{11} &= \frac{\bar{E}h^3}{24(1 - \nu^2)}\left\{\left[(1 - \bar{\nu}^2)\epsilon_1{}^0 + (\nu + \bar{\nu}^2)\epsilon_2{}^0 + \nu_\zeta(1 + \nu)w'\right]\right. \\
&\qquad + \frac{3h^2}{40}\left[(1 - \bar{\nu}^2)\epsilon_1{}'' + (\nu + \bar{\nu}^2)\epsilon_2{}'' + 2\nu_\zeta(1 + \nu)\frac{w''}{R_2}\right] \\
&\qquad \left. + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{3h^2}{20}(1 - \bar{\nu}^2)\left[a_1{}'\left(\frac{\epsilon_1{}^0}{R_1} - \epsilon_1{}'\right) + \frac{5h^2}{56R_1}\, a_1{}''\epsilon_1{}''\right]\right\} \\
P_{12} &= \frac{Gh^3}{24}\left\{\left(\beta_1{}^0 + \beta_2{}^0\right) + \frac{3h^2}{40}\left(\beta_1{}'' + \beta_2{}''\right) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{3h^2}{20}\left[a_1{}'\left(\frac{\beta_1{}^0}{R_1} - \beta_1{}'\right)\right.\right. \\
&\qquad \left.\left. + \frac{5h^2}{56R_1}\, a_1{}''\beta_1{}''\right]\right\}
\end{aligned}
\right\} \quad (87c)
$$

The expressions for the nine remaining resultants are obtained by permutation of subscripts.

The results listed in equations (87a) reduce to those of equations (47) if $w' = u_1{}'' = u_2{}'' = w'' = 0$, if $u_1{}'$ and $u_2{}'$ are determined from equations (43), if $\nu_\zeta = 0$, and then if terms involving powers of $h$ greater than three are neglected.

In this connection it is useful to write the expressions for $N_{11}$, $M_{11}$, and $P_{11}$ in the following alternative forms:

$$\left.\begin{aligned}
N_{11} &= \frac{Eh}{1-\nu^2}\Bigg[\Bigg\{\left(\epsilon_1{}^0 + \nu\epsilon_2{}^0\right) + \frac{h^2}{24}\left(\epsilon_1{}'' + \nu\epsilon_2{}''\right) \\
&\quad + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^2}{12}\left[a_1{}^0\left(\frac{\epsilon_1{}^0}{R_1} - \epsilon_1{}'\right) + \frac{3h^2}{40R_1}\,a_1{}'\epsilon_1{}''\right]\Bigg\} \\
&\quad + \nu^*\frac{E^*}{E}\Bigg(\left(w' + \frac{h^2}{12}\frac{w''}{R_2}\right) + \nu^*\Bigg\{\left(\epsilon_1{}^0 + \epsilon_2{}^0\right) + \frac{h^2}{24}\left(\epsilon_1{}'' + \epsilon_2{}''\right) \\
&\quad + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{h^2}{12}\left[a_1{}^0\left(\frac{\epsilon_1{}^0}{R_1} - \epsilon_1{}'\right) + \frac{3h^2}{40R_1}\,a_1{}'\epsilon_1{}''\right]\Bigg\}\Bigg)\Bigg] \\
M_{11} &= \frac{Eh^3}{12(1-\nu^2)}\Bigg[\Bigg\{\left(\epsilon_1{}' + \nu\epsilon_2{}'\right) - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\Bigg[a_1{}^0\epsilon_1{}^0 \\
&\quad - \frac{3h^2}{20}\,a_1{}'\left(\frac{\epsilon_1{}'}{R_1} - \frac{1}{2}\,\epsilon_1{}''\right)\Bigg]\Bigg\} + \nu^*\frac{E^*}{E}\Bigg(\left(\frac{w'}{R_2} + w''\right) + \nu^*\Bigg\{\left(\epsilon_1{}' + \epsilon_2{}'\right) \\
&\quad - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left[a_1{}^0\epsilon_1{}^0 - \frac{3h^2}{20}\,a_1{}'\left(\frac{\epsilon_1{}'}{R_1} - \frac{1}{2}\,\epsilon_1{}''\right)\right]\Bigg\}\Bigg)\Bigg] \\
P_{11} &= \frac{Eh^3}{24(1-\nu^2)}\Bigg[\Bigg\{\left(\epsilon_1{}^0 + \nu\epsilon_2{}^0\right) + \frac{3h^2}{40}\left(\epsilon_1{}'' + \nu\epsilon_2{}''\right) \\
&\quad + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{3h^2}{20}\left[a_1{}'\left(\frac{\epsilon_1{}^0}{R_1} - \epsilon_1{}'\right) + \frac{5h^2}{56R_1}\,a_1{}''\epsilon_1{}''\right]\Bigg\} \\
&\quad + \nu^*\frac{E^*}{E}\Bigg(\left(w' + \frac{3h^2}{20}\frac{w''}{R_2}\right) + \nu^*\Bigg\{\left(\epsilon_1{}^0 + \epsilon_2{}^0\right) + \frac{3h^2}{40}\left(\epsilon_1{}'' + \epsilon_2{}''\right) \\
&\quad + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{3h^2}{20}\left[a_1{}'\left(\frac{\epsilon_1{}^0}{R_1} - \epsilon_1{}'\right) + \frac{5h^2}{56R_1}\,a_1{}''\epsilon_1{}''\right]\Bigg\}\Bigg)\Bigg]
\end{aligned}\right\}\quad (88)$$

