NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2282

# AN IMPROVED APPROXIMATE METHOD FOR CALCULATING LIFT DISTRIBUTIONS DUE TO TWIST

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## SUMMARY

A new method is presented for calculating the lift distribution due to twist which gives a much closer approximation than the empirical method of Schrenk (NACA TM 948) and requires about the same amount of computing. The new method, based on lifting-line theory, makes use of the lift distribution due to angle of attack and takes into account the aspect ratio of the wing. The twist may be that of the washout incorporated in the wing design, of aeroelastic deformations, of deflected flaps or ailerons, or of downwash induced by another lifting surface or by the jet boundary of a wind tunnel. Examples are presented for the four main types of twist: symmetrical, antisymmetrical, continuous, and discontinuous. The applicability of the method to swept wings is also presented.

## INTRODUCTION

For many purposes it is desirable to calculate quickly an approximate spanwise lift distribution on a wing due to its twist. The twist may be that of the washout incorporated in the wing design, of aeroelastic deformations, of deflected flaps or ailerons, or of downwash induced by another lifting surface or by the jet boundary of a wind tunnel. One approximation, which has been used extensively in the past, is that of Schrenk (reference 1) modified by fairing out discontinuities as suggested in references 2 and 3. Although Schrenk's approximation may be adequate for some purposes, it is not as accurate as often desired since it does not take into account the aspect ratio of the wing and weights the twist angle arbitrarily according to the spanwise chord distribution. The ability to obtain a closer approximation with about the same amount of computing is therefore desirable. The method presented herein has been developed to provide such an approximation.

Gdaliahu, using lifting-line theory, has rigorously proved in reference 4 that the average angle of attack of a twisted wing can be obtained by a spanwise integration of the local angle of attack weighted according to the additional spanwise load distribution. The average angle multiplied by the three-dimensional lift-curve slope gives the lift coefficient. These results are proved herein in a somewhat different manner. It is also shown herein that the moment of the lift distribution of a twisted wing can be closely approximated by a simple expression. It is then reasoned that, if both the lift and moment given by the approximation agree with the lift and moment obtained from more exact calculations, then the approximate spanwise load distribution must agree fairly closely with the theoretical spanwise load distribution. A few examples given herein show this reasoning to be correct.

SYMBOLS

| | |
|---|---|
| $A$ | aspect ratio |
| $A_n$ | coefficients of trigonometric series for lift distribution |
| $B_n$ | coefficients of trigonometric series for lift distribution per unit angle of attack |
| $\Delta C_L$ | increment in lift coefficient due to twist |
| $C_n$ | coefficients of trigonometric series for lift distribution due to twist |
| $E$ | edge-velocity factor for symmetrical lift distributions $\left(\sqrt{1 + \frac{4 \cos^2\Lambda}{A^2}}\right)$ |
| $E'$ | edge-velocity factor for antisymmetrical lift distributions $\left(\sqrt{1 + \frac{16 \cos^2\Lambda}{A^2}}\right)$ |
| $a$ | three-dimensional lift-curve slope per degree |
| $a_\infty$ | two-dimensional lift-curve slope per degree |
| $a_o$ | section lift-curve slope per degree $\left(\frac{a_\infty \cos\Lambda}{E}\right)$ |
| $b$ | span of wing |
| $c$ | local chord of wing |
| $\bar{c}$ | mean geometric chord $(b/A)$ |

$c_l$ section lift coefficient

$c_{l_a}$ section lift coefficient for additional lift distribution

$c_{l_{a1}}$ section lift coefficient for additional lift distribution at a wing lift coefficient of unity

$c_{l_b}$ section lift coefficient for basic lift distribution

$c_{l_\epsilon}$ section lift coefficient for lift distribution due to twist

$pb/2V$ wing-tip helix angle, radians

$y$ spanwise coordinate

$y^*$ spanwise coordinate at position of discontinuity in twist

$\alpha$ angle of attack, degrees

$\alpha_i$ induced angle of attack, degrees

$\epsilon$ angle of twist, positive for washin, degrees

$\bar{\epsilon}$ average angle of twist, degrees

$\epsilon^*$ magnitude of discontinuity in twist, degrees

$\epsilon_c$ correction factor used in fairing discontinuity in twist, degrees

$\theta$ spanwise coordinate $\left(\cos^{-1} \frac{2y}{b}\right)$

$\Lambda$ sweep angle of wing quarter-chord line, degrees

## DEVELOPMENT OF METHOD

The approximate method developed herein for calculating lift distributions due to twist is based upon the determination of exact or approximate expressions for the area and moment of the lift distributions due to the twist in terms of the twist itself, the average twist, the additional lift distribution due to angle of attack, and the wing lift-curve slope. The lift distribution due to twist is designated as $c_{l_\epsilon}c/b$ which becomes equal to the basic lift distribution $c_{l_b}c/b$ when the wing lift coefficient is equal to zero. In general,

$$\frac{c_{l_\epsilon} c}{b} = \Delta C_L \frac{c_{l_{a1}} c}{b} + \frac{c_{l_b} c}{b} \qquad (1)$$

where

$$\Delta C_L = \frac{A}{2} \int_{-1}^{1} \frac{c_{l_\epsilon} c}{b} \, d\left(\frac{2y}{b}\right) \qquad (2)$$

Reference 4 and appendix A prove that

$$\int_{-1}^{1} \frac{c_{l_\epsilon} c}{b} \, d\left(\frac{2y}{b}\right) = a \int_{-1}^{1} \epsilon \frac{c_{l_{a1}} c}{b} \, d\left(\frac{2y}{b}\right) \qquad (3)$$

so that

$$\Delta C_L = \frac{Aa}{2} \int_{-1}^{1} \epsilon \frac{c_{l_{a1}} c}{b} \, d\left(\frac{2y}{b}\right) \qquad (4)$$

and the average twist

$$\bar{\epsilon} = \frac{\Delta C_L}{a}$$

$$= \frac{A}{2} \int_{-1}^{1} \epsilon \frac{c_{l_{a1}} c}{b} \, d\left(\frac{2y}{b}\right) \qquad (5)$$

## SYMMETRICAL TWIST DISTRIBUTIONS

For symmetrical twist distributions, the integral of equation (1) can be written as

$$\int_{0}^{1} \frac{c_{l_\epsilon} c}{b} \, d\left(\frac{2y}{b}\right) = \int_{0}^{1} \bar{\epsilon} \frac{c_{l_{a1}} c}{b} a \, d\left(\frac{2y}{b}\right) + \int_{0}^{1} (\epsilon - \bar{\epsilon}) \frac{c_{l_{a1}} c}{b} a \, d\left(\frac{2y}{b}\right) \qquad (6)$$

by using the preceding relationships. Since the last integral of equation (6) is the integral of the basic lift distribution, it is equal to zero and may be divided by any arbitrary constant without changing its value. Therefore,

$$\int_0^1 \frac{c_{l_\epsilon} c}{b} d\left(\frac{2y}{b}\right) = \int_0^1 \left(\bar{\epsilon} + \frac{\epsilon - \bar{\epsilon}}{F}\right) \frac{c_{l_{a1}} c}{b} a \, d\left(\frac{2y}{b}\right) \tag{7}$$

From a consideration of the moment of the lift distribution due to twist, appendix A shows that

$$\int_0^1 \frac{c_{l_\epsilon} c}{b} \frac{2y}{b} d\left(\frac{2y}{b}\right) \approx \int_0^1 \left(\bar{\epsilon} + \frac{\epsilon - \bar{\epsilon}}{F}\right) \frac{c_{l_{a1}} c}{b} a \frac{2y}{b} d\left(\frac{2y}{b}\right) \tag{8}$$

where

$$F = 1 + \frac{360}{\pi^2} \frac{a}{A}$$

Since the areas are equal (equation (7)) and the moments of the same areas are approximately equal (equation (8)), it is reasoned that the distributions are approximately equal. Therefore,

$$\frac{c_{l_\epsilon} c}{b} \approx \left(\bar{\epsilon} + \frac{\epsilon - \bar{\epsilon}}{F}\right) \frac{c_{l_{a1}} c}{b} a \tag{9}$$

A comparison of equations (1) and (9) shows that the additional part of the lift distribution due to twist is given by the exact expression

$$\frac{c_{l_a} c}{b} = \bar{\epsilon} \frac{c_{l_{a1}} c}{b} a \tag{10}$$

and the basic part of the lift distribution due to twist is given by the approximate expression

$$\frac{c_{l_b} c}{b} \approx \left(\frac{\epsilon - \bar{\epsilon}}{F}\right) \frac{c_{l_{a1}} c}{b} a \tag{11}$$

Since

$$F = 1 + \frac{360}{\pi^2} \frac{a}{A}$$

equation (11) may be written as

$$\frac{c_{l_b} c}{b} \approx \frac{\epsilon - \bar{\epsilon}}{1 + \frac{360}{\pi^2} \frac{a}{A}} \frac{c_{l_{a1}} c}{b} a \qquad (12)$$

By use of the further approximation (see reference 5) that

$$a \approx \frac{A}{AE + 2} a_\infty \approx \frac{A}{AE + 2} \frac{\pi^2}{90}$$

equation (12) may be written as

$$\frac{c_{l_b} c}{b} \approx \frac{A(\epsilon - \bar{\epsilon})}{AE + 6} \frac{c_{l_{a1}} c}{b} a_\infty \qquad (13)$$

