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A GENERAL THROUGH-FLOW THEORY OF FLUID FLOW WITH SUBSONIC OR SUPERSONIC VELOCITY IN TURBOMACHINES OF ARBITRARY HUB AND CASING SHAPES

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SUMMARY

A general steady through-flow theory of nonviscous fluid in turbomachines of arbitrary hub- and casing-wall shapes with subsonic or supersonic velocity is presented. The theory is applicable to both direct and inverse problems and is derived primarily for use in turbomachines having thin blades of high solidity with a simple approximate correction factor for blade-thickness effect. Through the use of the stream function, the continuity equation and the equation of motion in the radial direction are combined to form a principal equation for the present problem. The principal equation contains some terms that are either prescribed or to be determined by other equations defining the problem. Two forms of the principal equation are obtained for the two main groups of current compressor and turbine design in which the variation of tangential velocity and the variation of the ratio of tangential to axial velocity throughout the blade region are given. When the tangential velocity is given, the principal equation is elliptic or hyperbolic, depending on whether the meridional velocity is subsonic or supersonic, respectively. When a relation between the tangential and the axial velocity is given, however, the principal equation becomes hyperbolic when the relative velocity is supersonic. A general method of solution for both the elliptic and the hyperbolic case is outlined. Specific applications of the theory to several common types of compressor and turbine employing free-vortex, symmetrical-velocity-diagram, solid-rotation-type, nontwisted-blade, and radial-blade-element designs are discussed.

INTRODUCTION

With the increasing use of velocity diagrams other than free-vortex type, low inlet hub-tip-radius ratios, and high velocity of flow, the problem of three-dimensional flow in axial-flow turbomachines becomes more and more important. This problem is treated by Traupel, Meyer, and Marble (references 1 to 3) for incompressible fluid.

Compressible flow is treated in reference 4, in which methods for limiting solutions of zero and infinite-blade aspect ratio are obtained and a step-by-step method, as well as a simpler method based on an approximate knowledge of the shape of streamlines, is given for finite-blade aspect ratio. In reference 5, Reisner gives a method of blade design for compressible flow with the shape of neither hub nor casing wall specified in advance. The problem of supersonic flow in impellers of a given casing and blade shape is currently being investigated by Arthur W. Goldstein of the NACA Lewis laboratory.

The analysis made at the NACA Lewis laboratory and presented herein proposes a unified theory that is applicable to both direct and inverse problems for both subsonic and supersonic flows in compressors and turbines of arbitrary hub- and casing-wall shapes.

Equations of motion and energy for unsteady three-dimensional flow of a nonviscous compressible fluid are expressed in terms of some convenient quantities for analyzing flow in turbomachines. Entropy changes due to heat transfer in a cooled turbine and due to shock wave in supersonic flow can easily be accommodated. The condition under which irrotational-flow analysis is correct is also discussed.

The general equations are then simplified for steady through-flow in turbomachines having thin blades of high solidity. It is shown that, in the direct problem, just enough equations exist to determine all the variables; whereas in the inverse problem, after the inclusion of the integrability condition for the blade surface, either one variable or a relation between several variables can be specified by the designer.

In the solution of the problem, the continuity equation and the equation of motion in the radial direction are combined into a principal equation through the use of the stream function. This equation involves some terms that are either given or to be determined by other equations defining the present problem. Two forms of the principal equation are obtained for two main groups of designs in which the variation of tangential velocity and the variation of the ratio of tangential and axial velocities are prescribed by the designer. The criterions of whether the principal equation is elliptic or hyperbolic are obtained for both groups.

A general method of solving the set of equations for both the direct and the inverse problems is then described for turbomachines of arbitrary hub- and casing-wall shapes and for either an elliptic or hyperbolic principal equation.

The dimensionless forms of the principal and other equations for some typical designs are given. A simple approximate correction for blade thickness is also given.

