NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1909

EFFECT OF TRANSVERSE SHEAR AND ROTARY INERTIA ON THE NATURAL FREQUENCY OF A UNIFORM BEAM

By Edwin T. Kruszewski

Langley Aeronautical Laboratory
Langley Air Force Base, Va.

Washington
July 1949

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1909

EFFECT OF TRANSVERSE SHEAR AND ROTARY INERTIA ON THE NATURAL FREQUENCY OF A UNIFORM BEAM

By Edwin T. Kruszewski

SUMMARY

A theoretical analysis of the effect of transverse shear and rotary inertia on the natural frequencies of a uniform beam is presented. Frequency equations are derived for the cases of the cantilever beam, the symmetrically vibrating free-free beam, and the antisymmetrically vibrating free-free beam. Numerical results are presented in the form of curves giving the frequencies of the first three modes of the cantilever beam and the first six modes, three symmetrical and three antisymmetrical, of the free-free beam.

INTRODUCTION

In the dynamic analysis of aircraft, the determination of the natural frequency is of basic importance. A number of methods of vibrational analysis have been presented in the past. Tests on actual structures, however, have often shown discrepancies between the calculated and the observed values of the natural frequency.

One of the possible explanations for these discrepancies is that many of the previous calculations were based on methods in which only the elementary engineering theory of beam bending was used and no secondary effects were included. Among the secondary effects are shear lag, deformation of the webs due to transverse shear, and rotary inertia. The effect of shear lag and shear deformation of the web is to increase the flexibility of the beam because of the additional deflection that is introduced. The effect of rotary inertia is to increase the dynamic loading on the beam because of the additional inertia loading due to the rotational acceleration of the differential elements of the beam.

The effect of shear lag on the bending vibrations of box beams was discussed in reference 1 and is not considered herein. The effects of transverse shear and rotary inertia were discussed by Timoshenko (reference 2) but only to the extent of presenting the differential equation. In the present paper a general solution of this differential equation is given together with specific solutions for the cantilever and free-free beams. (See appendix.) Charts giving the frequencies of the first three modes of the cantilever and the first six modes (three symmetrical and three antisymmetrical) of the free-free beam are presented.

SYMBOLS

$A_S G$	shear stiffness, pounds
A_S	shear area, square inches
A_T	effective total cross-sectional area, square inches
C_1, C_2, C_3, C_4	constants of integration
EI	flexural stiffness, pound-inches^2
E	modulus of elasticity, psi
G	shear modulus, psi
I	moment of inertia, inches^4
L	length of cantilever beam and half-length of free-free beam, inches
M	moment, pound-inches
dM_{RI}	reversed effective moment due to rotational acceleration, pound-inches
V	shear, pounds
g	acceleration due to gravity, inches per second per second
k_B	frequency coefficient
k_{B_o}	frequency coefficient where shear and rotary inertia are neglected

k_S	coefficient of shear rigidity $\left(\frac{1}{L}\sqrt{\frac{EI}{A_S G}}\right)$
k_{RI}	coefficient of rotary inertia $\left(\frac{1}{L}\sqrt{\frac{I}{A_T}}\right)$
m	mass of beam per unit length, seconds^2 per inch^2
q	distributed loading, pounds per inch
t	time, seconds
w	weight of beam per unit length, pounds per inch
x	distance along span measured from root or center line, inches
y	deflection of beam, inches
α, β	constants defined with equation (5)
γ	shear strain
θ	rotation of cross section, radians
ω	natural frequency of beam, radians per second
ω_o	natural frequency of beam excluding secondary effects, radians per second
ξ	nondimensional coordinate
ρ	density of beam, pounds per cubic inch

RESULTS AND DISCUSSION

Theoretical.- The natural frequencies for the cantilever beams and the free-free beams are defined by the frequency equations derived in the appendix. These equations are in terms of the parameters k_B, k_S, and k_{RI}. In order to solve the equations, a trial-and-error process is used. For every combination of values of k_S and k_{RI} there are an infinite number of values of k_B that satisfy the frequency equations. The smallest value of k_B is associated with the first mode, the next larger value with the second mode, and so forth.