## Antisymmetrical Twist Distributions

For antisymmetrical twists, $\Delta C_L$ and $\bar{\epsilon}$ are both equal to zero, so that $\frac{c_{l_\epsilon} c}{b} = \frac{c_{l_b} c}{b}$. From a consideration of the lift for one semispan of the wing, appendix A shows that

$$\int_0^1 \frac{c_{l_b} c}{b} d\left(\frac{2y}{b}\right) \approx \int_0^1 \frac{\epsilon}{F'} \frac{c_{l_{a1}} c}{b} a \, d\left(\frac{2y}{b}\right) \qquad (14)$$

where

$$F' = 1 + \frac{180}{\pi^2} \frac{a}{A}$$

From a consideration of the moment of the lift distribution, it is also shown in appendix A that

$$\int_0^1 \frac{c_{l_b} c}{b} \frac{2y}{b} d\left(\frac{2y}{b}\right) \approx \int_0^1 \frac{\epsilon}{F'} \frac{c_{l_{a1}} c}{b} a \frac{2y}{b} d\left(\frac{2y}{b}\right) \qquad (15)$$

Again, since the areas are approximately equal (equation (14)) and the moments of the same areas are also approximately equal (equation (15)), it is reasoned that the distributions are approximately equal. Therefore,

$$\frac{c_{l_b} c}{b} \approx \frac{\epsilon}{F'} \frac{c_{l_{a1}} c}{b} a \qquad (16)$$

or

$$\frac{c_{l_b} c}{b} \approx \frac{\epsilon}{1 + \frac{180}{\pi^2} \frac{a}{A}} \frac{c_{l_{a1}} c}{b} a \qquad (17)$$

which may be further approximated as

$$\frac{c_{l_\epsilon} c}{b} \approx \frac{A\epsilon}{AE + 4} \frac{c_{l_{a1}} c}{b} a_\infty \qquad (18)$$

The use of the edge-velocity factor $E$ for symmetrical lift distributions is inherent in the method developed herein. For antisymmetrical twists, the use of the edge-velocity factor $E'$ has been shown to give results which are in closer agreement with those obtained by lifting-surface theory. (See reference 5.) In order to take this effect into account, equations (17) and (18) may be modified by multiplying the right-hand sides by the factor $\frac{AE + 4}{AE' + 4}$, or a fictitious lift-curve slope $a' = a \frac{AE + 2}{AE' + 2}$ may be used in equation (17) instead of the lift-curve slope $a$.

## ILLUSTRATIVE EXAMPLES

In order to illustrate the degree with which the lift distribution due to twist may be approximated by the method described, results of calculations are presented in figures 1 to 7 for the four main types of twist: symmetrical, antisymmetrical, continuous, and discontinuous. The wing chosen for four of the examples had a taper ratio of 0.5, an aspect ratio of 6.74, and a corresponding edge-velocity factor of 1.043. To show the effect of aspect ratio, an additional example is presented (fig. 3) for a wing with a taper ratio of 0.5, an aspect ratio of 13.92, and a corresponding edge-velocity factor of 1.010. A two-dimensional lift-curve slope of 0.1097 was assumed. The additional lift distribution (due to angle of attack) was obtained by the method of reference 6. Equations (12) and (17) were used inasmuch as they yielded slightly closer approximations than equations (13) and (18) for the comparison with results obtained by the more nearly exact methods of references 6 and 7 utilizing linear section lift curves. Also presented are results obtained by Schrenk's approximation as described in references 1 to 3 but modified by using the section lift-curve slope $a_o = \frac{a_\infty}{E}$ instead of the two-dimensional slope $a_\infty$. Therefore,

$$\frac{c_{l_b} c}{b} \approx \frac{\epsilon - \bar{\epsilon}}{2} \frac{c}{b} a_o \tag{19}$$

### Symmetrical Twist Distributions

Continuous twist.- The lift distributions shown in figure 1 are for the induced twist due to the jet boundary of a wind tunnel. The resulting distributions contain both the additional and basic parts. The main discrepancy between the two curves occurs at the root where the approximate method is unable to show the zero slope required by potential theory. The basic parts of these distributions are shown in figure 2 along with that calculated by Schrenk's method. Both approximate methods show the same discrepancy at the root but Schrenk's method shows a much larger discrepancy near the tip. The differences between Schrenk's method and the method of this paper are more clearly shown in figure 3 for the wing of aspect ratio 13.92 and the same twist as for figure 2.

Discontinuous twist.- The lift distributions due to the deflection of 50-percent-span flaps are shown in figure 4. The direct application of the method will give a discontinuity in the lift distribution which is impossible according to potential theory. In this figure, the discontinuity in twist distribution was faired with ellipses as indicated or in terms of a correction factor to be added to the discontinuous twist distributions:

For $0 < \frac{2y}{b} < \frac{2y^*}{b}$

$$\epsilon_c = -\frac{\epsilon^*}{2}\left[1 - \sqrt{1 - \left(\frac{2y/b}{2y^*/b}\right)^2}\right] \tag{20a}$$

For $\frac{2y^*}{b} < \frac{2y}{b} < 1$

$$\epsilon_c = \frac{\epsilon^*}{2}\left[1 - \sqrt{1 - \left(\frac{1 - \frac{2y}{b}}{1 - \frac{2y^*}{b}}\right)^2}\right] \tag{20b}$$

where $\epsilon^*$ is the magnitude of the discontinuity in twist at the spanwise position $2y^*/b$ of the discontinuity. This type of fairing is arbitrary and some other type might be used to give better results. The discrepancies shown in this figure are wholly due to discrepancies in basic lift distribution as shown in figure 5. The unfaired distributions are shown for comparison. The root bending moment of the unfaired curve obtained by the method described herein is less than 1 percent different from that of the more nearly exact curve. Furthermore, if the ordinates of the faired curve are multiplied by a constant to give the same root bending moment as that of the unfaired curve, a much closer approximation will be obtained. The closeness of the approximation obtained by Schrenk's method obviously depends upon proper fairing not only at the discontinuity but also near the tip.

## Antisymmetrical Twist Distributions

Continuous twist.- The lift distributions of a wing rolling with a tip helix angle of 0.01 radian or $0.573^\circ$ are shown in figure 6. In this case the agreement between the approximate method and the more nearly exact method is extremely good, whereas Schrenk's method is not nearly so good.

Discontinuous twist.- The lift distribution due to the deflection of 50-percent-span ailerons is shown in figure 7. As in the case of the partial-span flaps, both the unfaired and faired distributions are shown; the root bending moment of the unfaired curve is practically identical with that of the more nearly exact curve. The discontinuity in twist was faired in the same manner as for the case of the 50-percent-span flaps, the difference being that the signs of the correction factors were reversed. Again, if the ordinates of the faired curve are multiplied by a constant to give the same root bending moment as that of the unfaired curve, a much closer approximation will be obtained. The inadequacy of Schrenk's method is clearly shown.

## DISCUSSION

### Comparison of Methods

The examples shown were chosen to indicate some of the similarities as well as the differences between Schrenk's method and the method developed herein. In the case of an elliptical wing for which $\frac{c_{l_{a1}}c}{b} = \frac{c}{b}$, it is readily evident that Schrenk's original approximation, which uses the two-dimensional lift-curve slope

$$\frac{c_{l_b}c}{b} \approx \frac{\epsilon - \bar{\epsilon}}{2}\frac{c}{b}a_\infty$$

will give the same result as the improved approximation for an aspect ratio of approximately 6 for symmetrical twists and for an aspect ratio of approximately 4 for antisymmetrical twists. An apparent improvement in Schrenk's approximation would be the use of the appropriate aerodynamic induction factor, $\frac{A}{AE + 6}$ for symmetrical twists and $\frac{A}{AE' + 4}$ for antisymmetrical twists, rather than the factor of $\frac{1}{2}$. Such a method would then become that of the so-called strip theory modified by the aerodynamic induction factor. The factor $\frac{A}{AE + 6}$, however, should be used to obtain the basic lift distribution due to symmetrical twists rather than the factor $\frac{A}{AE + 2}$ commonly used.

In addition to taking into account the aspect ratio of the wing, the improved method makes use of the additional lift distribution instead of the chord distribution used by Schrenk. As the departure from elliptical lift and chord distributions becomes greater, the method developed herein becomes better as compared with Schrenk's method. Although the additional lift distributions are not quite as readily available as chord distributions, those of a large variety of plan forms can be found in reference 8. Even using the approximations of reference 1 or 9 for the additional lift distributions is better than using the chord distribution. Once the additional lift distribution has been obtained, the time required for computing the basic lift distribution is essentially the same for the improved method as for Schrenk's method.

## Application to Swept Wings

Inasmuch as the approximate method presented herein makes use of the additional lift distribution and three-dimensional lift-curve slope, it might be expected to yield satisfactory results for swept wings even though the method is based upon lifting-line theory. The adaptation of the formulas for swept wings is given in appendix B. Comparisons for two sweptback wings are presented in figures 8 to 10. For figure 8, the sweepback was $45^{\circ}$, the aspect ratio was 1.5, and the taper ratio was 0.5. For figures 9 and 10, the sweepback was $60^{\circ}$, the aspect ratio was 3.5, and the taper ratio was 0.5. The additional lift distributions and three-dimensional lift-curve slopes were obtained from reference 10 for use in equations (12) and (17). Curves from references 11 and 12 are used for comparison. In the application of Schrenk's approximation to the case of the sweptback wings, it was evident that using the two-dimensional lift-curve slope would give exaggerated values for the lift distribution. Equation (19) was therefore used, for which $a_o = \frac{a_\infty \cos \Lambda}{E}$, in order to provide a more nearly equal basis for comparison.

Figures 8 and 9 for linear symmetrical twist distributions indicate that the improved approximation does yield reasonably satisfactory results even for the extreme configurations selected. The fairly good agreement obtained in figure 9 with the modified Schrenk's approximation is fortuitous in the same manner as in figure 2 since $\frac{AE}{\cos \Lambda}$ was of the order of 6 for this configuration. For the configuration of figure 8, the value of $\frac{AE}{\cos \Lambda}$ was much less, and Schrenk's approximation is in error in the opposite direction from that shown in figure 3 for a high aspect ratio.