SYMBOLS

The following symbols are used in this report:

a	velocity of sound
${}_{n}^{m}B_{j}^{i}$	differentiation coefficients in equation (70) used to multiply function value at point x_j to give m^{th} derivative at x_i using polynomial of n^{th} degree
$b,f,J,K,$ L,M,N	functions of r and z
C	constant
c_p, c_v	specific heat of gas at constant pressure and volume, respectively
$\frac{D}{Dt}$	differentiation with respect to time following motion of fluid particle
$D^m q$	m^{th} derivative of q
F	blade force per unit mass of fluid
G	Green's function
H	total enthalpy per unit mass of fluid, $h + \frac{V^2}{2}$
h	enthalpy per unit mass of fluid
I	$= h + \frac{W^2}{2} - \frac{U^2}{2} = H - \omega(V_u r)$
k	thermal conductivity
l	characteristic curve
m	order of derivative

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n	degree of polynomial
p	pressure
Q	heat added to fluid particle along path of motion per unit mass per unit time
q	dependent variable
R	gas constant
${}_{n}^{m}R^{i}$	remainder term of m^{th} derivative at point x_i obtained by using polynomial of n^{th} degree
r,θ,z	cylindrical coordinates relative to stator
r,φ,z	cylindrical coordinates relative to rotor
s	entropy per unit mass
T	temperature
t	time
U	velocity of blade at radius r
u	internal energy per unit mass with 0° absolute as base temperature
V	absolute fluid velocity
W	relative fluid velocity
x	independent variable
β	$= \arctan \frac{W_u}{W_z}$
δ_r,δ_z	grid spacing in r- and z-directions, respectively
γ	ratio of specific heats
$\bar{\gamma}$	average value of γ

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η,ζ independent variables

λ slope of characteristic curve, $\frac{dr}{dz}$

μ = $\tan\beta = \frac{W_u}{W_z}$

ν value of x between x_o and x_n

ρ mass density

ψ stream function

ω angular velocity of blade

Subscripts:

b trailing edge of guide vane

c leading edge of rotor

d trailing edge of rotor

e leading edge of stator

h hub

i inlet

j any station

m meridional

o refers to position where blade element is radial or $F_r = 0$

r,u,z radial, circumferential, and axial components

T total state

t at tip or casing

Superscripts:

a,b, i,j,k,l grid points

* dimensionless values

GENERAL BASIC EQUATIONS

The three-dimensional compressible flow of a nonviscous gas through a turbomachine is governed by the following set of basic laws of aerothermodynamics: From the principle of conservation of matter, the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overline{W}) = 0 \tag{1}$$

or

$$\nabla \cdot \overline{W} + \frac{D(\log_e \rho)}{Dt} = 0 \tag{1a}$$

For a blade rotating at a constant angular speed $\overline{\omega}$, Newton's second law of motion gives

$$\frac{D\overline{W}}{Dt} - \omega^2 \overline{r} + 2\overline{\omega} \times \overline{W} = \frac{D\overline{V}}{Dt} = -\frac{1}{\rho} \nabla p \tag{2}$$

The first law of thermodynamics may be written as

$$\frac{Du}{Dt} + p \frac{D\left(\rho^{-1}\right)}{Dt} = Q \tag{3}$$

where u is related to the gas temperature T by

$$\frac{Du}{Dt} = c_v \frac{DT}{Dt} \tag{4}$$

and Q is given by the following equation if only conduction is considered:

$$Q = \rho^{-1} \nabla \cdot (k \nabla T) \tag{5}$$

For the range of gas temperature and pressure encountered in ordinary turbomachines, p, ρ, and T are accurately related by the following equation of state

$$p = R\rho T \tag{6}$$

Although the flow of gas through the turbomachine is completely defined by the preceding equations together with the known variations of c_v and k with temperature and the given boundary and initial conditions, in reference 4 it is found more convenient to express the state of the gas in terms of entropy, total enthalpy, or a quantity I of the gas, in addition to its velocity components. These quantities are defined as follows:

$$T\ ds = du + p\ d\left(\rho^{-1}\right) \tag{7}$$

$$H = h + \frac{V^2}{2} \tag{8}$$

$$I = h + \frac{W^2}{2} - \frac{U^2}{2} = H - \omega(V_u r) \tag{9}$$

where

$$h = u + p\rho^{-1} \tag{10}$$

When equations (4), (6), (7), (10), and the relation,

$$R = c_p - c_v = (\gamma-1)c_v$$

which follows from equations (4), (6), and (10), are used, there are obtained

$$T\ ds = dh - \frac{dp}{\rho} \tag{11}$$

$$T\ ds = \frac{\gamma}{\gamma-1}\ d\left(\frac{p}{\rho}\right) - \frac{dp}{\rho} \tag{11a}$$

