NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2065

A TRANSFORMATION THEORY OF THE PARTIAL DIFFERENTIAL EQUATIONS OF GAS DYNAMICS

By Charles Loewner

Syracuse University

Washington
April 1950

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2065

A TRANSFORMATION THEORY OF THE PARTIAL DIFFERENTIAL EQUATIONS OF GAS DYNAMICS

By Charles Loewner

SUMMARY

A transformation theory of systems of partial differential equations is developed which allows the construction of classes of pressure-density relations depending on arbitrarily many parameters for which the equations governing the flow can be transformed into an essentially simpler form, namely, into the Cauchy-Riemann equations in the subsonic region, into the system corresponding to the wave equation in the supersonic case, and finally into that form corresponding to the Tricomi equation in the transonic region. The transition from one system of differential equations to the other is always such that only the solving of ordinary differential equations is required in order to find solutions of the more complicated system from corresponding solutions of the simpler one.

INTRODUCTION

It has already been observed by Chaplygin (reference 1) that in the case of a pressure-density relation of the form $p = -\frac{a}{\rho} + b, a > 0$ the equations describing a steady irrotational flow in the plane can be reduced to the Cauchy-Riemann differential equations. This is achieved by transforming the equations into the hodograph plane, that is, the plane of the velocity components, and using suitable combinations of the latter as independent variables. Since any pressure-density relation can, in smaller regions of the variables involved, be approximated by the foregoing relation, approximation theories can be developed which make use of the Cauchy-Riemann equations. A series of recent papers, by Von Kármán, Tsien, Bers, Gelbart, Bartnoff, and Lin, follow this idea (references 2 to 9).

The investigations taken up in this report originated in the question whether the ideas developed by the Swedish mathematician Baecklund, in his transformation theory of partial differential equations of second order, could, after suitable modifications, be utilized for the study of the equations of compressible fluid flow. The fundamental problem of Baecklund was to find all pairs of surfaces Σ and Σ' in an x,y,z-space and an x',y',z'-space, respectively, and a one-to-one mapping between them such that, for corresponding points, four given equations

$$F_i(x,y,z,p,q,x',y',z',p',q') = 0 \qquad (i = 1,2,3,4) \tag{1}$$

are satisfied. As usual p and q designate the partial derivatives of z with respect to x and y and, similarly, p' and q' the derivatives of z' with respect to x' and y'. It is clear that, for a surface Σ given, in general no surface Σ' can be found which can be mapped onto Σ in such a way that equations (1) are satisfied. In order to find the necessary and sufficient conditions for the proper surface Σ' and mapping of Σ onto Σ' to exist, the following operations, according to Baecklund, must be performed (reference 10):

(a) Considering x' and y' as functions of x and y, each of equations (1) is to be differentiated up to the second order with respect to the two independent variables.

(b) From the equations thus obtained combined with equations (1), all quantities bearing a prime are to be eliminated.

In general the elimination result will consist of two partial differential equations of the third order in $z(x,y)$. They represent the conditions on the surface Σ. If the differentiations of first order in step (a) combined with equations (1) already allow for the elimination of the primed quantities, the resulting equation of second order represents the conditions on the surface Σ sought. Steps (a) and (b) may also be taken with reversed roles of surfaces Σ and Σ' and will lead to one or two partial differential equations in z' as a function of x' and y'.

These considerations show that equations (1) link solutions of two, in general, different systems of partial differential equations with each other. If one system is simpler than the other, assertions about the solutions of the more difficult system may be derived from knowledge of the solutions of the simpler one.

The steady irrotational flow of a compressible fluid may be described by the equations

$$\left.\begin{aligned} u_y - v_x &= 0 \\ (\rho u)_x + (\rho v)_y &= 0 \end{aligned}\right\} \tag{2}$$

where x and y are Cartesian coordinates in the physical plane and u and v the components of the velocity vector. The density ρ is to be thought of, according to Bernoulli's law, as a function of the speed $q = \sqrt{u^2 + v^2}$. The subscripts x and y in system (2) designate the respective differentiations.

Another description of the flow is obtained by introducing the potential function ϕ and the stream function ψ as follows: Equations (2), considered as integrability conditions, show that there exist functions $\phi(x,y)$ and $\psi(x,y)$ such that

$$u = \phi_x \qquad\qquad \rho u = \psi_y$$

$$v = \phi_y \qquad\qquad -\rho v = \psi_x$$

This leads to the equations

$$\left.\begin{aligned} \psi_x &= -\rho\phi_y \\ \psi_y &= \rho\phi_x \end{aligned}\right\} \tag{3}$$

in which the arguments u and v in ρ are to be replaced by ϕ_x and ϕ_y, respectively.

Both systems (2) and (3) are nonlinear in the case of compressible fluid flow because ρ is a variable function of u and v. As was recognized by Chaplygin, systems (2) and (3) take on essentially simpler form if the physical plane is replaced by the hodograph plane; that is, if the velocity components u and v are introduced as independent variables instead of x and y. Systems (2) and (3) then become linear. Each of the systems takes on the form

$$F_1^i\, \xi_u + G_1^i\, \xi_v + F_2^i\, \eta_u + G_2^i\, \eta_v = 0 \qquad (i = 1,2) \tag{4}$$

where ξ and η represent the coordinates x and y in the case of equations (2) and the potential and stream functions ϕ and ψ in the case of equations (3).

The coefficients F_k^i and G_k^i are definite functions of u and v and their forms depend on the equation of state adopted. For an incompressible fluid, equations (4) are identical with the Cauchy-Riemann equations.

Very often it is convenient to introduce coordinates other than u and v in the hodograph plane, for example, polar coordinates, or any other generalized coordinates s and t connected with u and v by a transformation

$$\left.\begin{aligned} s &= s(u,v) \\ t &= t(u,v) \end{aligned}\right\} \tag{5}$$

or

$$\left.\begin{aligned} u &= u(s,t) \\ v &= v(s,t) \end{aligned}\right\} \tag{5a}$$

where both transformation (5) and its inverse, transformation (5a), are continuously differentiable. If such a transformation is applied to equations (4), a system

$$f_1^i\, \xi_s + g_1^i\, \xi_t + f_2^i\, \eta_s + g_2^i\, \eta_t = 0 \qquad (i = 1,2) \tag{6}$$

of the same form as system (4) is obtained with new coefficients f_k^i and g_k^i which are now to be written as functions of the new coordinates s and t.

Following the ideas of Baecklund, the following question may be raised: Is it possible to interrelate two given systems of differential equations of the form of system (6) by a transformation of the Baecklund type? As an analogue to equations (1), a transformation of the "Baecklund type" will be defined here as given by a system of equations of the form

$$G_i\left(s,t,\xi,\eta,\xi_s,\xi_t,\eta_s,\eta_t,s',t',\xi',\eta',\xi'_{s'},\xi'_{t'},\eta'_{s'},\eta'_{t'}\right) = 0 \tag{7}$$

$$(i = 1,2, \ldots,6)$$

A solution of system (7) is represented by two pairs of functions

$$\left.\begin{aligned} \xi &= \xi(s,t) & \xi' &= \xi'(s',t') \\ \eta &= \eta(s,t) & \eta' &= \eta'(s',t') \end{aligned}\right\} \tag{8}$$

having as independent variables s,t and s',t', respectively, and a one-to-one mapping between the s,t- and s',t'-plane such that at corresponding points all equations (7) are satisfied.

If a geometrical interpretation is desired, two spaces P and P', each of four dimensions, with the coordinates (s,t,ξ,η) and (s',t',ξ',η'), respectively, have to be introduced. A pair of functions $\xi = \xi(s,t)$ and $\eta = \eta(s,t)$ represents a two-dimensional surface Σ in P and correspondingly a pair of functions $\xi' = \xi'(s',t')$ and $\eta' = \eta'(s',t')$ represents a surface Σ' in P'. To solve system (7) means to find all pairs of surfaces Σ and Σ' and a one-to-one mapping between them such that the quantities characterizing their tangent planes at corresponding points satisfy relations (7).

That the choice of six equations in system (7) is natural follows from the fact that if s and t are considered as independent variables and ξ,η,ξ',η',s', and t' as dependent variables, the number of equations coincides with the number of unknown functions.

Again, as in the case of equations (1), the following important question presents itself: What surfaces Σ (or Σ') belong to a solution of system (7)? It is to be expected that necessary and sufficient conditions may be found from operations corresponding to those applied to equations (1), that is,

(a) Considering s' and t' as functions of s and t, each of equations (7) is to be differentiated with respect to s and t up to the second order.

(b) All quantities bearing a prime are to be eliminated from the equations thus obtained combined with equations (7).

The result of the elimination will in general consist of four partial differential equations of third order. In particular cases a differentiation of first order in operation (a) suffices to yield necessary and sufficient conditions on Σ so that Σ' should exist, by the elimination of the primed quantities. This is the case with all Baecklund transformations studied and applied in this report.

The essential point in the preceding consideration is again the fact that, by a Baecklund transformation of the type of system (7), two, in general, different systems of partial differential equations containing two independent and two dependent variables are interconnected.

For the purposes of this investigation it is not necessary to study the most general type of Baecklund transformation such as system (7). Since the interest is directed to linear systems of the type of system (6), it is sufficient to consider only such equations (7) which are linear and homogeneous in the quantities ξ, η, ξ', and η' and their derivatives of first order. The further more restrictive assumption will be made that two of equations (7) describe a one-to-one mapping of a domain of the s,t-plane onto a domain of the s',t'-plane or, in other words, represent a transformation of the independent variables given by

$$\left.\begin{aligned} s' &= f(s,t) \\ t' &= g(s,t) \end{aligned}\right\} \tag{9}$$

Since transformations of any quantities resulting from such a coordinate transformation can easily be handled, there is no essential loss of generality if the assumption is henceforth made that equations (9) reduce to

$$\left.\begin{aligned} s' &= s \\ t' &= t \end{aligned}\right\} \tag{10}$$

System (7) can then be replaced by a system of only four equations, linear and homogeneous in ξ, η, ξ', and η', and their derivatives of first order with respect to s and t. The coefficients may be arbitrary functions of s and t satisfying the condition that they be twice continuously differentiable. Only the additional condition, that the four equations can be solved for the four first partial derivatives of ξ and η and also for the corresponding derivatives of ξ' and η', is assumed. Equations (7) can therefore be brought into the form

$$\left.\begin{aligned}
\xi'_s &= \rho_1^1\,\xi_s + \rho_2^1\,\eta_s + \tau_1^1\,\xi_t + \tau_2^1\,\eta_t + \alpha_1^1\,\xi + \alpha_2^1\,\eta + \gamma_1^1\,\xi' + \gamma_2^1\,\eta' \\
\xi'_t &= \sigma_1^1\,\xi_s + \sigma_2^1\,\eta_s + \omega_1^1\,\xi_t + \omega_2^1\,\eta_t + \beta_1^1\,\xi + \beta_2^1\,\eta + \delta_1^1\,\xi' + \delta_2^1\,\eta' \\
\eta'_s &= \rho_1^2\,\xi_s + \rho_2^2\,\eta_s + \tau_1^2\,\xi_t + \tau_2^2\,\eta_t + \alpha_1^2\,\xi + \alpha_2^2\,\eta + \gamma_1^2\,\xi' + \gamma_2^2\,\eta' \\
\eta'_t &= \sigma_1^2\,\xi_s + \sigma_2^2\,\eta_s + \omega_1^2\,\xi_t + \omega_2^2\,\eta_t + \beta_1^2\,\xi + \beta_2^2\,\eta + \delta_1^2\,\xi' + \delta_2^2\,\eta'
\end{aligned}\right\}\quad(11)$$

where the coefficients on the right are all functions of s and t, and the determinant of the coefficient matrix of equations (11) contained in the first four columns on the right is supposed to be different from zero.

Equations (11) can be brought into a simpler form by use of the matrix calculus notation. Set

$$A = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} \qquad B = \begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix} \qquad C = \begin{pmatrix} \gamma_1^1 & \gamma_2^1 \\ \gamma_1^2 & \gamma_2^2 \end{pmatrix}$$

$$R = \begin{pmatrix} \rho_1^1 & \rho_2^1 \\ \rho_1^2 & \rho_2^2 \end{pmatrix} \qquad D = \begin{pmatrix} \delta_1^1 & \delta_2^1 \\ \delta_1^2 & \delta_2^2 \end{pmatrix} \qquad S = \begin{pmatrix} \sigma_1^1 & \sigma_2^1 \\ \sigma_1^2 & \sigma_2^2 \end{pmatrix}$$

$$T = \begin{pmatrix} \tau_1^1 & \tau_2^1 \\ \tau_1^2 & \tau_2^2 \end{pmatrix} \qquad W = \begin{pmatrix} \omega_1^1 & \omega_2^1 \\ \omega_1^2 & \omega_2^2 \end{pmatrix}$$

Further designate the column matrix $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$ by ζ and $\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}$ by ζ'. If differentiation of a matrix means, as usual, differentiation of each of its elements, equations (11) can then be written in the simple form

$$\left.\begin{aligned} \zeta'_s &= R\zeta_s + T\zeta_t + A\zeta + C\zeta' \\ \zeta'_t &= S\zeta_s + W\zeta_t + B\zeta + D\zeta' \end{aligned}\right\} \tag{12}$$

A systematic study will now be made of the possibilities of transforming systems of equations of the form of system (6) into each other by transformations of the type described by equations (12), and applications to compressible fluid flow will be made.