Figure 10 for the antisymmetrical twist distribution of a rolling wing indicates less satisfactory results. The general shape and lateral center of pressure are in agreement with reference 12, but the magnitude is about 18 percent too great. Equation (17) was used directly for this calculation. The suggested modification of using the fictitious lift-curve slope $a'$ would decrease the error to about 10 percent. This error may be too large to be tolerated for the computation of the damping-in-roll derivative but may be sufficiently close for other purposes in which antisymmetrical twists are involved.

## CONCLUDING REMARKS

An improved approximate method has been developed for calculating the basic lift distribution due to any of the four main types of twist: symmetrical, antisymmetrical, continuous, or discontinuous. The method makes use of the additional lift distribution due to angle of attack and takes into account the aspect ratio of the wing. Although the method is based upon lifting-line theory, it gives reasonably good results even for swept wings.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., May 24, 1950

## APPENDIX A

## DERIVATION OF APPROXIMATIONS FOR AREA AND MOMENT OF LIFT DISTRIBUTIONS DUE TO TWIST

### General Equations from Lifting-Line Theory

In the following derivation, lifting-line theory is used and the section lift curves are assumed to be linear. The spanwise lift distribution is expressed as the trigonometric series (reference 6).

$$\frac{c_l c}{b} = \sum A_n \sin n\theta \tag{A1}$$

where $\theta$ is defined by the relation $\cos\theta = \frac{2y}{b}$. The lift distribution is also related to the section lift-curve slope by the expression

$$\frac{c_l c}{b} = \frac{a_o c}{b}\left(\alpha - \alpha_i\right) \tag{A2}$$

where the induced angle of attack, in degrees, is expressed by

$$\alpha_i = \frac{180}{4\pi \sin\theta} \sum nA_n \sin n\theta \tag{A3}$$

The lift distribution can be divided into two distributions by the substitution of $A_n = \alpha B_n + C_n$ in equation (A1). This substitution gives

$$\frac{c_l c}{b} = \alpha \sum B_n \sin n\theta + \sum C_n \sin n\theta \tag{A4}$$

where $\sum B_n \sin n\theta$ is the lift distribution per unit angle of attack and $\sum C_n \sin n\theta$ is the lift distribution due to twist. The additional lift distribution (that due to angle of attack for a wing lift coefficient of unity) is

$$\frac{c_{l_{a1}} c}{b} = \frac{1}{a} \sum B_n \sin n\theta \tag{A5}$$

The lift distribution due to twist is designated

$$\frac{c_{l_\epsilon} c}{b} = \sum C_n \sin n\theta \tag{A6}$$

An integration of equation (A5) along the span yields

$$\int_{-1}^{1} \frac{c_{l_{a1}} c}{b} d\left(\frac{2y}{b}\right) = \frac{1}{a} \int_{0}^{\pi} \sum B_n \sin n\theta \sin \theta \, d\theta \tag{A7}$$

$$\frac{2}{A} = \frac{\pi B_1}{2a}$$

or

$$B_1 = \frac{4a}{\pi A} \tag{A8}$$

The increment in lift coefficient due to twist is

$$\Delta C_L = \frac{A}{2} \int_{-1}^{1} \frac{c_{l_\epsilon} c}{b} d\left(\frac{2y}{b}\right)$$

$$= \frac{A}{2} \int_{0}^{\pi} \sum C_n \sin n\theta \sin \theta \, d\theta$$

$$\Delta C_L = \frac{A\pi C_1}{4} \tag{A9}$$

Substituting equation (A8) into equation (A9) yields

$$\Delta C_L = a \frac{C_1}{B_1} \tag{A10}$$

The average twist is

$$\bar{\epsilon} = \frac{\Delta C_L}{a} = \frac{C_1}{B_1} \tag{A11}$$

For a wing with twist, equation (A2) may be written

$$\frac{c_l c}{b} = \frac{a_o c}{b}\left(\alpha + \epsilon - \alpha_i\right) \tag{A12}$$

which can be combined with equations (A1) and (A3) to give

$$\sum A_n \sin n\theta = \frac{a_o c}{b}\left(\alpha + \epsilon - \frac{180}{4\pi \sin \theta} \sum nA_n \sin n\theta\right) \tag{A13}$$

Equation (A13) can be divided into the two equations:

$$\sum B_n \sin n\theta = \frac{a_o c}{b}\left(1 - \frac{45}{\pi \sin \theta} \sum nB_n \sin n\theta\right) \tag{A14}$$

and

$$\sum C_m \sin m\theta = \frac{a_o c}{b}\left(\epsilon - \frac{45}{\pi \sin \theta} \sum mC_m \sin m\theta\right) \tag{A15}$$

where $n$ has been replaced by $m$ for subsequent use. Equation (A15) can be rearranged in the following manner:

$$\epsilon \sin \theta - \frac{45}{\pi} \sum mC_m \sin m\theta = \frac{b}{a_o c} \sum C_m \sin m\theta \sin \theta \tag{A16}$$

Multiplying equation (A14) by equation (A16) yields after rearranging:

$$\epsilon \sum B_n \sin n\theta \sin \theta = \sum C_m \sin m\theta \sin \theta +$$

$$\frac{45}{\pi} \sum_n \sum_m (m - n) B_n C_m \sin n\theta \sin m\theta \qquad \text{(A17)}$$

which is the basic equation used hereinafter. Only odd values of the index $n$ exist because the lift distribution given by equation (A5) is symmetrical. Values of the index $m$ are odd for lift distributions due to symmetrical twist distributions and even for lift distributions due to antisymmetrical twist distributions.

## Symmetrical Twist Distributions

For symmetrical twist distributions, the same result will be obtained whether equation (A17) is integrated along the entire span or along the semispan. Integrating the double-summation term of equation (A17) gives

$$\int_0^{\pi/2} \frac{45}{\pi} \sum_n \sum_m (m - n) B_n C_m \sin n\theta \sin m\theta \, d\theta = 0$$

so that

$$\int_0^{\pi/2} \epsilon \sum B_n \sin n\theta \sin \theta \, d\theta = \int_0^{\pi/2} \sum C_m \sin m\theta \sin \theta \, d\theta \qquad \text{(A18)}$$

Therefore,

$$\int_0^1 \epsilon \frac{c_{l_{a1}} c}{b} a \, d\left(\frac{2y}{b}\right) = \int_0^1 \frac{c_{l_\epsilon} c}{b} \, d\left(\frac{2y}{b}\right) \qquad \text{(A19)}$$

which was proved in a different manner in reference 4. Equation (A19) can be readily shown to be valid for asymmetrical twist distributions if the integrations are performed along the entire span. If $\epsilon$ is replaced by $\bar{\epsilon}$ in equation (A18), the integration yields

$$\frac{\bar{\epsilon}\pi}{4} B_1 = \frac{\pi}{4} C_1$$

which is consistent with equation (A11).

Subtracting $\bar{\epsilon} \sum B_n \sin n\theta \sin \theta$ from both sides of equation (A17) gives

$$(\epsilon - \bar{\epsilon}) \sum B_n \sin n\theta \sin \theta = \sum C_m \sin m\theta \sin \theta - \frac{C_1}{B_1} \sum B_n \sin n\theta \sin \theta + \frac{45}{\pi} \sum_n \sum_m (m - n) B_n C_m \sin n\theta \sin m\theta \tag{A20}$$

This operation is equivalent to subtracting the additional part of the lift distribution due to twist and leaving only the basic part. In order to obtain a relationship involving the moment of the basic lift distribution, each term of equation (A20) is multiplied by $\cos \theta \, d\theta$ and then integrated along the semispan. These operations give

$$\int_0^{\pi/2} (\epsilon - \bar{\epsilon}) \sum B_n \sin n\theta \sin \theta \cos \theta \, d\theta = \int_0^{\pi/2} \sum C_m \sin m\theta \sin \theta \cos \theta \, d\theta - \int_0^{\pi/2} \frac{C_1}{B_1} \sum B_n \sin n\theta \sin \theta \cos \theta \, d\theta + \int_0^{\pi/2} \frac{45}{\pi} \sum_n \sum_m (m - n) B_n C_m \sin n\theta \sin m\theta \cos \theta \, d\theta \tag{A21}$$

The integrations of the right-hand terms of equation (A21) yield

$$\int_0^{\pi/2} \sum C_m \sin m\theta \sin \theta \cos \theta \, d\theta = \frac{C_1}{3} + \frac{C_3}{5} - \frac{C_5}{21} + \ldots$$

$$\int_0^{\pi/2} \frac{C_1}{B_1} \sum B_n \sin n\theta \sin \theta \cos \theta \, d\theta = \frac{C_1}{B_1}\left(\frac{B_1}{3} + \frac{B_3}{5} - \frac{B_5}{21} + \ldots\right)$$

and

$$\int_0^{\pi/2} \frac{45}{\pi} \sum_n \sum_m (m - n) B_n C_m \sin n\theta \sin m\theta \cos \theta \, d\theta = \frac{45}{\pi} \left[\frac{2}{5}\left(B_1C_3 - B_3C_1\right) - \frac{4}{21}\left(B_1C_5 - B_5C_1\right) + \ldots\right]$$

Therefore,

$$\int_0^{\pi/2} (\epsilon - \bar{\epsilon}) \sum B_n \sin n\theta \sin \theta \cos \theta \, d\theta = \left(\frac{C_3}{5} - \frac{C_5}{21} + \ldots - \frac{B_3C_1}{5B_1} + \frac{B_5C_1}{21B_1} + \ldots\right)\left(1 + \frac{90}{\pi} B_1\right) + \frac{45}{\pi}\left[-\frac{2}{21}\left(B_1C_5 - B_5C_1\right) + \ldots\right] \tag{A22}$$

