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SIMILARITY LAWS FOR TRANSONIC FLOW

ABOUT WINGS OF FINITE SPAN

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SUMMARY

Similarity laws for transonic flow about thin wings of finite span are obtained by considering the equations of transonic small-perturbation potential theory. By this means, functional relations for the pressure, lift, pitching moment, and pressure drag coefficients are derived and compared with the corresponding relations given by von Karman for two-dimensional transonic flow and with the relations for linearized subsonic and supersonic wing theory.

It is found, in contrast to the two-dimensional case where similitude depends upon the constancy of only one parameter, that two parameters must be held fixed to insure similitude for wings of finite span. It is further shown that the transonic similarity law coincides with one form of the similarity law of linearized subsonic and supersonic wing theory. The behavior at sonic speed of the two limiting cases of wings of infinite aspect ratio and wings of vanishingly small aspect ratio is discussed in terms of the similarity laws.

INTRODUCTION

The small-perturbation potential theory of transonic flow proposed apparently independently by Oswatitsch and Wieghardt, Sauer, Busemann and Guderley, von Kármán, Falkovich and others (references 1 through 8) is producing a marked change in the general concept of the nature of the transonic problem. It was the general opinion a few years ago that the interaction of the boundary layer and shock waves was of paramount importance and little hope was held for the usefulness of any potential theory of nonviscous transonic flow. Comparison of the early results of the transonic theory with those of experiments indicated, however, that perhaps the viscous effects were not always as dominant as had been supposed. In any case, investigators began to feel that a knowledge of transonic potential flow is necessary to the understanding of the viscous effects.

To date the applications of the transonic theory have dealt primarily with two-dimensional flows and, occasionally, with flows with axial

symmetry. Progress has been slowed, however, by the meager knowledge of the properties of mixed-elliptic-hyperbolic-type differential equations such as must be solved to obtain the velocity potential. Consequently, most of the early results were limited to simple two-dimensional flows, such as the Prandtl-Meyer expansion and the oblique shock wave, and to the flow in Laval nozzles. More recently, Guderley and Yoshihara have succeeded in obtaining a solution for a double-wedge airfoil at a Mach number of 1 (reference 9). In papers as yet unpublished, the corresponding problem for double-wedge airfoils in slightly supersonic flow has been treated by Vincenti of Ames Aeronautical Laboratory and the problem for finite wedges in slightly subsonic flow has been treated by Cole of California Institute of Technology. These results were shown by Liepmann and Bryson, in their paper "Transonic Flow Past Wedge Sections" presented at the July 1950 meeting of the Institute of Aeronautical Sciences, to be in good agreement with experimental pressure-distribution and drag data. Further support for the transonic potential-flow theory is provided in the recent work of Oswatitsch (reference 10) who investigated the flow around circular-arc airfoils in slightly subsonic flow and found that the theory predicted the presence of a shock wave which increased in intensity and moved rearwards with increasing Mach number and increasing airfoil-thickness ratio. Although these results have not been compared quantitatively with experimental results as yet, they are clearly in at least sensible agreement with observed physical phenomenon.

In contrast to the two-dimensional case where a moderate, but cogent, body of theory has been built up in the last few years, no analogous work has been done regarding transonic flows about wings of finite span. A major difficulty in the three-dimensional problem stems from the nonexistence of transformations enabling the linearization of the equations of motion as is accomplished in the two-dimensional case by the hodograph transformation. Because of these difficulties, the present study has been confined to the determination of the similarity laws of transonic flow about wings of finite span.

A knowledge of the similarity laws governing a given phenomenon is always of importance for the correlation and extrapolation of data. Some similarity rules are intuitively obvious (for instance, the concept of testing a model geometrically similar to the prototype); others are more obscure and can be discovered only by careful analysis of the fundamental aspects of the problem. An example of such a similarity law is the Prandtl-Glauert rule relating the flow of a compressible fluid around a given airfoil to the flow of an incompressible fluid around a second airfoil differing in geometry from the first airfoil in a manner prescribed by the similarity rules. The transonic similarity laws derived in this paper are of this type, two flows at different transonic Mach numbers being said to be dynamically similar when the geometry of the two wings is related in a prescribed manner.

It should be emphasized that similarity laws are always only approximate and that complete practical similarity never exists in nature unless the two systems are identical in every respect. Similarity rules are found from the equations representing the physical system and are therefore of the same order of approximation as the equations describing the phenomenon. Thus, for instance, the validity of the Prandtl-Glauert rule for the subsonic compressible flow about airfoils depends upon the flow being adequately described by the linearized equations of compressible flow, and, in the same manner, the transonic similarity laws depend upon the accuracy of the approximate differential equations of transonic flow. Therefore, similarity laws are useful not only for correlating the results of experiments, but also for inferring the validity of the basic equations, even though no actual solutions may be known.

The similarity laws for two-dimensional transonic flow about airfoils were first given by von Kármán to the Sixth International Congress for Applied Mechanics, Paris, September, 1946, and subsequently restated without proof in reference 6. Derivations of the results were published by von Kármán in reference 7 and by Kaplan in reference 11. The present paper represents an extension of the transonic similarity laws to include wings of finite span.

