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ANALYTICAL AND EXPERIMENTAL INVESTIGATION OF EFFECT OF TWIST ON VIBRATIONS OF CANTILEVER BEAMS

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SUMMARY

An analytical and experimental investigation was made of the effect of twist on the vibrations of cantilever beams. The analytical investigation was made by the use of Station Functions. General equations are developed for the coupled bending - bending-torsion vibrations of a cantilever beam, and it is shown how these equations reduce in simpler cases. The use of tabulated Station Numbers makes the analytical method presented herein particularly simple to apply.

An example is worked out in detail to illustrate the application of the method. Good agreement is obtained among the method presented, an exact theoretical solution developed herein, and experimental results.

It is shown that for a beam with a ratio of bending stiffness in the two principal directions equal to 144, the effect of coupling due to twist is to raise the value of the first natural frequency by a negligibly small amount, to decrease steadily the second frequency, and to lower the third frequency considerably.

INTRODUCTION

The failure of turbine and compressor blades due to vibrations has led to an increased interest in the study of the vibrations of these blades and in the determination of the natural modes and frequencies. In such theoretical studies, the compressor or turbine blade is usually assumed to act as a cantilever beam. The calculation of the uncoupled modes of arbitrarily shaped cantilever beams has been extensively investigated, but little work has as yet been done on calculating the coupled modes of such beams. If the geometry of the beam is such that coupling exists, the coupled modes are the actual vibrational modes that must be calculated.

There are three types of coupling that generally need to be considered: (1) coupling between bending and torsion, (2) coupling between bending in two directions due to natural twist, and (3) coupling between bending in two directions and torsion. Bending-torsion coupling has been studied by several investigators (references 1 to 4), but little work has been done on coupling involving twist.

The earliest work on the vibrations of twisted beams involved propeller blades and the general practice was to neglect the effect of twist. Later investigations include the effect of twist by applying the methods of static influence coefficients (references 5 and 8) or integral equations (references 6 and 7).

In the use of integral equations, great difficulty is encountered in the calculation of the higher modes of vibration because exact orthogonality conditions are required to prevent convergence back to the lowest mode. The influence coefficient method reduces the problem to one having a finite number of degrees of freedom. The accuracy of the higher modes, however, is poor. In using such a finite number of degrees of freedom, only the first third of the modes and first half of the frequencies obtained are within the usual engineering accuracy.

The present investigation was made at the NACA Lewis laboratory in order to present a straightforward accurate method for determining the coupled modes and frequencies of nonuniform twisted cantilever beams and to determine the effect of twist on the vibrational frequencies of such beams. The method is based on the use of Station Functions as developed for uncoupled and coupled bending-torsion vibrations in reference 4 and first discussed in reference 9. Incorporated in the method are the advantages of the continuous-function deflections of the Rayleigh-Ritz and Stodola methods together with the advantages of the finite number of degrees of freedom of the influence-coefficient method. Use is made of the Station Numbers derived in reference 4 and tabulated herein for convenience. An example is given and comparisons are made with exact theoretical and experimental values.

ANALYSIS

The present analysis of the vibrations of twisted cantilever beams is carried out in two ways: (1) an approximate solution using Station Functions and (2) an exact solution of the differential equations of equilibrium of the system.

The approximate solution using Station Functions for the general case of coupled bending - bending-torsion vibrations is presented herein. It is shown how the general equations reduce to the special cases of bending-bending, bending-torsion, bending, and torsion vibrations.

All symbols used in this section are listed and defined in appendix A. The exact solution and the derivation of the Station Function equations presented are discussed in appendixes B and C.

In the present analysis, matrix notation is used entirely. The use of matrix algebra reduces the amount of labor considerably. The only matrix operations needed are multiplication and addition. For those unfamiliar with the multiplication and addition of matrices, these simple processes are described in appendix D. The use of matrices, however, is not necessary. The scalar equations given in appendix B can be used for all the computations.

Bending - bending-torsion vibrations. - It is shown in appendix B that the deflections of the reference stations for a beam divided into n intervals of length δ, as shown in figure 1, are given by the following matric equation:

$$\begin{bmatrix} \psi \\ Y \\ X \end{bmatrix} = \frac{\omega^2 m_0 \delta^4}{EI_{x0}} \begin{bmatrix} \Gamma D_\theta & \epsilon \Gamma D_{\theta y} & \epsilon \Gamma D_{\theta x} \\ D_{y\theta} & D_y & D_{yx} \\ D_{x\theta} & D_{xy} & D_x \end{bmatrix} \begin{bmatrix} \psi \\ Y \\ X \end{bmatrix} \tag{1}$$

where the elements of these matrices are themselves matrices given as follows:

$$
\begin{aligned}
D_\theta &\equiv T_1 \, (KL + HM) \\
D_{\theta y} &\equiv T_1 \, (K_y \, L' + H_y \, M') \\
D_{\theta x} &\equiv T_1 \, (K_x \, L' + H_x \, M') \\
D_y &\equiv T_2 \, F_y \, P' - T_1 \, F_y \, Q' + T_3 \, G_y \, N' + T_4 \, G_y \, M' \\
D_x &\equiv T_2 \, F_x \, P' - T_1 \, F_x \, Q' + T_3 \, G_x \, N' + T_4 \, G_x \, M' \\
D_{yx} = D_{xy} &\equiv T_2 \, F_{xy} \, P' - T_1 \, F_{xy} \, Q' + T_3 \, G_{xy} \, N' + T_4 \, G_{xy} \, M' \\
D_{y\theta} &\equiv T_2 \, F_{y\theta} \, P - T_1 \, F_{y\theta} \, Q + T_3 \, G_{y\theta} \, N + T_4 \, G_{y\theta} \, M \\
D_{x\theta} &\equiv T_2 \, F_{x\theta} \, P - T_1 \, F_{x\theta} \, Q + T_3 \, G_{x\theta} \, N + T_4 \, G_{x\theta} \, M
\end{aligned}
\tag{2}
$$

The matrices L, M, N, P, Q, L', M', N', P', and Q' are matrices of Station Numbers. These matrices are functions only of n, the number of stations, and have been tabulated in tables I to VIII for n equal 1 to 8.

The matrices T_1, T_2, T_3, and T_4 are standard matrices listed in table IX. The first n rows and columns of these matrices are used, where n is the number of stations taken along the beam.

The H, K, F, and G matrices involve the physical properties of the beam. The element of the i^{th} row and j^{th} column of each of these matrices is given as follows:

Let

$$
\gamma_{ij} \equiv \begin{cases} 1 & \text{for } j > i \\ 0 & \text{for } j \leq i \end{cases}
$$

$$
\delta_{ij} \equiv \begin{cases} 1 & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases}
$$

Then

$$K_{ij} = \frac{I_i}{C_i} \delta_{ij}$$

$$K_{yij} = \frac{S_{y_i}}{C_i} \delta_{ij}$$

$$K_{xij} = \frac{S_{x_i}}{C_i} \delta_{ij}$$

$$H_{ij} = \frac{I_j}{C_i} \gamma_{ij}$$

$$H_{xij} = \frac{S_{x_j}}{C_i} \gamma_{ij}$$

$$H_{yij} = \frac{S_{y_j}}{C_i} \gamma_{ij}$$

$$F_{yij} = f_{yi} m_i \delta_{ij}$$

$$F_{xij} = f_{xi} m_i \delta_{ij}$$

$$F_{xyij} = - f_{xyi} m_i \delta_{ij}$$

$$G_{yij} = f_{yi} m_j \gamma_{ij}$$

$$G_{xij} = f_{xi} m_j \gamma_{ij}$$

$$G_{xyij} = - f_{xyi} m_j \gamma_{ij}$$

$$F_{y\theta ij} = (f_{yi} S_{yj} - f_{xyi} S_{xj}) \delta_{ij}$$

$$F_{x\theta ij} = (f_{xi} S_{xj} - f_{xyi} S_{yj}) \delta_{ij}$$

$$G_{y\theta ij} = (f_{yi} S_{yj} - f_{xyi} S_{xj}) \gamma_{ij}$$

$$G_{x\theta ij} = (f_{xi} S_{xj} - f_{xyi} S_{yj}) \gamma_{ij}$$

The matrices ψ, Y, and X are column matrices given by

$$\psi \equiv r_{y_0}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \theta_i \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad Y \equiv \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ y_i \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad X \equiv \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ x_i \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

The corresponding scalar equations, equations (B5) and (B6), can be found in appendix B.

The labor involved in the calculation of the D matrices of equation (2) can be considerably reduced if the equations are expressed in the following partitioned form:

$$D_\theta = T_1\begin{bmatrix} K & H \end{bmatrix}\begin{bmatrix} L \\ M \end{bmatrix}$$

$$D_{\theta y} = T_1\begin{bmatrix} K_y & H_y \end{bmatrix}\begin{bmatrix} L' \\ M' \end{bmatrix}$$

$$D_{\theta x} = T_1\begin{bmatrix} K_x & H_x \end{bmatrix}\begin{bmatrix} L' \\ M' \end{bmatrix}$$

$$D_y = \begin{bmatrix} T_2 & T_1 \end{bmatrix}\begin{bmatrix} F_y & 0 \\ 0 & F_y \end{bmatrix}\begin{bmatrix} P' \\ -Q' \end{bmatrix} + \begin{bmatrix} T_3 & T_4 \end{bmatrix}\begin{bmatrix} G_y & 0 \\ 0 & G_y \end{bmatrix}\begin{bmatrix} N' \\ M' \end{bmatrix}$$

$$D_x = \begin{bmatrix} T_2 & T_1 \end{bmatrix}\begin{bmatrix} F_x & 0 \\ 0 & F_x \end{bmatrix}\begin{bmatrix} P' \\ -Q' \end{bmatrix} + \begin{bmatrix} T_3 & T_4 \end{bmatrix}\begin{bmatrix} G_x & 0 \\ 0 & G_x \end{bmatrix}\begin{bmatrix} N' \\ M' \end{bmatrix}$$

$$D_{yx} = D_{xy} = \begin{bmatrix} T_2 & T_1 \end{bmatrix} \begin{bmatrix} F_{xy} & 0 \\ 0 & F_{xy} \end{bmatrix} \begin{bmatrix} P' \\ -Q' \end{bmatrix} + \begin{bmatrix} T_3 & T_4 \end{bmatrix} \begin{bmatrix} G_{xy} & 0 \\ 0 & G_{xy} \end{bmatrix} \begin{bmatrix} N' \\ M' \end{bmatrix}$$

$$D_{y\theta} = \begin{bmatrix} T_2 & T_1 \end{bmatrix} \begin{bmatrix} F_{y\theta} & 0 \\ 0 & F_{y\theta} \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix} + \begin{bmatrix} T_3 & T_4 \end{bmatrix} \begin{bmatrix} G_{y\theta} & 0 \\ 0 & G_{y\theta} \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix}$$

$$D_{x\theta} = \begin{bmatrix} T_2 & T_1 \end{bmatrix} \begin{bmatrix} F_{x\theta} & 0 \\ 0 & F_{x\theta} \end{bmatrix} \begin{bmatrix} P \\ -Q \end{bmatrix} + \begin{bmatrix} T_3 & T_4 \end{bmatrix} \begin{bmatrix} G_{x\theta} & 0 \\ 0 & G_{x\theta} \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} \qquad (2a)$$

With $\frac{1}{\omega^2} \frac{EI_{x0}}{m_0 \delta^4} \equiv \lambda$, the characteristic equation for the system becomes

$$\begin{vmatrix} \Gamma D_\theta - \lambda I & \epsilon \Gamma D_{\theta y} & \epsilon \Gamma D_{\theta x} \\ D_{y\theta} & D_y - \lambda I & D_{yx} \\ D_{x\theta} & D_{xy} & D_x - \lambda I \end{vmatrix} = 0 \qquad (3)$$

or

$$\left| D - \lambda I \right| = 0$$

where D is the matrix indicated, the dynamical matrix of the system, and I is the identity matrix. The corresponding scalar equations (B7) can be found in appendix B.

If the physical properties of the beam under consideration are known for each of the n intervals, the F, G, H, and K matrices denoting these physical properties can be written. The D matrices can be calculated by straightforward matrix computation using equations (2) and (2a) and tables I to IX. Any method available can then be used to solve equation (3) for the $3n$ values of λ.

Special cases. - For the cases where certain couplings between modes are negligible, equation (3) reduces to the following special cases:

(1) Bending-bending vibrations. With $\frac{1}{\omega^2}\frac{EI_{x0}}{m_0\delta^4} \equiv \lambda$,

$$\begin{vmatrix} D_y-\lambda I & D_{xy} \\ D_{yx} & D_x-\lambda I \end{vmatrix} = 0 \tag{3a}$$

(2) Bending-torsion vibrations. With $\frac{1}{\omega^2}\frac{EI_{x0}}{m_0\delta^4} \equiv \lambda$,

$$\begin{vmatrix} \Gamma D_\theta-\lambda I & \epsilon\,\Gamma D_{\theta y} \\ D_{y\theta} & D_y-\lambda I \end{vmatrix} = 0 \tag{3b}$$

(3) Uncoupled bending vibrations. With $\frac{1}{\omega^2}\frac{EI_{x0}}{m_0\delta^4} \equiv \lambda$,

$$\left| D_y - \lambda I \right| = 0 \tag{3c}$$

(4) Uncoupled torsion vibrations. With $\frac{1}{\omega^2}\frac{C_0}{I_0\delta^2} \equiv \lambda$,

$$\left| D_\theta - \lambda I \right| = 0 \tag{3d}$$

The determinantal equations (3) can be solved for the modes and the frequencies of the type vibration under consideration.