If the last term (in brackets) of equation (A22) is neglected, equation (A21) becomes

$$\int_0^{\pi/2} \frac{(\epsilon - \bar{\epsilon})}{F} \sum B_n \sin n\theta \sin \theta \cos \theta \, d\theta \approx \int_0^{\pi/2} \sum C_m \sin m\theta \sin \theta \cos \theta \, d\theta - \int_0^{\pi/2} \frac{C_1}{B_1} \sum B_n \sin n\theta \sin \theta \cos \theta \, d\theta$$

or

$$\int_0^1 \frac{c_{l_\epsilon} c}{b} \frac{2y}{b} \, d\left(\frac{2y}{b}\right) \approx \int_0^1 \left(\bar{\epsilon} + \frac{\epsilon - \bar{\epsilon}}{F}\right) \frac{c_{l_{a1}} c}{b} \, a \, \frac{2y}{b} \, d\left(\frac{2y}{b}\right) \tag{A23}$$

where

$$F = 1 + \frac{90}{\pi} B_1$$

$$= 1 + \frac{360}{\pi^2} \frac{a}{A}$$

## Antisymmetrical Twist Distributions

For antisymmetrical twist distributions, the average twist $\bar{\epsilon}$ is equal to zero, so that $\frac{c l_{\epsilon} c}{b} = \frac{c l_b c}{b}$. For this case, the integration of equation (A17) along the entire span is zero. In order to obtain a relationship involving the lift of one semispan, equation (A17) is integrated along the semispan. Thus,

$$\int_0^{\pi/2} \epsilon \sum B_n \sin n\theta \sin \theta \, d\theta = \int_0^{\pi/2} \sum C_m \sin m\theta \sin \theta \, d\theta +$$

$$\int_0^{\pi/2} \frac{45}{\pi} \sum_n \sum_m (m - n) B_n C_m \sin n\theta \sin m\theta \, d\theta \qquad \text{(A24)}$$

The integrations of the right-hand terms of equation (A24) yield

$$\int_0^{\pi/2} \sum C_m \sin m\theta \sin \theta \, d\theta = \frac{2C_2}{3} - \frac{4C_4}{15} + \frac{6C_6}{35} + \ldots$$

and

$$\int_0^{\pi/2} \frac{45}{\pi} \sum_n \sum_m (m - n)B_nC_m \sin n\theta \sin m\theta \, d\theta =$$

$$\frac{45}{\pi} C_2\left(\frac{2B_1}{3} + \frac{2B_3}{5} + \frac{2B_5}{7} + \ldots\right) - \frac{45}{\pi} C_4\left(\frac{4B_1}{5} + \frac{4B_3}{7} + \frac{4B_5}{9} + \ldots\right) +$$

$$\frac{45}{\pi} C_6\left(\frac{6B_1}{7} + \frac{6B_3}{9} + \frac{6B_5}{11} + \ldots\right) + \ldots$$

Therefore,

$$\int_0^{\pi/2} \epsilon \sum B_n \sin n\theta \sin \theta \, d\theta = \left(\frac{2C_2}{3} - \frac{4C_4}{15} + \frac{6C_6}{35} + \ldots\right)\left(1 + \frac{45}{\pi} B_1\right) +$$

$$\frac{45}{\pi}\left[C_2\left(\frac{2B_3}{5} + \frac{2B_5}{7} + \ldots\right) - C_4\left(\frac{8B_1}{15} + \frac{4B_3}{7} + \frac{4B_5}{9} + \ldots\right) + \right.$$

$$\left. C_6\left(\frac{24B_1}{35} + \frac{6B_3}{9} + \frac{6B_5}{11} + \ldots\right) + \ldots\right] \tag{A25}$$

If the last term (in brackets) of equation (A25) is neglected, equation (A24) becomes

$$\int_0^{\pi/2} \frac{\epsilon}{F'} \sum B_n \sin n\theta \sin \theta \, d\theta \approx \int_0^{\pi/2} \sum C_m \sin m\theta \sin \theta \, d\theta$$

or

$$\int_0^1 \frac{c_{l_b}c}{b} \, d\left(\frac{2y}{b}\right) \approx \int_0^1 \frac{\epsilon}{F'} \frac{c_{l_{a1}}c}{b} \, a \, d\left(\frac{2y}{b}\right) \tag{A26}$$

where

$$F' = 1 + \frac{45}{\pi} B_1$$

$$= 1 + \frac{180}{\pi^2} \frac{a}{A}$$

In order to obtain a relationship involving the moment of the basic lift distribution, each term of equation (A17) is multiplied by $\cos\theta\, d\theta$ and then integrated along the semispan. These operations give

$$\int_0^{\pi/2} \epsilon \sum B_n \sin n\theta \sin\theta \cos\theta\, d\theta = \int_0^{\pi/2} \sum C_m \sin m\theta \sin\theta \cos\theta\, d\theta +$$

$$\int_0^{\pi/2} \frac{45}{\pi} \sum_n \sum_m (m - n) B_n C_m \sin n\theta \sin m\theta \cos\theta\, d\theta \qquad \text{(A27)}$$

$$= \frac{\pi}{8} C_2 + \frac{45}{8}\left[\left(B_1 - B_3\right)C_2 + \left(B_3 - B_5\right)C_4 + \left(B_5 - B_7\right)C_6 + \ldots\right]$$

Therefore,

$$\int_0^{\pi/2} \epsilon \sum B_n \sin n\theta \sin\theta \cos\theta\, d\theta =$$

$$\frac{\pi}{8} C_2\left(1 + \frac{45}{\pi} B_1\right) - \frac{45}{8}\left[B_3\left(C_2 - C_4\right) + B_5\left(C_4 - C_6\right) + \ldots\right] \qquad \text{(A28)}$$

If the last term (in brackets) of equation (A28) is neglected, equation (A27) becomes

$$\int_0^{\pi/2} \frac{\epsilon}{F'} \sum B_n \sin n\theta \sin\theta \cos\theta\, d\theta \approx \int_0^{\pi/2} \sum C_m \sin m\theta \sin\theta \cos\theta\, d\theta$$

or

$$\int_0^1 \frac{c_{l_b} c}{b} \frac{2y}{b} \, d\left(\frac{2y}{b}\right) \approx \int_0^1 \frac{\epsilon}{F'} \frac{c_{l_{a1}} c}{b} \, a \, \frac{2y}{b} \, d\left(\frac{2y}{b}\right) \tag{A29}$$

# APPENDIX B

## ADAPTATION OF FORMULAS FOR SWEPT WINGS

For swept wings, both the edge-velocity factors and the three-dimensional lift-curve slope must be modified to include the effects of sweep. The edge-velocity factors are listed in the section entitled "Symbols." The formulas as listed are different from those given in reference 5 in that the $\cos^2\Lambda$ term used herein is given as a $\cos^4\Lambda$ term in reference 5. Although there is no rigorous proof that either factor is correct, the use of the $\cos^2\Lambda$ term has been found to yield answers in better agreement with those given in references 10 and 12. For example, the 10-percent error mentioned previously in connection with figure 10 would be increased to 16 percent by using the $\cos^4\Lambda$ term. The three-dimensional lift-curve slope can be written

$$a \approx \frac{A}{\frac{AE}{\cos\Lambda} + 2} a_\infty$$

$$\approx \frac{A}{\frac{AE}{\cos\Lambda} + 2} \frac{\pi^2}{90}$$

and the fictitious lift-curve slope used for antisymmetrical distributions can be written

$$a' = a \frac{\frac{AE}{\cos\Lambda} + 2}{\frac{AE'}{\cos\Lambda} + 2}$$

$$\approx \frac{A}{\frac{AE'}{\cos\Lambda} + 2} a_\infty$$

$$\approx \frac{A}{\frac{AE'}{\cos\Lambda} + 2} \frac{\pi^2}{90}$$

As a matter of interest, using the edge-velocity factors as defined herein yields the result that the term $\frac{AE}{\cos\Lambda}$ approaches the value of 2 and the term $\frac{AE'}{\cos\Lambda}$ approaches the value of 4 as the aspect ratio approaches 0. This result is in accordance with low-aspect-ratio theory (reference 13).

Using the relationships for $a$ and $a'$ in equations (12) and (17) yields the following equations:

For the symmetrical twists,

$$\frac{c_{l_b}c}{b} \approx \frac{\epsilon - \bar{\epsilon}}{1 + \frac{360}{\pi^2}\frac{a}{A}} \frac{c_{l_{a1}}c}{b} a \tag{B1}$$

or

$$\frac{c_{l_b}c}{b} \approx \frac{A(\epsilon - \bar{\epsilon})}{\frac{AE}{\cos\Lambda} + 6} \frac{c_{l_{a1}}c}{b} a_\infty \tag{B2}$$

where

$$\bar{\epsilon} = A\int_0^1 \epsilon \frac{c_{l_{a1}}c}{b} d\left(\frac{2y}{b}\right) \tag{B3}$$

For antisymmetrical twists,

$$\frac{c_{l_b}c}{b} \approx \frac{\epsilon}{1 + \frac{180}{\pi^2}\frac{a'}{A}} \frac{c_{l_{a1}}c}{b} a' \tag{B4}$$

or

$$\frac{c_{l_b}c}{b} \approx \frac{A\epsilon}{\frac{AE'}{\cos\Lambda} + 4} \frac{c_{l_{a1}}c}{b} a_\infty \tag{B5}$$

For discontinuous twists, the discontinuity must be faired out. For convenience, this fairing may be accomplished through the use of ellipses as indicated in equations (20).