Numerical.- The frequency equations were used to calculate natural frequencies for the first three modes of the cantilever and the first six modes (three symmetrical and three antisymmetrical) of the free-free beam. The results of these calculations are given in chart form in figures 1 to 3. In these charts the ratio of the natural frequency ω to the natural frequency ω_o obtained by neglecting secondary effects is plotted as a function of the shear-stiffness parameter k_S and the rotary-inertia parameter k_{RI}. In order to obtain ω from the ratio ω/ω_o, ω_o must first be calculated from the formula

$$\omega_o = k_{B_o}\sqrt{\frac{EI}{mL^4}}$$

where k_{B_o} is a constant for a particular mode and is given on the chart for that mode in figures 1 to 3. The values of ω/ω_o are given only for values of k_S from 0 to 0.24 and for values of k_{RI} in increments of 0.05 from 0 to 0.20 inasmuch as values above these are out of the range of probable design. The frequency equations derived in the appendix, however, can be used if values of ω/ω_o are needed for values of k_S or k_{RI} greater than those shown in the figures.

Inspection of the curves shown in figures 1 to 3 shows that in extreme cases for which the coefficients k_S and k_{RI} are large, reductions in frequency as high as 50 percent can be had in the first mode; whereas reductions as high as 80 percent can be had in the third mode.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., May 16, 1949

APPENDIX

DERIVATIONS AND SOLUTIONS OF FREQUENCY EQUATIONS

In the solutions for the natural bending frequencies of beams based on the elementary engineering theory, the beam deflection is considered as a function only of the manner of support and the flexural stiffness EI and, also, dynamic loading is considered as a function only of the translatory acceleration of the particles. The deflection, however, is also influenced by the shear stiffness $A_S G$ of the beam and the dynamic loading is influenced by the rotational acceleration of the cross section of the beam.

The effect of the shear deformation is introduced into the solution in the equation for the slope of the deflection curve. In the engineering theory of bending, the slope of the deflection curve $\frac{dy}{dx}$ is assumed to depend only on the rotation of the cross sections of the beam associated with flexure θ. The slope, however, depends also on the shear strain γ which when included gives

$$\frac{dy}{dx} = \theta + \gamma = \int \frac{M}{EI} dx + \frac{V}{A_S G}$$

where M is the moment and V, the shear. The coordinate system is shown in figure 4.

The effect of the rotary inertia is introduced into the solution in the summation of moments of the differential element shown in figure 5. The elements of the beam perform not only a translatory motion but also a rotation. The rotation θ is due to bending alone since the shear just superimposes a sliding of adjacent cross sections with respect to one another. This rotation varies with time. The angular velocity and the acceleration of each element are given by $\frac{\partial \theta}{\partial t}$ and $\frac{\partial^2 \theta}{\partial t^2}$, respectively. The reversed effective moment of the forces due to this angular acceleration on a length dx is expressed as

$$dM_{RI} = -\frac{I\rho}{g} \frac{\partial^2 \theta}{\partial t^2} dx$$

where ρ is the density of the beam and g is the acceleration due to gravity. The moment is taken as positive in the clockwise direction as shown in figure 5.

The inclusion of both of these effects in the equilibrium equations of a uniform beam in free harmonic vibration results in the following differential equation:

$$EI\frac{d^4y}{dx^4} + EI\left(\frac{m\omega^2}{A_SG} + \frac{m\omega^2}{A_TE}\right)\frac{d^2y}{dx^2} - \left(m\omega^2 - \frac{Im^2\omega^4}{A_TA_SG}\right)y = 0 \qquad (1)$$

where

A_S effective shear-carrying area

A_T effective total cross-sectional area (w/ρ)

w weight of the beam per unit length

m mass of the beam per unit length

ω natural frequency

The terms not containing A_S and A_T are the terms that make up the standard differential equation when the effects of shear stiffness and rotary inertia are neglected. The middle term is composed entirely of secondary effects. The last term contains an interaction effect of the two secondary effects.