SYMBOLS

A	aspect ratio
a	speed of sound
a*	critical speed of sound
b	semispan
C _D	drag coefficient
C _L	lift coefficient
C _m	pitching-moment coefficient about most forward point of wing
C _p	pressure coefficient
c	wing chord
D	drag function
$f\left(\frac{x}{c}, \frac{y}{b}\right)$	ordinate-distribution function

L	lift function
M	pitching-moment function
M	Mach number
P	pressure function
p	static pressure
S	wing area
s	stretching factors defined in equation (9) (The subscripts denote the reference quantity, e.g., $s_x = x'/x$.)
t	maximum thickness of wing
U_0	free-stream velocity
u,v,w	velocity components in the x,y,z directions, respectively
x,y,z	Cartesian coordinates where x extends in the direction of the free-stream velocity
α	angle of attack
Γ	$\gamma + 1$
γ	ratio of specific heats, for air $\gamma = 1.4$
λ	arbitrary constant
ρ	mass density
τ	ordinate-amplitude parameter
Φ	velocity potential
ϕ	perturbation velocity potential of transonic theory ($\phi = \Phi - a^*x$)
$\bar{\phi}$	perturbation velocity potential of linearized theory ($\bar{\phi} = \phi - U_0x$)

Subscripts

- o conditions in the free stream
 z=0 conditions at the z=0 plane

BASIC EQUATIONS

Differential Equations

To the accuracy of the present theory, it is sufficient to consider the flow field as isentropic and irrotational. The velocity potential Φ then satisfies the following quasi-linear partial differential equation:

$$\left(1 - \frac{\Phi_x^2}{a^2}\right) \Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right) \Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right) \Phi_{zz} - 2 \frac{\Phi_x \Phi_y}{a^2} \Phi_{xy} - 2 \frac{\Phi_y \Phi_z}{a^2} \Phi_{yz} - 2 \frac{\Phi_z \Phi_x}{a^2} \Phi_{zx} = 0 \quad (1)$$

in which the local speed of sound a is given by

$$a^2 + \frac{\gamma-1}{2} (\Phi_x^2 + \Phi_y^2 + \Phi_z^2) = \frac{\gamma+1}{2} a^{*2} \quad (2)$$

In equation (2), a^* is the critical velocity of sound, that is, the speed of sound at points where the local Mach number is 1.

In linearized theory, either subsonic or supersonic, it is assumed that the velocity at any point in the field differs only slightly in magnitude and direction from the free-stream velocity. The differences between the local velocities and the free-stream velocity are considered as perturbation velocities; thus the perturbation potential $\bar{\Phi}$ of linearized theory is defined by

$$\bar{\Phi} = \Phi - U_0 x \quad (3)$$

Then equation (1) reduces, through the neglect of second and higher powers of the perturbation velocities, to the well-known Prandtl equation of linearized compressible flow

$$(1-M_0^2) \frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial y^2} + \frac{\partial^2 \bar{\phi}}{\partial z^2} = 0 \quad (4)$$

where the free-stream velocity is directed along the positive x axis as shown in figure 1 and where M_0 is the Mach number of the free stream. This equation has been found to lead to satisfactory wing theories and has been widely used for both subsonic and supersonic flows. As sonic velocity is approached, however, these theories often lead to absurdities. To review briefly: For wings of infinite span, the linearized-theory value of the longitudinal component of the perturbation velocity on the surface of a fixed airfoil is proportional to $(|1-M_0^2|)^{-1/2}$; consequently, u approaches infinity as M_0 approaches 1 and the theory is clearly inapplicable. For wings of finite span, however, u may be large or small at sonic velocity depending on the particular problem, as discussed in detail in reference 12. Specifically, for three-dimensional lifting surfaces of zero thickness the velocities remain finite everywhere except at the leading edges, their magnitudes generally increasing with increasing aspect ratio and angle of attack. For wings of nonzero thickness, however, u generally becomes large logarithmically as $1-M_0^2$ approaches zero. Thus, linearized theory of three-dimensional flows is applicable to lifting surfaces of small or moderate aspect ratios at transonic speeds, but cannot be applied in this speed range to wings of finite thickness.

These difficulties led to the abandoning of the restrictions of linear theory and to the search for a more exact equation for transonic flows. In the ensuing small-perturbation transonic theory, it is assumed that all resultant velocities, including the free-stream velocity, differ only slightly from the critical velocity of sound. The differences between the local velocities and the critical velocity of sound are considered as perturbation velocities; thus the perturbation potential is defined by

$$\phi = \bar{\phi} - a^*x \quad (5)$$

In contrast to the linearized theory where second- and higher-order terms in the perturbations are neglected, it is necessary in the small perturbation transonic theory to retain the second-order terms and to neglect the third- and higher-order terms. To this degree of approximation, equation (1) reduces, for the case of three-dimensional transonic flow with small inclination, to the following expression:

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\gamma+1}{a^*} \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x^2} \quad (6)$$

It may be noted that this differential equation is not only quasi-linear but is also of mixed type, changing from a hyperbolic equation in the supersonic regions where $\partial\phi/\partial x$ is positive to an elliptic equation in the subsonic regions where $\partial\phi/\partial x$ is negative. Direct application of this equation presents an extremely difficult problem.

Although equation (6) has received little attention, the corresponding equation for the flow in a plane (i.e., where $\partial^2\phi/\partial y^2 = 0$) has formed the basis for a considerable body of theory relating to airfoils and two-dimensional nozzles. A great aid in this work is the fact that the two-dimensional counterpart of equation (6) can be transformed into a linear equation, although still of mixed type, by means of the hodograph transformation. The resulting expression has been the subject of extensive study by Tricomi (reference 13) and others.