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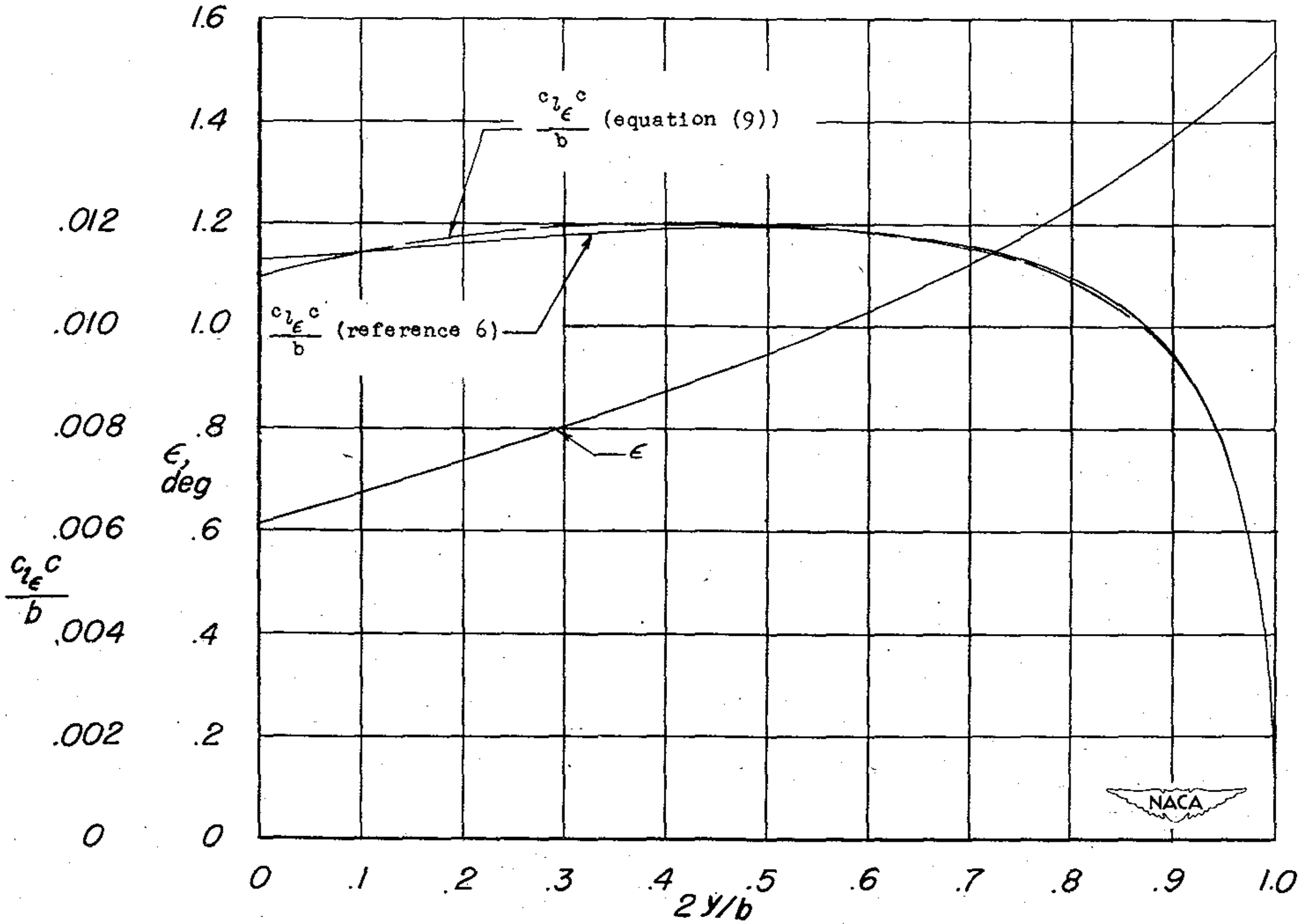


Figure 1.- Lift distribution due to jet-boundary-induced washin (symmetrical continuous twist). Taper ratio, 0.5; aspect ratio, 6.74; edge-velocity factor, 1.043.

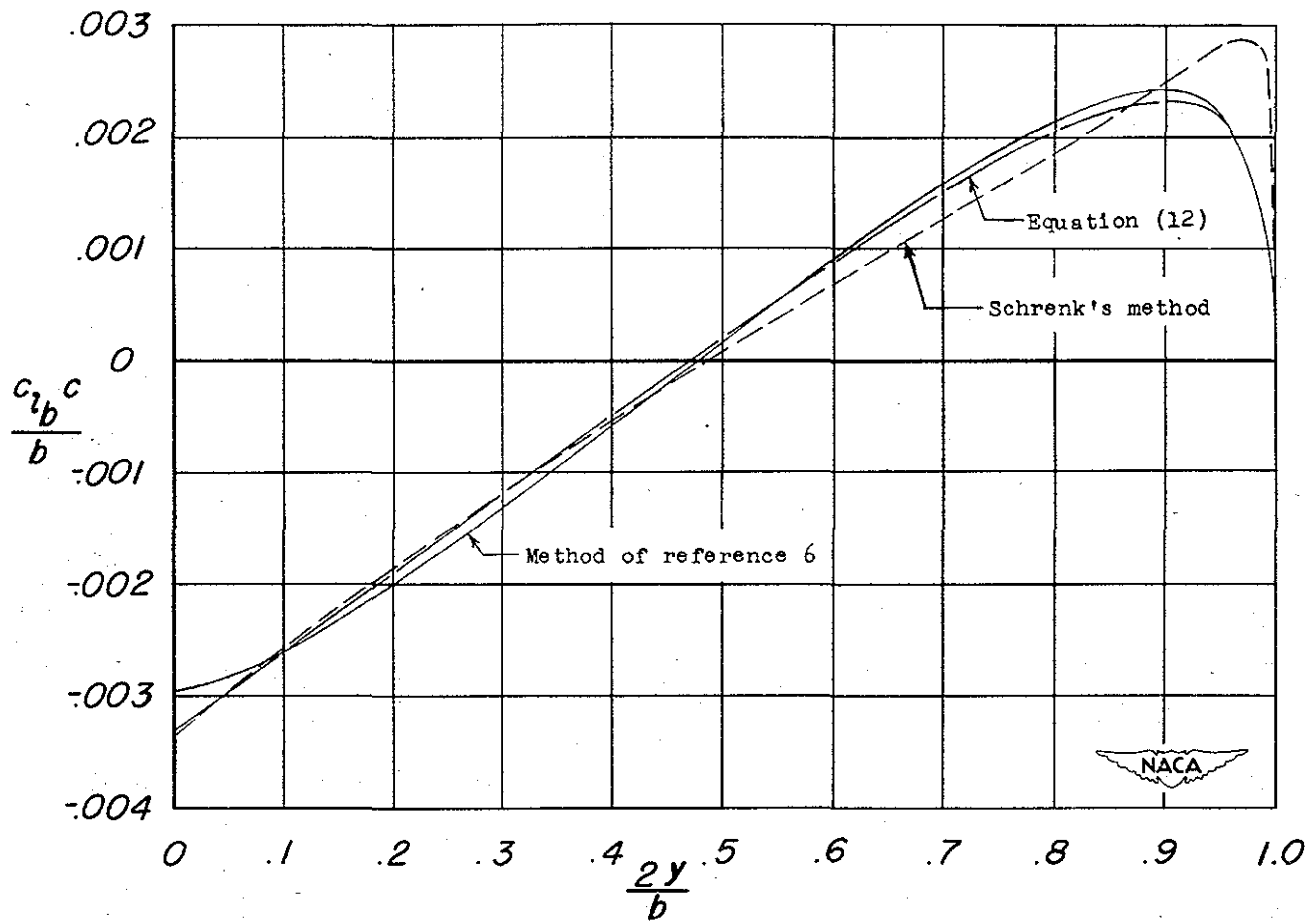


Figure 2.- Basic part of lift distribution due to jet-boundary-induced washin (symmetrical continuous twist). Taper ratio, 0.5; aspect ratio, 6.74; edge-velocity factor, 1.043.