Division of the differential equation by $m\omega^2$ and the substitution of the nondimensional coordinate $\xi = \frac{x}{L}$ change equation (1) to

$$\frac{1}{k_B^2}\frac{d^4y}{d\xi^4} + \left(k_S^2 + k_{RI}^2\right)\frac{d^2y}{d\xi^2} - \left(1 - k_B^2k_S^2k_{RI}^2\right)y = 0 \qquad (2)$$

where

$$k_B = \omega\sqrt{\frac{mL^4}{EI}} \qquad (3a)$$

$$k_S = \frac{1}{L}\sqrt{\frac{EI}{A_S G}} \tag{3b}$$

and

$$k_{RI} = \frac{1}{L}\sqrt{\frac{I}{A_T}} \tag{3c}$$

The quantity k_B can be thought of as a frequency constant, as can be seen by transposing equation (3a) to

$$\omega = k_B\sqrt{\frac{EI}{mL^4}} \tag{4}$$

For a given uniform beam the quantity $\sqrt{\frac{EI}{mL^4}}$ is constant; therefore, the natural frequency of the beam is a function of the coefficient k_B.

The solution of the differential equation can be written in the form

$$y = C_1 \cosh k_B\alpha\xi + C_2 \sinh k_B\alpha\xi + C_3 \cos k_B\beta\xi + C_4 \sin k_B\beta\xi \tag{5}$$

where

$$\alpha = \sqrt{\frac{-\left(k_S^2 + k_{RI}^2\right) + \sqrt{\left(k_S^2 - k_{RI}^2\right)^2 + \frac{4}{k_B^2}}}{2}}$$

$$\beta = \sqrt{\frac{\left(k_S^2 + k_{RI}^2\right) + \sqrt{\left(k_S^2 - k_{RI}^2\right)^2 + \frac{4}{k_B^2}}}{2}}$$

and C_1, C_2, C_3, and C_4 are constants of integration.

Boundary conditions.- The boundary conditions for the vibrating-beam problems considered in this paper will be specified at the root and at the tip.

For the conditions at the root, the equations are as follows:

For the cantilever beam the conditions of zero deflection and slope equal to shear strain are expressed by

$$[y]_{\xi=0} = 0 \tag{6}$$

$$\frac{1}{L}\left[\frac{dy}{d\xi}\right]_{\xi=0} = \frac{1}{A_S G}[V]_{\xi=0} = -\frac{1}{L}\,\frac{k_S{}^2 k_B{}^2}{1 - k_S{}^2 k_{RI}{}^2 k_B{}^2}\left[\frac{1}{k_B{}^2}\frac{d^3y}{d\xi^3} + \left(k_S{}^2 + k_{RI}{}^2\right)\frac{dy}{d\xi}\right]_{\xi=0} \tag{7}$$

For the symmetrically vibrating free-free beam the conditions of zero shear and, therefore, zero slope are expressed by

$$\left[\frac{dy}{d\xi}\right]_{\xi=0} = 0 \tag{8}$$

$$\left[\frac{1}{k_B{}^2}\frac{d^3y}{d\xi^3} + \left(k_S{}^2 + k_{RI}{}^2\right)\frac{dy}{d\xi}\right]_{\xi=0} = 0 \tag{9}$$

For the antisymmetrically vibrating beam the conditions of zero deflection and zero moment are expressed by

$$[y]_{\xi=0} = 0 \tag{10}$$

$$\left[\frac{d^2y}{d\xi^2}\right]_{\xi=0} = 0 \tag{11}$$

For the conditions at the tip for all beams, the conditions of zero moment and zero shear are expressed by the following equations:

$$\left[\frac{1}{k_B{}^2}\frac{d^2y}{d\xi^2} + k_S{}^2y\right]_{\xi=1} = 0 \qquad (12)$$

$$\left[\frac{1}{k_B{}^2}\frac{d^3y}{d\xi^3} + \left(k_S{}^2 + k_{RI}{}^2\right)\frac{dy}{d\xi}\right]_{\xi=1} = 0 \qquad (13)$$

The complete deflection curve of a beam is obtained by substituting equation (5) in the proper boundary conditions. This substitution results in a set of four homogeneous linear equations in C_1, C_2, C_3, and C_4. In order that solutions other than zero exist (that is, that vibration can occur), the determinant of the coefficients of the C's must be equal to zero.