This investigation was conducted at Syracuse University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

The author is indebted to Messrs. Helmut Aulbach and Samuel Schechter for valuable assistance.

SYMBOLS

a,b,c	constants
$\alpha^i_k, \beta^i_k, \gamma^i_k, \delta^i_k,$ $\rho^i_k, \sigma^i_k, \tau^i_k, \omega^i_k$	functions of s and t defined by equations (11)
A,B,C,D,R,S,T,W	matrices having the previous functions as respective elements
F_i,G_i	functions defined by equations (1) and (7), respectively
H	matrix defined by equation (14)
I	unit matrix

K	constant matrix
$k(s),k_1(s)$	functions defined in theorem 3 (appendix)
$\bar{k}(\sigma)$	majorant function used in theorem 1 (appendix)
M	Mach number
p	pressure
q	speed
s,t,s^*,t^*	curvilinear coordinates in hodograph plane
u,v	velocity components
x,y	Cartesian coordinates in physical plane
W^{-1}	inverse of matrix W
W^*	"adjoint" of matrix W
α,β	constants
γ	exponent in pressure-density relation
η_k^i	elements of the matrix H
$\delta^i,\beta^i,\omega^i,\eta^i$	functions defined by equations (35a)
ϕ	velocity potential
ψ	stream function
ρ	density
σ	function defined by equation (87)
τ	variable defined by equation (80)
θ	angle between velocity vector and x-axis
κ	auxiliary variable defined by relation (74)
ζ,ζ'	column vectors $\begin{pmatrix}\xi\\ \eta\end{pmatrix}$, $\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}$, respectively
ω	function defined by equations (49)

$\omega^*(s^*)$	function defined by formula (61)
$\tau(M)$	trace of matrix M
$\omega_1,\omega_2,\omega_3$	functions of the form given by equation (111) which approximate ω
ω^*_1,ω^*_2	functions which approximate ω^*
s	as subscript, partial derivative with respect to s; any other variable used as subscript, partial derivative with respect to that variable

ANALYSIS

1. Baecklund Transformations Connecting Two Systems of Partial-Differential Equations of First Order

A steady irrotational flow in the plane is described by a system of differential equations of the form of system (6). A study of the application of Baecklund transformations to such systems must start with answering the following fundamental question: Which of the Baecklund transformations of the form of equations (12) transform two systems of partial differential equations of the form of system (6) into each other? A complete answer will be given in this section, and it will appear in the form of a system of partial differential equations in the matrix coefficients in equations (12).

In order to simplify the writing, system (6) will always be written in a matrix form. After setting

$$\begin{pmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{pmatrix} = F \qquad\qquad \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix} = G$$

system (6) may be written in the form

$$F\zeta_s + G\zeta_t = 0 \qquad (13)$$

(Henceforth, a system of equations written in matrix form as a single relation will also be referred to as an equation.) Without loss of generality it may be assumed that the determinant of F is never zero. As can be easily seen, this means that a line $t = \text{Constant}$ can never be touched by a characteristic curve of equation (13). This can always be achieved by a suitable choice of the coordinates s and t. Equation (13) can now be written in the form

$$\zeta_s = H\zeta_t \tag{14}$$

where $H = -F^{-1}G$ represents a matrix function of s and t.

In order to find the conditions ζ must satisfy if it is to belong to a solution of system (12), the latter equations must be differentiated, and the primed quantities eliminated. If the first equation is differentiated with respect to t and the second with respect to s, subtraction of the ensuing equations yields

$$\begin{aligned} 0 = {} & S\zeta_{ss} + (W - R)\zeta_{st} - T\zeta_{tt} + \\ & (S_s - R_t + B)\zeta_s + (W_s - T_t - A)\zeta_t + (B_s - A_t)\zeta + \\ & D\zeta'_s - C\zeta'_t + \left(D_s - C_t\right)\zeta' \end{aligned} \tag{15}$$

If, further, the expressions for ζ'_s and ζ'_t from equations (12) are substituted in equation (15), the resulting equation is

$$\begin{aligned} 0 = {} & S\zeta_{ss} + (W - R)\zeta_{st} - T\zeta_{tt} + \\ & \left(S_s - R_t + B + DR - CS\right)\zeta_s + \left(W_s - T_t - A + DT - CW\right)\zeta_t + \\ & \left(DA - CB + B_s - A_t\right)\zeta + \left(DC - CD + D_s - C_t\right)\zeta' \end{aligned} \tag{16}$$

If the assumption is introduced that the coefficient of ζ' in equation (16) is identically zero, or

$$D_s - C_t + DC - CD = 0 \tag{17}$$

then equation (16) is already free of primed quantities and therefore represents a condition on ζ. In general, equation (16) is of second order. Since only equations of the form of equation (13) are studied here, the further assumptions that

$$\left.\begin{aligned} W &= R \\ S &= T = 0 \end{aligned}\right\} \tag{18}$$

$$B_s - A_t + DA - CB = 0 \tag{19}$$

will be made. Transformation (12) then reduces to

$$\left.\begin{aligned} \zeta'_s &= W\zeta_s + A\zeta + C\zeta' \\ \zeta'_t &= W\zeta_t + B\zeta + D\zeta' \end{aligned}\right\} \tag{20}$$

and equation (16) to

$$(-W_t + B + DW)\zeta_s + (W_s - A - CW)\zeta_t = 0 \tag{21}$$

The latter procedure leading to equation (21) may be subordinated to the derivation of the integrability conditions of a "Mayer-Lie system" of partial differential equations. (See reference 11.) Indeed, if ζ is considered as given, then equations (20) represent such a system for the determination of ζ'. A Mayer-Lie system expresses all derivatives of first order of the unknown functions in terms of the independent and the dependent variables themselves. The principal theorem of the theory of Mayer-Lie systems applied here asserts that, if equations (17) and (19) are satisfied, equation (21) is not only a necessary but also a sufficient condition for the existence of a ζ' that can be connected with ζ by equations (20). It furthermore asserts that there exists a ζ' having an arbitrarily preassigned initial value ζ'_0 at a given point (s_0,t_0), and ζ' is uniquely determined by ζ'_0. The computation of ζ' requires the solving of a system of ordinary differential equations with given initial conditions.

It is now easy to find further conditions on the coefficient of equations (12) so that also ζ' will satisfy a system of the form of equation (13). The roles of the quantities ζ and ζ' have to be reversed. To attain this, equations (20) have first to be solved for ζ_s and ζ_t. The assumption that this is possible was introduced from the beginning and means, evidently, that the determinant of W is nowhere zero or that W has an inverse, $W' = W^{-1}$. The inversion yields

$$\left.\begin{aligned} \zeta_s &= W'\zeta'_s + A'\zeta' + C'\zeta \\ \zeta_t &= W'\zeta'_t + B'\zeta' + D'\zeta \end{aligned}\right\} \tag{22}$$

with

$$\left.\begin{array}{ll} A' = -W^{-1}C, \; C' = -W^{-1}A & \\ & W' = W^{-1} \\ B' = -W^{-1}D, \; D' = -W^{-1}B & \end{array}\right\} \tag{23}$$

The conditions sought will now be obtained by replacing all quantities in equations (17) and (19) by the corresponding primed quantities. This leads to

$$D'_s - C'_t + D'C' - C'D' = 0 \tag{24}$$

$$B'_s - A'_t + D'A' - C'B' = 0 \tag{25}$$

and the equation satisfied by ζ' is, corresponding to equation (21),

$$\left(-W'_t + B' + D'W'\right)\zeta'_s + \left(W'_s - A' - C'W'\right)\zeta'_t = 0 \tag{26}$$

The final result obtained may be stated as follows: If all the differential equations (17), (19), (24), and (25) in A, B, C, D, and W are satisfied, the Baecklund transformation, equations (20), transforms system (21) into system (26).

If the system of equations consisting of equations (17), (19), (24), and (25) is written in scalar form, the number of unknown scalar functions exceeds the number of equations by four. It is therefore to be expected that, if four of the unknowns are arbitrarily chosen or, more generally, four additional relations between the unknowns are added to the system, there will still exist infinitely many solutions. In particular, it is to be expected that the equation in ζ or ζ' may arbitrarily be prescribed and still be transformable into infinitely many other equations by suitable Baecklund transformations.

Consider the problem of finding all Baecklund transformations such that a preassigned equation

$$\zeta'_s = H\zeta'_t \tag{27}$$

implies equation (26). (Naturally, at all points (s,t) where the coefficients of equation (26) are not both singular, that is, do not have vanishing determinants, equations (26) and (27) express the same relation.) A comparison of equations (27) and (26) shows that then

$$W'_s - A' - C'W' = \left(W'_t - B' - D'W'\right)H \tag{28}$$

This equation must be added to equations (17), (19), (24), and (25). The whole system of equations can be essentially simplified by the following steps: If the second of equations (20) is multiplied by H from the left and then subtracted from the first, the relation

$$0 = W\zeta_s - HW\zeta_t + \left(A - HB\right)\zeta + \left(C - HD\right)\zeta' \tag{29}$$

is obtained. If the coefficients of equation (26) are not both singular, equation (29) will hold for any ζ and ζ' connected by the Baecklund transformation, equations (20). An arbitrary constant can be chosen for ζ, and for a given point (s,t) a corresponding value of ζ' can be preassigned. A substitution into equation (29) then shows that

$$A = HB \quad \text{and} \quad C = HD \tag{30}$$

In the degenerate case where the coefficients of equation (26) are singular, equations (30) will be assumed to be true too. System (29) becomes now

$$\zeta_s = W^{-1}HW\zeta_t \tag{31}$$

Thus, relations (30) and (28) are obtained and are to be added to equations (17), (19), (24), and (25).

It will now be shown that equations (24) and (25) are already consequences of equations (17), (19), (28), and (30). Indeed, equations (23) imply that

$$D'_s = -W'_sB - W'B_s, \quad C'_t = -W'_tA - W'A_t$$

A substitution into the left side L of equation (24) leads to

$$L = D'_s - C'_t + D'C' - C'D' = -W'_sB + W'_tA - W'\left(B_s - A_t\right) + W'BW'A - W'AW'B$$

and by use of relation (19)

$$L = -W'_sB + W'_tA + W'(DA - CB) + W'BW'A - W'AW'B$$

$$= \left(W'_t - B' - D'W'\right)A - \left(W'_s - A' - C'W'\right)B$$

But this expression is zero on account of equation (28) and the first of equations (30). In a similar way it can be verified that equation (25) is also a consequence of equations (17), (19), (28), and (30).

It is easy to eliminate the primed quantities from equation (28) by direct substitution or, more simply, by observing that the new equation may be obtained from equation (28) by dropping the primes from the latter and replacing H by $W^{-1}HW$ which corresponds to an interchange of the roles of ζ and ζ'. This leads to

$$W_s - A - CW = \left(W_t - B - DW\right)W^{-1}HW \tag{28a}$$

Equations (30) allow the elimination of A and C from equations (17), (19), and (28a). If this is done the following final system for B, D, and W is obtained:

$$\left.\begin{aligned} &D_s - (HD)_t + DHD - HD^2 = 0 \\ &B_s - (HB)_t + DHB - HDB = 0 \\ &W_s - HB - HDW = \left(W_t - B - DW\right)W^{-1}HW \end{aligned}\right\} \tag{32}$$

If equations (32) are satisfied, the Baecklund transformation described by equations (20), with A and C defined by equation (30), transforms the equation $\zeta_s = W^{-1}HW\zeta_t$ into the equation $\zeta'_s = H\zeta'_t$.

Some general remarks will now be made about the nature of equations (32): The first equation contains only the unknown D, the second contains the unknown B in addition to D, and finally the third contains all three unknowns D, B, and W. If the equations are solved successively, one of the unknown matrices has to be determined at every step. Since the whole system of equations (32) is of the Cauchy-Kowalewski type and of first order, the general solution will depend on 12 arbitrary functions of 1 variable which arise from the initial conditions. This shows that, in general, each of a manifold of systems of equations depending on 12 arbitrary functions of 1 variable can be transformed into a given system.

A second remark refers to the possibility of generalizing the transformations thus far considered. Transformations (20) do not form a group. This means that, if two such transformations are composed, the resulting transformation will in general not be contained among transformations (20), but it is exactly this fact that allows for the construction of an extended class of transformations which make possible the linking of systems of differential equations not transformable into each other before. By repeating the composition process several times, the class of transformations can be widened more and more and an increasing flexibility can then be achieved.