Returning to wings of finite span and to equation (6), it is important to note that for many lifting-surface problems involving pointed low-aspect-ratio wings it is not necessary to solve the complete equation since, as discussed previously, the right side remains small and can be neglected. In these instances, the resulting equation is of parabolic type in the number of dimensions considered. Since this device does not succeed in general, however, it appears necessary to study equation (6) in its entirety. The first step in such a study is to determine the similarity rules, if any exist, of three-dimensional transonic flows about wings.

Boundary Conditions

The determination of similarity rules depends not only upon the differential equation, as expressed in the present instance by equation (6), but also on the boundary conditions. Consider the boundary conditions at infinity where the velocity ratio u/a is equal to the free-stream Mach number M_0 , then equations (2) and (5) yield, to the order of the perturbation analysis, the following:

$$\left(\frac{\partial \phi}{\partial x}\right)_0 = U_0^{-a^*} = -\frac{2a_0^2}{\gamma+1} \left(\frac{1-M_0^2}{U_0+a^*}\right) \approx -\frac{a^*}{\gamma+1} (1-M_0^2)$$

$$\left(\frac{\partial \phi}{\partial y}\right)_0 = 0 \quad \left(\frac{\partial \phi}{\partial z}\right)_0 = 0 \quad (7)$$

The boundary conditions at the wing surface will be assumed to be replaceable by boundary conditions on the $z=0$ plane. This approximation, used almost universally in all small-perturbation wing theories, actually becomes better as the Mach number approaches 1 because of the small change of streamline cross section at sonic speed. At $z=0$ the vertical component of velocity is given by

$$\left(\frac{\partial \phi}{\partial z}\right)_{z=0} = U_0 \left(\frac{\partial z}{\partial x}\right)_{z=0} = U_0 \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \approx a^* \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \quad (8)$$

where the shape of the wing profile is given by

$$z = c \tau f(x/c, y/b)$$

where $f(x/c, y/b)$ represents the ordinate-distribution function and τ is an ordinate-amplitude parameter. Note that, in general, a variation of τ results in a simultaneous change of the thickness ratio, camber, and angle of attack. In the special case of a nonlifting wing having symmetrical sections, τ is proportional to the thickness ratio; for inclined flat-plate wings of vanishing thickness, τ is proportional to the angle of attack.

DERIVATION OF SIMILARITY LAWS

If equation (6) is now transformed into a system with primed quantities and the proportionality or stretching factors are denoted by s with appropriate subscripts such that

$$\left. \begin{aligned} x' &= s_x x & y' &= s_y y & z' &= s_z z \\ \phi' &= s_\phi \phi & a^{*'} &= s_a^* a^* \\ (\gamma+1)' &= \Gamma' = s_\Gamma \Gamma = s_\Gamma (\gamma+1) \end{aligned} \right\} \quad (9)$$

we find

$$\frac{s_y^2}{s_\phi} \left(\frac{\partial^2 \phi'}{\partial y'^2}\right) + \frac{s_z^2}{s_\phi} \left(\frac{\partial^2 \phi'}{\partial z'^2}\right) = \frac{s_a^* s_x^3}{s_\Gamma s_\phi^2} \left[\frac{(\gamma+1)'}{a^{*'}} \frac{\partial \phi'}{\partial x'} \frac{\partial^2 \phi'}{\partial x'^2} \right] \quad (10)$$

The flow in the primed system is similar to that in the original system if φ' satisfies the same differential equation and boundary conditions as φ . Consequently, for similarity to exist, it is necessary first of all that the differential equations for the two flows (equations (6) and (10)) be the same. Therefore, the following relations between the stretching factors must be satisfied:

$$\frac{s_y}{s_z} = 1; \quad \frac{s_\Gamma s_\varphi s_y^2}{s_{a^*} s_x^3} = 1 \quad (11)$$

After adjusting the stretching factors as described above, the next step is to investigate the boundary conditions satisfied by φ' in order to determine the kinematics of the flow and the geometry of the wing in the primed system. The flow at infinity in the primed system may be expressed in two ways: First,

$$\left(\frac{\partial \varphi'}{\partial x'}\right)_0 = -\frac{a^{*'} (\gamma+1)'}{(\gamma+1)'} (1-M_0^2) = -\frac{s_{a^*} a^*}{s_\Gamma (\gamma+1)} (1-M_0^2) \quad (12a)$$

and second,

$$\left(\frac{\partial \varphi'}{\partial x'}\right)_0 = \frac{s_\varphi}{s_x} \left(\frac{\partial \varphi}{\partial x}\right)_0 = -\frac{s_\varphi a^*}{s_x (\gamma+1)} (1-M_0^2) \quad (12b)$$

The relationship between the flows at infinity in the two systems may be found by equating the end forms of equations (12a) and (12b), and simplifying the result by means of equation (11), thus

$$(1-M_0^2)' = \frac{s_\Gamma s_\varphi}{s_{a^*} s_x} (1-M_0^2) = \frac{s_x^2}{s_y^2} (1-M_0^2) = \frac{s_x^2}{s_z^2} (1-M_0^2) \quad (13)$$

The boundary conditions at the $z=0$ plane, important for fixing the thickness and angle of attack of the new wing in the primed system, are given similarly, on the basis of equation (8), by

$$\left(\frac{\partial \varphi'}{\partial z'}\right)_{z'=0} = a^{*'} \tau' \frac{\partial}{\partial (x'/c')} f' \left(\frac{x'}{c'}, \frac{y'}{b'}\right) = s_{a^*} a^* \tau' \frac{\partial}{\partial (x'/c')} f' \left(\frac{x'}{c'}, \frac{y'}{b'}\right) \quad (14a)$$

and by

$$\left(\frac{\partial\phi'}{\partial z'}\right)_{z'=0} = \frac{s_\phi}{s_z} \left(\frac{\partial\phi}{\partial z}\right)_{z=0} = \frac{s_\phi}{s_z} a^* \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \quad (14b)$$