<u>Frequency equations</u>.- The equations obtained by setting the determinant equal to zero give the relationship between k_S, k_{RI}, and k_B required to determine the natural frequency which is contained in k_B. These equations for each beam treated in this paper are as follows:

For the cantilever beam

$$2 - \frac{k_B\left(k_S{}^2 + k_{RI}{}^2\right)}{\sqrt{1 - k_S{}^2k_{RI}{}^2k_B{}^2}} \sin k_B\beta \sinh k_B\alpha$$

$$+ \left[k_B{}^2\left(k_S{}^2 - k_{RI}{}^2\right)^2 + 2\right] \cos k_B\beta \cosh k_B\alpha = 0 \qquad (14)$$

For the symmetrically vibrating free-free beam

$$\beta\left(\beta^2 - k_S^2\right)\tanh k_B\alpha + \alpha\left(\alpha^2 + k_S^2\right)\tan k_B\beta = 0 \tag{15}$$

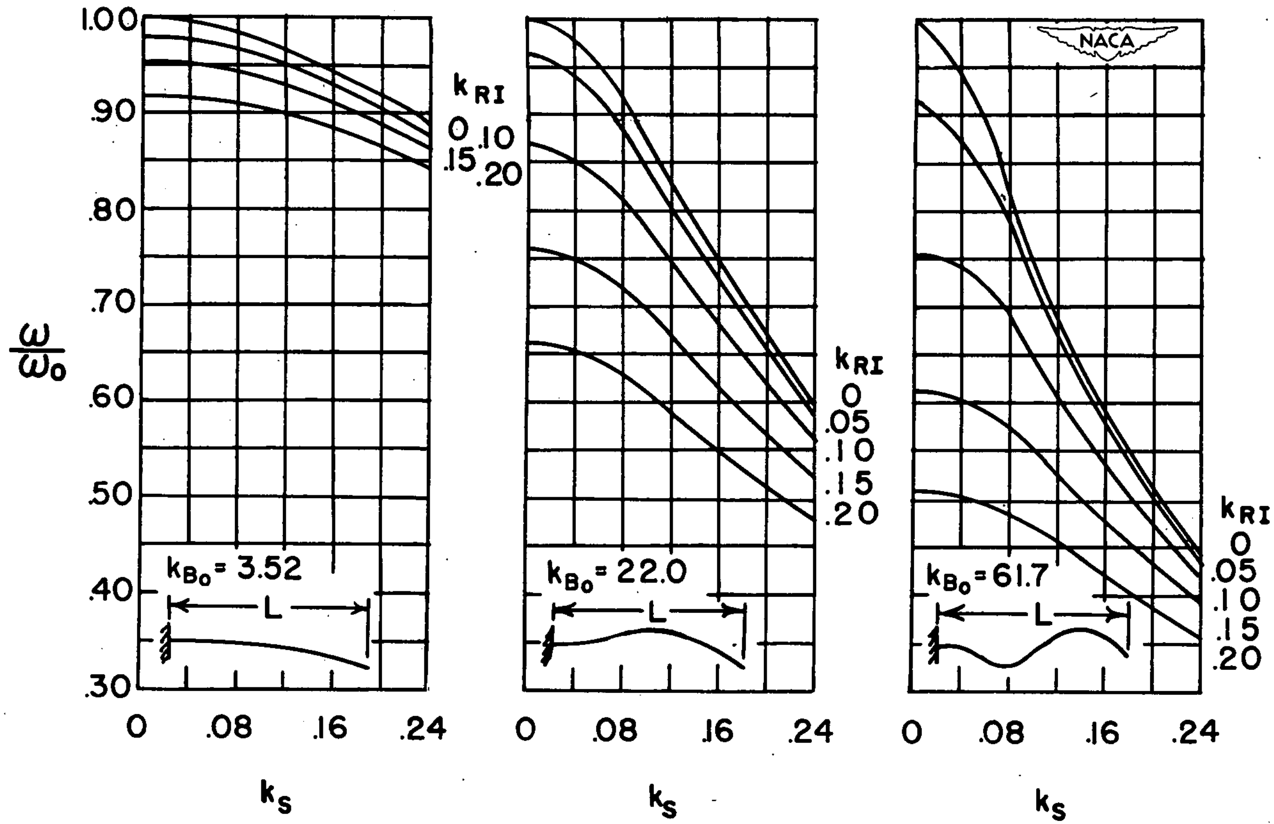
For the antisymmetrically vibrating free-free beam