EXAMPLES

In applying the previously discussed method, the elements of the dynamical matrix must be determined for a given beam. These quantities depend upon the physical properties of the beam and upon the number of stations chosen. If the physical properties of the beam are known, the dynamical matrix can be calculated directly from equations (2) or (2a). The matrices T_1, T_2, T_3, T_4, L, M, N, P, Q, L', M', N', P', and Q' appearing in equations (2) depend only upon the number of stations n used and can be read directly from tables I to IX. Then, one of equations (3) can be solved for the modes and the frequencies of vibration by any method desired. The following detailed example illustrates the method for coupled bending - bending-torsion vibrations:

Consider a cantilever beam with I_η/I_ξ, the ratio of bending stiffness in the ξ-direction to bending stiffness in the η-direction, equal to 100, and having linear twist, with total angle of twist equal to 1 radian. Let $\Gamma \equiv \frac{1}{\delta^2} \frac{I_0}{C_0} \frac{EI_{x0}}{m_0} = 0.03$, let r_η/r_ξ equal 10, and let ϵ, the coefficient of coupling between torsion and bending in the η-direction, $r_{y0}{}^2/r_{g0}{}^2$, equal 0.2.

Let the number of stations chosen be two $(n = 2)$. The calculation of the elements of the F, G, H, and K matrices is shown in table X. The multiplication of these matrices by the appropriate L, M, N, P, Q, L', M', N', P', and Q' matrices from table II is also shown, and the determinantal equation is calculated. The roots of this 6×6 determinant can be computed in any manner desired.

APPLICATIONS AND RESULTS

The method was applied to determine the effect of twist upon the natural frequencies of a cantilever beam. Calculations were performed upon a beam such as that shown in figure 2. The ratio I_η/I_ξ was chosen as 144. Calculations were made for several values of total twist, ranging from 0 radians to 1.0 radian. The first, second, and third frequencies of vibration were obtained in each case.

The procedure for obtaining the exact theoretical values is derived in appendix C. A determination of the first natural frequency was made for I_η/I_ξ equal to 144, and having φ_t, the total angle of twist, equal to 0.70 radian.

Several beams having length equal to $6\frac{1}{2}$ inches and $I_\eta/I_\xi = 144$ were machined, mounted, and vibrated to determine their natural frequencies. These frequencies were determined to within ± 1 percent. Several values of total twist were taken.

The results of the calculations performed are plotted in figure 3 for the first, second, and third modes of vibration. The effect of coupling due to twist on the first mode for this beam is seen to raise the natural frequency to a value slightly higher than it would have if there were no twist; for a total angle of twist of 0.75 radian, the percentage increase is 0.5 percent. This difference is less than the experimental error.

Coupling due to twist has a more pronounced effect upon the higher modes of vibration. For the second mode, the natural frequency decreases to approximately 82 percent of its untwisted value for a total angle of twist equal to 0.75 radian and decreases to approximately 72 percent of its untwisted value for a total angle of twist equal to 1.0 radian.

The natural frequency of the third mode also decreases. For a total angle of twist of 0.75 radian, the frequency is approximately 80 percent of the untwisted value. For this mode, the percentage decrease first becomes larger and then smaller as the total angle of twist is increased through the range investigated.

The experimental results are also shown in figure 3. The beams tested had total angles of twist equal to 0.11, 0.39, 0.6, and 0.76 radian. The first mode changes very little. For the second mode, the percentages of the untwisted value for the frequency are 101, 93, 88, and 75 percent. The third-mode frequencies could not be obtained with any degree of certainty.

In addition, experimental data taken from reference 10 have been interpolated for a value of $I_\eta/I_\xi = 144$ and are plotted in figure 3. Agreement between the theoretical and experimental results, in general, is very good.

CONCLUDING REMARKS

Calculations based on the method presented to determine the effect of coupling due to twist on the natural frequencies of twisted cantilever beams were performed. For a beam with ratio of bending stiffness in the two principal directions equal to 144, the first natural frequency is raised by a negligibly small amount; for the second mode, the natural frequency decreases greatly as the twist is increased; the effect of coupling due to twist for the third mode is also to lower its frequency considerably.

A comparison among the method presented, an exact theoretical solution, and experimental results was made for a specific case; good agreement resulted.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, October 17, 1950.

APPENDIX A

SYMBOLS

The following symbols are used in the report:

C_0	torsional stiffness of beam at root
C_i	ratio of average torsional stiffness of i^{th} interval to torsional stiffness at root
E	Young's modulus
$f_j(\zeta)$, $g_j(\zeta)$	Station Function for torsion and bending, respectively, associated with j^{th} station
f_x, f_y, f_{xy}	$I_x/EI_\xi I_\eta$, $I_y/EI_\xi I_\eta$, and $I_{xy}/EI_\xi I_\eta$, respectively
f_{xi}, f_{yi}, f_{xyi}	average value of $I_xI_{x0}/I_\xi I_\eta$, $I_yI_{x0}/I_\xi I_\eta$, and $I_{xy}I_{x0}/I_\xi I_\eta$ over i^{th} interval
I	identity matrix
I_0	mass moment of inertia per unit length of beam at root
I_i	ratio of average mass moment of inertia per unit length of beam of i^{th} interval to mass moment of inertia per unit length at root
I_x, I_y, I_ξ, I_η	moment of inertia of cross section of beam about x-, y-, ξ-, and η-axes, respectively, function of z
I_{x0}	moment of inertia of cross section of beam about x-axis at root
i, j, k, r	summation indices
l	length of beam
m	mass per unit length of beam, function of z
m_0	mass per unit length of beam at root

m_i	ratio of average mass per unit length of i^{th} interval to mass per unit length at root
n	number of intervals
r_y, r_x, r_η, r_ξ	absolute magnitude of projection of distance from elastic axis to center of gravity on perpendicular to y, x, η, and ξ bending directions, respectively
r_{y0}	value of r_y at root of beam
S_x, S_y	static mass unbalance associated with x- and y-deflections, respectively, mr_x, mr_y, function of z
S_{xi}, S_{yi}	ratio of average static mass unbalance associated with x- and y-deflections, respectively, of i^{th} interval, to static mass unbalance associated with y-deflection at root
x,y	bending deflections in direction of x- and y-axis, respectively, of Cartesian coordinate system
x_i, y_i	bending deflection in direction of x- and y-axis, respectively, at i^{th} station
ξ, η	bending deflection in direction of minor and major axes of inertia, respectively, at cross section of beam
z	distance from root of beam
Γ	$(1/\delta^2)\ (I_0/C_0)\ (EI_{x0}/m_0)$
δ	length of interval between stations
ϵ	coupling coefficient, ${r_{y0}}^2/{r_{g0}}^2$
θ	angle of torsional vibration
λ	frequency parameter, latent root of dynamical matrices
ζ	dimensionless distance, $\frac{z}{\delta}$

APPENDIX B

STATION FUNCTIONS AND DETERMINANTAL EQUATIONS

In the usual influence-coefficient methods for solving dynamical problems, a continuous body having an infinite number of degrees of freedom is replaced by a body having a finite number of degrees of freedom. Two principal assumptions are then made, which introduce inaccuracies into the solution, particularly in the higher modes: (1) The inertial loads of all the infinitesimal masses in a finite interval can be replaced by a resultant that passes through the center of gravity of that interval; and (2) a concentrated load that is the resultant of a distributed load produces the same deflection as the distributed load.

In order to eliminate these assumptions, Rauscher (reference 9) introduced the concept of Station Functions. In this approach, it is assumed that the inertial loads and, consequently, the deflections are continuous functions along the beam. The value of these continuous deflection functions at n reference stations must equal the deflections of these reference stations. The loading on the beam is therefore a continuous function of the deflections of the reference stations. Inasmuch as the deflections of the reference stations can be computed from the loading on the beam, which in turn is available from the deflections, the deflections are therefore obtained as functions of themselves.

In the method employed here, the deflection of a continuous beam is approximated by the sum of a set of continuous Station Functions. Each of these functions is chosen: (1) to satisfy the boundary conditions of the problem, (2) to vanish at all stations other than the one with which it is associated, and (3) to equal 1 at the station with which it is associated. In terms of these functions, the deflections are

$$\left.\begin{aligned} \theta &= \sum_{j=1}^{n} f_j(\zeta)\ \theta_j \\ y &= \sum_{j=1}^{n} g_j(\zeta)\ y_j \\ x &= \sum_{j=1}^{n} g_j(\zeta)\ x_j \end{aligned}\right\} \qquad \text{(B1)}$$

where

ζ	dimensionless distance along beam, z/δ
θ, y, x	torsional deflection, bending deflections in y- and x-directions, respectively, functions of ζ
θ_j, y_j, x_j	torsional deflection, bending deflections in y- and x-directions, respectively, at j^{th} station
$f_j(\zeta)$, $g_j(\zeta)$	Station Functions in torsion and bending, respectively, associated with j^{th} station

where

$$
\left.
\begin{aligned}
f_j(\zeta) &= \frac{\prod\limits_{j \neq i} (\zeta - j)\,\zeta\,(\zeta - a_1)}{\prod\limits_{j \neq i} (i - j)\,i\,(i - a_1)} \\
g_j(\zeta) &= \frac{\prod\limits_{j \neq i} (\zeta - j)\;\zeta^2\;(\zeta^2 + a_2\zeta + a_3)}{\prod\limits_{j \neq i} (i - j)\;i^2\;(i^2 + a_2 i + a_3)}
\end{aligned}
\right\} \qquad \text{(B2)}
$$

where $\prod\limits_{j \neq i}$ denotes the product for all values of j except $j = i$, and the constants a_1, a_2, and a_3 are determined to satisfy the boundary conditions at the free end of the beam, where the moments and the shear forces are zero.

For a given frequency of vibration, the influence coefficients for the system can be derived. The following equations are applicable at any of the stations:

$$\theta_i = \delta^2 \int_0^i q_t(\zeta) \int_0^\zeta \frac{d\zeta_1}{C(\zeta_1)} d\zeta + \delta^2 \int_i^n q_t(\zeta) \int_0^i \frac{d\zeta_1}{C(\zeta_1)} d\zeta$$

$$\begin{aligned}
y_i = {} & \delta^4 \int_0^i q_y(\zeta) \int_0^\zeta f_y (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1 + \\
& \delta^4 \int_i^n q_y(\zeta) \int_0^i f_y (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1 - \\
& \delta^4 \int_0^i q_x(\zeta) \int_0^\zeta f_{xy} (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1 - \\
& \delta^4 \int_i^n q_x(\zeta) \int_0^i f_{xy} (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1
\end{aligned} \tag{B3}$$

$$\begin{aligned}
x_i = {} & -\delta^4 \int_0^i q_y(\zeta) \int_0^\zeta f_{xy} (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1 - \\
& \delta^4 \int_i^n q_y(\zeta) \int_0^i f_{xy} (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1 + \\
& \delta^4 \int_0^i q_x(\zeta) \int_0^\zeta f_x (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1 + \\
& \delta^4 \int_i^n q_x(\zeta) \int_0^i f_x (\zeta_1 - \zeta)(i - \zeta)\, d\zeta\, d\zeta_1
\end{aligned}$$

where the loading functions $q_t(\zeta)$, $q_x(\zeta)$, and $q_y(\zeta)$ are

$$
\begin{aligned}
q_t &= I\omega^2\theta + S_x\omega^2 x + S_y\omega^2 y \\
q_x &= m\omega^2 x + S_x\omega^2\theta \\
q_y &= m\omega^2 y + S_y\omega^2\theta
\end{aligned}
\tag{B4}
$$

where

$C(\zeta_1)$ torsional stiffness of beam, function of ζ

I mass moment of inertia per unit length of beam, function of ζ

S_x, S_y static mass unbalance associated with x- and y-deflections, respectively, function of ζ