where

$$\left.\begin{aligned} \nu^* &= \frac{\nu_\zeta E}{(1-\nu)E_\zeta} \\ E^* &= \frac{(1-\nu^2)E_\zeta}{1-2\nu_\zeta\nu^*} \end{aligned}\right\} \tag{89}$$

The coefficients of $\nu^*$ in equations (88) represent the contributions of the transverse normal stress $\sigma_\zeta$ and are zero when $\sigma_\zeta$ vanishes. In accordance with the remarks made in the preceding section, it is evident that if the effects of the transverse normal stress are to be taken into account in a reasonably accurate way, the terms $w'$ and $w''$ must not be excluded, in general, unless the orthotrophy is such that $\nu_\zeta$ is negligibly small.

It appears, however, that the contributions of the terms $u_1''$ and $u_2''$ are in general of small importance. In fact, it is seen that these terms appear principally in combinations of the form $u_1{}^0 + \frac{h^2}{24}u_1''$ and $u_2{}^0 + \frac{h^2}{24}u_2''$, such combinations representing the average values of the assumed displacements $U_1$ and $U_2$ over the thickness. Thus, it appears that neglect of the terms $u_1''$ and $u_2''$ can be largely compensated by interpreting the terms $u_1{}^0$ and $u_2{}^0$ as average values rather than as values assumed at the middle surface. The equations herein listed afford the basis of a more detailed investigation of the importance of the terms $u_1''$ and $u_2''$.

If these terms are neglected there remain seven displacement functions and, consequently, seven boundary conditions are to be satisfied along an edge of the plate or shell. Thus two conditions in addition to the usual five conditions suggested by statics are to be imposed. In particular, if the boundary conditions involve the stresses, the two additional conditions prescribe the integrated first and second moments S and T of the edge transverse shearing stresses, in addition to the transverse shear resultant Q. If the terms $u_1''$ and $u_2''$ are also retained, the second moments of the edge normal and shear stresses in the direction of the middle surface are also to be prescribed.

It is of some interest to investigate in what way the number of boundary conditions is reduced when the effect of transverse shear is neglected by taking $G_\zeta = \infty$. Reference to equations (87) then shows that in order that the resultants Q, S, and T remain finite there must be taken

$$\mu_1{}^0 = \mu_1{}' = \mu_1{}'' = 0 \tag{90}$$

and hence, in accordance with equations (84c), there must follow

$$\left.\begin{aligned} u_1{}' &= \frac{u_1}{R_1} - \frac{1}{\alpha_1}\frac{\partial w}{\partial \xi_1} \qquad & u_2{}' &= \frac{u_2}{R_2} - \frac{1}{\alpha_2}\frac{\partial w}{\partial \xi_2} \\ u_1{}'' &= -\frac{1}{\alpha_1}\frac{\partial w'}{\partial \xi_1} \qquad & u_2{}'' &= -\frac{1}{\alpha_2}\frac{\partial w'}{\partial \xi_2} \end{aligned}\right\} \tag{91a}$$

and also

$$\frac{1}{R_1}\frac{\partial w'}{\partial \xi_1} - \frac{\partial w''}{\partial \xi_1} = \frac{1}{R_2}\frac{\partial w'}{\partial \xi_2} - \frac{\partial w''}{\partial \xi_2} = 0 \tag{91b}$$