$$\alpha\left(\alpha^2 + k_S^2\right)\tanh k_B\alpha - \beta\left(\beta^2 - k_S^2\right)\tan k_B\beta = 0 \tag{16}$$

REFERENCES

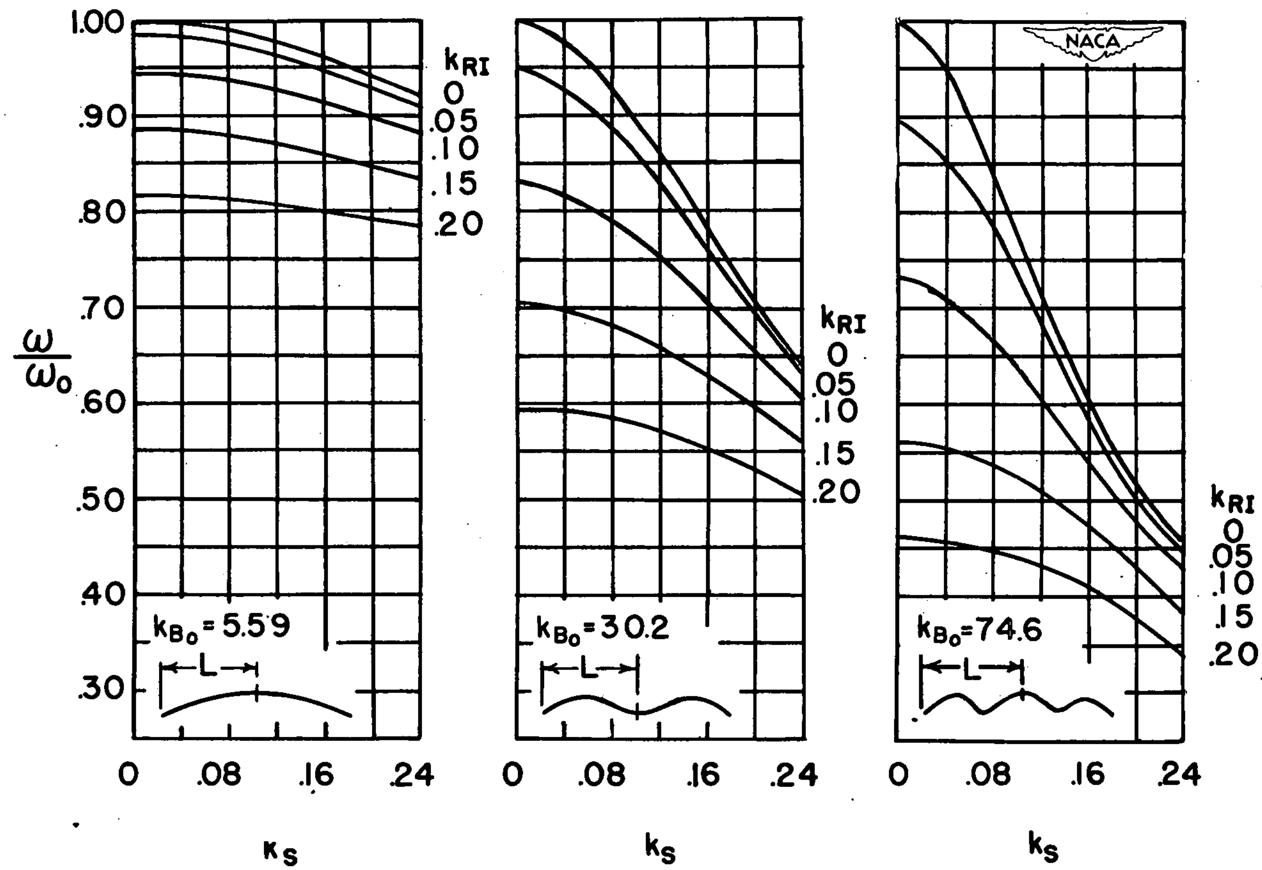
1. Anderson, Roger A., and Houbolt, John C.: Effect of Shear Lag on Bending Vibration of Box Beams. NACA TN 1583, 1948.