The integrals of equation (B3) may be written as the sum of integrals over each interval by substituting these values for the loading functions into equations (B3), using equations (B1) for the deflections, and assuming constant values of m, S_x, I, S_y, f_x, f_y, and f_{xy} over each interval. The order of summation can then be changed to yield

$$
\left.
\begin{aligned}
\psi_i &= \omega^2 \frac{m_0\delta^4}{EI_{x0}}\left(\Gamma\sum_{j=1}^{n} D_{\theta ij}\psi_i + \epsilon\Gamma\sum_{j=1}^{n} D_{\theta yij}\, y_j + \epsilon\Gamma\sum_{j=1}^{n} D_{\theta xij}\, x_j\right) \\
y_i &= \omega^2 \frac{m_0\delta^4}{EI_{x0}}\left(\sum_{j=1}^{n} D_{y\theta ij}\psi_i + \sum_{j=1}^{n} D_{yij}\, y_j + \sum_{j=1}^{n} D_{yxij}\, x_j\right) \\
x_i &= \omega^2 \frac{m_0\delta^4}{EI_{x0}}\left(\sum_{j=1}^{n} D_{x\theta ij}\psi_i + \sum_{j=1}^{n} D_{xyij}\, y_j + \sum_{j=1}^{n} D_{xij}\, x_j\right)
\end{aligned}
\right\}
\tag{B5}
$$

where

$$\psi_i \equiv r_{y0}\,\theta_i$$

$$D_{\theta ij} \equiv \sum_{k=1}^{i} \frac{1}{C_k}\left(I_k L_{kj} + \sum_{r=k+1}^{n} I_r M_{rj}\right)$$

$$D_{\theta yij} \equiv \sum_{k=1}^{i} \frac{1}{C_k}\left(S_{yk} L_{kj} + \sum_{r=k+1}^{n} S_{yr} M_{rj}\right) \tag{B6}$$

$$D_{\theta xij} \equiv \sum_{k=1}^{i} \frac{1}{C_k}\left(S_{xk} L_{kj} + \sum_{r=k+1}^{n} S_{xr} M_{rj}\right)$$

$$D_{yij} \equiv \sum_{k=1}^{i} f_{yk} \left\{ m_k(iP'_{kj}-Q'_{kj}) + \sum_{r=k+1}^{n} m_r \left[(i-k+1/2)N'_{rj} + \left(\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right) M'_{rj} \right] \right\}$$

$$D_{xij} \equiv \sum_{k=1}^{i} f_{xk} \left\{ m_k(iP'_{kj}-Q'_{kj}) + \sum_{r=k+1}^{n} m_r \left[(i-k+1/2)N'_{rj} + \left(\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right) M'_{rj} \right] \right\}$$

$$D_{yxij} \equiv - \sum_{k=1}^{i} f_{xyk} \left\{ m_k(iP'_{kj}-Q'_{kj}) + \sum_{r=k+1}^{n} m_r \left[(i-k+1/2)N'_{rj} + \left(\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right) M'_{rj} \right] \right\}$$

$$D_{y\theta ij} = - \sum_{k=1}^{i} f_{xyk} \left\{ S_{xk}(iP_{kj}-Q_{kj}) + \sum_{r=k+1}^{n} S_{xr} \left[(i-k+1/2)N_{rj} + \left(\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right) M_{rj} \right] \right\}$$

$$+ \sum_{k=1}^{i} f_{yk} \left\{ S_{yk}(iP_{kj}-Q_{kj}) + \sum_{r=k+1}^{n} S_{yr} \left[(i-k+1/2)N_{rj} + \left(\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right) M_{rj} \right] \right\}$$

$$D_{x\theta ij} = - \sum_{k=1}^{i} f_{xyk} \left\{ S_{yk}(iP_{kj}-Q_{kj}) + \sum_{r=k+1}^{n} S_{yr} \left[(i-k+1/2)N_{rj} + \left(\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right) M_{rj} \right] \right\}$$

$$+ \sum_{k=1}^{i} f_{xk} \left\{ S_{xk}(iP_{kj}-Q_{kj}) + \sum_{r=k+1}^{n} S_{xr} \left[(i-k+1/2)N_{rj} + \left(\frac{k^3-(k-1)^3}{3} - \frac{(2k-1)i}{2} \right) M_{rj} \right] \right\}$$

The station numbers L_{ij}, M_{ij}, N_{ij}, P_{ij}, Q_{ij}, L'_{ij}, M'_{ij}, N'_{ij}, P'_{ij}, and Q'_{ij} are the elements in the i^{th} row and j^{th} column of the Station Number matrices L, M, N, P, Q, L', M', N', P', and Q', respectively, which have been tabulated in tables I to VIII.

Equation (B5) can be written for n values of ψ_i, y_i, and x_i. This procedure yields $3n$ homogeneous equations; the determinant of the coefficients can then be equated to zero.

Where $\frac{1}{\omega^2} \frac{EI_{x0}}{m_0 \delta^4} \equiv \lambda$,

$$\begin{vmatrix}
\Gamma D_{\theta 11}-\lambda & \cdots & \Gamma D_{\theta 1n} & \epsilon\Gamma D_{\theta y11} & \cdots & \epsilon\Gamma D_{\theta y1n} & \epsilon\Gamma D_{\theta x11} & \cdots & \epsilon\Gamma D_{\theta x1n} \\
\Gamma D_{\theta 21} & \cdots & \Gamma D_{\theta 2n} & \epsilon\Gamma D_{\theta y21} & \cdots & \epsilon\Gamma D_{\theta y2n} & \epsilon\Gamma D_{\theta x21} & \cdots & \epsilon\Gamma D_{\theta x2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\Gamma D_{\theta n1} & \cdots & \Gamma D_{\theta nn}-\lambda & \epsilon\Gamma D_{\theta yn1} & \cdots & \epsilon\Gamma D_{\theta ynn} & \epsilon\Gamma D_{\theta yn1} & \cdots & \epsilon\Gamma D_{\theta ynn} \\
D_{y\theta 11} & \cdots & D_{y\theta 1n} & D_{y11}-\lambda & \cdots & D_{y1n} & D_{yx11} & \cdots & D_{yx1n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
D_{y\theta n1} & \cdots & D_{y\theta nn} & D_{yn1} & \cdots & D_{ynn}-\lambda & D_{yxn1} & \cdots & D_{yxnn} \\
D_{x\theta 11} & \cdots & D_{x\theta 1n} & D_{xy11} & \cdots & D_{xy1n} & D_{x11}-\lambda & \cdots & D_{x1n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
D_{x\theta n1} & \cdots & D_{x\theta nn} & D_{xyn1} & \cdots & D_{xynn} & D_{xn1} & \cdots & D_{xnn}-\lambda
\end{vmatrix} = 0 \qquad \text{(B7)}$$

With the matrices D_y, D_x, D_{yx}, D_{xy}, $D_{y\theta}$, $D_{x\theta}$, D_θ, $D_{\theta y}$, and $D_{\theta x}$ defined as in equation (2) of the analysis, the 3n homogeneous equations obtained from equation (B5) can be written in matric form (equation (1) of "Analysis"); the T-matrices are obtained when the scalar form is transformed.

APPENDIX C

EXACT SOLUTION

The differential equations of equilibrium of the cantilever beam can be derived from consideration of the forces and the moments acting on an elemental section of the beam. These forces and moments are shown in figure 4.

With $\sin d\varphi = d\varphi$ and $\cos d\varphi = 1$ and second-order differentials neglected, from the equilibrium of bending moments

$$\left.\begin{aligned} \frac{\partial M_\xi}{\partial z} - M_\eta \frac{d\varphi}{dz} - V_\xi &= 0 \\ \frac{\partial M_\eta}{\partial z} + M_\xi \frac{d\varphi}{dz} - V_\eta &= 0 \end{aligned}\right\} \tag{C1}$$

where

M_ξ, M_η bending moment about η- and ξ-axes, respectively, function of z

M_θ torsional moment, function of z

V_ξ, V_η shear force in ξ- and η-directions, respectively, function of z

From the equilibrium of forces,

$$\left.\begin{aligned} \frac{\partial V_\xi}{\partial z} &= m\omega^2 \xi + S_\xi \omega^2 \theta + V_\eta \varphi' \\ \frac{\partial V_\eta}{\partial z} &= m\omega^2 \eta + S_\eta \omega^2 \theta - V_\xi \varphi' \end{aligned}\right\} \tag{C2}$$

From the equilibrium of torsional moments,

$$\frac{\partial M_\theta}{\partial z} + I\omega^2 \theta + S_\xi \omega^2 \xi + S_\eta \omega^2 \eta = 0 \tag{C3}$$

Upon manipulation of equations (C1) and (C2) and the substitution of the relations

$$
\left.
\begin{aligned}
M_{\xi} &= EI_{\eta}\,\frac{\partial^2 \xi}{\partial z^2} \\
M_{\eta} &= EI_{\xi}\,\frac{\partial^2 \eta}{\partial z^2} \\
M_{\theta} &= C\,\frac{\partial \theta}{\partial z}
\end{aligned}
\right\}
\qquad (C4)
$$

where C is the torsional stiffness, a function of z,

$$
\left.
\begin{aligned}
&\frac{\partial^2}{\partial z^2}\left(EI_{\eta}\,\frac{\partial^2 \xi}{\partial z^2}\right) - \left(\frac{\partial \varphi}{\partial z}\right)^2 EI_{\eta}\,\frac{\partial^2 \xi}{\partial z^2} - m\omega^2 \xi - S_{\xi}\omega^2 \theta - \frac{2\partial \varphi}{\partial z}\,\frac{\partial}{\partial z}\left(EI_{\xi}\,\frac{\partial^2 \eta}{\partial z^2}\right) - \\
&\qquad \frac{\partial^2 \varphi}{\partial z^2}\,EI_{\xi}\,\frac{\partial^2 \eta}{\partial z^2} = 0 \\
&\frac{\partial^2}{\partial z^2}\left(EI_{\xi}\,\frac{\partial^2 \eta}{\partial z^2}\right) - \left(\frac{\partial \varphi}{\partial z}\right)^2 EI_{\xi}\,\frac{\partial^2 \eta}{\partial z^2} - m\omega^2 \eta - S_{\eta}\omega^2 \theta + \frac{2\partial \varphi}{\partial z}\,\frac{\partial}{\partial z}\left(EI_{\eta}\,\frac{\partial^2 \xi}{\partial z^2}\right) + \\
&\qquad \frac{\partial^2 \varphi}{\partial z^2}\,EI_{\eta}\,\frac{\partial^2 \varphi}{\partial z^2} = 0 \\
&\frac{\partial}{\partial z}\left(C\,\frac{\partial \theta}{\partial z}\right) + I\omega^2 \theta + S_{\xi}\omega^2 \xi + S_{\eta}\omega^2 \eta = 0
\end{aligned}
\right\}
\qquad (C5)
$$

Equations (C5) are the differential equations of equilibrium of a cantilever beam vibrating in coupled bending - bending-torsion motion.

<u>Exact solution.</u> - A uniform, linearly twisted cantilever beam, vibrating in coupled bending - bending-torsion motion is considered.

Using the change of variables

$$\left.\begin{aligned} Y_1 &\equiv EI_\eta\,\xi \\ Y_2 &\equiv EI_\xi\,\eta \\ Y_3 &\equiv EI_\eta r_\xi\,\theta \end{aligned}\right\} \tag{C6}$$

and letting

$$\varphi_t \equiv \frac{l}{z}\varphi$$

$$\zeta \equiv \frac{z}{l}$$

$$\frac{\omega^2}{EI_\xi/ml^4} = 12.36\,\frac{\omega^2}{\omega_\xi^{\;2}} \equiv \Omega$$

(The value of the constant is taken from reference 11.)

$$\frac{I_\eta}{I_\xi} \equiv 1/\gamma_b$$

$$\frac{Il^2/C}{ml^4/EI_\eta} = \left(\frac{\pi}{2}\right)^2 \frac{1}{12.36}\,\frac{\omega^2{}_\xi}{\omega_t^{\;2}} = 0.2\,\frac{\omega^2{}_\xi}{\omega_t^{\;2}} \equiv 1/\gamma_t$$

$$r_\eta/r_\xi \equiv R$$

$$r_\xi^{\;2}/r_g^{\;2} \equiv \epsilon$$

reduces equations (C5) to yield

$$\left.\begin{aligned} &Y_1{}^{iv} - \varphi_t^{\;2}Y_1{}'' - \Omega Y_1 - 2\varphi_t Y_2{}''' - \Omega Y_3 = 0 \\ &Y_2{}^{iv} - \varphi_t^{\;2}Y_2{}'' - \frac{\Omega}{\gamma_b}Y_2 + 2\varphi_t Y_1{}''' - R\Omega Y_3 = 0 \\ &Y_3{}'' + \frac{\Omega}{\gamma_t}Y_3 + \epsilon\frac{\Omega}{\gamma_t}Y_1 + R\epsilon\frac{\Omega}{\gamma_t\gamma_b}Y_2 = 0 \end{aligned}\right\} \tag{C7}$$

where primes denote successive differentiation with respect to ζ.

Let $Y_1 = D_i e^{\lambda_i \zeta}$, $Y_2 = E_i e^{\lambda_i \zeta}$, and $Y_3 = F_i e^{\lambda_i \zeta}$ be solutions to equation (C7).