$$d\left(\frac{s}{R}\right) = \frac{1}{\gamma-1}\ d(\log_e p) - \frac{\gamma}{\gamma-1}\ d(\log_e \rho) \tag{12}$$

$$d\left(\frac{s}{R}\right) = \frac{1}{\gamma-1}\ d(\log_e T) - d(\log_e \rho) \tag{12a}$$

and the equation of continuity can be written as

$$\nabla \cdot \overline{W} + \frac{1}{\gamma-1}\frac{D(\log_e T)}{Dt} - \frac{D}{Dt}\left(\frac{s}{R}\right) = 0 \tag{13}$$

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From equations (9) and (11),

$$\frac{1}{\rho}\nabla p + \frac{1}{2}\nabla W^2 - \omega^2\overline{r} = \nabla I - T\nabla s$$

With this equation and the relation

$$\frac{D\overline{W}}{Dt} = \frac{\partial\overline{W}}{\partial t} + (\overline{W}\cdot\nabla)\overline{W} = \frac{\partial\overline{W}}{\partial t} + \frac{1}{2}\nabla\overline{W}^2 - \overline{W}\times(\nabla\times\overline{W})$$

the equation of motion (2) can be written as

$$\frac{\partial\overline{W}}{\partial t} - \overline{W}\times(\nabla\times\overline{W}) + 2\overline{\omega}\times\overline{W} = -\nabla I + T\nabla s \tag{14}$$

An alternate form of equation (14) that involves the vorticity of the absolute motion is obtained as follows: Using the cylindrical coordinate system with the z-axis parallel to $\overline{\omega}$ yields

$$\overline{V} = \overline{W} + \overline{\omega}\times\overline{r} \tag{15}$$

$$\nabla\times\overline{V} = \nabla\times\overline{W} + \nabla\times(\overline{\omega}\times\overline{r})$$

but

$$\nabla\times(\overline{\omega}\times\overline{r}) = (\overline{r}\cdot\nabla)\overline{\omega} - (\overline{\omega}\cdot\nabla)\overline{r} + \overline{\omega}(\nabla\cdot\overline{r}) - \overline{r}(\nabla\cdot\overline{\omega}) = 2\overline{\omega}$$

Hence,

$$\nabla\times\overline{V} = \nabla\times\overline{W} + 2\overline{\omega} \tag{15a}$$

This relation can also be seen from the following expressions of relative and absolute vorticity expressed in terms of cylindrical coordinates r,φ,z and r,θ,z, which refer to the rotor and stator, respectively:

$$\left.\begin{aligned}
(\nabla\times\overline{W})_r &= \frac{1}{r}\frac{\partial W_z}{\partial\varphi} - \frac{\partial W_u}{\partial z}\\
(\nabla\times\overline{W})_u &= \frac{\partial W_r}{\partial z} - \frac{\partial W_z}{\partial r}\\
(\nabla\times\overline{W})_z &= \frac{1}{r}\frac{\partial(W_u r)}{\partial r} - \frac{1}{r}\frac{\partial W_r}{\partial\varphi}
\end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned}(\nabla\times\overline{V})_r &= \frac{1}{r}\frac{\partial V_z}{\partial\theta} - \frac{\partial V_u}{\partial z}\\(\nabla\times\overline{V})_u &= \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\\(\nabla\times\overline{V})_z &= \frac{1}{r}\frac{\partial(V_u r)}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\end{aligned}\right\}\tag{17}$$

and the relation

$$\frac{\partial(V_u r)}{\partial r} = \frac{\partial(W_u r)}{\partial r} + 2\omega r \tag{18}$$

When equation (15a) is used, the alternate form of equation (14) is

$$\frac{\partial\overline{W}}{\partial t} - \overline{W}\times(\nabla\times\overline{V}) = -\nabla I + T\nabla s \tag{14a}$$