The differential equations connected with the study of a steady irrotational flow are of a special type. As a consequence, it is not necessary to consider here the most general transformation of the form of equations (20). The proper restrictions to be made on these transformations will be discussed in the following section.

2. Specialized Systems of Differential Equations and Specialized Baecklund Transformations

The systems of differential equations $\zeta_s = H\zeta_t$ occurring in the study of a steady irrotational flow have, after a suitable choice of coordinates s and t in the hodograph plane, the following two properties: (a) The elements in the principal diagonal of the matrix H are both identically zero and (b) H depends only on one of the variables s and t, say, s.

Property (a) expresses the fact that the equations can be derived from a problem in the calculus of variations.

Properties (a) and (b) suggest the consideration of only such of transformations (20) which preserve both properties (a) and (b). Property (b) is certainly not destroyed if the assumption is made that all coefficients in transformation (20) depend only on s. This leads immediately to a remarkable simplification of system (32). The resulting system of ordinary differential equations is

$$\left.\begin{aligned} D_s + DHD - HD^2 &= 0 \\ B_s + DHB - HDB &= 0 \\ W_s - HB - HDW &= -(B + DW)W^{-1}HW \end{aligned}\right\} \tag{33}$$

The subscript s indicates the ordinary derivative with respect to the only independent variable s.

Before conditions are introduced insuring preservation also of property (a), some consequences will be drawn from system (33).

The following notation of matrix theory will frequently be used:

(1) The trace $\tau(M)$ of a matrix $M = \begin{pmatrix} m_1^1 & m_2^1 \\ m_1^2 & m_2^2 \end{pmatrix}$ is defined as the sum of the elements lying in its principal diagonal, that is,

$$\tau(M) = m_1^1 + m_2^2$$

The trace τ is an additive function of M, that is, for any two matrices M_1 and M_2,

$$\tau(M_1 + M_2) = \tau(M_1) + \tau(M_2)$$

and the trace of a scalar multiple λM of M is

$$\tau(\lambda M) = \lambda\tau(M)$$

It is further easily verified that for a product of the matrices M_1 and M_2

$$\tau(M_1M_2) = \tau(M_2M_1)$$

or that

$$\tau(M_1M_2 - M_2M_1) = 0$$

(2) The "adjoint" M^* of a matrix M is defined as the matrix

$$M^* = \begin{pmatrix} m_2^2 & -m_2^1 \\ -m_1^2 & m_1^1 \end{pmatrix}$$

Evidently $MM^* = M^*M = \delta(M)I$, where I represents the unit matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\delta(M) = \begin{vmatrix} m_1^1 & m_2^1 \\ m_1^2 & m_2^2 \end{vmatrix}$$

is the determinant of M. If $\delta(M) \neq 0$, $\frac{M^*}{\delta(M)} = M^{-1}$.

The simple facts of matrix theory reviewed under definitions (1) and (2) will now be used to draw consequences from equations (33). If the trace is taken of the left side of the first of equations (33), the following equation is obtained:

$$\tau(D_s) = \tau\left(HD^2 - DHD\right) = \tau(HD \cdot D - D \cdot HD) = 0$$

or

$$\tau(D_s) = \left[\tau(D)\right]_s = 0$$

This shows that the trace of D is constant.

If the same equation is multiplied, from the right side, by D^* and the trace is then taken, the resulting equation is

$$\tau(D_S D^*) = \delta(D)\tau(HD - DH) = 0$$

However, an easy computation shows that

$$\tau(D_S D^*) = \left(\delta_1^1\right)_s \delta_2^2 - \left(\delta_2^1\right)_s \delta_1^2 - \left(\delta_1^2\right)_s \delta_2^1 + \left(\delta_2^2\right)_s \delta_1^1 = \left[\delta(D)\right]_s$$

This leads to the fact that the determinant of D is also constant.

In a similar way, multiplying the second of equations (33) by B^* and the third by W^* from the right and computing the trace afterward lead, as the reader will easily verify, to the further result that the determinants of B and W are constant too.

If a solution of system (33) is known, a four-parameter family of new solutions may be obtained by leaving D unchanged but replacing B and W by new matrices $\overline{B} = BK$ and $\overline{W} = WK$ with a common constant right-hand matrix factor K, whose determinant $\delta(K) \neq 0$. Indeed, if substitution of $\overline{B}$ and $\overline{W}$ in the second and third of equations (33) is made, the matrix K appears in each term as a common right-hand factor, and the equations obtained are therefore consequences of equations (33).

A final remark concerns some particular solutions of equations (33): If D represents a solution of the first of equations (33) and $\delta(D) \neq 0$, then any scalar multiples of D

$$B = \lambda D \quad \text{and} \quad W = \mu D$$

with constant factors λ and μ, $\mu \neq 0$, satisfy the second and third of equations (33). The proof is immediately obtained by substitution.

The general solution of system (33) depends on 12 arbitrary constants, but only those solutions which guarantee the preservation of property (a) are of interest. By the addition of this new postulate, a subclass of the solutions of system (33) will be obtained. Without trying to characterize the latter completely, a six-parameter family of solutions of this subclass will be constructed by a specialization of the form of the matrices involved.

A simple method of assuring the preservation of property (a) is to assume W to be of the form

$$W = \begin{pmatrix} \omega_1^1 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \tag{34}$$

Indeed, the matrix H of the primed equation $\zeta'_s = H\zeta'_t$ is supposed to have the form

$$H = \begin{pmatrix} 0 & \eta_2^1 \\ \eta_1^2 & 0 \end{pmatrix}$$

and the matrix $W^{-1}HW$ of the transformed equation $\zeta_s = W^{-1}HW\zeta_t$ is then

$$W^{-1}HW = \begin{pmatrix} 0 & \eta_2^1 \dfrac{\omega_2^2}{\omega_1^1} \\ \eta_1^2 \dfrac{\omega_1^1}{\omega_2^2} & 0 \end{pmatrix}$$

which shows that property (a) is preserved.

In order to obtain solutions of system (33) with the matrix W of the form of equation (34), an adjustment of the forms of D and B must be made by setting

$$D = \begin{pmatrix} 0 & \delta_2^1 \\ \delta_1^2 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & \beta_2^1 \\ \beta_1^2 & 0 \end{pmatrix} \tag{35}$$

It will now be proved that system (33) has solutions of the form of equations (34) and (35) with arbitrarily preassigned initial values of the six available matrix elements δ_2^1, δ_1^2, β_2^1, β_1^2, ω_1^1, and ω_2^2.

If the same equation is multiplied, from the right side, by D^* and the trace is then taken, the resulting equation is

$$\tau(D_S D^*) = \delta(D)\tau(HD - DH) = 0$$

However, an easy computation shows that

$$\tau(D_S D^*) = \left(\delta_1^1\right)_s \delta_2^2 - \left(\delta_2^1\right)_s \delta_1^2 - \left(\delta_1^2\right)_s \delta_2^1 + \left(\delta_2^2\right)_s \delta_1^1 = \left[\delta(D)\right]_s$$

This leads to the fact that the determinant of D is also constant.

In a similar way, multiplying the second of equations (33) by B^* and the third by W^* from the right and computing the trace afterward lead, as the reader will easily verify, to the further result that the determinants of B and W are constant too.

If a solution of system (33) is known, a four-parameter family of new solutions may be obtained by leaving D unchanged but replacing B and W by new matrices $\overline{B} = BK$ and $\overline{W} = WK$ with a common constant right-hand matrix factor K, whose determinant $\delta(K) \neq 0$. Indeed, if substitution of $\overline{B}$ and $\overline{W}$ in the second and third of equations (33) is made, the matrix K appears in each term as a common right-hand factor, and the equations obtained are therefore consequences of equations (33).

A final remark concerns some particular solutions of equations (33): If D represents a solution of the first of equations (33) and $\delta(D) \neq 0$, then any scalar multiples of D

$$B = \lambda D \quad \text{and} \quad W = \mu D$$

with constant factors λ and μ, $\mu \neq 0$, satisfy the second and third of equations (33). The proof is immediately obtained by substitution.

The general solution of system (33) depends on 12 arbitrary constants, but only those solutions which guarantee the preservation of property (a) are of interest. By the addition of this new postulate, a subclass of the solutions of system (33) will be obtained. Without trying to characterize the latter completely, a six-parameter family of solutions of this subclass will be constructed by a specialization of the form of the matrices involved.

A simple method of assuring the preservation of property (a) is to assume W to be of the form

$$W = \begin{pmatrix} \omega_1^1 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \tag{34}$$

Indeed, the matrix H of the primed equation $\zeta'_s = H\zeta'_t$ is supposed to have the form

$$H = \begin{pmatrix} 0 & \eta_2^1 \\ \eta_1^2 & 0 \end{pmatrix}$$

and the matrix $W^{-1}HW$ of the transformed equation $\zeta_s = W^{-1}HW\zeta_t$ is then

$$W^{-1}HW = \begin{pmatrix} 0 & \eta_2^1 \dfrac{\omega_2^2}{\omega_1^1} \\ \eta_1^2 \dfrac{\omega_1^1}{\omega_2^2} & 0 \end{pmatrix}$$

which shows that property (a) is preserved.

In order to obtain solutions of system (33) with the matrix W of the form of equation (34), an adjustment of the forms of D and B must be made by setting

$$D = \begin{pmatrix} 0 & \delta_2^1 \\ \delta_1^2 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & \beta_2^1 \\ \beta_1^2 & 0 \end{pmatrix} \tag{35}$$

It will now be proved that system (33) has solutions of the form of equations (34) and (35) with arbitrarily preassigned initial values of the six available matrix elements δ_2^1, δ_1^2, β_2^1, β_1^2, ω_1^1, and ω_2^2.

In order to simplify the necessary computations the following abbreviations will be used:

$$\left.\begin{matrix} \delta_2^1 = \delta^1 & \beta_2^1 = \beta^1 & \eta_2^1 = \eta^1 & \omega_1^1 = \omega^1 \\ \delta_1^2 = \delta^2 & \beta_1^2 = \beta^2 & \eta_1^2 = \eta^2 & \omega_2^2 = \omega^2 \end{matrix}\right\} \tag{35a}$$

The first equation of system (33) will now be written in an explicit form. A simple computation shows that

$$DHD = \begin{pmatrix} 0 & \eta^2(\delta^1)^2 \\ \eta^1(\delta^2)^2 & 0 \end{pmatrix} \qquad HD^2 = \begin{pmatrix} 0 & \eta^1\delta^1\delta^2 \\ \eta^2\delta^1\delta^2 & 0 \end{pmatrix}$$

and, since

$$D_s = \begin{pmatrix} 0 & (\delta^1)_s \\ (\delta^2)_s & 0 \end{pmatrix}$$

the first of equations (33) is equivalent to the pair of scalar equations

$$\left.\begin{aligned} (\delta^1)_s + \eta^2(\delta^1)^2 - \eta^1\delta^1\delta^2 = 0 \\ (\delta^2)_s + \eta^1(\delta^2)^2 - \eta^2\delta^1\delta^2 = 0 \end{aligned}\right\} \tag{36}$$

The integration of the system of differential equations, equations (36), can be reduced to the integration of only one equation with one unknown function. As was observed before, the determinant of D, given here by $\delta(D) = -\delta^1\delta^2$, is constant. This can be verified again by multiplying the first of equations (36) by δ^2 and the second by δ^1 and adding the resulting equations. This leads to $(\delta^1\delta^2)_s = 0$, which shows the constancy of $\delta(D) = -\delta^1\delta^2$. If the abbreviation $-\delta^1\delta^2 = a$ is used, the first equation of system (36) becomes

$$\left(\delta^1\right)_s + \eta^2\left(\delta^1\right)^2 + a\eta^1 = 0 \tag{37}$$

This is a differential equation of first order in δ^1 alone and is of the Riccati type. If $a \neq 0$, the second of equations (36) is automatically satisfied by $\delta^2 = \frac{-a}{\delta^1}$. If $a = 0$ but $\delta^1 \neq 0$, then $\delta^2 = 0$. All this can easily be verified by the reader.