Then

$$\left.\begin{aligned}
&(\lambda_i^4 - \varphi_t^2 \lambda_i^2 - \Omega)\, D_i - 2\varphi_t \lambda_i^3 E_i - \Omega F_i = 0 \\
&2\varphi_t \lambda_i^3 D_i + (\lambda_i^4 - \varphi_t^2 \lambda_i^2 - \Omega/\gamma_b)\, E_i - R\Omega F_i = 0 \\
&\frac{\epsilon}{\gamma_t}\Omega D_i + \frac{R\epsilon}{\gamma_b \gamma_t}\Omega E_i + \left(\lambda_i^2 + \frac{1}{\gamma_t}\Omega\right) F_i = 0
\end{aligned}\right\} \qquad \text{(C8)}$$

A tenth-order equation in λ^2 is obtained by equating the determinant of the coefficients of equation (C8) to zero. For the case where $R = 0$, that is, the center of torsion is on the axis of minimum moment of inertia, this equation reduces to a quintic in λ_i^2.

$$\begin{aligned}
&\lambda_i^{10} + (\Omega/\gamma_t + 2\varphi_t^2)\lambda_i^8 + \left[\varphi_t^4 - \Omega(1 + 1/\gamma_b - 2\varphi_t^2/\gamma_t)\right]\lambda\lambda_i^6 + \\
&\quad \Omega\left[\varphi_t^2 (1 + 1/\gamma_b) + \frac{\varphi_t^4 - (1 + 1/\gamma_b)\Omega}{\gamma_t} + \frac{\epsilon}{\gamma_t}\Omega\right]\lambda_i^4 + \\
&\quad \Omega^2\left[1/\gamma_b + \frac{\varphi_t^2(1 + 1/\gamma_b)}{\gamma_t} - \frac{\varphi_t^2\epsilon}{\gamma_t}\right]\lambda_i^2 + \frac{1-\epsilon}{\gamma_t\gamma_b}\Omega^3 = 0
\end{aligned} \qquad \text{(C9)}$$

Let the roots of equation (C9) be $\lambda_1, \lambda_{-1}, \lambda_2, \lambda_{-2}, \lambda_3, \lambda_{-3}, \lambda_4, \lambda_{-4}, \lambda_5$, and λ_{-5}, where

$$\lambda_i = -\lambda_{-i}, \quad i = 1, 2, \ldots 5$$

Then

$$\left.\begin{aligned}
Y_1 &= \sum_{\substack{i=-5 \\ i\neq 0}}^{5} D_i e^{\lambda_i \zeta} \\
Y_2 &= \sum_{\substack{i=-5 \\ i\neq 0}}^{5} E_i e^{\lambda_i \zeta} \\
Y_3 &= \sum_{\substack{i=-5 \\ i\neq 0}}^{5} F_i e^{\lambda_i \zeta}
\end{aligned}\right\} \tag{C10}$$

where

$$\left.\begin{aligned}
E_i &= \frac{-2\varphi_t \lambda_i^3}{(\lambda_i^4 - \varphi_t^2 \lambda^2 - \Omega/\gamma_b)} D_i \equiv G_i D_i \\
F_i &= \frac{-\epsilon\Omega}{\gamma_t \lambda_i^2 + \Omega} D_i \equiv H_i D_i
\end{aligned}\right\} \tag{C11}$$

$$i = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$$

The boundary conditions are

at $\zeta = 0$

$$Y_1(0) = Y_1'(0) = Y_2(0) = Y_2'(0) = Y_3(0) = 0$$

at $\zeta = 1$

$$Y_1''(1) = Y_1'''(1) = Y_2''(1) = Y_2'''(1) = Y_3'(1) = 0 \tag{C12}$$

The substitution of equation (C10) into the appropriate equations of (C12) will yield 10 homogeneous equations in the 10 variables D_i. For a nontrivial solution, the determinant of the coefficients must vanish. Thus:

$$
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\lambda_1 & -\lambda_1 & \lambda_2 & -\lambda_2 & \lambda_3 & -\lambda_3 & \lambda_4 & -\lambda_4 & \lambda_5 & -\lambda_5 \\
G_1 & G_{-1} & G_2 & G_{-2} & G_3 & G_{-3} & G_4 & G_{-4} & G_5 & G_{-5} \\
G_1\lambda_1 & -G_{-1}\lambda_1 & G_2\lambda_2 & -G_{-2}\lambda_2 & G_3\lambda_3 & -G_3\lambda_3 & G_4\lambda_4 & -G_4\lambda_4 & G_5\lambda_5 & -G_5\lambda_5 \\
H_1 & H_{-1} & H_2 & H_{-2} & H_3 & H_{-3} & H_4 & H_{-4} & H_5 & H_{-5} \\
\lambda_1^2 e^{\lambda_1} & \lambda_1^2 e^{-\lambda_1} & \lambda_2^2 e^{\lambda_2} & \lambda_2^2 e^{-\lambda_2} & \lambda_3^2 e^{\lambda_3} & \lambda_3^2 e^{-\lambda_3} & \lambda_4^2 e^{\lambda_4} & \lambda_4^2 e^{-\lambda_4} & \lambda_5^2 e^{\lambda_5} & \lambda_5^2 e^{-\lambda_5} \\
\lambda_1^3 e^{\lambda_1} & -\lambda_1^3 e^{-\lambda_1} & \lambda_2^3 e^{\lambda_2} & -\lambda_2^3 e^{-\lambda_2} & \lambda_3^3 e^{\lambda_3} & -\lambda_3^3 e^{-\lambda_3} & \lambda_4^3 e^{\lambda_4} & \lambda_4^3 e^{-\lambda_4} & \lambda_5^3 e^{\lambda_5} & \lambda_5^3 e^{-\lambda_5} \\
G_1\lambda_1^2 e^{\lambda_1} & G_{-1}\lambda_1^2 e^{-\lambda_1} & G_2\lambda_2^2 e^{\lambda_2} & G_{-2}\lambda_2^2 e^{-\lambda_2} & G_3\lambda_3^2 e^{\lambda_3} & G_{-3}\lambda_3^3 e^{-\lambda_3} & G_4\lambda_4^2 e^{\lambda_4} & G_{-4}\lambda_4^2 e^{-\lambda_4} & G_5\lambda_5^2 e^{\lambda_5} & G_{-5}\lambda_5^2 e^{-\lambda_5} \\
G_1\lambda_1^3 e^{\lambda_1} & -G_{-1}\lambda_1^3 e^{-\lambda_1} & G_2\lambda_2^3 e^{\lambda_2} & -G_{-2}\lambda_2^3 e^{-\lambda_2} & G_3\lambda_3^3 e^{\lambda_3} & -G_{-3}\lambda_3^3 e^{-\lambda_3} & G_4\lambda_4^3 e^{\lambda_4} & -G_{-4}\lambda_4^3 e^{-\lambda_4} & G_5\lambda_5^3 e^{\lambda_5} & -G_{-5}\lambda_5^3 e^{-\lambda_5} \\
H_1\lambda_1 e^{\lambda_1} & -H_{-1}\lambda_1 e^{-\lambda_1} & H_2\lambda_2 e^{\lambda_2} & -H_{-2}\lambda_2 e^{-\lambda_2} & H_3\lambda_3 e^{\lambda_3} & -H_{-3}\lambda_3 e^{-\lambda_3} & H_4\lambda_4 e^{\lambda_4} & -H_{-4}\lambda_4 e^{-\lambda_4} & H_5\lambda_5 e^{\lambda_5} & -H_{-5}\lambda_5 e^{-\lambda_5}
\end{vmatrix} = 0 \tag{C13}
$$

The process of solution is as follows: The value of the determinant in equation (C13) must be plotted against frequency; the value of frequency for which this determinant becomes zero is thereby obtained. This procedure involves first solving the quintic equation (C9) for each assumed value of frequency parameter Ω, and then calculating the elements of the determinant from equation (C11). The value of frequency for which the determinant of equation (C13) equals zero is the natural frequency of vibration of the beam.

The exact solution is obviously long and laborious. Numerical calculations have been made for the case where there is no torsional coupling with the bending modes. In this case, equation (C12) may be replaced by a biquadratic equation in λ_i^2, and the order of the determinant of equation (C16) is reduced to 8. The results of these calculations are included in the body of the report, where a comparison is made with values given by the approximate solution.

APPENDIX D

ELEMENTARY PROPERTIES OF MATRICES

A matrix is a set of numbers or other elements arranged in rows and columns that obey certain rules of addition and multiplication. Thus

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -4 & 9 \end{bmatrix}$$

is a matrix of numbers containing two rows and three columns.

$$\begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

is a matrix containing four rows and one column. It is called a column matrix. Similarly, a set of numbers arranged in a single row is designated a row matrix. A $n \times m$ matrix is one containing n rows and m columns.

Let an $n \times m$ matrix be designated A. Then the element in the i^{th} row and j^{th} column of A will be designated a_{ij} (the first subscript designating the row, the second subscript the column).

The operations of addition and multiplication of matrices are now defined as follows:

If a matrix A is added to a matrix B (B must have the same number of rows and columns as A), the result is a matrix C given by

$$C = A + B \quad \text{where} \quad c_{ij} = a_{ij} + b_{ij}$$

If an $n \times m$ matrix A is multiplied by a matrix B, the result is a matrix C given by

$$C = AB \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

that is, the element in the i^{th} row and j^{th} column of C is obtained by taking the sum of the products obtained by multiplying the elements of the i^{th} row of A with the corresponding elements of the j^{th} column of B.

It is to be noted that the order of multiplication is important, that is, in general, $BA \neq AB$, and that in order to multiply two matrices together, the number of columns of the first must equal the number of rows of the second.

Examples:

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -13 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 35 \\ -7 & -10 \end{bmatrix}$$

For a complete discussion of the properties of matrices, see reference 12.

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TABLE I - STATION NUMBERS, n = 1

$$L_{jk} = \begin{bmatrix}5/12\end{bmatrix} \quad M_{jk} = \begin{bmatrix}2/3\end{bmatrix} \quad N_{jk} = \begin{bmatrix}5/12\end{bmatrix} \quad P_{jk} = \begin{bmatrix}3/20\end{bmatrix} \quad Q_{jk} = \begin{bmatrix}7/180\end{bmatrix}$$

$$L'_{jk} = \begin{bmatrix}13/45\end{bmatrix} \quad M'_{jk} = \begin{bmatrix}2/5\end{bmatrix} \quad N'_{jk} = \begin{bmatrix}13/45\end{bmatrix} \quad P'_{jk} = \begin{bmatrix}71/630\end{bmatrix} \quad Q'_{jk} = \begin{bmatrix}31/1008\end{bmatrix}$$

TABLE II - STATION NUMBERS, n = 2

$$L = \begin{bmatrix}8/15 & -31/240 \\ 7/60 & 94/240\end{bmatrix} \quad M = \begin{bmatrix}11/12 & -13/48 \\ 5/12 & 29/48\end{bmatrix} \quad N = \begin{bmatrix}8/15 & -31/240 \\ 8/15 & 239/240\end{bmatrix}$$

$$P = \begin{bmatrix}0.183333 & -0.037500 \\ .025000 & .143750\end{bmatrix} \quad Q = \begin{bmatrix}0.046032 & -0.008135 \\ .029365 & .181448\end{bmatrix}$$

$$L' = \begin{bmatrix}0.367100 & -0.034875 \\ .224675 & .310011\end{bmatrix} \quad M' = \begin{bmatrix}0.536364 & -0.060795 \\ .627273 & .448674\end{bmatrix}$$

$$N' = \begin{bmatrix}0.367100 & -0.034875 \\ .851948 & .758685\end{bmatrix} \quad P' = \begin{bmatrix}0.137933 & -0.011252 \\ .057955 & .118463\end{bmatrix}$$

$$Q' = \begin{bmatrix}0.036616 & -0.002614 \\ .069733 & .150415\end{bmatrix}$$

TABLE III - STATION NUMBERS, n = 3

L

$$\begin{bmatrix} 0.545833 & -0.241667 & 0.106019 \\ .137500 & .450000 & -.078240 \\ -.020833 & .141667 & .376388 \end{bmatrix}$$

M

$$\begin{bmatrix} 0.950000 & -0.525000 & 0.235185 \\ .450000 & .725000 & -.153704 \\ -.050000 & .475000 & .568519 \end{bmatrix}$$

N

$$\begin{bmatrix} 0.545833 & -0.241667 & 0.106019 \\ .587500 & 1.175000 & -.231944 \\ -.120833 & 1.091667 & 1.513426 \end{bmatrix}$$

P

$$\begin{bmatrix} 0.186310 & -0.068452 & 0.029563 \\ .032143 & .160714 & -.023677 \\ -.005357 & .031548 & .139749 \end{bmatrix}$$

Q

$$\begin{bmatrix} 0.046577 & -0.014583 & 0.006222 \\ .038244 & .202083 & -.028963 \\ -.011756 & .068750 & .316408 \end{bmatrix}$$

L'

$$\begin{bmatrix} 0.399646 & -0.083406 & 0.022325 \\ .175000 & .391964 & -.038816 \\ -.045142 & .224317 & .308509 \end{bmatrix}$$

M'

$$\begin{bmatrix} 0.596268 & -0.149356 & 0.040630 \\ .533205 & .602896 & -.072928 \\ -.097426 & .625418 & .445812 \end{bmatrix}$$

N'

$$\begin{bmatrix} 0.399646 & -0.083406 & 0.022325 \\ .708205 & .994860 & -.111744 \\ -.239994 & 1.475153 & 1.200133 \end{bmatrix}$$

P'

$$\begin{bmatrix} 0.148013 & -0.026378 & 0.006972 \\ .042560 & .143948 & -.012081 \\ -.012798 & .057937 & .118007 \end{bmatrix}$$

Q'

$$\begin{bmatrix} 0.038884 & -0.006034 & 0.001579 \\ .050843 & .181698 & -.014830 \\ -.028318 & .127659 & .267865 \end{bmatrix}$$

TABLE IV - STATION NUMBERS, n = 4

L

$$\begin{bmatrix} 0.576455 & -0.292857 & 0.229365 & -0.096544 \\ .128307 & .459226 & -.126190 & .045874 \\ -.023545 & .169643 & .407144 & -.055919 \\ .009789 & -.036606 & .162699 & .365434 \end{bmatrix}$$

M

$$\begin{bmatrix} 1.022222 & -0.647917 & 0.522222 & -0.221701 \\ .429630 & .747917 & -.255556 & .094850 \\ -.051852 & .518750 & .633333 & -.105961 \\ .022222 & -.085417 & .522222 & .543924 \end{bmatrix}$$

N

$$\begin{bmatrix} 0.576455 & -0.292857 & 0.229365 & -0.096544 \\ .557937 & 1.207143 & -.381746 & .140724 \\ -.127249 & 1.207143 & 1.673810 & -.267841 \\ .076455 & -.292857 & 1.729365 & 1.997206 \end{bmatrix}$$

P

$$\begin{bmatrix} 0.194478 & -0.081920 & 0.062798 & -0.026267 \\ .029597 & .163021 & -.037401 & .013391 \\ -.006581 & .041295 & .148512 & -.017322 \\ .002612 & -.009598 & .037202 & .136803 \end{bmatrix}$$