The use of equations (2), (9), and (11) yields

$$\begin{aligned}\frac{DI}{Dt} &= T\frac{Ds}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}(\overline{W}\cdot\nabla)p + \overline{W}\cdot\frac{D\overline{W}}{Dt} - \overline{U}\cdot\frac{D\overline{U}}{Dt}\\&= T\frac{Ds}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \overline{W}\cdot(\omega^2\overline{r} - 2\overline{\omega}\times\overline{W}) - \overline{U}\cdot\frac{D\overline{U}}{Dt}\\&= T\frac{Ds}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial t}\end{aligned}\tag{19}$$

Hence, energy equation (3) can be written as

$$Q = T\frac{Ds}{Dt} = \frac{DI}{Dt} - \frac{1}{\rho}\frac{\partial p}{\partial t} \tag{20}$$

The preceding equations lead to several important general considerations.

If the blade rows are not placed too close together, the pressure of gas at a fixed point relative to the blade can be taken as constant with respect to time. Consequently, according to equation (20), the entropy and the quantity I of the gas stay constant along its relative streamlines for adiabatic flow. The constancy of I means that the rate of change of total enthalpy along the streamline is equal to the angular speed of the blade multiplied by the rate of change of angular momentum about the machine axis of the gas particle along its streamline, which is the well-known Euler turbine equation usually 2057 derived under less general conditions. In a cooled turbine where the heat transfer may be large, the rate of change of s and I along the streamline can be obtained by using equation (20). Again, for steady relative flow, equation (14a) shows that the vanishing of absolute vorticity requires both gradient I and gradient s to vanish or the difference between ∇I and $T\nabla s$ to vanish. When both gradient I and gradient s are zero upstream of the blade row and the flow is adiabatic, s is uniform in passing through the blade row; p is then a function only of ρ (according to equation (12)), and consequently, according to Kelvin's circulation theorem, the absolute vorticity will remain zero in passing through the blade row and the flow can then be treated on the basis of irrotational absolute flow.

For flow through a stationary blade row, equation (14a) becomes

$$\frac{\partial \overline{V}}{\partial t} - \overline{V}\times(\nabla\times\overline{V}) = -\nabla H + T\nabla s \tag{14b}$$

which agrees with similar relations previously obtained by Vazsonyi (reference 6), and Hicks, Guenther, and Wasserman (reference 7). It is interesting to see that, for relative flow in a rotating blade row, $\overline{V}\times(\nabla\times\overline{V})$ becomes $\overline{W}\times(\nabla\times\overline{V})$ and H becomes I.

When it is assumed that the fluid enters the machine with uniform H and s and zero vorticity, the adiabatic flow through the inlet guide vanes can be treated on the basis of irrotational flow. When the guide vanes impart a radial variation of the tangential velocity of the fluid downstream of the vanes similar to that in a potential vortex, the circulation is constant along the blade span and the fluid maintains a uniform H and s and a zero vorticity going into the following rotor blade row. If the rotor blade row is situated far away from the inlet guide vanes, the fluid enters the rotor with a uniform I in the circumferential direction as well as in the radial direction, and the flow through the rotor blades can again be treated on the basis of zero absolute vorticity. If the rotor is close to the guide vanes, the fluid entering the rotor blades is circumferentially nonuniform, which condition is balanced by the unsteady term of the relative velocity, and the flow through the rotor blades should theoretically be treated on the basis of unsteady flow with zero absolute vorticity. When the guide vanes impart a radial variation of tangential velocity of the fluid downstream of the vanes different from that in a free vortex, however, the circulation varies along the span of the guide vanes, thereby shedding vortices from the trailing edge to the fluids downstream, and the fluid enters the following rotor blades with a uniform s and H, a nonuniform I, and a nonzero value of absolute vorticity. Consequently, the flow through the rotor blade row cannot be treated on the basis of zero absolute vorticity.

From the preceding discussion, the choice of H or I and s as the basic thermodynamic variables of the gas is apparent. Compressor and turbine rotors are usually designed to add or subtract the same amount of energy radially to or from the gas; hence, H is usually radially constant throughout the machine. When the circumferential velocity of gas upstream of the blade row is zero or varies inversely with radius, I is then radially constant throughout the machine when the heat transfer is zero or is uniform radially. These facts will be utilized in the following developments.