2. Timoshenko, S.: Vibration Problems in Engineering. Second ed., D. Van Nostrand Co., Inc., 1937, p. 337.



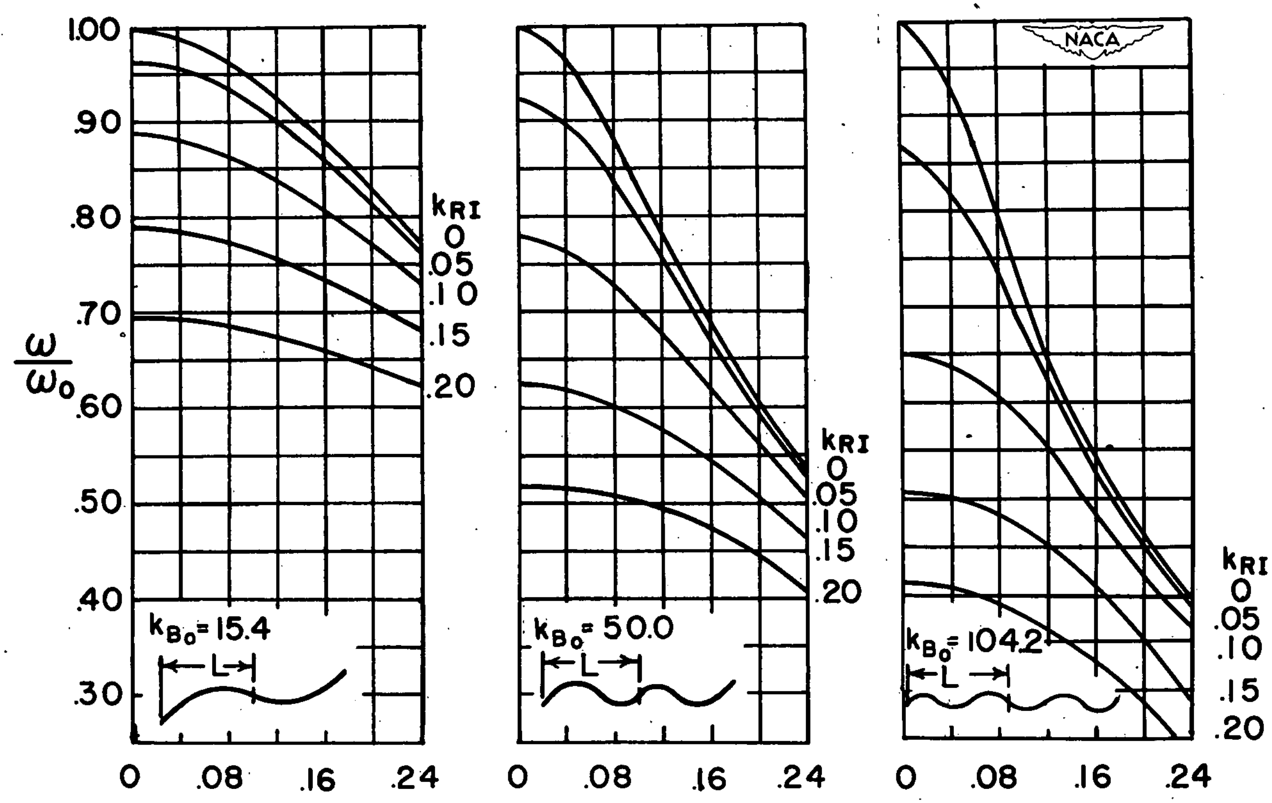
(a) First mode. (b) Second mode. (c) Third mode.