Q

$$\begin{bmatrix} 0.048240 & -0.017295 & 0.013040 & -0.005428 \\ .035167 & .204828 & -.045624 & .016301 \\ -.014547 & .090642 & .335791 & -.038574 \\ .008359 & -.030688 & .118397 & .446729 \end{bmatrix}$$

L'

$$\begin{bmatrix} 0.413738 & -0.116662 & 0.065879 & -0.019023 \\ .164798 & .424050 & -.086085 & .021577 \\ -.037937 & .181143 & .381486 & -.034723 \\ .020126 & -.052641 & .233289 & .305180 \end{bmatrix}$$

M'

$$\begin{bmatrix} 0.623188 & -0.211987 & 0.122052 & -0.035456 \\ .511882 & .667412 & -.166738 & .042628 \\ -.082891 & .544025 & .582158 & -.064723 \\ .042276 & -.112648 & .643846 & .438962 \end{bmatrix}$$

N'

$$\begin{bmatrix} 0.413738 & -0.116662 & 0.065879 & -0.019023 \\ .676680 & 1.091462 & -.252823 & .064205 \\ -.203719 & 1.269193 & 1.545802 & -.164169 \\ .146954 & -.390585 & 2.164827 & 1.622066 \end{bmatrix}$$

P'

$$\begin{bmatrix} 0.152256 & -0.036502 & 0.020304 & -0.005836 \\ .039616 & .153469 & -.026235 & .006481 \\ -.010651 & .044508 & .140822 & -.010869 \\ .005795 & -.015000 & .060554 & .117037 \end{bmatrix}$$

Q'

$$\begin{bmatrix} 0.039818 & -0.008281 & 0.004551 & -0.001303 \\ .047267 & .193310 & -.032114 & .007917 \\ -.023551 & .097745 & .318707 & -.024222 \\ .018630 & -.048203 & .194016 & .382752 \end{bmatrix}$$

TABLE V - STATION NUMBERS, n = 5

L

$$\begin{bmatrix} 0.603222 & -0.373049 & 0.329315 & -0.233780 & 0.090928 \\ .118738 & .482705 & -.152232 & .088740 & -.032326 \\ -.013316 & .157904 & .414830 & -.081796 & .025602 \\ .009561 & -.041899 & .197171 & .379612 & -.043348 \\ -.005965 & .022188 & -.055209 & .181299 & .356940 \end{bmatrix}$$

M

$$\begin{bmatrix} 1.097991 & -0.839550 & 0.762798 & -0.548214 & 0.214236 \\ .408755 & .799339 & -.313591 & .187897 & -.069026 \\ -.040898 & .493783 & .651687 & -.159325 & .050904 \\ .019866 & -.089550 & .575298 & .576786 & -.080137 \\ -.013120 & .049339 & -.126091 & .562897 & .525349 \end{bmatrix}$$

N

$$\begin{bmatrix} 0.608222 & -0.373049 & 0.329315 & -0.233780 & 0.090928 \\ .527493 & 1.282044 & -.465823 & .276637 & -.101352 \\ -.100112 & 1.145470 & 1.718204 & -.400446 & .127410 \\ .069159 & -.310549 & 1.923065 & 2.109970 & -.283759 \\ -.058445 & .219544 & -.559573 & 2.432887 & 2.458336 \end{bmatrix}$$

P

$$\begin{bmatrix} 0.202887 & -0.103119 & 0.089079 & -0.062665 & 0.024289 \\ .026908 & .169599 & -.044602 & .025479 & -.009225 \\ -.005074 & .037952 & .150488 & -.024868 & .007667 \\ .002776 & -.011949 & .049347 & .140460 & -.013645 \\ -.001627 & .006007 & -.014691 & .042295 & .134478 \end{bmatrix}$$

Q

$$\begin{bmatrix} 0.049943 & -0.021583 & 0.018334 & -0.012809 & 0.004952 \\ .031910 & .212792 & -.054326 & .030950 & -.011196 \\ -.011210 & .083245 & .340126 & -.055302 & .017031 \\ .008927 & -.038398 & .157843 & .458397 & -.044065 \\ -.006836 & .025243 & -.061693 & .177006 & .573757 \end{bmatrix}$$

L'

$$\begin{bmatrix} 0.427616 & -0.139170 & 0.105126 & -0.062435 & 0.017789 \\ .155389 & .439216 & -.116132 & .058056 & -.015600 \\ -.031892 & .171445 & .407319 & -.071204 & .016526 \\ .016702 & -.046382 & .195713 & .369286 & -.030662 \\ -.011611 & .027911 & -.064950 & .243406 & .301832 \end{bmatrix}$$

M'

$$\begin{bmatrix} 0.649902 & -0.255330 & 0.197103 & -0.117990 & 0.033722 \\ .492141 & .699256 & -.228783 & .117132 & -.031711 \\ -.070298 & .523828 & .633812 & -.136452 & .032307 \\ .034939 & -.099723 & .573549 & .557359 & -.056459 \\ -.024007 & .058134 & -.137821 & .664560 & .432107 \end{bmatrix}$$

N'

$$\begin{bmatrix} 0.427616 & -0.139170 & 0.105126 & -0.062435 & 0.017789 \\ .647530 & 1.138472 & -.344915 & .175188 & -.047311 \\ -.172488 & 1.219101 & 1.674943 & -.344108 & .081140 \\ .121519 & -.345551 & 1.916360 & 2.041363 & -.200039 \\ -.107639 & .260447 & -.616234 & 2.901646 & 2.030260 \end{bmatrix}$$

P'

$$\begin{bmatrix} 0.156411 & -0.043239 & 0.032115 & -0.018958 & 0.005389 \\ .036908 & .157833 & -.034985 & .017186 & -.004593 \\ -.008903 & .041704 & .148520 & -.021868 & .005002 \\ .004824 & -.013166 & .048803 & .137234 & -.009675 \\ -.003375 & .008078 & -.018600 & .063512 & .116059 \end{bmatrix}$$

Q'

$$\begin{bmatrix} 0.040729 & -0.009758 & 0.007151 & -0.004201 & 0.001192 \\ .043977 & .198610 & -.042759 & .020954 & -.005596 \\ -.019678 & .091532 & .335798 & -.048664 & .011120 \\ .015509 & -.042299 & .156076 & .447980 & -.031249 \\ -.014228 & .034055 & -.078385 & .267043 & .495663 \end{bmatrix}$$

TABLE VI - STATION NUMBERS, n = 6

L

$$\begin{bmatrix} 0.638800 & -0.466718 & 0.489124 & -0.390902 & 0.242612 & -0.086982 \\ .110755 & .506999 & -.193096 & .126869 & -.071822 & .024796 \\ -.014236 & .145645 & .434869 & -.098416 & .045687 & -.014745 \\ .006331 & -.032547 & .183203 & .386367 & -.059026 & .016304 \\ -.005048 & .021896 & -.063832 & .221845 & .359875 & -.035287 \\ .004127 & -.016541 & .040989 & -.076357 & .198216 & .350031 \end{bmatrix}$$

M

$$\begin{bmatrix} 1.172073 & -1.066598 & 1.150584 & -0.930965 & 0.581796 & -0.209220 \\ .391101 & .853106 & -.404200 & .273028 & -.156399 & .054246 \\ -.032371 & .468124 & .693794 & -.194854 & .092907 & -.030239 \\ .013323 & -.070505 & .546418 & .592473 & -.111954 & .031561 \\ -.010149 & .044513 & -.133366 & .624591 & .537351 & -.064035 \\ .008879 & -.035782 & .089627 & -.171416 & .599157 & .510543 \end{bmatrix}$$

N

$$\begin{bmatrix} 0.638800 & -0.466718 & 0.489124 & -0.390902 & 0.242612 & -0.086982 \\ .501856 & 1.360105 & -.597296 & .399897 & -.228221 & .079042 \\ -.078978 & 1.081893 & 1.822457 & -.488124 & .231501 & -.075223 \\ .046300 & -.244062 & 1.822457 & 2.163786 & -.394888 & .110987 \\ -.045644 & .199948 & -.597296 & 2.720209 & 2.509279 & -.291427 \\ .048522 & -.195451 & .489124 & -.933437 & 3.194001 & 2.902746 \end{bmatrix}$$

P

$$\begin{bmatrix} 0.210893 & -0.127634 & 0.130870 & -0.103635 & 0.063996 & -0.022893 \\ .024685 & .176358 & -.055956 & .036020 & -.020218 & .006957 \\ -.003894 & .034411 & .156259 & -.029594 & .013474 & -.004323 \\ .001823 & -.009203 & .045293 & .142229 & -.018261 & .004969 \\ -.001498 & .006452 & -.018484 & .056698 & .134578 & -.011243 \\ .001143 & -.004561 & .011225 & -.020561 & .046991 & .132560 \end{bmatrix}$$

Q

$$\begin{bmatrix} 0.051551 & -0.026505 & 0.026719 & -0.021011 & 0.012925 & -0.004616 \\ .029221 & .220969 & -.068060 & .043690 & -.024496 & .008425 \\ -.008595 & .075401 & .352909 & -.065761 & .029896 & -.009587 \\ .005860 & -.029565 & .144808 & .464052 & -.058920 & .016022 \\ -.006322 & .027216 & -.077925 & .238215 & .574036 & -.047575 \\ .005949 & -.023740 & .058410 & -.106938 & .243745 & .698254 \end{bmatrix}$$

L'

$$\begin{bmatrix} 0.441269 & -0.164215 & 0.141177 & -0.110263 & 0.062735 & -0.017248 \\ .146890 & .454690 & -.138234 & .089227 & -.046982 & .012510 \\ -.026501 & .161768 & .420956 & -.092878 & .040903 & -.010245 \\ .012624 & -.039301 & .186000 & .390035 & -.058770 & .012684 \\ -.009161 & .024150 & -.059634 & .211076 & .358339 & -.027291 \\ .007658 & -.018844 & .040279 & -.079137 & .253477 & .298735 \end{bmatrix}$$

M'

$$\begin{bmatrix} 0.676394 & -0.303948 & 0.267118 & -0.210538 & 0.120308 & -0.033141 \\ .474177 & .731991 & -.275585 & .182630 & -.097119 & .025961 \\ -.059129 & .503742 & .662165 & -.180822 & .081568 & -.020586 \\ .026582 & -.085145 & .553477 & .598464 & -.110987 & .024431 \\ -.018685 & .049793 & -.126314 & .604424 & .535339 & -.049677 \\ .015649 & -.038661 & .083246 & -.166643 & .685021 & .425817 \end{bmatrix}$$

N'

$$\begin{bmatrix} 0.441269 & -0.164215 & 0.141177 & -0.110263 & 0.062735 & -0.017248 \\ .621067 & 1.186681 & -.413819 & .271857 & -.144101 & .038471 \\ -.144759 & 1.169252 & 1.745286 & -.454522 & .204039 & -.051417 \\ .092370 & -.294736 & 1.846431 & 2.185427 & -.391731 & .085977 \\ -.083901 & .223322 & -.564890 & 2.628772 & 2.499695 & -.225999 \\ .085903 & -.212149 & .456509 & -.912352 & 3.678582 & 2.427820 \end{bmatrix}$$

P'

$$\begin{bmatrix} 0.160476 & -0.050692 & 0.042839 & -0.033225 & 0.018841 & -0.005173 \\ .034473 & .162263 & -.041308 & .026154 & -.013674 & .003631 \\ -.007337 & .038896 & .152473 & -.028221 & .012212 & -.003042 \\ .003631 & -.011103 & .045980 & .143453 & -.018226 & .003877 \\ -.002691 & .007043 & -.017100 & .053359 & .133988 & -.008676 \\ .002241 & -.005503 & .011713 & -.022768 & .066470 & .115150 \end{bmatrix}$$

Q'

$$\begin{bmatrix} 0.041616 & -0.011384 & 0.009490 & -0.007320 & 0.004140 & -0.001135 \\ .041021 & .203987 & -.050434 & .031846 & -.016634 & .004415 \\ -.016206 & .085309 & .344556 & -.062753 & .027119 & -.006753 \\ .011674 & -.035665 & .146998 & .468015 & -.058815 & .012502 \\ -.011350 & .029701 & -.072063 & .224100 & .571521 & -.036708 \\ .011694 & -.028708 & .061097 & -.118729 & .345986 & .606979 \end{bmatrix}$$

TABLE VII - STATION NUMBERS, n = 7

L

$$\begin{bmatrix} 0.667840 & -0.570270 & 0.701415 & -0.664780 & 0.464325 & -0.252961 & 0.083943 \\ .104206 & .530293 & -.240662 & .187716 & -.119635 & .061768 & -.019995 \\ -.011357 & .135439 & .455571 & -.124526 & .065155 & -.030681 & .009557 \\ .004254 & -.025248 & .168587 & .404281 & -.070887 & .027335 & -.007949 \\ -.002897 & .014482 & -.049531 & .205850 & .365984 & -.045435 & -.011270 \\ .003070 & -.014123 & .040797 & -.089027 & .244525 & .344753 & -.029689 \\ -.003082 & .013524 & -.036117 & .067092 & -.099849 & .213881 & .344229 \end{bmatrix}$$

M

$$\begin{bmatrix} 1.243487 & -1.321299 & 1.672922 & -1.605312 & 1.129029 & -0.617807 & 0.205449 \\ 0.376396 & 0.905437 & -.511106 & 0.409927 & -.264373 & .137401 & -.044613 \\ -.026266 & .446479 & .737737 & -.250374 & .134585 & -.064103 & .020059 \\ .009112 & -.055674 & .516672 & .629063 & -.136596 & .054068 & -.015852 \\ -.005896 & .029812 & -.104856 & .592219 & .551252 & -.084474 & .021382 \\ .006025 & -.027896 & .081487 & -.182666 & .668960 & .507772 & -.053078 \\ -.006513 & .028701 & -.077078 & .144688 & -.220971 & .632193 & .498306 \end{bmatrix}$$