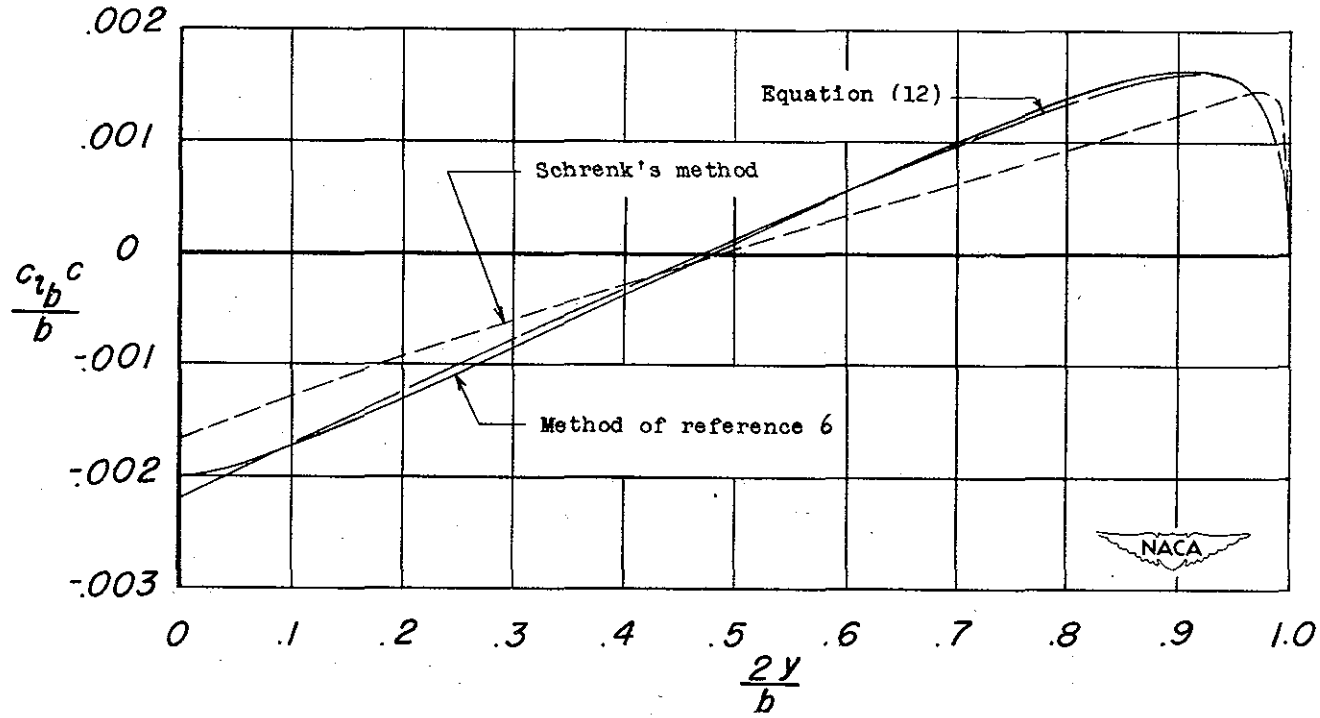


Figure 3.- Basic lift distribution for symmetrical continuous twist. Taper ratio, 0.5; aspect ratio, 13.92; edge-velocity factor, 1.010.

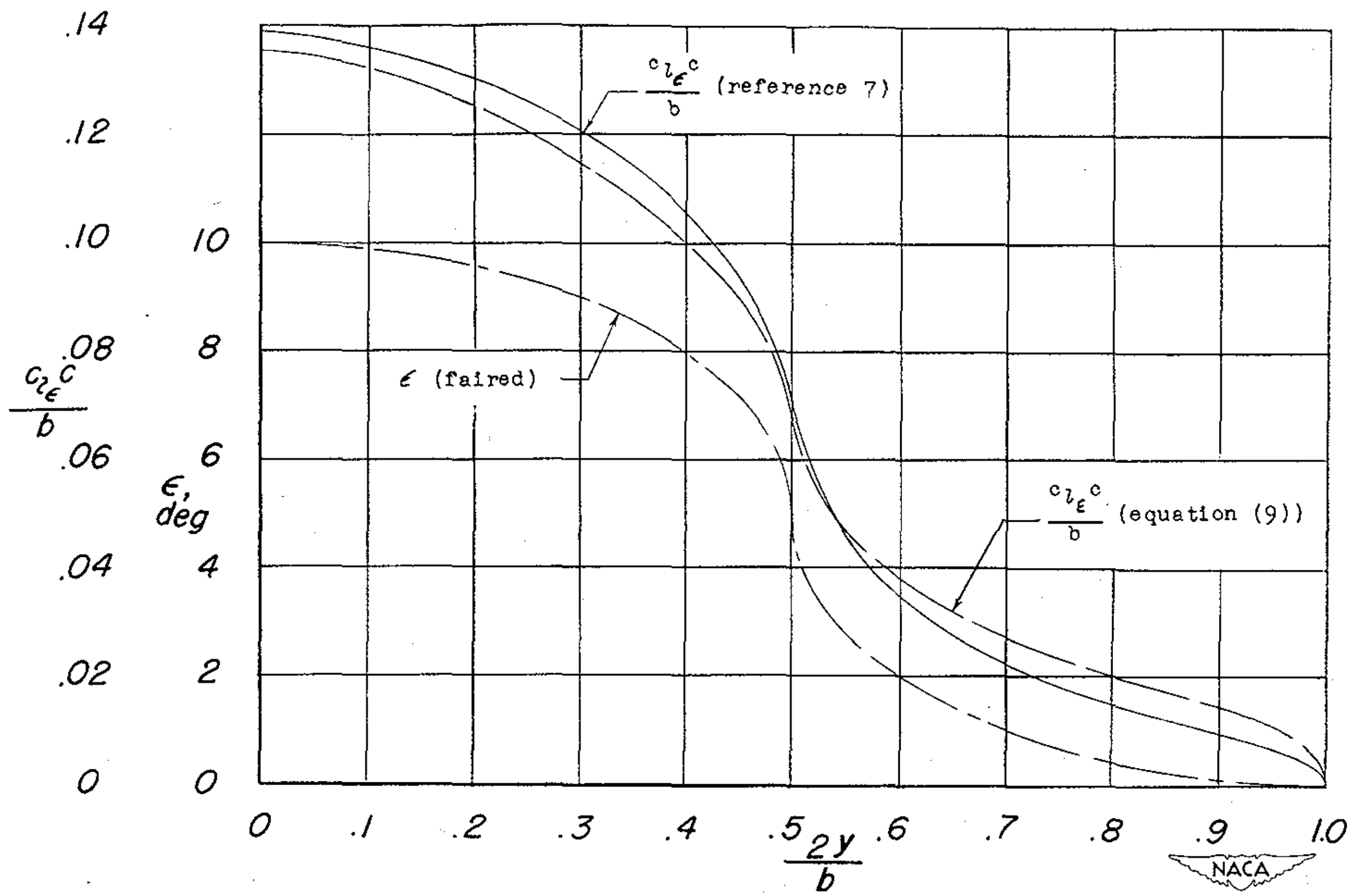


Figure 4.- Lift distribution due to deflection of 0.50-span flaps (symmetrical discontinuous twist). Taper ratio, 0.5; aspect ratio, 6.74; edge-velocity factor, 1.043.

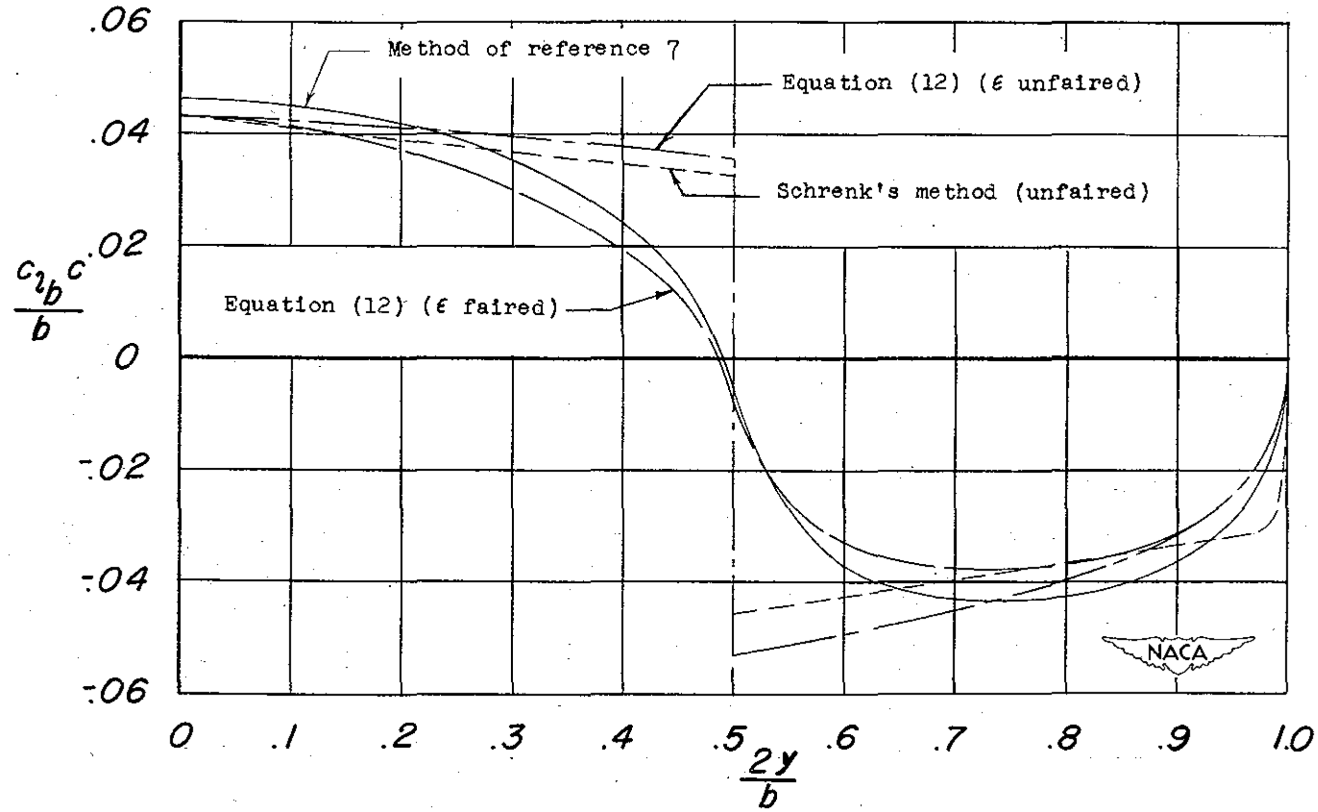


Figure 5.- Basic lift distribution for 0.50-span flaps (symmetrical discontinuous twist). Taper ratio, 0.5; aspect ratio, 6.74; edge-velocity factor, 1.043.