After equation (37) has been integrated, β^1 and β^2 are to be determined from the second equation of system (33). A simple computation yields

$$DHB - HDB = \begin{pmatrix} 0 & \left(\delta^1\eta^2 - \delta^2\eta^1\right)\beta^1 \\ -\left(\delta^1\eta^2 - \delta^2\eta^1\right)\beta^2 & 0 \end{pmatrix}$$

and the equations sought are therefore

$$\left.\begin{aligned} \left(\beta^1\right)_s + \left(\delta^1\eta^2 - \delta^2\eta^1\right)\beta^1 = 0 \\ \left(\beta^2\right)_s - \left(\delta^1\eta^2 - \delta^2\eta^1\right)\beta^2 = 0 \end{aligned}\right\} \tag{38}$$

A comparison of system (38) with system (36) shows that in the general case where $a \neq 0$, that is, where both δ^1 and δ^2 are not zero, β^1 and δ^1 and also β^2 and δ^2 differ only by a constant factor, that is,

$$\left.\begin{aligned} \beta^1 = \delta^1 c^1 \\ \beta^2 = \delta^2 c^2 \end{aligned} \quad \left(c^1, c^2, \text{Constant}\right)\right\} \tag{39}$$

This is in agreement with a result obtained before, which showed that in general, if $\delta(D) \neq 0$, B is obtained from D by multiplying the latter by a constant matrix from the right. Indeed, system (39) can be expressed in the form

$$B = D\begin{pmatrix} c^2 & 0 \\ 0 & c^1 \end{pmatrix}$$

In the degenerate case where $a = 0$ and, for example, $\delta^1 = 0$ but $\delta^2 \neq 0$, equations (39) have to be replaced by

$$\left.\begin{aligned} \beta^1 &= \frac{c^1}{\delta^2} \\ & \qquad\qquad (c^1,\ c^2 \text{ constant}) \\ \beta^2 &= \delta^2 c^2 \end{aligned}\right\} \tag{39a}$$

If δ^1 and δ^2 are both zero, equations (39) are to be replaced by

$$\left.\begin{aligned} \beta^1 &= c^1 \\ & \qquad\qquad (c^1,\ c^2 \text{ constant}) \\ \beta^2 &= c^2 \end{aligned}\right\} \tag{39b}$$

as can be easily verified. The essential result is that, once the δ's are known, the β's can be written down immediately without any integration.

The differential equations in ω^1 and ω^2 will now be derived. The terms occurring in the third of equations (33) are

$$HB = \begin{pmatrix} 0 & \eta^1 \\ \eta^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta^1 \\ \beta^2 & 0 \end{pmatrix} = \begin{pmatrix} \eta^1\beta^2 & 0 \\ 0 & \eta^2\beta^1 \end{pmatrix}$$

$$HDW = \begin{pmatrix} 0 & \eta^1 \\ \eta^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \delta^1 \\ \delta^2 & 0 \end{pmatrix}\begin{pmatrix} \omega^1 & 0 \\ 0 & \omega^2 \end{pmatrix} = \begin{pmatrix} \eta^1\delta^2\omega^1 & 0 \\ 0 & \eta^2\delta^1\omega^2 \end{pmatrix}$$

$$W^{-1}HW = \begin{pmatrix} \frac{1}{\omega^1} & 0 \\ 0 & \frac{1}{\omega^2} \end{pmatrix}\begin{pmatrix} 0 & \eta^1 \\ \eta^2 & 0 \end{pmatrix}\begin{pmatrix} \omega^1 & 0 \\ 0 & \omega^2 \end{pmatrix} = \begin{pmatrix} 0 & \eta^1 \frac{\omega^2}{\omega^1} \\ \eta^2 \frac{\omega^1}{\omega^2} & 0 \end{pmatrix}$$

$$B\left(W^{-1}HW\right) = \begin{pmatrix} 0 & \beta^1 \\ \beta^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \eta^1 \frac{\omega^2}{\omega^1} \\ \eta^2 \frac{\omega^1}{\omega^2} & 0 \end{pmatrix} = \begin{pmatrix} \beta^1\eta^2 \frac{\omega^1}{\omega^2} & 0 \\ 0 & \beta^2\eta^1 \frac{\omega^2}{\omega^1} \end{pmatrix}$$

$$DHW = \begin{pmatrix} 0 & \delta^1 \\ \delta^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \eta^1 \\ \eta^2 & 0 \end{pmatrix}\begin{pmatrix} \omega^1 & 0 \\ 0 & \omega^2 \end{pmatrix} = \begin{pmatrix} \delta^1\eta^2\omega^1 & 0 \\ 0 & \delta^2\eta^1\omega^2 \end{pmatrix}$$

The differential equations are therefore given by

$$\left.\begin{aligned} \left(\omega^1\right)_s - \eta^1\beta^2 - \eta^1\delta^2\omega^1 + \beta^1\eta^2 \frac{\omega^1}{\omega^2} + \delta^1\eta^2\omega^1 = 0 \\ \left(\omega^2\right)_s - \eta^2\beta^1 - \eta^2\delta^1\omega^2 + \beta^2\eta^1 \frac{\omega^2}{\omega^1} + \delta^2\eta^1\omega^2 = 0 \end{aligned}\right\} \qquad (40)$$

As before, the system of equations (40) can be reduced to one equation of the Riccati type in ω^1 or ω^2. As was found before, the determinant $\delta(W) = \omega^1\omega^2$ is constant. This can be verified immediately by multiplying the first of equations (40) by ω^2 and the second by ω^1 and adding. The result is that $\omega^1_s\,\omega^2 + \omega^2_s\,\omega^1 = 0$ or that $\left(\omega^1\omega^2\right)_s = 0$ which implies that $\omega^1\omega^2 = b$ is a constant. The constant b is different from zero because it was always assumed that W is nonsingular. Replacing ω^2 in the first of equations (40) by $\dfrac{b}{\omega^1}$ yields

$$\left(\omega^1\right)_s + \frac{\beta^1\eta^2}{b}\left(\omega^1\right)^2 + \left(\delta^1\eta^2 - \delta^2\eta^1\right)\omega^1 - \eta^1\beta^2 = 0 \tag{41}$$

The second of equations (40) is an automatic consequence of equation (41). This may be verified by replacing ω^2 by $\omega^2 = \dfrac{b}{\omega^1}$ in the second equation of system (40). Equation (41) is of the Riccati type as was asserted before.

3. Equations Transformable into the Cauchy-Riemann Equations or into Those Corresponding to the Wave or Tricomi Equation

The essential objective of this investigation is to find equations of state, or the corresponding density-speed relations, which lead to equations transformable into a well-known canonical form. In view of this, it is of particular interest to study those equations which can be transformed into the Cauchy-Riemann equations or into those corresponding to the wave or Tricomi equation by transformations discussed in the preceding section. Corresponding to each of these three problems, the matrix H of equation (27) becomes, respectively,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -s & 0 \end{pmatrix} \tag{42}$$

In each case it is required to solve equations (36), (38), and (40), where the values of η^1 and η^2 from the chosen canonical form have been substituted. Since all of equations (36), (38), and (40) are of the first order, their general solution depends on six arbitrary constants or, in other words, there exists a six-parameter family of transformations leading to the given canonical form. It would be incorrect to conclude, however, that the equations transformable into the given canonical form also constitute a six-parameter family. Indeed it was already observed that if B and W are multiplied by the same constant matrix from the right, the resulting matrices together with the unaltered D again represent a solution of equations (36), (38), and (40). In particular this matrix factor may be chosen as a constant scalar multiple cI of the unit matrix I. In this case, however, $(cW)^{-1}H(cW) = W^{-1}HW$ so that the corresponding differential equations are identical. This shows that, in general, a five-parameter family of equations can be transformed into a given canonical form.

Consider first the case where the given canonical form is that of the Cauchy-Riemann equations. The values of η^1 and η^2 are then $\eta^1 = 1$ and $\eta^2 = -1$. Equations (37) and (41), to which equations (36) and (40) were previously reduced, now become

$$\delta^1_s - \left(\delta^1\right)^2 + a = 0 \tag{43}$$

$$\omega^1_s - \frac{\beta^1}{b}\left(\omega^1\right)^2 - \left(\delta^1 + \delta^2\right)\omega^1 - \beta^2 = 0 \tag{44}$$

If the constant a is chosen different from zero, the substitutions into equation (44) of $\delta^2 = -\frac{a}{\delta^1}$, $\beta^1 = \delta^1 c^1$, and $\beta^2 = \delta^2 c^2$ are to be made with c^1 and c^2 as arbitrary constants. In the degenerate case where $a = 0$, the substitutions to be performed are given by equations (39a) or (39b) of the preceding section.

Equation (43) can be solved in terms of elementary functions, but equation (44), it seems, cannot be solved in general in an elementary way. In order to obtain solutions expressible in terms of elementary functions, the restriction that $\delta^1 = \delta^2 = 0$ will now be made. According to equation (39b), β^1 and β^2 are then constants, and equation (44) reduces to

$$\omega^1_s - \lambda^1\left(\omega^1\right)^2 - \lambda^2 = 0 \tag{45}$$

$$\left(\lambda^1 = \frac{\beta^1}{b}\ ,\ \lambda^2 = \beta^2\right)$$

where λ^1 and λ^2 are arbitrary constants.

Equation (45) can be solved by elementary methods, and the reader will easily verify the following statements:

(a) If λ^1 and λ^2 have different signs, the solutions of equation (45) have the form

$$\omega^1 = A \coth \alpha(s - s_o) \quad \text{or} \quad \omega^1 = A \tanh \alpha(s - s_o) \tag{46}$$

with arbitrary constants A, α, and s_o.

(b) If λ^1 and λ^2 have the same sign, the solutions are of the form

$$\omega^1 = A \tan \alpha(s - s_o) \tag{47}$$

with arbitrary constants A, α, and s_o.

(c) If λ^2 or λ^1 is zero, the solutions have the form

$$\omega^1 = \frac{1}{\alpha s + \beta} \quad \text{or} \quad \omega^1 = \alpha s + \beta \tag{48}$$

with constant values α and β.

The computation of the differential equations $\zeta_s = W^{-1}HW\zeta_t$ which are transformable into the Cauchy-Riemann differential equations follows immediately. A simple computation shows that

$$W^{-1}HW = \begin{pmatrix} \frac{1}{\omega^1} & 0 \\ 0 & \frac{1}{\omega^2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega^1 & 0 \\ 0 & \omega^2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\omega^2}{\omega^1} \\ -\frac{\omega^1}{\omega^2} & 0 \end{pmatrix}$$

and the equations therefore always have the form

$$\left.\begin{aligned} \xi_s &= \omega\eta_t \\ & \qquad\qquad \left(\omega = \frac{\omega^2}{\omega^1}\right) \\ \eta_s &= -\frac{1}{\omega}\xi_t \end{aligned}\right\} \tag{49}$$

Since $\omega^1\omega^2 = b$ is a constant, the function $\omega(s)$ in system (49) takes on one of the following forms:

$$\left.\begin{aligned}
&\omega = C\tanh^2\alpha(s - s_o) \quad \text{or} \quad \omega = C\coth^2\alpha(s - s_o) \\
&\omega = C\tan^2\alpha(s - s_o) \\
&\omega = \pm(\alpha s + \beta)^2 \quad \text{or} \quad \omega = \pm\frac{1}{(\alpha s + \beta)^2}
\end{aligned}\right\} \tag{50}$$

where C in the first two of equations (50) is also an arbitrary constant. Each formula of system (50) gives a three- or two-parameter family of differential equations of the form of equations (49) which can be transformed into the Cauchy-Riemannn differential equations.

Similar considerations lead to differential equations which can be transformed into those connected with the wave equation (where $\eta^1 = \eta^2 = 1$). Since these equations can be computed in a manner completely analogous to the preceding computation, only the final result will be stated:

All equations of the form

$$\left.\begin{aligned}
\xi_s &= \omega\eta_t \\
\eta_s &= \frac{1}{\omega}\xi_t
\end{aligned}\right\} \tag{51}$$

with an ω again represented by one of the functions given by equations (50), can be transformed into equations connected with the wave equation.

The study of equations transformable into a form corresponding to the Tricomi equation (where $\eta^1 = 1$, $\eta^2 = -s$) is of particular interest since it can be utilized in the investigation of transonic flows. As in the previous discussion, only the case $\delta^1 = \delta^2 = 0$ will be considered in more detail. Equation (41) here takes on the following form:

$$\omega^1_s - \lambda^1 s\left(\omega^1\right)^2 - \lambda^2 = 0 \tag{52}$$

$$\left(\lambda^1 = \frac{\beta^1}{b},\ \lambda^2 = \beta^2\right)$$

where λ^1 and λ^2 are arbitrary constants. It seems impossible to express the solutions of equation (52) in terms of elementary functions except when $\lambda^1 = 0$ or $\lambda^2 = 0$.

If $\lambda^1 = 0$, the solutions of equation (52) are given by

$$\omega^1 = \alpha s + \beta \tag{53}$$

where α and β are arbitrary constants.

If $\lambda^2 = 0$ the solutions are given by

$$\omega^1 = \frac{1}{\alpha s^2 + \beta} \tag{54}$$

where α and β are arbitrary constants.

The differential equation $\zeta_s = W^{-1}HW\zeta_t$ which is transformed into equation (27) can here again be computed immediately, and it is seen that

$$W^{-1}HW = \begin{pmatrix} \frac{1}{\omega^1} & 0 \\ 0 & \frac{1}{\omega^2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -s & 0 \end{pmatrix} \begin{pmatrix} \omega^1 & 0 \\ 0 & \omega^2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\omega^2}{\omega^1} \\ -s\frac{\omega^1}{\omega^2} & 0 \end{pmatrix}$$

The equations therefore have the form

$$\left.\begin{aligned} \xi_s &= \omega\eta_t \\ \eta_s &= -\frac{s}{\omega}\xi_t \end{aligned} \quad \left(\omega = \frac{\omega^2}{\omega^1}\right) \right\} \tag{55}$$

where ω may take on one of the forms

$$\left.\begin{aligned} \omega &= \pm \frac{1}{(\alpha s + \beta)^2} \\ \omega &= \pm(\alpha s^2 + \beta)^2 \end{aligned}\right\} \tag{56}$$

with arbitrary constants α and β, according as ω^1 is given by expression (53) or (54).