N

$$\begin{bmatrix} 0.667840 & -0.570270 & 0.701415 & -0.664780 & 0.464325 & -0.252961 & 0.083943 \\ .480602 & 1.435730 & -.751768 & .597643 & -.384008 & .199169 & -.064608 \\ -.063889 & 1.028397 & 1.931045 & -.625274 & .334325 & -.158887 & .049675 \\ .031590 & -.192270 & 1.718603 & 2.291470 & -.480675 & .189539 & -.055505 \\ -.026481 & .133730 & -.468955 & 2.574726 & 2.570992 & -.383331 & .096798 \\ .033195 & -.153603 & .448232 & -1.002357 & 3.589325 & 2.883613 & -.295079 \\ -.042160 & .185730 & -.498585 & .935220 & -1.425675 & 4.007039 & 3.334065 \end{bmatrix}$$

P

$$\begin{bmatrix} 0.218415 & -0.154452 & 0.185835 & -0.174510 & 0.121269 & -0.065852 & 0.021819 \\ .022882 & .182773 & -.069049 & .052757 & -.033334 & .017129 & -.005533 \\ -.003069 & .031489 & .162183 & -.037056 & .019010 & -.008878 & .002756 \\ .001211 & -.007052 & .040990 & .147488 & -.021700 & .008211 & -.002373 \\ -.000853 & .004233 & -.014225 & .051989 & .136202 & -.014243 & .003481 \\ .000925 & -.004239 & .012162 & -.026098 & .063551 & .130010 & -.009551 \\ -.000863 & .003780 & -.010056 & .018551 & -.027154 & .051386 & .130932 \end{bmatrix}$$

Q

$$\begin{bmatrix} 0.053049 & -0.031847 & 0.037666 & -0.035122 & 0.024312 & -0.013170 & 0.004358 \\ .027042 & .228718 & -.083877 & .063907 & -.040333 & .020712 & -.006689 \\ -.006769 & .068932 & .366024 & -.082279 & .042147 & -.019672 & .006106 \\ .003890 & -.022639 & .130958 & .480975 & -.069981 & .026453 & -.007644 \\ -.003599 & .017853 & -.059956 & .218352 & .580855 & -.060230 & .014713 \\ .004829 & -.022136 & .063490 & -.136173 & .330701 & .684756 & -.049981 \\ -.005357 & .023463 & -.062408 & .115110 & -.168426 & .318021 & .820708 \end{bmatrix}$$

L'

$$\begin{bmatrix} 0.454474 & -0.191650 & 0.186093 & -0.164912 & 0.120971 & -0.064447 & 0.016993 \\ .139454 & -.470082 & -.163300 & .119521 & -.080299 & .040918 & -.010571 \\ -.022185 & .152887 & .435294 & -.110017 & .060805 & -.028584 & .007141 \\ .009385 & -.032714 & .175539 & .402291 & -.074625 & .029249 & -.006865 \\ -.006253 & .018392 & -.050868 & .201292 & .375245 & -.049377 & .009965 \\ .005679 & -.015410 & .036116 & -.075258 & .226101 & .348824 & -.024541 \\ -.005486 & .014288 & -.031254 & .056382 & -.094568 & .263233 & .295919 \end{bmatrix}$$

M'

$$\begin{bmatrix} 0.702228 & -0.357636 & 0.355047 & -0.317567 & 0.234133 & -0.125078 & 0.033023 \\ .458303 & .764868 & -.329159 & .247432 & -.168109 & .086131 & -.022307 \\ -.050122 & .485193 & .692136 & -.216700 & .122950 & -.058376 & .014642 \\ .019989 & -.071706 & .532099 & .623577 & -.143020 & .057414 & -.013579 \\ -.012820 & .038132 & -.108454 & .584350 & .568583 & -.092005 & .018939 \\ .011370 & -.031029 & .073511 & -.157315 & .634247 & .516450 & -.044219 \\ -.011099 & .028988 & -.063669 & .115706 & -.197770 & .704705 & .420133 \end{bmatrix}$$

N'

$$\begin{bmatrix} 0.454474 & -0.191650 & 0.186093 & -0.164912 & 0.120971 & -0.064447 & 0.016993 \\ .597757 & 1.234950 & -.492459 & .366953 & -.248408 & .127049 & -.032878 \\ -.122429 & 1.123273 & 1.819566 & -.543417 & .306705 & -.145336 & .036425 \\ .069352 & -.247832 & 1.771836 & 2.273022 & -.503685 & .201491 & -.047602 \\ -.057533 & .170920 & -.484684 & 2.538692 & 2.649577 & -.417397 & .085721 \\ .062529 & -.170555 & .403671 & -.861833 & 3.397336 & 2.931074 & -.245636 \\ -.072080 & .188216 & -.413268 & .750618 & -1.281188 & 4.491463 & 2.816717 \end{bmatrix}$$

P'

$$\begin{bmatrix} 0.164382 & -0.058807 & 0.056122 & -0.049381 & 0.036084 & -0.019183 & 0.005053 \\ .032358 & .166641 & -.048434 & .034761 & -.023166 & .011759 & -.003033 \\ -.006088 & .036328 & .156613 & -.033166 & .017983 & -.008393 & .002091 \\ .002679 & -.009168 & .042913 & .147041 & -.022914 & .008826 & -.002060 \\ -.001831 & .005346 & -.014528 & .050504 & .139081 & -.015450 & .003074 \\ .001689 & -.004565 & .010624 & -.021773 & .057851 & .131144 & -.007854 \\ -.001614 & .004199 & -.009166 & .016464 & -.027322 & .069349 & .114318 \end{bmatrix}$$

Q'

$$\begin{bmatrix} 0.042465 & -0.013146 & 0.012374 & -0.010826 & 0.007887 & -0.004186 & 0.001102 \\ .038456 & .209297 & -.059075 & .042284 & -.028149 & .014281 & -.003682 \\ -.013441 & .079621 & .353730 & -.073708 & .039907 & -.018616 & .004637 \\ .008611 & -.029438 & .137128 & .479557 & -.073907 & .028441 & -.006638 \\ -.007724 & .022542 & -.061218 & .212061 & .593022 & -.065330 & .012992 \\ .008815 & -.023823 & .055439 & -.113560 & .300912 & .690659 & -.041094 \\ -.010037 & .026110 & -.056987 & .102355 & -.169818 & .430362 & .716958 \end{bmatrix}$$

TABLE VIII - STATION NUMBERS, n = 8

L

$$\begin{bmatrix}
0.695399 & -0.682210 & 0.967933 & -1.075644 & 0.890699 & -0.546005 & 0.263837 & -0.081470 \\
.098774 & .552335 & -.293058 & .268272 & -.202760 & .117811 & -.054953 & .016673 \\
-.009295 & .127088 & .475377 & -.154856 & .096189 & -.050989 & .022543 & -.006669 \\
.002983 & -.020104 & .156433 & .422771 & -.089536 & .038914 & -.015684 & .004462 \\
-.001713 & .009719 & -.038364 & .189080 & .382419 & -.054493 & .017966 & -.004764 \\
.001553 & -.008094 & .026960 & -.069027 & .226647 & .350388 & -.036525 & .008241 \\
-.002035 & .010130 & -.031067 & .067276 & -.117271 & .265711 & .332647 & -.025572 \\
.002418 & -.011643 & .034080 & -.068232 & .101348 & -.125543 & .228566 & .339239
\end{bmatrix}$$

M

$$\begin{bmatrix}
1.312192 & -1.600390 & 2.337500 & -2.630070 & 2.192995 & -1.350286 & 0.654484 & -0.202414 \\
.364019 & .955663 & -.630523 & -.593584 & -.454018 & .265559 & -.124395 & .037821 \\
-.021829 & .428499 & .780389 & -.315718 & .201514 & -.108064 & .048081 & -.014268 \\
.006490 & -.045077 & .491625 & .667195 & -.175130 & .078152 & -.031856 & .009104 \\
-.003545 & .020351 & -.082648 & .558819 & .584107 & -.102878 & .034801 & -.009307 \\
.003081 & -.016189 & .054514 & -.143453 & .633389 & .520374 & -.066829 & .015388 \\
-.003931 & .019610 & -.060500 & .132430 & -.237005 & .709714 & .484484 & -.045167 \\
.005039 & -.024337 & .071477 & -.143916 & .215982 & -.274441 & .662747 & .487926
\end{bmatrix}$$

N

$$\begin{bmatrix}
0.695399 & -0.682210 & 0.967933 & -1.075644 & 0.890699 & -0.546005 & 0.263837 & -0.081470 \\
.462793 & 1.507998 & -.923581 & .861856 & -.656778 & .383370 & -.179348 & .054494 \\
-.052953 & .984086 & 2.036155 & -.786292 & .499217 & -.267117 & .118705 & -.035205 \\
.022453 & -.155335 & 1.631308 & 2.424356 & -.614926 & .273370 & -.111252 & .031774 \\
-.015893 & .091123 & -.368956 & 2.424356 & 2.718847 & -.466005 & .157170 & -.041992 \\
.016958 & -.089039 & .299530 & -.786292 & 3.393592 & 2.952258 & -.370670 & .085181 \\
-.025621 & .127790 & -.394067 & .861856 & -1.539301 & 4.523995 & 3.239551 & -.296574 \\
.037691 & -.182002 & .534419 & -1.075644 & 1.613222 & -2.046630 & 4.867795 & 3.754721
\end{bmatrix}$$

P

$$\begin{bmatrix}
0.225483 & -0.183160 & 0.254178 & -0.279850 & 0.230544 & -0.140865 & 0.067914 & -0.020947 \\
.021401 & .188781 & -.083331 & .074709 & -.055975 & .032367 & -.015052 & .004560 \\
-.002485 & .029123 & .167794 & -.045646 & .027793 & -.014611 & .006430 & -.001898 \\
.000839 & -.005550 & .037442 & .152881 & -.027134 & .011568 & -.004626 & .001312 \\
-.000499 & .002808 & -.010886 & .046981 & .141097 & -.016907 & .005472 & -.001443 \\
.000464 & -.002412 & .007975 & -.020073 & .058226 & .131525 & -.011570 & .002573 \\
-.000622 & .003082 & -.009418 & .020253 & -.034728 & .070024 & .126311 & -.008294 \\
.000684 & -.003289 & .009602 & -.019152 & .028253 & -.034431 & .055545 & .129520
\end{bmatrix}$$

Q

$$\begin{bmatrix}
0.054447 & -0.037524 & 0.051181 & -0.055949 & 0.045912 & -0.027983 & 0.013468 & -0.004150 \\
.025255 & .235968 & -.101109 & .090392 & -.067648 & .039093 & -.018172 & .005504 \\
-.005476 & .063696 & .378439 & -.101283 & .061577 & -.032351 & .014232 & -.004201 \\
.002694 & -.017807 & .119546 & .498324 & -.087460 & .037251 & -.014889 & .004221 \\
-.002103 & .011838 & -.045861 & .197214 & .601611 & -.071468 & .023114 & -.006093 \\
.002422 & -.012593 & .041626 & -.104712 & .302906 & .692634 & -.060517 & .013452 \\
-.003869 & .019183 & -.058605 & .126011 & -.215987 & .434540 & .791745 & -.051711 \\
.004929 & -.023705 & .069206 & -.138032 & .203600 & -.248035 & .399381 & .941457
\end{bmatrix}$$

L'

$$\begin{bmatrix}
0.467152 & -0.221089 & 0.240602 & -0.240468 & 0.199636 & -0.134575 & 0.066783 & -0.016869 \\
.132995 & .485054 & -.190947 & .157691 & -.119822 & .076760 & -.037002 & .009215 \\
-.018808 & .145081 & .449649 & -.129719 & .081035 & -.047434 & .021830 & -.005321 \\
.007073 & -.027389 & .165809 & .415511 & -.088018 & .042322 & -.017943 & .004228 \\
-.004196 & .013699 & -.042396 & .189994 & .386395 & -.061356 & .021637 & -.004785 \\
.003523 & -.010590 & .027676 & -.064551 & .216254 & .362797 & -.042253 & .008020 \\
-.003811 & .010924 & -.026353 & .052121 & -.092688 & .240572 & .340545 & -.022283 \\
.004154 & -.011572 & .026810 & -.049497 & .076212 & -.111015 & .272632 & .293354
\end{bmatrix}$$

M'

$$\begin{bmatrix}
0.727233 & -0.415706 & 0.462586 & -0.466670 & 0.389428 & -0.263362 & 0.130945 & -0.033107 \\
.444367 & .797175 & -.388839 & .329868 & -.253524 & .163432 & -.079057 & .019722 \\
-.043004 & .468735 & .722423 & -.258293 & .165704 & -.098083 & .045388 & -.011093 \\
.015242 & -.060778 & .512112 & .650770 & -.170613 & .084195 & -.036054 & .008527 \\
-.008698 & .028715 & -.091426 & .561588 & .591124 & -.115917 & .041862 & -.009326 \\
.007082 & -.021414 & .056607 & -.135711 & .614158 & .543778 & -.077799 & .015058 \\
-.007522 & .021627 & -.052469 & .104892 & -.191506 & .662638 & .500211 & -.039785 \\
.008334 & -.023284 & .054058 & -.100193 & .155413 & -.230699 & .723527 & .414996
\end{bmatrix}$$