The relationship between the ordinate-amplitude parameters necessary for transonic similitude may be found by equating the end forms of equations (14a) and (14b) and simplifying the result by means of equations (11) and (13). Thus, if the two wings have the same ordinate-distribution function (i.e., $f'(x'/c', y'/b') = f(x/c, y/b)$), we have

$$\tau' = \frac{s_\phi}{s_{a^*} s_z} \quad \tau = \frac{s_x^3}{s_\Gamma s_z^3} \quad \tau = \frac{1}{s_\Gamma} \left[\frac{(1-M_0^2)'}{(1-M_0^2)} \right]^{3/2} \quad (15)$$

The pressure coefficient is given to first-order terms by

$$C_p = \frac{p-p_0}{\frac{1}{2}\rho_0 U_0^2} \approx -\frac{2(u-U_0)}{U_0} = -\frac{2[\partial\phi/\partial x + a^* - U_0]}{U_0} \approx -\frac{2}{a^*} \left[\frac{\partial\phi}{\partial x} - \left(\frac{\partial\phi}{\partial x}\right)_0 \right] \quad (16)$$

The pressure coefficients on the wing in the primed system are then related to those on the original wing as follows:

$$C_p' = -\frac{2}{a^{*'}} \left[\frac{\partial\phi'}{\partial x'} - \left(\frac{\partial\phi'}{\partial x'}\right)_0 \right] = \frac{s_\phi}{s_{a^*} s_x} \left\{ -\frac{2}{a^*} \left[\frac{\partial\phi}{\partial x} - \left(\frac{\partial\phi}{\partial x}\right)_0 \right] \right\}$$

$$= \frac{s_\phi}{s_{a^*} s_x} C_p = \frac{1}{s_\Gamma} \frac{(1-M_0^2)'}{(1-M_0^2)} C_p = \frac{1}{s_\Gamma^{1/3}} \left(\frac{\tau'}{\tau}\right)^{2/3} C_p \quad (17)$$

Equations (13), (15), and (17) may be considered as representing the similitude conditions for transonic flows about wings. Since the significance of these equations is difficult to visualize in their present form, they will be recast into a form more familiar to aerodynamicists. Equation (13), expressing the coordinate transformation, may be interpreted as the relationship between the aspect ratios of the two wings, thus

$$\sqrt{(1-M_0^2)'} A' = \sqrt{1-M_0^2} A \quad (18)$$

Equation (15), expressing the relationship between the thickness parameters of the two wings, may be rewritten as

$$\frac{\sqrt{(1-M_0^2)}^\tau}{[(\gamma+1)^\tau \tau^\tau]^{1/3}} = \frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}} \quad (19)$$

Finally, the pressure coefficients at corresponding points on the two wings having their geometry related according to equations (18) and (19) are given by

$$C_p' \left\{ \frac{\sqrt{(1-M_0^2)}^\tau}{[(\gamma+1)^\tau \tau^\tau]^{1/3}}, \sqrt{(1-M_0^2)}^\tau A'; \frac{x', y'}{c', b'} \right\} = \left. \begin{aligned} & \left[\frac{\gamma+1}{(\gamma+1)^\tau} \right]^{1/3} \left(\frac{\tau'}{\tau} \right)^{2/3} C_p \left\{ \frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_0^2} A; \frac{x, y}{c, b} \right\} \end{aligned} \right\} \quad (20)$$

This relationship may be expressed in still another form, less suitable for comparing the pressures on two wings but more appropriate for correlating the results of tests on a family of wings, as follows:

$$C_p = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} P \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_0^2} A; \frac{x, y}{c, b} \right] \quad (21)$$

The lift coefficient is given similarly by

$$C_L = \frac{1}{S} \oint C_p \, dx dy = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} L \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_0^2} A \right] \quad (22)$$

the moment coefficient by

$$C_m = \frac{1}{S c} \oint C_p \, x \, dx dy = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} M \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_0^2} A \right] \quad (23)$$

and the drag coefficient by

$$C_D = \frac{1}{S} \oint C_p \frac{dz}{dx} \, dx dy = \frac{\tau^{5/3}}{(\gamma+1)^{1/3}} D \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_0^2} A \right] \quad (24)$$

where P , L , M , and D are undetermined functions of the indicated arguments.¹ Recall that the ordinate-amplitude parameter τ is a rather general quantity proportional, for example, to the thickness ratio for the limiting case of nonlifting wings having symmetrical airfoils and to the angle of attack for the limiting case of inclined flat-plate wings.

A typical example of the use of the foregoing similarity rules is illustrated in figure 2 by sketches of two related wings in transonic flow. Figure 2(a) shows a triangular wing having an aspect ratio of 2.0, thickness ratio t/c of 10 percent, and angle of attack α of 10° . If this wing is in an air stream with a Mach number of 0.90, the related wing for another air stream ($\gamma'=\gamma$) with a Mach number of 0.95 is as shown in figure 2(b). The second wing has an aspect ratio of 2.78, a thickness ratio of 3.75 percent, and an angle of attack of 3.75° . Both wings have triangular sections. The lift, pitching-moment, and drag coefficients are related as indicated in the figure.