Equations of the form of equations (55) may be brought into an, at times, more suitable form by a transformation of the independent variables given by

$$\left.\begin{aligned} s^* &= s^*(s) \\ t^* &= t \end{aligned}\right\} \tag{57}$$

This transformation will, if suitably chosen, simplify the first of equations (55) to the form $\xi_{s^*} = \eta_{t^*}$. The transformed equations are

$$\left.\begin{aligned} \xi_{s^*} &= \omega\frac{ds}{ds^*}\eta_{t^*} \\ \eta_{s^*} &= \frac{-s}{\omega}\frac{ds}{ds^*}\xi_{t^*} \end{aligned}\right\} \tag{58}$$

The desired simplification is then achieved if $\omega\frac{ds}{ds^*} = 1$, that is, if

$$s^* = \int \omega(s)\ ds + \text{Constant} \tag{59}$$

Equations (58) then take on the form

$$\left.\begin{aligned} \xi_{s^*} &= \eta_{t^*} \\ \eta_{s^*} &= -\omega^*(s^*)\xi_{t^*} \end{aligned}\right\} \tag{60}$$

where

$$\omega^*(s^*) = \frac{s}{(\omega)^2} \tag{61}$$

If ω is given by the first of equations (56), the resulting transformation is

$$s^* = \pm \frac{s}{\beta(\alpha s + \beta)} + \text{Constant} \tag{62}$$

A further simple computation, which may be left to the reader, yields

$$\omega^*(s^*) = \pm \frac{\beta^6(s^* - s_o^*)}{\left[1 \mp \alpha\beta(s^* - s_o^*)\right]^5} \tag{63}$$

where s_o^* is an arbitrary constant. If ω is given by the second of equations (56), the form obtained for $\omega^*(s^*)$ becomes more complicated. Since the latter form is not used in what follows, it will not be written down.

4. Density-Speed Relations Leading to Systems of Differential Equations Transformable into the Canonical Forms of the Preceding Section

If the linearized equation of state $p = -\frac{a}{\rho} + b$ is used as an approximation to the actual equation of state, then the potential function ϕ and the stream function ψ satisfy the Cauchy-Riemann differential equations in the hodograph plane, after suitable independent variables have been chosen. The results of the preceding sections supply a method for constructing more general equations of state, or the corresponding relations between density ρ and speed q, where a transition to a simple canonical form is made possible. This is achieved by combining the transformations discussed in the preceding sections with suitable simple coordinate transformations in the hodograph plane.

If the polar coordinates q and θ are used in the hodograph plane, the equations in ϕ and ψ are

$$\left.\begin{aligned} \phi_q &= -\frac{1 - M^2}{\rho q}\psi_\theta \\ \phi_\theta &= \frac{q}{\rho}\psi_q \end{aligned}\right\} \tag{64}$$

The local Mach number M and the density ρ are functions of q alone and, according to Bernoulli's law,

$$M^2 = -\frac{q}{\rho}\frac{d\rho}{dq} \tag{65}$$

Equations (64) can be brought into a more suitable form by a coordinate transformation of the form

$$\left.\begin{aligned} s &= s(q) \\ t &= -\theta \end{aligned}\right\} \tag{66}$$

It may be noted here that the change in the sign of θ is introduced only to eliminate the minus sign from the first of equations (64).

The following considerations will first be restricted to the subsonic region ($M < 1$), where equations (64) are of elliptic type. It is well known that equations (64) can then be brought into the symmetric form of equations (49) by a suitable transformation of the form of equations (66). Indeed after such a transformation, equations (64) become

$$\left.\begin{aligned} \phi_s &= \frac{1 - M^2}{\rho q}\frac{dq}{ds}\psi_t \\ \phi_t &= -\frac{q}{\rho}\frac{ds}{dq}\psi_s \end{aligned}\right\} \tag{67}$$

Equations (67) are made symmetric by the choice of a transformation, equations (66), which has the property that

$$\frac{1 - M^2}{\rho q}\frac{dq}{ds} = \frac{q}{\rho}\frac{ds}{dq} \tag{68}$$

This leads to the equation

$$\left(\frac{ds}{dq}\right)^2 = \frac{1 - M^2}{q^2} \tag{69}$$

or

$$\frac{ds}{dq} = \frac{\sqrt{1 - M^2}}{q} = \frac{\sqrt{1 + \frac{q}{\rho}\frac{d\rho}{dq}}}{q} \tag{70}$$

which implies that

$$s = \int \frac{\sqrt{1 - M^2}}{q} dq + \text{Constant} \tag{71}$$

The function $\omega(s)$ which appears in the now symmetrized form of equations (67)

$$\left.\begin{aligned} \phi_s &= \omega(s)\psi_t \\ \psi_s &= -\frac{1}{\omega(s)}\phi_t \end{aligned}\right\} \tag{72}$$

is given by

$$\omega(s) = \frac{\sqrt{1 - M^2}}{\rho} = \frac{q}{\rho}\frac{ds}{dq} \tag{73}$$

where the right-hand term is to be thought of as expressed in terms of s, as given by equation (71).

A combination of these results with those of the preceding section allows the construction of such density-speed relations which lead to equations transformable into the Cauchy-Riemann equations. Each formula in system (50) represents a three-parameter family of functions $\omega(s)$ for which equations (49) can be transformed into the Cauchy-Riemann equations. If the function $\omega(s)$ of equation (73) is now identified with a function of one of the families, equations (50), the equation thus obtained together with equation (70) represents a system of ordinary differential equations of first order in ρ and s as functions of q. By solving this system, with unrestricted values of the involved three parameters, a five-parameter family of pairs of functions $\rho = \rho(q)$ and $s = s(q)$ is obtained. The functions $\rho = \rho(q)$ are then the ones sought. Since the addition of a constant to $s(q)$ (with $\rho(q)$ left unchanged) leads again to a solution of the differential system, a four-parameter family of speed-density relations, which leads to equations transformable into the Cauchy-Riemann equations, is thus obtained.

A constant function is included as a special case of the functions $\omega(s)$ given by system (50). It is to be expected that the functions $\omega(s) = \text{Constant}$ will lead to the density-speed relation which corresponds to the linearized pressure-volume relation. This will now be verified.

If, instead of q,

$$\kappa = \log q \tag{74}$$

is introduced as the independent variable and the constant $\frac{1}{\omega}$ is called c, equation (73) can be written in the form

$$\frac{d\rho}{d\kappa} = \rho\left(\frac{\rho^2}{c^2} - 1\right) \tag{75}$$

The integration of equation (75) by the method of separation of variables leads to the solution

$$\rho = \frac{bc}{\sqrt{q^2 + b^2}} \tag{76}$$

where b is a constant, as the reader will easily verify.

A substitution of function (76) into Bernoulli's equation finally yields

$$p = \frac{-(bc)^2}{\rho} + \text{Constant} \tag{77}$$

which is the linearized pressure-volume relation.

For a general function $\omega(s)$ the integration of the system consisting of equations (70) and (73) can, by the elimination of ρ, be reduced to an integration of a differential equation of second order in $s(q)$ alone. Indeed, equation (73) shows that

$$\rho = \frac{\frac{ds}{d\kappa}}{\omega(s)} \tag{78}$$

which, when substituted into equation (70), leads to

$$\frac{d^2s}{d\kappa^2} = \left(\frac{ds}{d\kappa}\right)^3 + \frac{1}{\omega(s)}\frac{d\omega}{ds}\left(\frac{ds}{d\kappa}\right)^2 - \frac{ds}{d\kappa} \tag{79}$$

after a simple computation. If any solution of equation (79) is substituted into equation (78), a desired function $\rho = \rho(q)$ is obtained.

Equation (79) becomes essentially simpler if the roles of the independent variable κ and the dependent variable s are interchanged. If the abbreviation

$$\tau = \frac{d\kappa}{ds} \tag{80}$$

is introduced into equation (79), a simple computation yields

$$\frac{d\tau}{ds} = \tau^2 - \frac{d \log \omega}{ds}\tau - 1 \tag{81}$$

Any solution of the Riccati type equation, equation (81), gives a possible function $\kappa = \kappa(s)$ when integrated.

The function ω corresponding to the real equation of state has a special analytic character, which is useful to know in the discussion of approximating functions of ω. The pressure-density relation of gas dynamics, upon which $\omega(s)$ depends, is of the form $p = a\rho^{\gamma} + b$, where a, b, and γ are constants and $\gamma \geqq 1$. If dimensionless variables are introduced such that the density and velocity of sound are both 1 at $q = 0$, the following formulas are obtained:

$$\rho = \left(1 - \frac{\gamma - 1}{2}q^2\right)^{\frac{1}{\gamma-1}} \qquad (\gamma \neq 1) \tag{82}$$

$$\rho = e^{-\frac{q^2}{2}} \qquad (\gamma = 1) \tag{83}$$

and

$$M^2 = \frac{q^2}{1 - \frac{\gamma - 1}{2}q^2} \tag{84}$$

In addition, the function ω is given by

$$\omega = \frac{\left(1 - \frac{\gamma + 1}{2}q^2\right)^{1/2}}{\left(1 - \frac{\gamma - 1}{2}q^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \qquad (\gamma \neq 1) \tag{85}$$

and

$$\omega = \sqrt{1 - q^2}\, e^{\frac{q^2}{2}} \qquad (\gamma = 1)$$

These formulas show that all three functions ρ, M^2, and ω are analytic functions of q^2 and can be represented by power series in q^2 which converge in the whole subsonic region $q^2 < \frac{2}{\gamma + 1}$. In particular, the power series for ω begins with the terms

$$\omega = 1 - \frac{\gamma + 1}{8} q^4 + \ldots \tag{86}$$

Formula (71) shows further that if σ is defined as

$$\sigma = e^{2s} \tag{87}$$

then the latter function can also be represented by a power series in q^2. If the constant of integration in equation (71) is suitably chosen, the power series for σ begins with the terms

$$\sigma = q^2 - \frac{q^4}{2} + \ldots \tag{88}$$

From a combination of equations (86) and (88), it follows that ω can be represented by a power series in σ starting with the terms

$$\omega = 1 - \frac{\gamma + 1}{8} \sigma^2 + \ldots \tag{89}$$

The question of how far the function $\omega(s)$ can be approximated by a function of the family, equations (50), can now be discussed. If the approximation is desired for a range of q starting with $q = 0$, the best possible adjustment at $q = 0$ is achieved with a function $\omega_1 = C \tanh^2 \alpha(s - s_o)$, with the following choice of the constants:

$$\left.\begin{aligned} C &= 1 \\ \alpha &= 2 \\ 4e^{-4s_o} &= \frac{\gamma + 1}{8} \end{aligned}\right\} \tag{90}$$

This shows that

$$\omega_1 = \left(\frac{1 - \frac{\gamma + 1}{32} \sigma^2}{1 + \frac{\gamma + 1}{32} \sigma^2} \right)^2 \tag{91}$$

The speed-density relations which correspond to $\omega = \omega_1$ have now to be studied. In order to find these relations, it is necessary that the function ω_1 be substituted for ω in equation (81). If a solution of the resulting differential equation is integrated with respect to s, a possible "$s-\kappa$" relation is obtained. A desired speed-density relation is then obtained by a subsequent substitution into equation (78).

It is convenient here to use the quantity $\sigma = e^{2s}$ of equation (87) as the independent variable instead of s. Equation (81), in which ω has been replaced by ω_1, then takes on the form

$$2\sigma \frac{d\tau}{d\sigma} = \tau^2 + \frac{\frac{\gamma + 1}{2} \sigma^2}{1 - \left(\frac{\gamma + 1}{32} \right)^2 \sigma^4} \tau - 1 \tag{92}$$

That particular solution τ_1 which allows for the best approximation to the "real" speed-density relation, equation (82) or (83), at $q = 0$ is now to be determined from among the infinitely many solutions of equation (92).

It can easily be deduced from equation (88) that the function τ which corresponds to the real equation of state has a power-series expansion in σ starting with the terms

$$\tau = 1 + \frac{1}{2}\sigma + . . . \tag{93}$$

It will be shown in the appendix that equation (92) has one and only one analytic solution in the neighborhood of $\sigma = 0$ which satisfies the initial conditions

$$\left.\begin{aligned} \tau &= 1 \\ \frac{d\tau}{d\sigma} &= c_1 \end{aligned}\right\} \quad \text{for} \quad \sigma = 0 \tag{94}$$

where c_1 is an arbitrary constant.