N'

$$\begin{bmatrix}
0.467152 & -0.221089 & 0.240602 & -0.240468 & 0.199636 & -0.134575 & 0.066783 & -0.016869 \\
.577362 & 1.282229 & -.579786 & .487559 & -.373346 & .240192 & -.116059 & .028937 \\
-.104816 & 1.082551 & 1.894495 & -.646305 & .412443 & -.243600 & .112606 & -.027507 \\
.052799 & -.209723 & 1.702145 & 2.367821 & -.599857 & .294907 & -.126105 & .029809 \\
-.038988 & .128559 & -.408100 & 2.436346 & 2.750891 & -.525024 & .189085 & -.042089 \\
.038933 & -.117660 & .310711 & -.743106 & 3.287044 & 3.081687 & -.431248 & .083310 \\
-.048943 & .140686 & -.341167 & .681473 & -1.241724 & 4.216400 & 3.341811 & -.260993 \\
.062492 & -.174560 & .405216 & -.750848 & 1.164103 & -1.725908 & 5.337321 & 3.198326
\end{bmatrix}$$

P'

$$\begin{bmatrix}
0.168111 & -0.067464 & 0.072148 & -0.071591 & 0.059201 & -0.039810 & 0.019727 & -0.004979 \\
.030533 & .170868 & -.056240 & .045534 & -.034316 & .021885 & -.010524 & .002618 \\
-.005119 & .034087 & .160734 & -.038818 & .023783 & -.013810 & .006331 & -.001541 \\
.002003 & -.007611 & .040069 & .150902 & -.026822 & .012656 & -.005328 & .001252 \\
-.001220 & .003953 & -.012016 & .047159 & .142374 & -.019020 & .006593 & -.001450 \\
.001045 & -.003129 & .008120 & -.018614 & .054969 & .135367 & -.013324 & .002493 \\
-.001145 & .003272 & -.007864 & .015449 & -.027027 & .062213 & .128649 & -.007173 \\
.001228 & -.003421 & .007911 & -.014572 & .022340 & -.032195 & .072134 & .113558
\end{bmatrix}$$

Q'

$$\begin{bmatrix}
0.043271 & -0.015018 & 0.015838 & -0.015627 & 0.012882 & -0.008646 & 0.004280 & -0.001080 \\
.036245 & .214419 & -.068533 & .055337 & -.041657 & .026552 & -.012763 & .003174 \\
-.011294 & .074659 & .362854 & -.086223 & .052749 & -.030611 & .014029 & -.003414 \\
.006433 & -.024428 & .127977 & .491981 & -.086477 & .040765 & -.017155 & .004029 \\
-.005146 & .016664 & -.050617 & .197950 & .606915 & -.080395 & .027846 & -.006122 \\
.005456 & -.016330 & .042369 & -.097075 & .285874 & .712714 & -.069680 & .013034 \\
-.007123 & .020358 & -.061530 & .096084 & -.168026 & .385904 & .806281 & -.044710 \\
.008863 & -.024694 & .057107 & -.105178 & .161236 & -.232314 & .519815 & .825790
\end{bmatrix}$$

TABLE IX - T_i MATRICES

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 \\ 5 & 5 & 5 & 5 & 5 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 & 6 & 6 & 0 & 0 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 0 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \end{bmatrix}$$

$$T_3 = 1/2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 7 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\ 9 & 7 & 5 & 3 & 1 & 0 & 0 & 0 \\ 11 & 9 & 7 & 5 & 3 & 1 & 0 & 0 \\ 13 & 11 & 9 & 7 & 5 & 3 & 1 & 0 \\ 15 & 13 & 11 & 9 & 7 & 5 & 3 & 1 \end{bmatrix}$$

$$T_4 = -1/6 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 13 & 7 & 0 & 0 & 0 & 0 & 0 \\ 10 & 22 & 22 & 10 & 0 & 0 & 0 & 0 \\ 13 & 31 & 37 & 31 & 13 & 0 & 0 & 0 \\ 16 & 40 & 52 & 52 & 40 & 16 & 0 & 0 \\ 19 & 49 & 67 & 73 & 67 & 49 & 19 & 0 \\ 22 & 58 & 82 & 94 & 94 & 82 & 58 & 22 \end{bmatrix}$$

TABLE X - CALCULATION SETUP FOR BENDING - BENDING-TORSION VIBRATIONS

[Uniform beam, $m_i = C_i = I_i = I_{\xi i} = 1$; $I_\eta/I_\xi = 100$; $\Gamma = 0.03$; $r_\eta/r_\xi = 10$; $\epsilon = 0.2$; let n = 2].

1	2	3	4	5	6	7	8	9	10	11	12	13	14
i	m_i	C_i	I_i	$I_{\xi i}$	$I_{\eta i}$	φ_i	$\sin \varphi_i$	$\cos \varphi_i$	f_{xi}	f_{yi}	f_{xyi}	S_{xi}	S_{yi}
1	1	1	1	1	100	0.25	0.24740	0.96891	0.07059	0.93940	-0.23731	-0.15051	0.99365
2	1	1	1	1	100	.75	.68164	.73169	.46999	.54002	-.49376	-.60847	.79985

$$f_{xi} = \frac{I_{x0}}{I_{\xi i} I_{\eta i}} \left(I_{\eta i} \sin^2 \varphi_i + I_{\xi i} \cos^2 \varphi_i \right)$$

$$f_{yi} = \frac{I_{x0}}{I_{\xi i} I_{\eta i}} \left(I_{\xi i} \sin^2 \varphi_i + I_{\eta i} \cos^2 \varphi_i \right)$$

$$f_{xyi} = \frac{I_{x0}}{I_{\xi i} I_{\eta i}} \left(I_{\eta i} - I_{\xi i} \right) \sin \varphi_i \cos \varphi_i$$

$$S_{xj} = \left(\frac{r_\xi}{r_\eta} \right) \cos \varphi_i - \sin \varphi_i$$

$$S_{yj} = \left(\frac{r_\xi}{r_\eta} \right) \sin \varphi_i + \cos \varphi_i$$

TABLE X - CALCULATION SETUP FOR BENDING - BENDING-TORSION VIBRATIONS - Continued

$$D_y = \left[T_2T_1\right] \begin{bmatrix} F_y & 0 \\ 0 & F_y \end{bmatrix} \begin{bmatrix} P' \\ -Q' \end{bmatrix} + \left[T_3T_4\right] \begin{bmatrix} G_y & 0 \\ 0 & G_y \end{bmatrix} \begin{bmatrix} N' \\ M' \end{bmatrix}, \text{ and so forth}$$

$$D_y = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.93940 & 0 & 0 & 0 \\ 0 & .54002 & 0 & 0 \\ 0 & 0 & .93940 & 0 \\ 0 & 0 & 0 & .54002 \end{bmatrix} \begin{bmatrix} 0.137933 & -0.011252 \\ .057955 & .118463 \\ -.036616 & .002614 \\ -.069733 & -.150415 \end{bmatrix} +$$

$$\begin{bmatrix} 1/2 & 0 & -1/6 & 0 \\ 3/2 & 1/2 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0.93940 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .93940 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.367100 & -0.034875 \\ .851948 & .758685 \\ .536364 & -.060795 \\ .627273 & .448674 \end{bmatrix}$$

$$D_x = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.07059 & 0 & 0 & 0 \\ 0 & .46999 & 0 & 0 \\ 0 & 0 & .07059 & 0 \\ 0 & 0 & 0 & .46999 \end{bmatrix} \begin{bmatrix} 0.137933 & -0.011252 \\ .057955 & .118463 \\ -.036616 & .002614 \\ -.069733 & -.150415 \end{bmatrix} +$$

$$\begin{bmatrix} 1/2 & 0 & -1/6 & 0 \\ 3/2 & 1/2 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0.07059 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .07059 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.367100 & -0.034875 \\ .851948 & .758685 \\ .536364 & -.060795 \\ .627273 & .448674 \end{bmatrix}$$

TABLE X - CALCULATION SETUP FOR BENDING - BENDING-TORSION VIBRATIONS - Continued

$$F_x = \begin{bmatrix} f_{x1m1} & 0 \\ 0 & f_{x2m2} \end{bmatrix} = \begin{bmatrix} 0.07059 & 0 \\ 0 & .46999 \end{bmatrix} \qquad F_y = \begin{bmatrix} 0.93940 & 0 \\ 0 & .54002 \end{bmatrix} \qquad F_{xy} = \begin{bmatrix} 0.23731 & 0 \\ 0 & .49376 \end{bmatrix}$$

$$F_{xij} = f_{xi} m_j \delta_{ij} \qquad F_{yij} = f_{yi} m_j \delta_{ij} \qquad F_{xyij} = - f_{xyi} m_j \delta_{ij}$$

$$G_x = \begin{bmatrix} 0 & f_{x1m2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.07059 \\ 0 & 0 \end{bmatrix} \qquad G_y = \begin{bmatrix} 0 & 0.93940 \\ 0 & 0 \end{bmatrix} \qquad G_{xy} = \begin{bmatrix} 0 & 0.23731 \\ 0 & 0 \end{bmatrix}$$

$$G_{xij} = f_{xi} m_j \gamma_{ij} \qquad G_{yij} = f_{yi} m_j \gamma_{ij} \qquad G_{xyij} = - f_{xyi} m_j \gamma_{ij}$$

$$K_y = \begin{bmatrix} \frac{S_{x1}}{C_1} & 0 \\ 0 & \frac{S_{x2}}{C_2} \end{bmatrix} = \begin{bmatrix} 0.99365 & 0 \\ 0 & .79985 \end{bmatrix} \qquad K_x = \begin{bmatrix} -0.15051 & 0 \\ 0 & -.60847 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_{xij} = \frac{S_{xj}}{C_i} \delta_{ij} \qquad K_{yij} = \frac{S_{yj}}{C_i} \delta_{ij} \qquad K_{ij} = \frac{I_j}{C_i} \delta_{ij}$$

$$H_y = \begin{bmatrix} 0 & \frac{S_{x2}}{C_2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.99365 \\ 0 & 0 \end{bmatrix} \qquad H_x = \begin{bmatrix} 0 & -0.15051 \\ 0 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$H_{xij} = \frac{S_{xj}}{C_i} \gamma_{ij} \qquad H_{yij} = \frac{S_{yj}}{C_i} \gamma_{ij} \qquad H_{ij} = \frac{I_j}{C_i} \gamma_{ij}$$

TABLE X - CALCULATION SETUP FOR BENDING - BENDING-TORSION VIBRATIONS - Continued

$$\theta = \begin{bmatrix} f_{y1}S_{y1} - f_{xy1}S_{x1} & 0 \\ 0 & f_{y2}S_{y2} - f_{xy2}S_{x2} \end{bmatrix} = \begin{bmatrix} 0.89772 & 0 \\ 0 & .13150 \end{bmatrix} \qquad F_{x\theta} = \begin{bmatrix} 0.22518 & 0 \\ 0 & .10896 \end{bmatrix}$$

$$F_{y\theta ij} = (f_{yi}S_{yj} - f_{xyi}S_{xj})\delta_{ij} \qquad F_{x\theta ij} = (f_{xi}S_{xj} - f_{xyi}S_{yj})\delta_{ij}$$

$$y\theta = \begin{bmatrix} 0 & f_{y1}S_{y2} - f_{xy1}S_{x2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.60698 \\ 0 & 0 \end{bmatrix} \qquad G_{x\theta} = \begin{bmatrix} 0 & 0.14686 \\ 0 & 0 \end{bmatrix}$$

$$G_{y\theta ij} = (f_{yi}S_{yj} - f_{xyi}S_{xj})\gamma_{ij} \qquad G_{x\theta ij} = (f_{xi}S_{xj} - f_{xyi}S_{yj})\gamma_{ij}$$

$$D_\theta = T_1 \begin{bmatrix} K & H \end{bmatrix} \begin{bmatrix} L \\ M \end{bmatrix}, \text{ and so forth}$$

$$D_\theta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.533333 & -0.129167 \\ .116667 & .391667 \\ 11/12 & -13/48 \\ 5/12 & 29/48 \end{bmatrix}$$

$$D_{\theta y} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.99365 & 0 & 0 & 0.99365 \\ 0 & .79985 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.367100 & -0.034875 \\ .224675 & .310011 \\ .536364 & -.060795 \\ .627273 & .448674 \end{bmatrix}$$

$$D_{\theta x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -0.15051 & 0 & 0 & -0.15051 \\ 0 & -.60847 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.367100 & -0.034875 \\ .224675 & .310011 \\ .536364 & -.060795 \\ .627273 & .448674 \end{bmatrix}$$