If the preceding equations (equations (21) through (24)) are applied to airfoils in two-dimensional flow, $\sqrt{1-M_0^2} A$ disappears from the arguments and the corresponding expressions for the pressure, lift, pitching moment, and drag coefficients are as follows:

$$C_p = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} P \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}}, \frac{x}{c} \right] \quad (25)$$

$$C_L = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} L \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}} \right] \quad (26)$$

$$C_m = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} M \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}} \right] \quad (27)$$

$$C_D = \frac{\tau^{5/3}}{(\gamma+1)^{1/3}} D \left[\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}} \right] \quad (28)$$

¹After completing this paper, the author became aware of another treatment of the same problem by Sune B. Berndt in report KTH Aero. TN 14 (1950) of the Royal Institute of Technology, Stockholm, Sweden.

Berndt's analysis differs from that given here because of the inclusion of some higher-order terms in his statement of the equations of transonic small-perturbation potential theory. In this respect, Berndt's analysis departs from commonly accepted practice in transonic-flow theory. The results of the two papers, although different in general, become identical at sonic speed, that is, as $\sqrt{1-M_0^2} \rightarrow 0$. It is not clear at the present time whether the inclusion of the higher-order terms produces any greater accuracy in the results.

These results are equivalent to those given by von Kármán in reference 7 but differ slightly from those given by Kaplan (reference 11) and previously by von Kármán (reference 6) as a result of minor differences in the perturbation analysis. In particular, $1-M_0^2$ is sometimes factored and approximated in the following manner:

$$1-M_0^2 = (1+M_0)(1-M_0) \approx 2(1-M_0) \quad (29)$$

The choice between the two representations appears quite arbitrary at the present time. In the present analysis, the results are expressed in terms of $1-M_0^2$ rather than $1-M_0$ for the purposes of obtaining unity with the similarity rules of subsonic and supersonic wing theory.

DISCUSSION OF RESULTS

Wings of Infinite Span

The transonic similarity laws for two-dimensional flow (equations (25) through (28)) imply that, if a force or moment coefficient for a single airfoil is known as a function of Mach number throughout the transonic range, it is also known for related airfoils of arbitrary thickness parameter. Since the extension of airfoil data in this manner depends on a single similarity parameter $\sqrt{1-M_0^2}/[(\gamma+1)\tau]^{1/3}$, the results of transonic experiments with a family of related airfoils should be correlated by plotting, as a function of the similarity parameter, the appropriate force or moment coefficient multiplied by the reciprocal of the coefficient of the undetermined function P , L , M , or D . For example, lift results would be given by plotting $[(\gamma+1)^{1/3}/\tau^{2/3}]C_L$ as a function of $\sqrt{1-M_0^2}/[(\gamma+1)\tau]^{1/3}$; drag results by plotting $[(\gamma+1)^{1/3}/\tau^{5/3}]C_D$ versus $\sqrt{1-M_0^2}/[(\gamma+1)\tau]^{1/3}$, and so forth. To the accuracy of the present small-perturbation transonic theory, airfoil data so plotted should all fall along a single line.

If equations (25) through (28) are applied to airfoils in a flow having a free-stream Mach number of one (i.e., when $\sqrt{1-M_0^2}/[(\gamma+1)\tau]^{1/3}=0$), the functions L , M , and D are all equal to constants and P depends only on the coordinate x/c at which it is evaluated. Therefore, as pointed out in reference 7, airfoils having similar thickness distributions have pressure, lift, and pitching-moment coefficients proportional to the two-thirds power and drag coefficient proportional to the five-thirds power of the thickness parameter τ . All the coefficients are inversely proportional to the one-third power of $(\gamma+1)$.

Wings of Finite Span

The transonic similarity laws for wings of finite span (equations (21) through (24)) imply that similitude depends upon two parameters rather than only one as was the case for two-dimensional flow. Consequently, if data are correlated as described in the preceding section, the results will no longer fall on a single line but will form a number of lines each corresponding to a particular value of the second similarity parameter $\sqrt{1-M_0^2} A$.

Since the dependence of similitude on two parameters introduces the possibility of indeterminate forms as M_0 approaches unity, no statements can be made regarding the nature of wing characteristics at sonic speed as was done for the two-dimensional case. In addition, the conclusions of two-dimensional transonic flow analyses may apply only to wings of extremely large aspect ratio since it is not only necessary that the aspect ratio itself be large but also that the similarity parameter $\sqrt{1-M_0^2} A$ be large. Since such a condition cannot hold, even approximately, for actual wings of finite span as M_0 approaches unity, it would be of interest to seek a solution of equation (6) for the transonic flow about a three-dimensional wing.