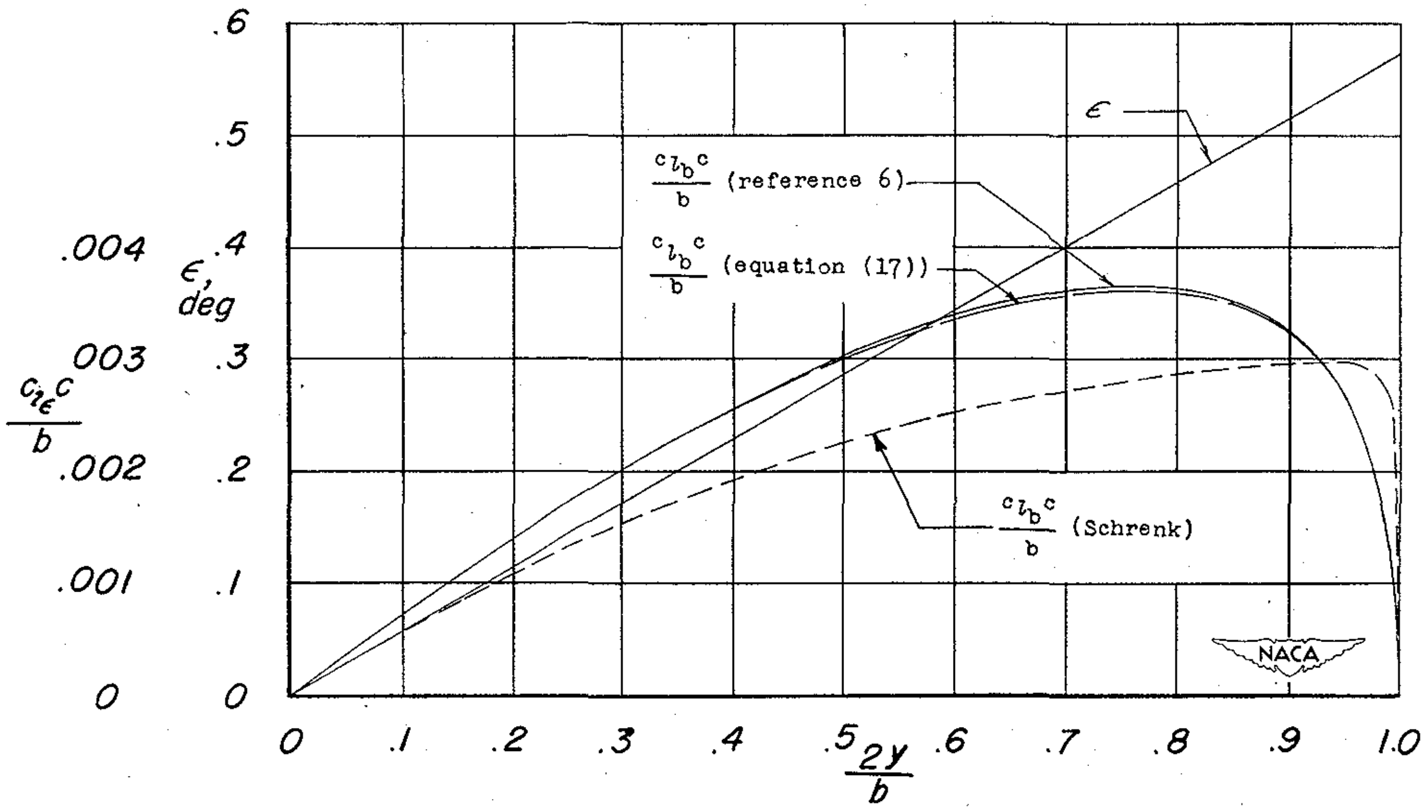


Figure 6.- Basic lift distribution for antisymmetrical continuous twist due to rolling. $\frac{pb}{2V}$ = 0.01 radian; taper ratio, 0.5; aspect ratio, 6.74; edge-velocity factor, 1.043.

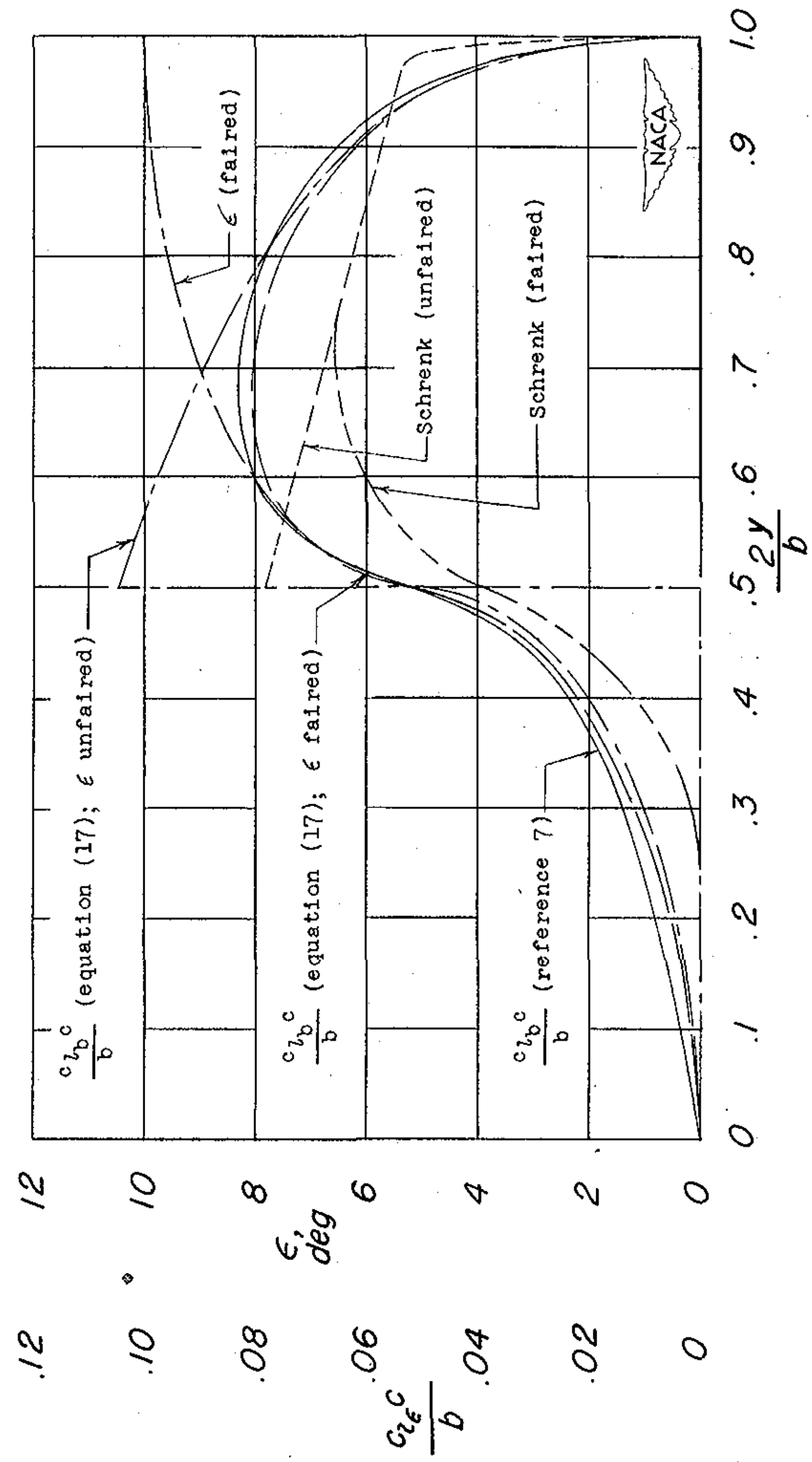


Figure 7.- Basic lift distribution for 0.50-span ailerons (antisymmetrical discontinuous twist). Taper ratio, 0.5; aspect ratio, 6.74; edge-velocity factor, 1.043.