The particular solution of equation (92) corresponding to $c_1 = \frac{1}{2}$ therefore gives the best possible adaptation to the real equation of state at $q = 0$. The $s-\kappa$ relation is finally obtained by an integration of this τ with respect to s. The constant of integration has to be adjusted such that $\left(\frac{d\sigma}{dq^2}\right)_{q=0} = 1$ since this is the case with the real equation of state as shown by equation (88).

The discussed approximation method is not the only possible one. One may, for instance, introduce the requirement that the approximating function $\omega_1(s)$ coincide with the function ω, which is deduced from the real equation of state, at three preassigned points. Numerical results of the mentioned approximations will be discussed at the end of this section.

The foregoing investigation has so far been restricted to subsonic flows, and will now be extended to the transonic region. The canonical form, equations (49), must now be abandoned since it can arise only for an elliptic-type system of differential equations. This form will be replaced, as is usual for a mixed flow, by a new one for which the second equation of system (67) becomes

$$\phi_t = -\psi_s \tag{95}$$

This is achieved by a transformation, equations (66), which satisfies the condition that

$$\frac{ds}{dq} = \frac{\rho}{q} \tag{96}$$

or

$$s = \int \frac{\rho}{q}\, dq + \text{Constant} \tag{97}$$

The first equation of system (67) takes on the form

$$\phi_s = \omega^*(s)\psi_t \tag{98}$$

with

$$\omega^*(s) = \frac{1 - M^2}{\rho^2} = \frac{1 + \frac{q}{\rho}\frac{d\rho}{dq}}{\rho^2} \tag{99}$$

In the case where the speed-density relation, equation (82), is used, then $\omega^*(s)$ becomes

$$\omega^*(s) = \frac{1 - \frac{\gamma + 1}{2}q^2}{\left(1 - \frac{\gamma - 1}{2}q^2\right)^{\frac{\gamma+1}{\gamma-1}}} \qquad (\gamma \neq 1) \tag{100}$$

or

$$\omega^*(s) = \left(1 - q^2\right)e^{q^2} \qquad (\gamma = 1) \tag{100a}$$

<u>Remark</u>.- A comparison of equations (100) and (85) shows that in the subsonic region and for equal values of q the function ω^* is the square of ω. This relation could also have been obtained by a consideration of invariants of the differential equations under coordinate transformations.

The following procedure is now parallel to that followed in the subsonic region. It is desired to construct speed-density relations which lead to equations transformable into those connected with the Tricomi equation. This is achieved by identifying the function ω^* of equation (99) with a function of formula (63) in which s^* is now replaced by s. Equations (96) and (99) represent then a system of two ordinary differential equations of first order in ρ and s as functions of q or $\kappa = \log q$. Again this system can be reduced to a differential equation of second order in s as a function of κ by substituting $\rho = \frac{ds}{d\kappa}$ from equation (96) into equation (99). The result is

$$\frac{d^2s}{d\kappa^2} + \frac{ds}{d\kappa} - \omega^*(s)\left(\frac{ds}{d\kappa}\right)^3 = 0 \tag{101}$$

This again is simplified by choosing s as the independent variable. If the substitution of $\tau = \frac{d\kappa}{ds}$ is performed as before, equation (101) reduces to

$$\frac{d\tau}{ds} = \tau^2 - \omega^*(s) \tag{102}$$

Should $\bar{\tau} = \frac{1}{\tau}$ be used as the dependent variable, then equation (102) would reduce to

$$\frac{d\bar{\tau}}{ds} = \omega^*\bar{\tau}^2 - 1 \tag{102a}$$

The question of choosing the most suitable approximation function ω^*, call it ω^*_1, from among the functions of formula (63), will now be discussed. An approximation is desired in the neighborhood of the sonic speed $q_s = \sqrt{\frac{2}{\gamma + 1}}$. Without loss of generality it can be assumed that the value of s corresponding to this sonic speed is zero. The constant s_o in formula (63) must then also be set equal to zero. The best approximation of the function $\omega^*(s)$ of formula (100) by $\omega^*_1(s)$ will be achieved if the remaining two constants α and β are so chosen that

$$\left.\begin{aligned} \left(\frac{d\omega^*}{ds}\right)_{s=0} &= \left(\frac{d\omega^*_1}{ds}\right)_{s=0} \\ \left(\frac{d^2\omega^*}{ds^2}\right)_{s=0} &= \left(\frac{d^2\omega^*_1}{ds^2}\right)_{s=0} \end{aligned}\right\} \tag{103}$$

By the use of formulas (100), (97), and (82), a simple computation yields

$$\left.\begin{aligned} \left(\frac{d\omega^*}{ds}\right)_{s=0} &= -2\left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+2}{\gamma-1}} \\ \left(\frac{d^2\omega^*}{ds^2}\right)_{s=0} &= -2(2\gamma + 5)\left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+3}{\gamma-1}} \end{aligned}\right\} \tag{104}$$

A comparison with formula (63) written in the form

$$\omega^*_1 = -\frac{as}{(1 - bs)^5} \tag{105}$$

shows that a and b are

$$\left.\begin{aligned} a &= 2\left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+2}{\gamma-1}} \\ b &= \frac{1}{10}(2\gamma + 5)\left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}} \end{aligned}\right\} \tag{106}$$

After the computation of $\omega^*_1(s)$ has been completed, equation (102) must be solved. The initial condition to be satisfied is obtained from the condition that the relation between s and q and the relation obtained from the real equation of state must agree at the sonic speed $q = q_s$ up to the second order.

Equation (97) shows that

$$\tau = \frac{1}{\rho} = \frac{1}{\left(1 - \frac{\gamma - 1}{2}q^2\right)^{\frac{1}{\gamma-1}}} \tag{107}$$

and the initial condition for equation (102) is obtained by a substitution of $q^2 = {q_s}^2 = \frac{2}{\gamma + 1}$. This yields the condition that

$$\tau = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}} \quad \text{for} \quad s = 0 \tag{108}$$

By an integration of the corresponding solution of equation (102) with respect to s, the function $\kappa = \kappa(s)$ is obtained. The constant of integration must again be adjusted to the real equation of state. This requires that

$$\kappa = \frac{1}{2} \log \frac{2}{\gamma + 1} \quad \text{for} \quad s = 0 \tag{109}$$

The previous considerations contain constructions of approximating speed-density relations in the subsonic and transonic regions leading to simplified equations of motion. A similar procedure would lead to approximation formulas in a pure supersonic region. Transformations which lead to the canonical form connected with the wave equation would then have to be used. Since the discussion in such a case would be absolutely parallel to the one for the subsonic region, it is omitted here.

In order to have a picture of the approximations which can be achieved, numerical computations have been made in the subsonic and transonic cases. The results are given in tables 1 and 2 and illustrated in figures 1 and 2, respectively.

Table 1 contains the tabulation of four functions ω, ω_1, ω_2, and ω_3. The first one is the function given by formula (85) with $\gamma = 1.405$. The variable

$$\sigma_1 = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 - 1}\right)^{\frac{1}{2}\sqrt{\frac{\gamma+1}{\gamma-1}}} \sqrt{\sigma} \tag{110}$$

is chosen as the independent variable instead of σ. The variable σ_1 is normalized by the chosen constant factor so that it ranges in the total subsonic region from 0 to 1. The values of ω are taken from reference 4. All the functions ω_1, ω_2, and ω_3 are approximating functions of ω of the form of the first of equations (50) and all can be written in the form

$$\omega = C\left(\frac{1 - \alpha\sigma_1{}^{\beta}}{1 + \alpha\sigma_1{}^{\beta}}\right)^2 \tag{111}$$

The function ω_1 is the approximating function given by formula (91), and it gives the best possible approximation to ω at $q = 0$. The functions ω_2 and ω_3 are interpolation functions of the form of equation (111) determined by the conditions

$$\omega = \omega_2 \text{ for } \sigma_1 = 0 (M = 0),\ \sigma_1 = 0.63 (M = 0.4702),\text{ and } \sigma_1 = 0.80 (M = 0.6344) \tag{112a}$$

$$\omega = \omega_3 \text{ for } \sigma_1 = 0 (M = 0),\ \sigma_1 = 0.63 (M = 0.4702),\text{ and } \sigma_1 = 0.96 (M = 0.8698) \tag{112b}$$

In the range, for example, from $\sigma_1 = 0$ to $\sigma_1 = 0.80$, which corresponds to a range of M from 0 to 0.6344, the deviation of ω_2 from ω is never greater than one unit of the third decimal place, with the exception of the last entry where the deviation becomes three units of the third decimal place.

Table 2 contains the tabulation of the function ω^* given by formula (100) which occurs in the discussion of the transonic region and the approximation function ω^*_1 given by formula (105), for the value $\gamma = 1.5$. The approximation of ω^* usually used is that given by the linear function $\omega^*_2(s)$ giving the tangent of the ω^*-curve at the zero point which corresponds to the sonic speed. For comparison, all three functions ω^*, ω^*_1, and ω^*_2 are represented in figure 2. It is seen that down to $s = -0.22$, which corresponds to a Mach number of about $M = 0.70$, ω^*_1 gives a far better approximation to ω^* than ω^*_2. Also in the supersonic region up to, for example, $s = 0.11$, which corresponds to a Mach number $M = 1.29$, the function ω^*_1 is strongly superior to ω^*_2.

CONCLUDING REMARKS

The present investigation contains the study of a class of transformations of the Baecklund type which transform systems of partial differential equations into each other where simple point transformations ordinarily fail. Corresponding solutions of the two systems are derived from each other by the solving of a system of ordinary differential equations, a process, which, both from a theoretical and practical point of view, is essentially simpler than solving a system of partial differential equations. An application of this study to gas dynamics leads to a five-parameter family of speed-density relations for which the corresponding equations for the stream and potential functions can be transformed into the Cauchy-Riemann equations or into those connected with the wave or Tricomi equation.

The methods developed allow for a far-reaching extension explained as follows.

The class of transformations studied does not form a group; that is, a transformation obtained by a composition of two such transformations is in general not contained in the original class of transformations. This represents an advantage, since the process of composition can be repeated arbitrarily often, and the class of transformations can be extended more and more. This iteration process allows the construction of a family of speed-density relations, which contain arbitrarily many parameters and which lead to equations transformable into one of the three canonical forms used.

It is of great theoretical and practical interest to determine all systems of partial differential equations whose form can be brought arbitrarily close to one of three canonical forms by such composite transformations.

Syracuse University
Syracuse, N. Y., October 27, 1947

APPENDIX

PROOFS OF THEOREMS

The theorem used in section 4 will be proved in the following generalized form:

<u>Theorem 1</u>. Consider the differential equation

$$2\sigma \frac{d\tau}{d\sigma} = \tau^2 + k(\sigma)\tau - 1 \tag{A1}$$

where the coefficient $k(\sigma)$ is an analytic function of σ in the neighborhood $\sigma = 0$ and has a zero at $\sigma = 0$ of at least the second order. The power-series expansion of $k(\sigma)$ is therefore of the form

$$k(\sigma) = a_2\sigma^2 + a_3\sigma^3 + \ldots \tag{A2}$$

If $k(\sigma)$ satisfies these conditions, then there exists a solution of equation (A1), analytic about $\sigma = 0$, which satisfies the initial conditions

$$\tau = 1, \qquad \left(\frac{d\tau}{d\sigma}\right)_{\sigma=0} = c_1 \tag{A3}$$

where c_1 is arbitrary.

<u>Proof</u>.— The required solution must be expansible in a power series of the form

$$\tau = 1 + c_1\sigma + c_2\sigma^2 + \ldots \tag{A4}$$

By substituting series (A4) into equation (A1) and comparing coefficients of equal powers on both sides of equation (A1), the relation

$$2(n - 1)c_n = \sum_{i=1}^{n-1} c_i c_{n-i} + \sum_{i=1}^{n-1} a_i c_{n-i} + a_n \tag{A5}$$

$$(n = 2, 3, . . .)$$

is obtained, where $a_1 = 0$.

Equation (A5) represents a recurrence formula which allows for a successive computation of all coefficients c_i in series (A4). It will be proved now that series (A4) has a positive radius of convergence. Series (A2) has a majorant of the form

$$\overline{k}(\sigma) = \frac{A\sigma^2}{1 - \alpha\sigma} = \overline{a_2}\sigma^2 + \overline{a_3}\sigma^3 + . . . \tag{A6}$$

having a radius of convergence which can be assumed arbitrarily close to that of series (A2).

The values of the coefficients in series (A6) are

$$\overline{a}_i = A\alpha^{i-2}(i = 2, 3, . . .), \quad \overline{a}_1 = 0 \tag{A7}$$

If in formula (A5) the a_i's are replaced by the $\overline{a}_i$'s and the factor $n - 1$ on the left side is dropped, a modified recurrence formula

$$2\overline{c}_n = \sum_{i=1}^{n-1} \overline{c}_i\overline{c}_{n-i} + \sum_{i=1}^{n-1} \overline{a}_i\overline{c}_{n-i} + \overline{a_n} \qquad (n = 2, 3, . . .) \tag{A8}$$

is obtained, which again determines $\overline{c_n}$ uniquely once $\overline{c_1}$ has been chosen.