Although no solution of equation (6) for a wing of finite aspect ratio has been found as yet, the case of a pointed wing of zero thickness and of vanishing aspect ratio can be treated. This example, when taken together with the infinite-aspect-ratio case will serve to give further insight into the nature of transonic similarity laws. It has been shown previously (references 12 and 14) that the slender-pointed-wing theory of R. T. Jones (reference 15) yields values for the differential pressures, lift, drag, and pitching moment of moderate-aspect-ratio wings at sonic speed that are consistent with the assumptions of linearized theory as exemplified by equation (3). It can be shown further that this same solution satisfies the equation for transonic flow (equation (6)) in the limiting case of vanishing aspect ratio since, in approaching the limit, the magnitude of the right-hand side of the equation diminishes much more rapidly than does the left-hand side. The lift, pitching-moment, and drag coefficients at sonic speed of pointed wings of vanishing aspect ratio are then given by the following expressions where, since the wings are considered to be flat lifting surfaces of vanishing thickness, the parameter τ represents the angle of attack:

$$C_L = \frac{\pi}{2} A\tau \quad (30)$$

$$C_m = -\frac{\pi}{3} A\tau \quad (31)$$

$$C_D = \frac{\pi}{4} A\tau^2 \quad (32)$$

The nature of the functions L , M , and D of equations (22), (23), and (24) can be determined for the present special case by rewriting the above expressions in terms of the similarity parameters, thus

$$C_L = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} \quad L = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} \left\{ \frac{\pi}{2} \frac{[(\gamma+1)\tau]^{1/3}}{\sqrt{1-M_0^2}} (\sqrt{1-M_0^2} A) \right\} \quad (33)$$

$$C_m = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} \quad M = - \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} \left\{ \frac{\pi}{3} \frac{[(\gamma+1)\tau]^{1/3}}{\sqrt{1-M_0^2}} (\sqrt{1-M_0^2} A) \right\} \quad (34)$$

$$C_D = \frac{\tau^{5/3}}{(\gamma+1)^{1/3}} \quad D = \frac{\tau^{5/3}}{(\gamma+1)^{1/3}} \left\{ \frac{\pi}{4} \frac{[(\gamma+1)\tau]^{1/3}}{\sqrt{1-M_0^2}} (\sqrt{1-M_0^2}) \right\} \quad (35)$$

These results furnish an interesting contrast with the two-dimensional results described previously. Consider, for instance, the lift and drag coefficients of very thin flat-plate wings. The two-dimensional results of von Kármán show that, at sonic speed, the lift is proportional to the two-thirds power and the drag to the five-thirds power of the angle of attack. With wings of low aspect ratio, however, the above example shows the more familiar relationship of the lift being proportional to the angle of attack and the drag to the square of the angle of attack.

COMPARISON WITH SIMILARITY LAWS OF LINEARIZED SUBSONIC AND SUPERSONIC WING THEORY

It is of interest to compare the transonic similarity rules with the similarity rules of linearized subsonic and supersonic wing theory. The analysis for linearized theory proceeds as follows. The differential equation is

$$(1-M_0^2) \frac{\partial^2 \bar{\Phi}}{\partial x^2} + \frac{\partial^2 \bar{\Phi}}{\partial y^2} + \frac{\partial^2 \bar{\Phi}}{\partial z^2} = 0 \quad (36)$$

The boundary conditions at infinity are given by

$$\left(\frac{\partial \bar{\phi}}{\partial x}\right)_0 = \left(\frac{\partial \bar{\phi}}{\partial y}\right)_0 = \left(\frac{\partial \bar{\phi}}{\partial z}\right)_0 = 0 \quad (37)$$

and at the wing surface, again approximated by specifying conditions in the $z=0$ plane, by

$$\left(\frac{\partial \bar{\phi}}{\partial z}\right)_{z=0} = U_0 \left(\frac{\partial z}{\partial x}\right)_{z=0} = U_0 \tau \frac{\partial}{\partial (x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \quad (38)$$

The pressure coefficient is given by

$$C_p = -\frac{2}{U_0} \frac{\partial \bar{\phi}}{\partial x} \quad (39)$$

Once again the procedure consists of transforming the differential equation and boundary conditions into a system with primed quantities related to the original quantities as follows:

$$\left. \begin{aligned} x' &= s_x x & y' &= s_y y & z' &= s_z z \\ \bar{\phi}' &= s_{\bar{\phi}} \bar{\phi} & \sqrt{(1-M_0^2)'} &= s_{\beta} \sqrt{1-M_0^2} & U_0' &= s_U U_0 \end{aligned} \right\} \quad (40)$$

From the differential equation in the primed system

$$\frac{s_x^2}{s_{\beta}^2 s_{\bar{\phi}}} \left[(1-M_0^2)' \frac{\partial^2 \bar{\phi}'}{\partial x'^2} \right] + \frac{s_y^2}{s_{\bar{\phi}}} \frac{\partial^2 \bar{\phi}'}{\partial y'^2} + \frac{s_z^2}{s_{\bar{\phi}}} \frac{\partial^2 \bar{\phi}'}{\partial z'^2} = 0 \quad (41)$$

two relationships between the stretching factors may be found. They are

$$\frac{s_y s_{\beta}}{s_x} = 1 \quad \frac{s_y}{s_z} = 1 \quad (42)$$

An immediate consequence of this transformation is that the aspect ratios of wings in similar flow fields are related according to the following expression, identical to equation (18) for the transonic case:

$$\sqrt{(1-M_0^2)'} A' = \sqrt{1-M_0^2} A \quad (43)$$

Since $\bar{\phi}'$ is proportional to $\bar{\phi}$, the boundary conditions at infinity are automatically satisfied. The boundary conditions at the wing may be given in either of two forms

$$\left(\frac{\partial \bar{\phi}'}{\partial z'}\right)_{z'=0} = \frac{s\bar{\phi}}{s_z} \left(\frac{\partial \bar{\phi}}{\partial z}\right)_{z=0} = \frac{s\bar{\phi}}{s_z} U_0 \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \quad (44a)$$