The nature of the relevant boundary conditions is determined by considering the form assumed by the variation of the expression $\pi_{le}$ which represents the energy associated with the edge loads. If use is made of equations (74) and (91a), this variation becomes

$$\begin{aligned} \delta\pi_{le} = -\oint \Bigg[ & \overline{N}_{nn}\delta u_n + \overline{M}_{nn}\delta\left(\frac{u_n}{R_n} - \frac{1}{\alpha_n}\frac{\partial w}{\partial \xi_n}\right) - \frac{1}{\alpha_n}\overline{P}_{nn}\delta\frac{\partial w'}{\partial \xi_n} \\ & + \overline{N}_{nt}\delta u_t + \overline{M}_{nt}\left(\frac{1}{R_t}\delta u_t - \frac{1}{\alpha_t}\frac{\partial \delta w}{\partial \xi_t}\right) - \frac{1}{\alpha_t}\overline{P}_{nt}\frac{\partial \delta w'}{\partial \xi_t} \\ & + \overline{Q}_n\delta w + \overline{S}_n\delta w' + \overline{T}_n\delta w''\Bigg]\alpha_t\, d\xi_t \end{aligned}$$

The terms involving tangential differentiation can be integrated by parts to give the result

$$\delta\pi_{le} = -\oint\left[\overline{N}_{nn}\delta u_n + \overline{M}_{nn}\delta\left(\frac{u_n}{R_n} - \frac{1}{\alpha_n}\frac{\partial w}{\partial\xi_n}\right) + \left(\overline{N}_{nt} + \frac{\overline{M}_{nt}}{R_t}\right)\delta u_t \right.$$
$$+ \left(\overline{Q}_n + \frac{1}{\alpha_t}\frac{\partial\overline{M}_{nt}}{\partial\xi_t}\right)\delta w + \left(\overline{S}_n + \frac{1}{\alpha_t}\frac{\partial\overline{P}_{nt}}{\partial\xi_t}\right)\delta w'$$
$$\left. - \frac{1}{\alpha_n}\overline{P}_{nn}\delta\frac{\partial w'}{\partial\xi_n} + \overline{T}_n\delta w''\right]\alpha_t\, d\xi_t \qquad (92)$$

By considering the results of the application of the principle of minimum potential energy, it is concluded that either the quantities whose variations appear in equation (92) must be prescribed or the quantities which appear as their coefficients must be prescribed edge values of relevant resultants. Thus, in particular, if transverse shear effects are neglected, the three resultants $Q_n$, $N_{nt}$, and $M_{nt}$ cannot in general be separately prescribed at a boundary, but only "effective" shear resultants of the form

$$\left.\begin{aligned} \tilde{N}_{nt} &= N_{nt} + \frac{M_{nt}}{R_t} \\ \tilde{Q}_n &= Q_n + \frac{1}{\alpha_t}\frac{\partial M_{nt}}{\partial\xi_t} \end{aligned}\right\} \qquad (93)$$

can be specified. This result is in accordance with known facts first established by Kirchoff for the flat plate and by Basset (reference 10) for the circular cylindrical and spherical shell.

## CONCLUDING REMARKS

From a survey of various systems of equations given in the literature for the analysis of small deflections of thin elastic shells, it appears that the question concerning the best form of the basic system of equations of shell theory has not yet been decided, even in the small-deflection theory.

The present authors believe that their approach by way of the principle of minimum potential energy represents definite progress for the following reasons: (1) It is now possible to have a succession of two-dimensional theories of varying degrees of exactness, depending on the number of terms which are retained in the series of the displacement components and (2) the equations which are obtained include the possibility of analyzing boundary-layer effects not only when the boundary layer is of order $\sqrt{Rh}$ but also when the boundary layer is of order $h$, where $R$ is the radius of curvature and $h$ is the thickness.

Massachusetts Institute of Technology
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