Figure 1.— Change in the natural frequency of a cantilever beam due to shear and rotary inertia. $\omega_o = k_{B_o}\sqrt{\frac{EI}{mL^4}}$; $k_S = \frac{1}{L}\sqrt{\frac{EI}{A_S G}}$.



(a) First mode. (b) Second mode. (c) Third mode.

Figure 2.— Change in the natural frequency of a symmetrically vibrating free-free beam due to shear and rotary inertia. $\omega_o = k_{B_o}\sqrt{\frac{EI}{mL^4}}$; $k_S = \frac{1}{L}\sqrt{\frac{EI}{A_S G}}$.



(a) First mode. (b) Second mode. (c) Third mode.

Figure 3.— Change in the natural frequency of an antisymmetrically vibrating free-free beam due to shear and rotary inertia. $\omega_o = k_{B_o}\sqrt{\frac{EI}{mL^4}}$; $k_S = \frac{1}{L}\sqrt{\frac{EI}{A_S G}}$.

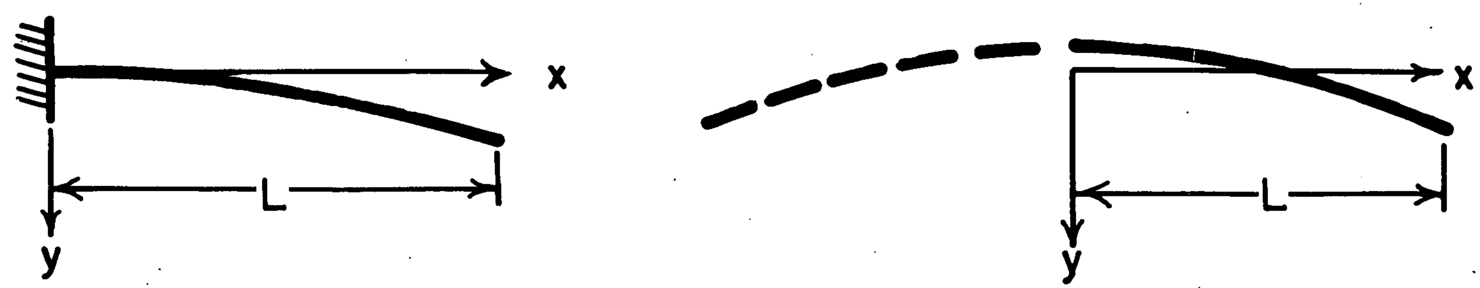


Figure 4.— Coordinate system.

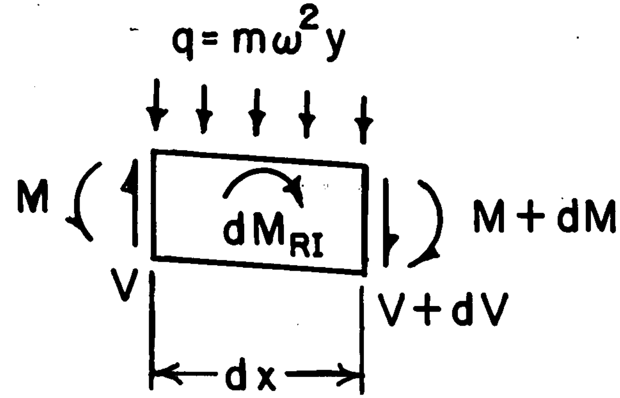


Figure 5.— Forces on a differential element.