$$\left(\frac{\partial \bar{\phi}'}{\partial z'}\right)_{z'=0} = U_0' \tau' \frac{\partial}{\partial(x'/c')} f'\left(\frac{x'}{c'}, \frac{y'}{b'}\right) = s_U U_0 \tau' \frac{\partial}{\partial(x'/c')} f\left(\frac{x'}{c'}, \frac{y'}{b'}\right) \quad (44b)$$

whence, if the two wings have the same ordinate-distribution functions, that is, $f\left(\frac{x}{c}, \frac{y}{b}\right) = f'\left(\frac{x'}{c'}, \frac{y'}{b'}\right)$, the ordinate-amplitude parameters are related as follows:

$$\tau' = \frac{s\bar{\phi}}{s_U s_z} \tau = \frac{s\bar{\phi}}{s_U s_x} \frac{\sqrt{(1-M_0^2)'}}{\sqrt{1-M_0^2}} \tau = \frac{\lambda}{\lambda'} \frac{\sqrt{(1-M_0^2)'}}{\sqrt{1-M_0^2}} \tau \quad (45)$$

where λ/λ' is a constant equal to $s\bar{\phi}/s_U s_x$. The relationship between the pressure coefficients at corresponding points is given by

$$C_p' = -\frac{2}{U_0'} \frac{\partial \bar{\phi}'}{\partial x'} = \frac{s\bar{\phi}}{s_U s_x} \left(-\frac{2}{U_0} \frac{\partial \bar{\phi}}{\partial x}\right) = \frac{s\bar{\phi}}{s_U s_x} C_p = \frac{\lambda}{\lambda'} C_p \quad (46)$$

or more completely by

$$C_p' \left(\frac{\sqrt{(1-M_0^2)'}}{\lambda' \tau'}, \sqrt{(1-M_0^2)'} A'; \frac{x'}{c'}, \frac{y'}{b'}\right) = \frac{\lambda}{\lambda'} C_p \left(\frac{\sqrt{1-M_0^2}}{\lambda \tau}, \sqrt{1-M_0^2} A; \frac{x}{c}, \frac{y}{b}\right) \quad (47)$$

In this analysis, λ has remained a completely arbitrary coefficient to be selected as best suits the particular problem at hand. For instance, the compressible-flow relationships between two wings having

identical pressure distributions are found by setting $\lambda=1$. If, on the other hand, λ is set equal to $\sqrt{1-M_0^2}$, the thickness ratios, camber, and angle of attack of the two wings are equal. This degree of arbitrariness in the similarity laws for the subsonic and supersonic flow about wings is in contrast to the case for transonic flow in which no undetermined coefficient like λ is to be found.

To facilitate comparison with the transonic similarity rules, the similarity rules given by linearized theory for the pressure, lift, pitching-moment, and drag coefficients of wings in subsonic or supersonic flow will be expressed in a form analogous to that of equations (21) through (24), thus

$$C_p = \frac{1}{\lambda} \bar{P} \left(\frac{\sqrt{1-M_0^2}}{\lambda\tau}, \sqrt{1-M_0^2} A; \frac{x}{c}, \frac{y}{b} \right) \quad (48)$$

$$C_p = \frac{1}{\lambda} \bar{L} \left(\frac{\sqrt{1-M_0^2}}{\lambda\tau}, \sqrt{1-M_0^2} A \right) \quad (49)$$

$$C_m = \frac{1}{\lambda} \bar{M} \left(\frac{\sqrt{1-M_0^2}}{\lambda\tau}, \sqrt{1-M_0^2} A \right) \quad (50)$$

$$C_D = \frac{\tau}{\lambda} \bar{D} \left(\frac{\sqrt{1-M_0^2}}{\lambda\tau}, \sqrt{1-M_0^2} A \right) \quad (51)$$

It is seen that the similarity rules of linearized subsonic and supersonic wing theory may be expressed in a wide variety of forms depending upon the choice of the parameter λ . The question then arises: Amongst the many possible representations of the similarity rules of subsonic and supersonic wing theory, is there one which coincides with the transonic similarity rule? It would be useful if there were such a representation since then the results of experiments with families of related wings could be correlated throughout the entire Mach number range by plotting the results as a function of two parameters $\sqrt{1-M_0^2} A$ and $[(\gamma+1)\tau]^{1/3}/\sqrt{1-M_0^2}$.

It can indeed be shown that there is such a representation of the similarity rules of subsonic and supersonic flow. Thus, by letting

$$\lambda = \frac{(\gamma+1)^{1/3}}{\tau^{2/3}} \quad (52)$$

it can be seen that the expressions for the pressure, lift, pitching-moment, and drag coefficients are identical in form to the corresponding equations for transonic flow (equations (21) through (24)).

CONCLUDING REMARKS

Similarity laws for transonic flow about thin wings of finite span have been obtained by considering the equations of transonic small-perturbation potential theory. By this means, functional relations for the pressure, lift, pitching-moment, and drag coefficients were derived and compared with the corresponding relations given by von Kármán for two-dimensional transonic flow.

It was shown that similarity of transonic flows about wings of finite span depended on two parameters: First, $\sqrt{1-M_0^2}/[(\gamma+1)\tau]^{1/3}$ as for two-dimensional flows and second, $\sqrt{1-M_0^2} A$. The similarity laws reduced for wings of infinite aspect ratio to the form given by von Kármán and showed, for example, that at sonic speed the lift of thin, flat airfoils is proportional to the two-thirds power and the drag to the five-thirds power of the angle of attack. For thin, flat wings of vanishing aspect ratio, however, the more familiar relationships of the lift being proportional to the first power and the drag to the second power of the angle of attack were found. Equally simple results for wings of intermediate aspect ratio could not be ascertained and only the similarity rules were given.