If a value for $\overline{c_1}$ is chosen such that

$$|c_1| \leqq \overline{c_1} \tag{A9}$$

then a comparison of formulas (A5) and (A8) shows immediately that

$$|c_n| \leqq \overline{c_n} \qquad (n = 1, 2, \ldots) \tag{A10}$$

It is, however, easy to show that the series

$$\overline{c_1}\sigma + \overline{c_2}\sigma^2 + \ldots \tag{A11}$$

has a positive radius of convergence. Indeed just as the c_n's, given by recurrence formula (A5), are the coefficients of a solution to the differential equation, equation (A1), equation (A8) expresses the requirement that the $\overline{c_n}$'s are the coefficients of a generating function

$$\overline{\tau}(\sigma) \doteq \sum_{n=1}^{\infty} \overline{c_n}\sigma^n \tag{A12}$$

which satisfies the algebraic relation

$$2\overline{\tau} = \overline{\tau}^2 + \frac{A\sigma^2}{1 - \alpha\sigma^2}(1 + \overline{\tau}) \tag{A13}$$

or

$$\overline{\tau}^2 + \left(\frac{A\sigma^2}{1 - \alpha\sigma^2} - 2\right)\overline{\tau} + \frac{A\sigma^2}{1 - \alpha\sigma^2} = 0 \tag{A14}$$

and which vanishes for $\sigma = 0$. This determines $\overline{\tau}$ to be

$$\overline{\tau} = 1 - \frac{A\sigma^2}{2\left(1 - \alpha\sigma^2\right)} - \sqrt{\left[1 - \frac{A\sigma^2}{2\left(1 - \alpha\sigma^2\right)}\right]^2 - \frac{A\sigma^2}{1 - \alpha\sigma^2}} \tag{A15}$$

But function (A15) can be expanded into a power series in σ with a positive radius of convergence. This completes the proof of theorem 1.

It is of interest to know how far the solution $\tau(\sigma)$ of equation (A1) satisfying initial conditions (A3) is represented by the power series, series (A4). A definite answer can be given under the restrictive conditions

$$a_n \geqq 0 \qquad (n = 2, 3, \ldots) \tag{A16}$$

$$c_1 \geqq 0 \tag{A17}$$

It will be shown later that these conditions are fulfilled when equation (81) contains an ω corresponding to the real equation of state, provided that $\gamma \geqq 1$, or when it contains an $\omega = \omega_1$, a function which was used as an approximation to the former and given by equation (91). The following theorem is given in answer to the foregoing question:

> Theorem 2. If the restrictive conditions (A16) and (A17) are satisfied and one knows that the solution τ exists in a range $0 \leqq \sigma < \sigma_o$, then the radius of convergence of series (A4) is at least equal to σ_o.

Proof.- Now recurrence formula (A5) shows immediately that

$$c_n \geqq 0 \qquad (n = 1, 2, 3, \ldots) \tag{A18}$$

An immediate consequence of inequalities (A18) is that if σ_1 represents the radius of convergence of series (A4), all derivatives of $\tau(\sigma)$ in the interval $0 \leqq \sigma < \sigma_1$ are non-negative. An analytic continuation by power series shows further that this is true not only for $0 \leqq \sigma < \sigma_1$ but in the whole interval $0 \leqq \sigma < \sigma_o$.

A theorem of S. Bernstein (reference 12) is now used to complete the proof. Bernstein's theorem states:

> If a function $\tau(\sigma)$ has derivatives of arbitrarily high order and all are non-negative in the interval $0 \leqq \sigma < \sigma_o$ (the function is then said to be absolutely monotonic in the interval) then it is analytic there and can be represented by a power series in σ whose radius of convergence is at least equal to σ_o.

In the special cases considered in section 4, the value of c_1 was equal to $\frac{1}{2}$. Condition (A17) is therefore satisfied. But condition (A16) is also fulfilled. This is expressed in

<u>Theorem 3</u>. Condition (A16) is satisfied both for $k(s) = -\frac{d}{ds}\log\omega$, where ω is represented by formula (85) with $\gamma \geqq 1$ and for $k_1(s) = -\frac{d}{ds}\log\omega_1$ with ω_1 given by formula (91).

<u>Proof</u>.— That the theorem is true for k_1 is immediately seen from the expression for k_1 in equation (92). In order to verify it for $k(s)$, it must be shown that the function k and all its derivatives with respect to σ are non-negative.

Consider first k itself:

$$k = -\frac{d\log\omega}{ds} = -\frac{d\log\omega}{dq^2}\frac{dq^2}{ds} \tag{A19}$$

According to equations (85) and (71),

$$-\frac{d\log\omega}{dq^2} = \frac{\gamma+1}{4}\frac{q^2}{\left(1-\frac{\gamma-1}{2}q^2\right)\left(1-\frac{\gamma+1}{2}q^2\right)} \tag{A20}$$

$$\frac{dq^2}{ds} = 2q^2\sqrt{\frac{1-\frac{\gamma-1}{2}q^2}{1-\frac{\gamma+1}{2}q^2}} \tag{A21}$$

Functions (A20) and (A21) have power-series expansions in q^2 with non-negative coefficients. This is immediately clear for function (A20). In order to prove it for function (A21), consider

$$\log\sqrt{\frac{1-\frac{\gamma-1}{2}q^2}{1-\frac{\gamma+1}{2}q^2}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{\gamma+1}{2}\right)^n - \left(\frac{\gamma-1}{2}\right)^n\right]q^{2n} \tag{A22}$$

All the coefficients of series (A22) are certainly non-negative since $\gamma \geqq 1$, and therefore those of the expansion of function (A21) and those of equation (A19) are also non-negative. Thus the function k is non-negative.

It will now be shown that

$$\frac{dk}{d\sigma} = \frac{dk}{dq^2} \cdot \frac{dq^2}{ds} \frac{1}{2\sigma} \tag{A23}$$

is also non-negative. The first two factors of the right-hand member of equation (A23) have power-series expansions in q^2 with non-negative coefficients. It will now be shown that $\frac{1}{\sigma}$, which is obtained as a function of q^2 from equation (71), has an expansion of the form

$$\frac{1}{\sigma} = \frac{1}{q^2}\left(1 + \gamma_1 q^2 + \gamma_2 q^4 + \ldots\right) \tag{A24}$$

all of whose coefficients γ_n are non-negative. It is sufficient to prove that all the coefficients of the expansion of $\sqrt{\dfrac{1 - \frac{\gamma + 1}{2}q^2}{1 - \frac{\gamma - 1}{2}q^2}}$ are non-positive with the exception of the first. Indeed,

$$\frac{1 - \frac{\gamma + 1}{2}q^2}{1 - \frac{\gamma - 1}{2}q^2} = 1 - \sum_{n=1}^{\infty} \left(\frac{\gamma - 1}{2}\right)^{n-1} q^{2n} = 1 - U$$

where all the coefficients of the expansion for U are non-negative. Now

$$\sqrt{\frac{1 - \frac{\gamma + 1}{2}q^2}{1 - \frac{\gamma - 1}{2}q^2}} = \sqrt{1 - U} = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} U^n$$

evidently involves only non-positive coefficients in q^2 with the exception of the first term 1. By a multiplication of the series

In the special cases considered in section 4, the value of c_1 was equal to $\frac{1}{2}$. Condition (A17) is therefore satisfied. But condition (A16) is also fulfilled. This is expressed in

Theorem 3. Condition (A16) is satisfied both for $k(s) = -\frac{d}{ds} \log \omega$, where ω is represented by formula (85) with $\gamma \geqq 1$ and for $k_1(s) = -\frac{d}{ds} \log \omega_1$ with ω_1 given by formula (91).

Proof.- That the theorem is true for k_1 is immediately seen from the expression for k_1 in equation (92). In order to verify it for $k(s)$, it must be shown that the function k and all its derivatives with respect to σ are non-negative.

Consider first k itself:

$$k = -\frac{d \log \omega}{ds} = -\frac{d \log \omega}{dq^2} \frac{dq^2}{ds} \tag{A19}$$

According to equations (85) and (71),

$$-\frac{d \log \omega}{dq^2} = \frac{\gamma + 1}{4} \frac{q^2}{\left(1 - \frac{\gamma - 1}{2}q^2\right)\left(1 - \frac{\gamma + 1}{2}q^2\right)} \tag{A20}$$

$$\frac{dq^2}{ds} = 2q^2\sqrt{\frac{1 - \frac{\gamma - 1}{2}q^2}{1 - \frac{\gamma + 1}{2}q^2}} \tag{A21}$$

Functions (A20) and (A21) have power-series expansions in q^2 with non-negative coefficients. This is immediately clear for function (A20). In order to prove it for function (A21), consider

$$\log\sqrt{\frac{1 - \frac{\gamma - 1}{2}q^2}{1 - \frac{\gamma + 1}{2}q^2}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\left[\left(\frac{\gamma + 1}{2}\right)^n - \left(\frac{\gamma - 1}{2}\right)^n\right]q^{2n} \tag{A22}$$

All the coefficients of series (A22) are certainly non-negative since $\gamma \geqq 1$, and therefore those of the expansion of function (A21) and those of equation (A19) are also non-negative. Thus the function k is non-negative.

It will now be shown that

$$\frac{dk}{d\sigma} = \frac{dk}{dq^2} \cdot \frac{dq^2}{ds} \frac{1}{2\sigma} \tag{A23}$$

is also non-negative. The first two factors of the right-hand member of equation (A23) have power-series expansions in q^2 with non-negative coefficients. It will now be shown that $\frac{1}{\sigma}$, which is obtained as a function of q^2 from equation (71), has an expansion of the form

$$\frac{1}{\sigma} = \frac{1}{q^2}\left(1 + \gamma_1 q^2 + \gamma_2 q^4 + \ . \ . \ .\right) \tag{A24}$$

all of whose coefficients γ_n are non-negative. It is sufficient to prove that all the coefficients of the expansion of $\sqrt{\dfrac{1 - \frac{\gamma + 1}{2}q^2}{1 - \frac{\gamma - 1}{2}q^2}}$ are non-positive with the exception of the first. Indeed,

$$\frac{1 - \frac{\gamma + 1}{2}q^2}{1 - \frac{\gamma - 1}{2}q^2} = 1 - \sum_{n=1}^{\infty} \left(\frac{\gamma - 1}{2}\right)^{n-1} q^{2n} = 1 - U$$

where all the coefficients of the expansion for U are non-negative. Now

$$\sqrt{\frac{1 - \frac{\gamma + 1}{2}q^2}{1 - \frac{\gamma - 1}{2}q^2}} = \sqrt{1 - U} = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} U^n$$

evidently involves only non-positive coefficients in q^2 with the exception of the first term 1. By a multiplication of the series expansions of the factors in the right-hand member of equations (A23), the result is obtained that $\frac{dk}{d\sigma}$ has an expansion in q^2 with non-negative coefficients and is therefore non-negative.

By a continuation of these considerations in the same manner it can evidently be proved that all derivatives of k with respect to σ have power-series expansions in q^2 with non-negative coefficients and are therefore non-negative.

REFERENCES

1. Chaplygin, S. A.: On Gas Jets. Scientific Memoirs, Moscow Univ., Math. Phys. Sec., vol. 21, 1902, pp. 1–121. (Eng. translation published by Brown Univ., 1944; also NACA TM 1063, 1944.)

2. Von Kármán, Th.: Compressibility Effects in Aerodynamics. Jour. Aero. Sci., vol. 8, no. 9, July 1941, pp. 337–356.

3. Tsien, Hsue-Shen: Two-Dimensional Subsonic Flow of Compressible Fluids. Jour. Aero. Sci., vol. 6, no. 10, Aug. 1939, pp. 399–407.

4. Bers, Lipman: On a Method of Constructing Two-Dimensional Subsonic Compressible Flows around Closed Profiles. NACA TN 969, 1945.

5. Bers, Lipman: On the Circulatory Subsonic Flow of a Compressible Fluid past a Circular Cylinder. NACA TN 970, 1945.

6. Bers, Lipman: Velocity Distribution on Wing Sections of Arbitrary Shape in Compressible Potential Flow. I – Symmetric Flows Obeying the Simplified Density-Speed Relation. NACA TN 1006, 1946.

7. Gelbart, Abe: On Subsonic Compressible Flows by a Method of Correspondence. I – Methods for Obtaining Subsonic Circulatory Compressible Flows about Two-Dimensional Bodies. NACA TN 1170, 1947.

8. Bartnoff, Shepard, and Gelbart, Abe: On Subsonic Compressible Flows by a Method of Correspondence. II – Application of Methods to Studies of Flow with Circulation about a Circular Cylinder. NACA TN 1171, 1947.

9. Lin, C. C.: On an Extension of the Von Kármán-Tsien Method to Two-Dimensional Subsonic Flows with Circulation around Closed Profiles. Quart. Appl. Math., vol. IV, no. 3, Oct. 1946.