TABLE X - CALCULATION SETUP FOR BENDING - BENDING-TORSION VIBRATIONS - Continued

$$D_{xy} = D_{yx} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.23731 & 0 & 0 & 0 \\ 0 & .49376 & 0 & 0 \\ 0 & 0 & .23731 & 0 \\ 0 & 0 & 0 & .49376 \end{bmatrix} \begin{bmatrix} 0.137933 & -0.011252 \\ .057955 & .118463 \\ -.036616 & .002614 \\ -.069733 & -.150415 \end{bmatrix} +$$

$$\begin{bmatrix} 1/2 & 0 & -1/6 & 0 \\ 3/2 & 1/2 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0.23731 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .23731 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.367100 & -0.034875 \\ .851948 & .758685 \\ .536364 & -.060795 \\ .627273 & .448674 \end{bmatrix}$$

$$D_{y\theta} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.89772 & 0 & 0 & 0 \\ 0 & .13150 & 0 & 0 \\ 0 & 0 & .89772 & 0 \\ 0 & 0 & 0 & .13150 \end{bmatrix} \begin{bmatrix} 0.183333 & -0.037500 \\ .025000 & .143750 \\ -.046032 & .008135 \\ -.029365 & -.181448 \end{bmatrix} +$$

$$\begin{bmatrix} 1/2 & 0 & -1/6 & 0 \\ 3/2 & 1/2 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0.60698 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .60698 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 8/15 & -31/240 \\ 8/15 & 239/240 \\ 11/12 & -13/48 \\ 5/12 & 29/48 \end{bmatrix}$$

$$D_{x\theta} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.22518 & 0 & 0 & 0 \\ 0 & .10896 & 0 & 0 \\ 0 & 0 & .22518 & 0 \\ 0 & 0 & 0 & .10896 \end{bmatrix} \begin{bmatrix} 0.183333 & -0.037500 \\ .025000 & .143750 \\ -.046032 & .008135 \\ -.029365 & -.181448 \end{bmatrix} +$$

$$\begin{bmatrix} 1/2 & 0 & -1/6 & 0 \\ 3/2 & 1/2 & -2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 0 & 0.14686 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .14686 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 8/15 & -31/240 \\ 8/15 & 239/240 \\ 11/12 & -13/48 \\ 5/12 & 29/48 \end{bmatrix}$$

TABLE X - CALCULATION SETUP FOR BENDING - BENDING-TORSION VIBRATIONS - Concluded

$$D_{\theta} = \begin{bmatrix} 0.95 & 0.475 \\ 1.0667 & .86667 \end{bmatrix} \qquad D_{\theta y} = \begin{bmatrix} 0.98806 & 0.41170 \\ 1.1678 & .65913 \end{bmatrix}$$

$$D_{\theta x} = \begin{bmatrix} -0.14966 & 0.062281 \\ -.28637 & -.25091 \end{bmatrix} \qquad D_{y} = \begin{bmatrix} 0.39713 & 0.27799 \\ 1.0573 & .81610 \end{bmatrix}$$

$$D_{x} = \begin{bmatrix} 0.029842 & 0.020889 \\ .099280 & .098473 \end{bmatrix} \qquad D_{yx} = D_{xy} = \begin{bmatrix} 0.10032 & 0.070226 \\ .28360 & .23708 \end{bmatrix}$$

$$D_{y\theta} = \begin{bmatrix} 0.24297 & 0.24175 \\ .60753 & .61612 \end{bmatrix} \qquad D_{x\theta} = \begin{bmatrix} 0.059879 & 0.051724 \\ .151141 & .15672 \end{bmatrix}$$

$$\Gamma = 0.03, \quad \epsilon\Gamma = 0.006 \qquad \frac{1}{\omega^2}\,\frac{EI_{x0}}{m_0\delta^4} \equiv \lambda$$

$$\begin{vmatrix} 0.028500-\lambda & 0.014250 & 0.0059284 & 0.0024702 & -0.00089796 & -0.00037369 \\ .032001 & .026000-\lambda & .0070068 & .0039548 & -.0017182 & -.0015055 \\ .24297 & .24175 & .39713-\lambda & .27799 & .10032 & .070226 \\ .60753 & .61612 & 1.0573 & .81610-\lambda & .28360 & .23708 \\ .059879 & .051724 & .10032 & .070226 & .029842-\lambda & .020889 \\ .151141 & .15672 & .28360 & .23708 & .099280 & .098473-\lambda \end{vmatrix} = 0$$

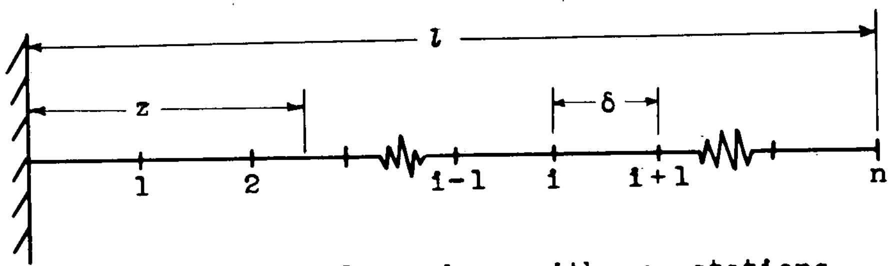


Figure 1. - Cantilever beam with n stations.

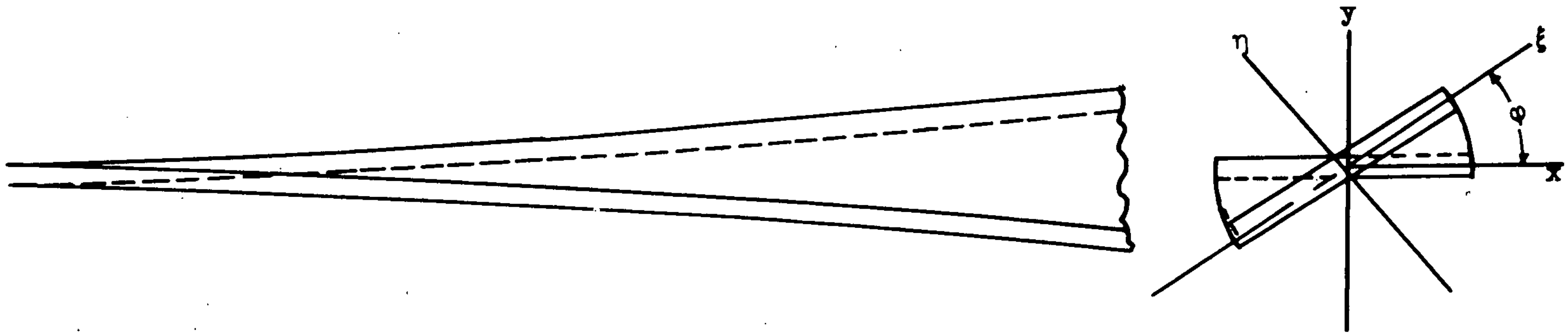


Figure 2. - Uniform naturally twisted cantilever beam.

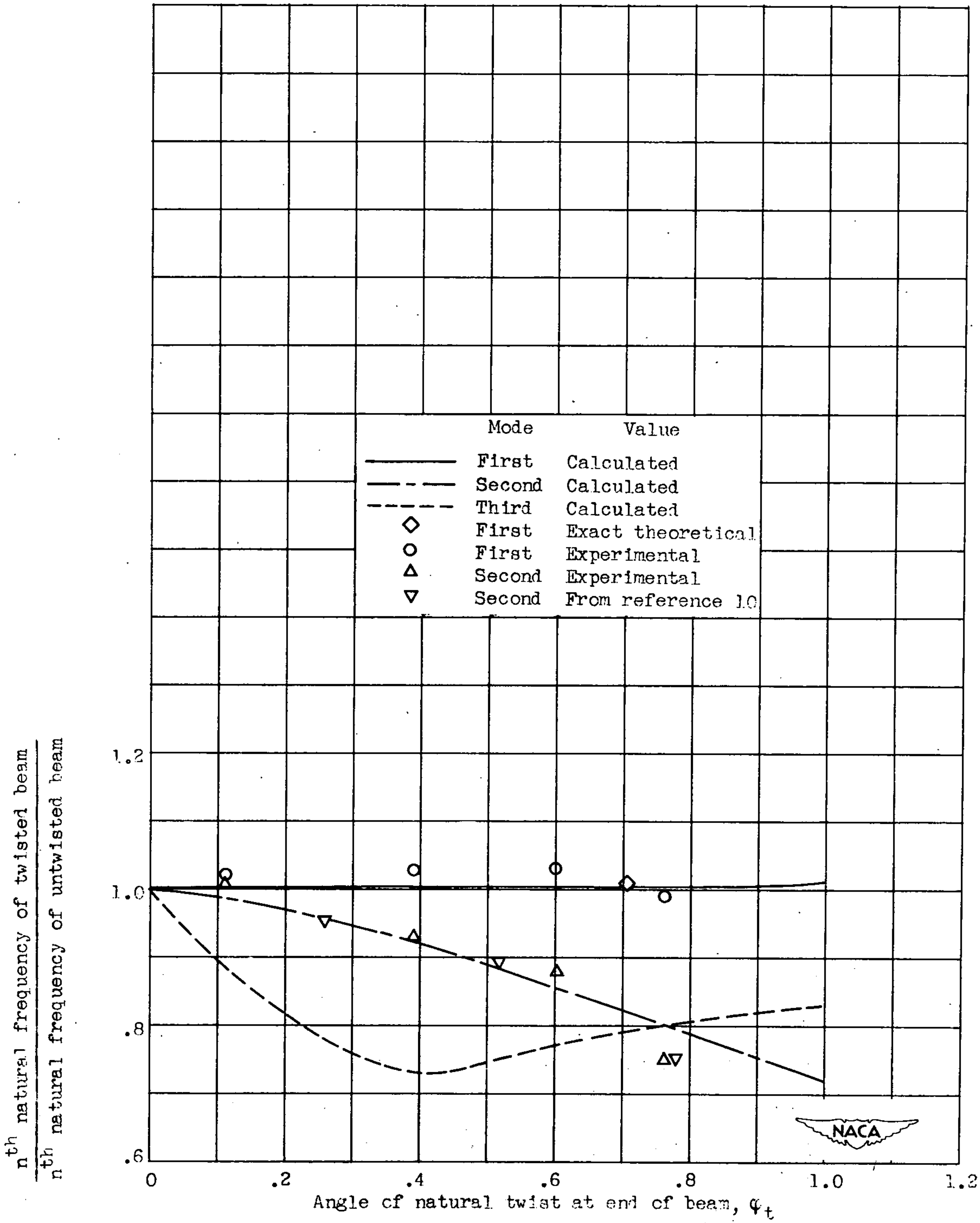
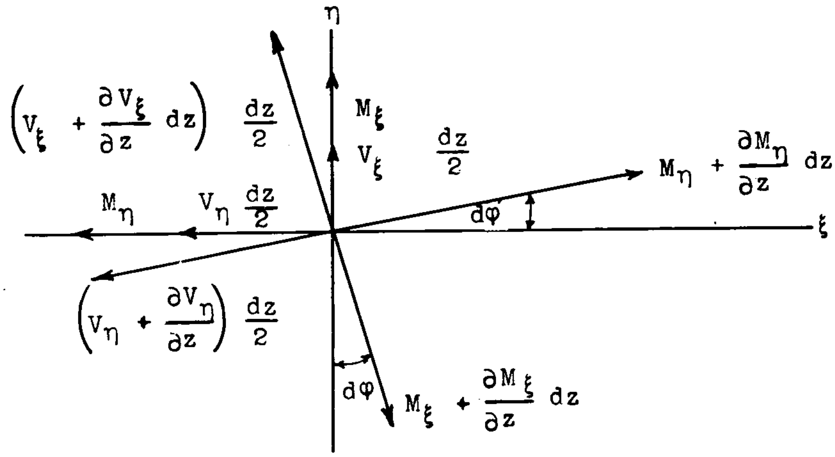
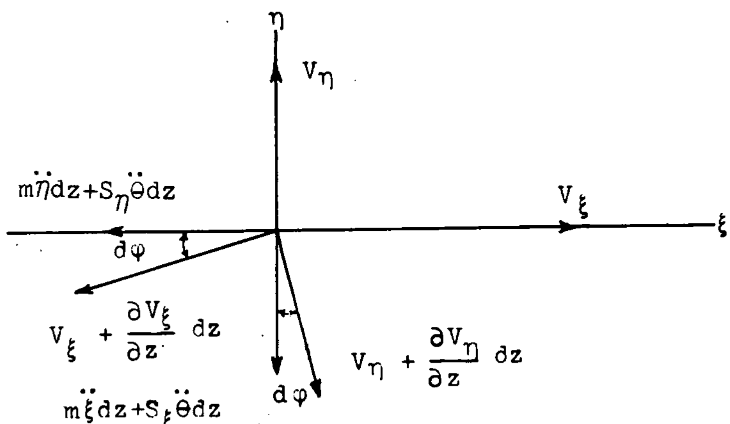


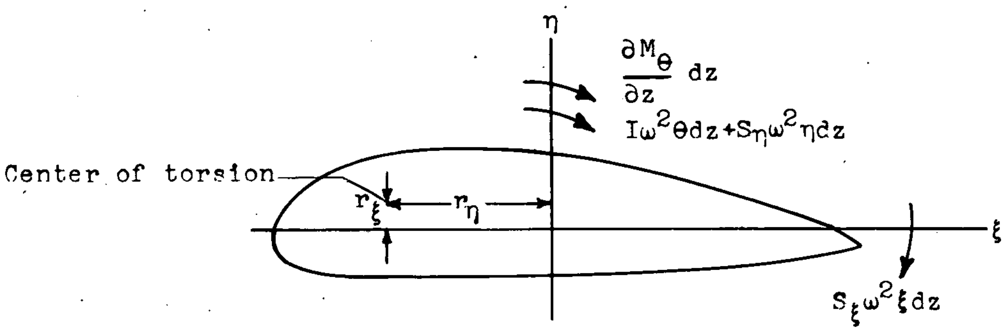
Figure 3. - Effect of angle of twist on natural frequencies, $I_\eta/I_\xi = 144$.



(a) Moments acting at section of vibrating beam.



(b) Forces acting at section of vibrating beam.



(c) Torsional moments acting at section of vibrating beam.

Figure 4. - Naturally twisted cantilever beam vibrating in coupled flexural motion.