The corresponding similarity laws of linearized subsonic and supersonic wing theory were also derived and compared with the transonic similarity rules. The similarity laws of linearized theory contained an arbitrary parameter and could therefore be expressed in many ways. It was shown that one of these forms coincides with the transonic similarity laws.

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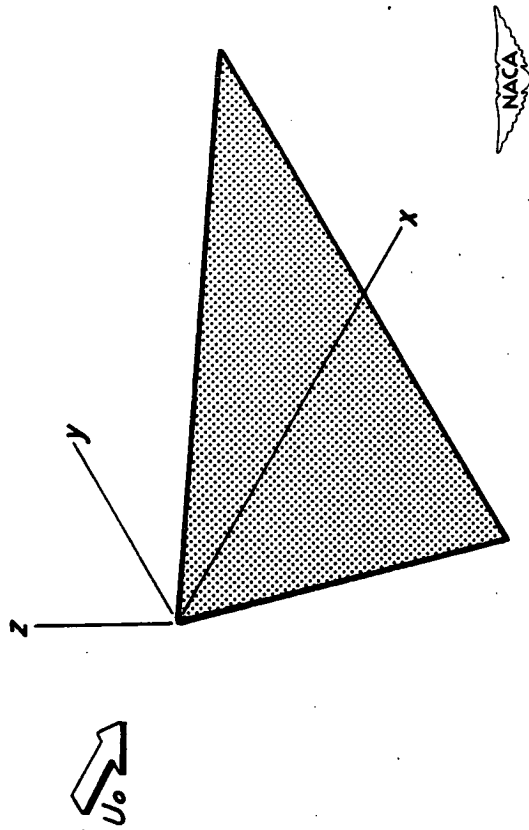
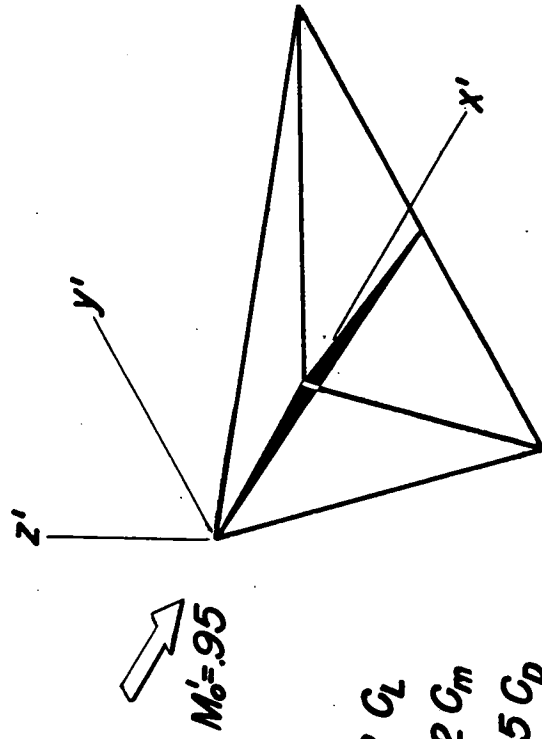


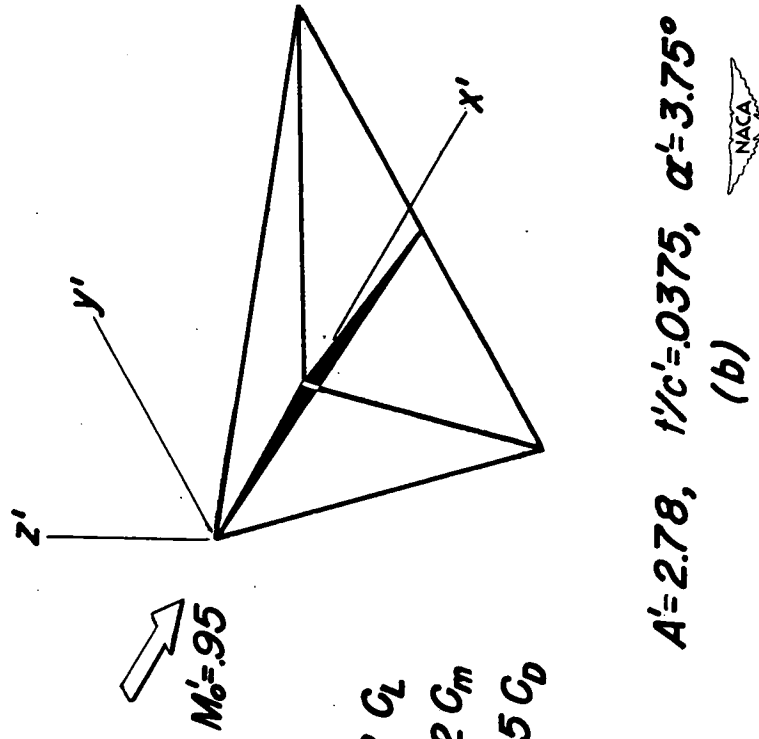
Figure 1.- View of wing showing coordinate axes.



$C_L' = .52 C_L$
 $C_m' = .52 C_m$
 $C_D' = .195 C_D$

$A = 2.0, t/c = .10, \alpha = 10^\circ$

(a)



$A = 2.78, t/c = .0375, \alpha = 3.75^\circ$

(b)



Figure 2.- Related wings in transonic flow.