10. Darboux, Gaston: Leçons sur la théorie générale des surfaces. T. III. Gauthier-Villars (Paris), 1894, pp. 438–444.

11. Weyl, Hermann: Mathematische Analyse des Raumproblems. J. Springer (Berlin), 1923, pp. 66–68.

12. Widder, David Vernon: The Laplace Transform. Princeton Math. Ser., vol. 6, Princeton Univ. Press, 1941, p. 146.

TABLE 1.– VALUES OF ω, ω_1, ω_2, AND ω_3

[ω given by equation (85), where $\frac{\sigma_1}{1.45} = e^s$, relationship between s and q given by equations (71) and (84), and $\gamma = 1.405$. ω_2 and ω_3 obtained from $\left(\frac{1 - \alpha\sigma_1{}^\beta}{1 + \alpha\sigma_1{}^\beta}\right)^2$ which is an approximation formula of the first of equations (50). Constants α and β evaluated for $\sigma_1 = 0.63$ (Mach number, 0.4702) and $\sigma_1 = 0.80$ (Mach number, 0.6344) in ω_2 and for $\sigma_1 = 0.63$ (Mach number, 0.4702) and $\sigma_1 = 0.96$ (Mach number, 0.8698) in ω_3.]

σ_1	ω	ω_1	ω_2	ω_3
0.02	1.0000	1.0000	1.0000	1.0000

.16	1.0000			
.18	.9999			
.20	.9999			
.22	.9998	1.0000		
.24	.9998	.9996	.9999	
.26	.9997	.9996	.9999	
.28	.9996	.9996	.9998	
.30	.9994	.9996	.9998	.9999
.32	.9992	.9992	.9996	.9999
.34	.9990	.9992	.9994	.9998
.36	.9987	.9988	.9992	.9996
.38	.9984	.9988	.9990	.9996
.40	.9980	.9988	.9986	.9994
.42	.9975	.9980	.9982	.9990
.44	.9968	.9976	.9978	.9988
.46	.9961	.9968	.9972	.9982
.48	.9953	.9964	.9964	.9976
.50	.9944	.9956	.9954	.9968
.52	.9933	.9948	.9942	.9956
.54	.9920	.9940	.9930	.9944
.56	.9905	.9932	.9914	.9926
.58	.9888	.9924	.9894	.9906
.60	.9868	.9912	.9872	.9878
.62	.9847	.9900	.9849	.9947
.64	.9818	.9884	.9819	.9809
.66	.9791	.9872	.9785	.9761
.68	.9749	.9853	.9746	.9706
.70	.9709	.9837	.9702	.9639
.72	.9660	.9817	.9651	.9559
.74	.9602	.9799	.9596	.9465
.76	.9535	.9775	.9532	.9357
.78	.9455	.9752	.9460	.9229
.80	.9350	.9724	.9380	.9082
.82	.9238	.9696	.9291	.8913
.84	.9113	.9667	.9193	.8720
.86	.8948	.9631	.9086	.8503
.88	.8740	.9596	.8966	.8257
.90	.8492	.9561	.8836	.7987
.92	.8153	.9520	.8694	.7681
.94	.7720	.9479	.8540	.7350
.96	.7037	.9436	.8372	.6984
.98	.5891	.9388	.8192	.6595
1.00	.0000	.9339	.7998	.6173

NACA

TABLE 2.– VALUES OF ω^* GIVEN BY FORMULA (100)

AND ω^*_1 GIVEN BY FORMULA (105)

[In formula (100), $s = \int \frac{\rho}{q} dq$; $\gamma = 1.5$. In formula (105), a and b determined by equations (106) for $\gamma = 1.5$.]

s	ω^*	ω^*_1
−0.30	0.8832	0.5821
−.28	.8687	.5955
−.26	.8520	.6072
−.24	.8329	.6165
−.22	.8108	.6227
−.20	.7849	.6250
−.18	.7548	.6223
−.16	.7195	.6132
−.14	.6779	.5962
−.12	.6283	.5690
−.10	.5690	.5292
−.09	.5349	.5037
−.08	.4973	.4738
−.07	.4559	.4389
−.06	.4101	.3986
−.05	.3597	.3522
−.04	.3028	.2989
−.03	.2394	.2380
−.02	.1686	.1686
−.01	.0895	.0897
.00	.0000	.0000
.01	−.1021	−.1016
.02	−.2183	−.2164
.03	−.3503	−.3464
.04	−.5034	−.4930
.05	−.6789	−.6585
.06	−.8853	−.8449
.07	−1.1270	−1.0552
.08	−1.4127	−1.2921
.09	−1.7536	−1.5588
.10	−2.1659	−1.8594
.12	−3.2931	−2.5792
.14	−5.0852	−3.4935
.16	−8.1820	−4.6564
.18	−14.2794	−6.1398
.20	−29.0994	−8.0379

NACA

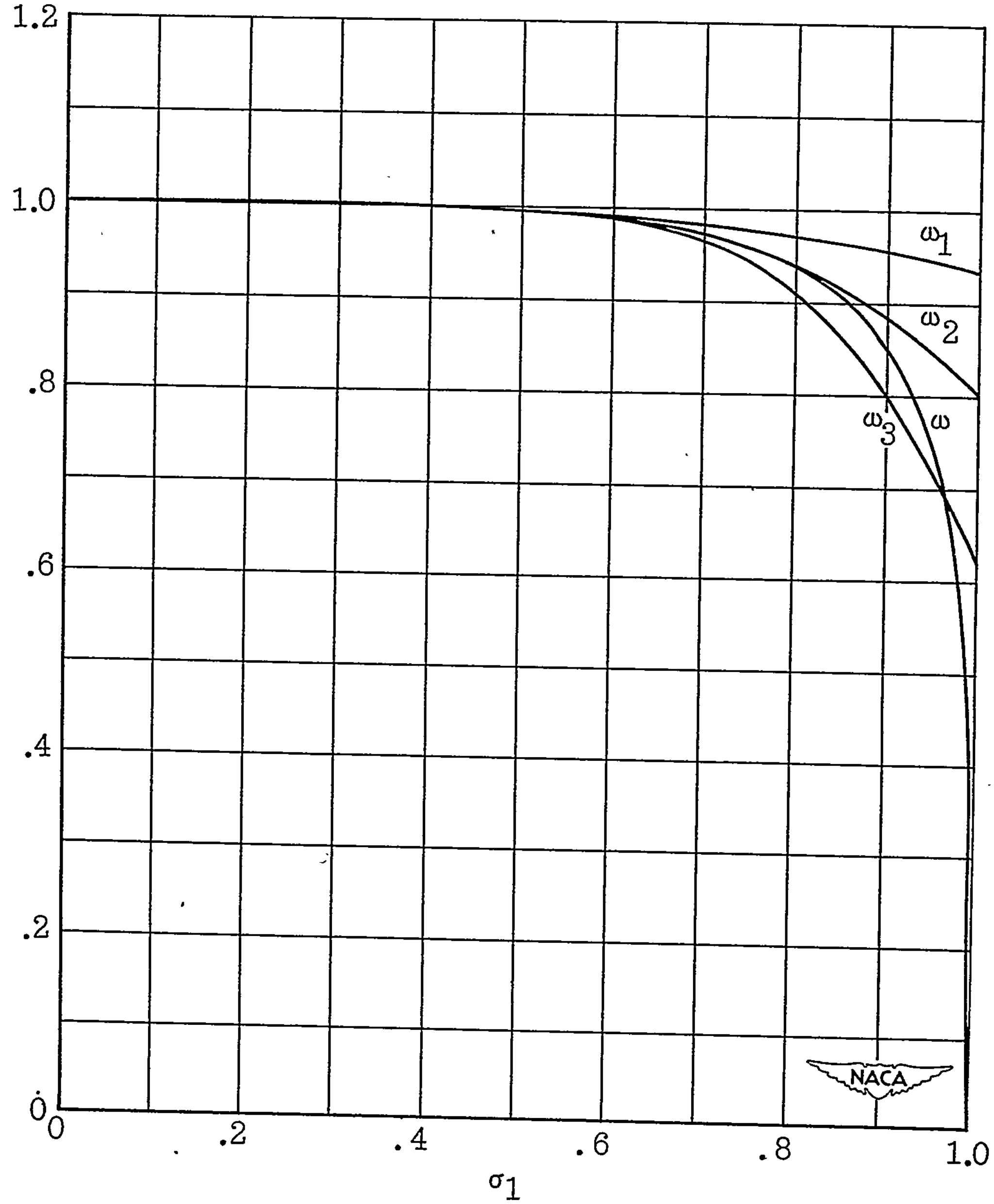

Figure 1.- Results of computations in subsonic case.

$$\omega_1 = \left(\frac{1 - 0.01713\sigma_1{}^4}{1 + 0.01713\sigma_1{}^4}\right)^2; \quad \omega_2 = \left(\frac{1 - 0.0558\sigma_1{}^{5.6}}{1 + 0.0558\sigma_1{}^{5.6}}\right)^2; \quad \omega_3 = \left(\frac{1 - 0.12\sigma_1{}^{7.2}}{1 + 0.12\sigma_1{}^{7.2}}\right)^2.$$

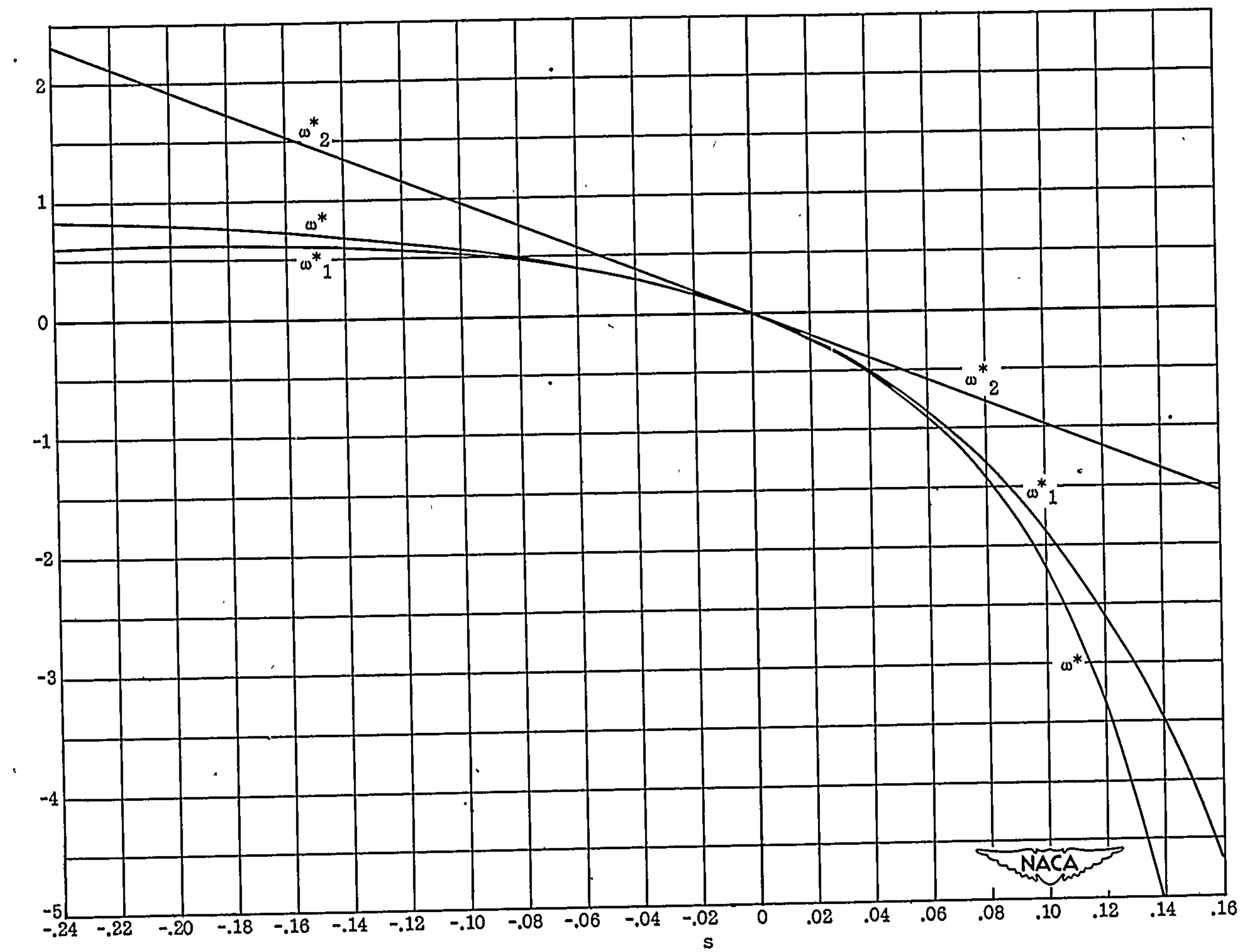

Figure 2.- Results of computations in transonic case. $\omega^*_1 = \dfrac{-9.54s}{(1 - 1.25s)^5}$; $\omega^*_2 = -9.54s$.