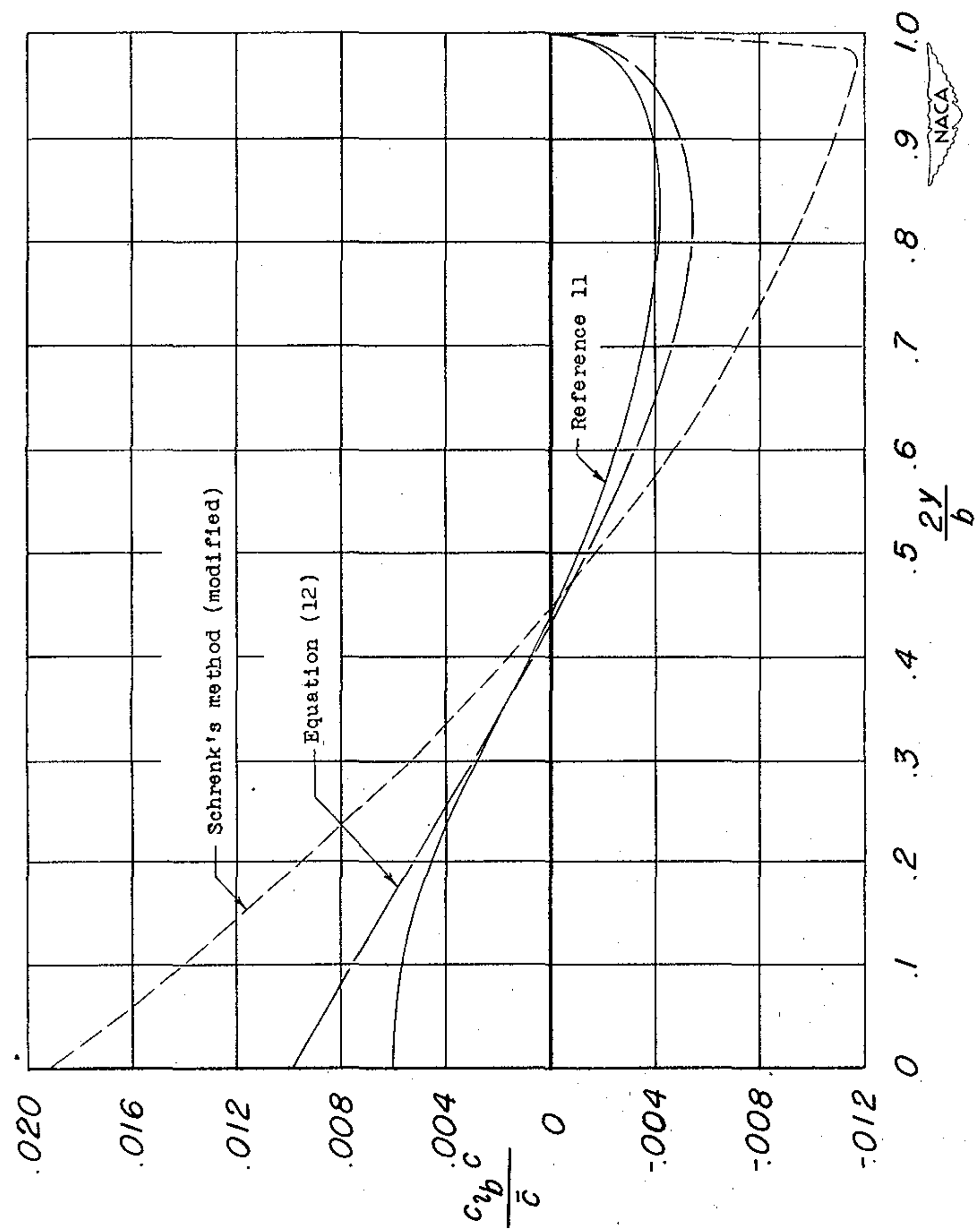


Figure 8.- Basic lift distribution per degree of linear twist. Sweep, 45°; taper ratio, 0.5; aspect ratio, 1.5.

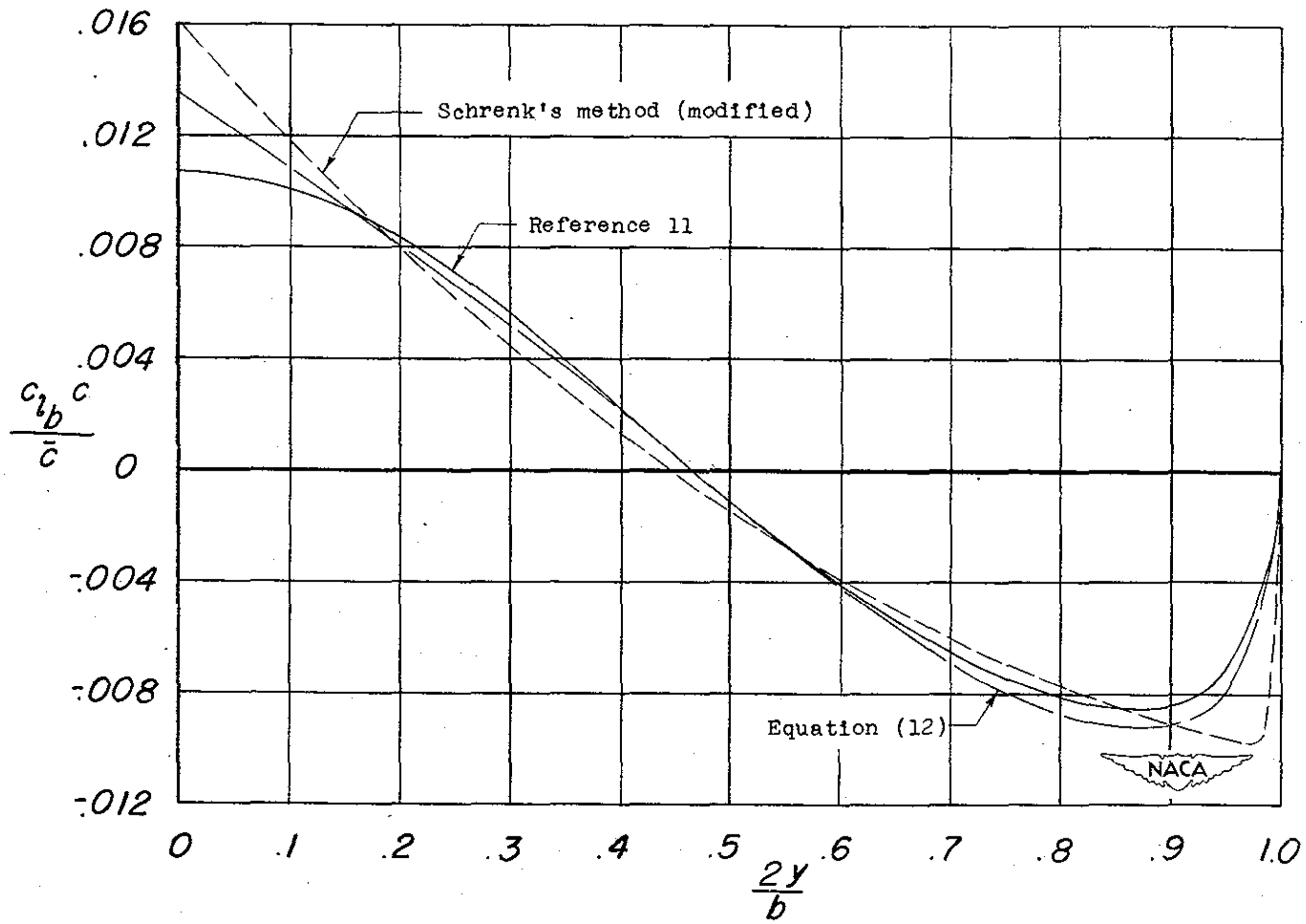


Figure 9.- Basic lift distribution per degree of linear twist. Sweep, 60°; taper ratio, 0.5; aspect ratio, 3.5.

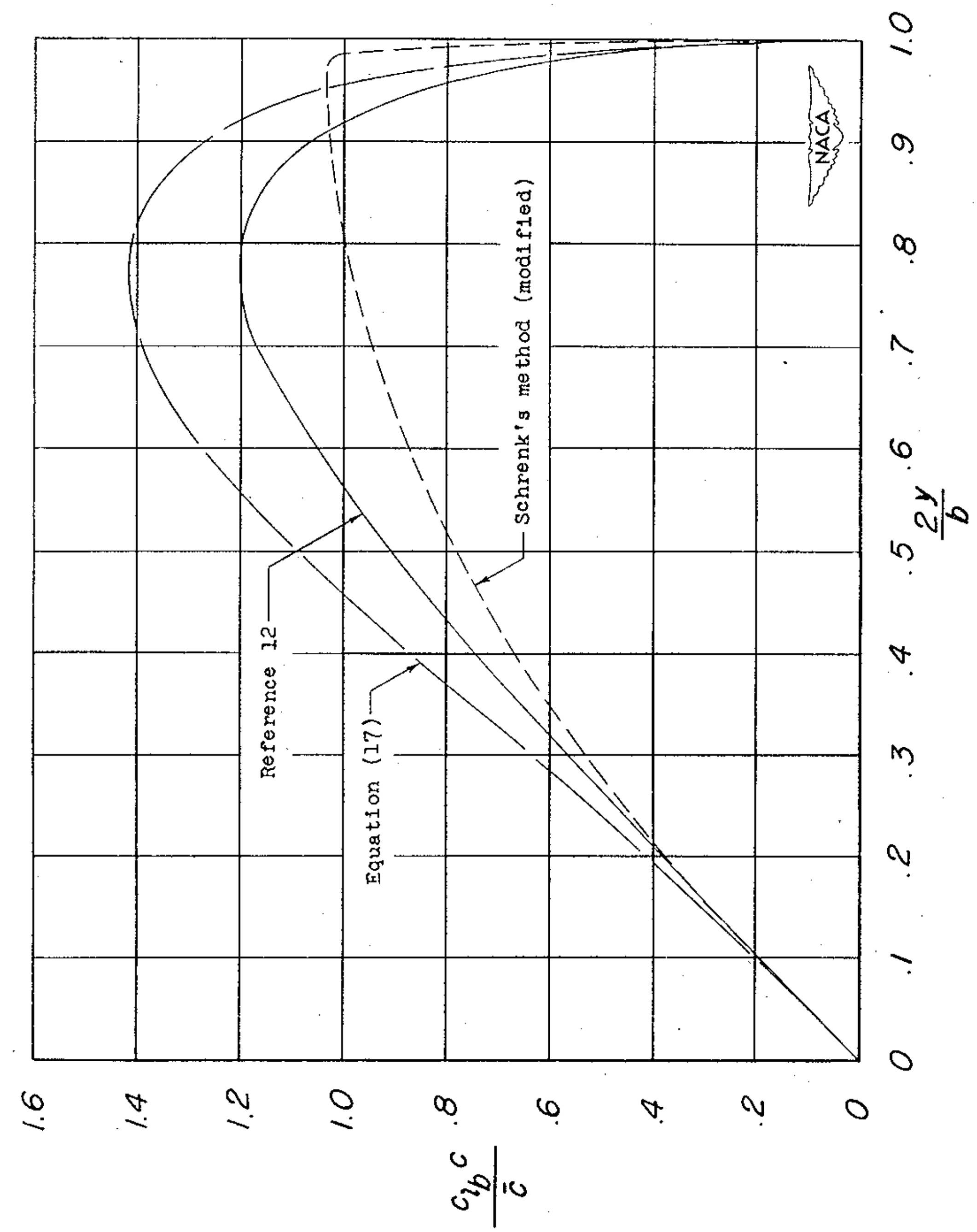


Figure 10.- Basic lift distribution for rolling wing. $\frac{pb}{2V} = 1.0$ radian; sweep, $60^{\circ}$; taper ratio, 0.5; aspect ratio, 3.5.