The continuity equation (13) can also be put into a form containing H or I by use of a constant value of γ, that is, $\overline{\gamma}$ for the range of temperatures involved in the process. By using $\overline{\gamma}$, equation (12a) can be written as

$$d(\log_e \rho) = d\left[\log_e \left(T^{\frac{1}{\overline{\gamma}-1}} e^{-\frac{s}{R}}\right)\right] \tag{21}$$

Integrating from the inlet total state yields

$$\rho^* = \frac{\rho}{\rho_{T,i}} = \left(\frac{T}{T_{T,i}}\right)^{\frac{1}{\overline{\gamma}-1}} e^{-\Delta s^*} = h^{*\frac{1}{\overline{\gamma}-1}} e^{-\Delta s^*}$$

$$= \left(\frac{H - \frac{V^2}{2}}{H_i}\right)^{\frac{1}{\overline{\gamma}-1}} e^{-\Delta s^*} = \left(\frac{I - \frac{W^2}{2} + \frac{\omega^2 r^2}{2}}{H_i}\right)^{\frac{1}{\overline{\gamma}-1}} e^{-\Delta s^*} \tag{22}$$

where $\Delta s^* = s^* - s^*_{T,i}$. The continuity equation then takes the following form:

$$\frac{\partial}{\partial t}\left(h^{*\frac{1}{\overline{\gamma}-1}} e^{-\Delta s^*}\right) + \nabla\cdot\left(h^{*\frac{1}{\overline{\gamma}-1}} e^{-\Delta s^*} \overline{W}\right) = 0 \tag{23}$$

The solution of the three-dimensional-flow problem thus consists of the simultaneous solution of equations (1) or (23), (14), and (20).

GENERAL THROUGH-FLOW THEORY

Because of the enormous mathematical difficulty in solving the preceding set of general equations, the essential feature of the three-dimensional flow in turbomachines will be investigated with the following two simplifying conditions: (1) The blade rows will be assumed to be placed so far apart that the relative flow through any blade row is steady. Under this condition, the equations of continuity, motion, and energy in the scalar forms are:

Continuity equation, from equation (1),

$$\frac{1}{r}\frac{\partial(\rho W_r r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho W_u)}{\partial \varphi} + \frac{\partial(\rho W_z)}{\partial z} = 0 \tag{24}$$

or, from equation (23),

$$\frac{1}{r}\frac{\partial\left(h^{*\frac{1}{\gamma-1}} e^{-\Delta s^*} W_r r\right)}{\partial r} + \frac{1}{r}\frac{\partial\left(h^{*\frac{1}{\gamma-1}} e^{-\Delta s^*} W_u\right)}{\partial \varphi} + \frac{\partial\left(h^{*\frac{1}{\gamma-1}} e^{-\Delta s^*} W_z\right)}{\partial z} = 0 \tag{24a}$$

The three equations of motion, from equations (14),

$$-\frac{W_u}{r}\left[\frac{\partial(V_u r)}{\partial r} - \frac{\partial W_r}{\partial \varphi}\right] + W_z\left(\frac{\partial W_r}{\partial z} - \frac{\partial W_z}{\partial r}\right) = -\frac{\partial I}{\partial r} + T\frac{\partial s}{\partial r} \tag{25}$$

$$\frac{W_r}{r}\left[\frac{\partial(V_u r)}{\partial r} - \frac{\partial W_r}{\partial \varphi}\right] - W_z\left(\frac{1}{r}\frac{\partial W_z}{\partial \varphi} - \frac{\partial W_u}{\partial z}\right) = -\frac{1}{r}\frac{\partial I}{\partial \varphi} + \frac{T}{r}\frac{\partial s}{\partial \varphi} \tag{26}$$

$$-W_r\left(\frac{\partial W_r}{\partial z} - \frac{\partial W_z}{\partial r}\right) + W_u\left(\frac{1}{r}\frac{\partial W_z}{\partial \varphi} - \frac{\partial W_u}{\partial z}\right) = -\frac{\partial I}{\partial z} + T\frac{\partial s}{\partial z} \tag{27}$$

and the energy equation, from equation (20),

$$Q = T\left(W_r\frac{\partial s}{\partial r} + \frac{W_u}{r}\frac{\partial s}{\partial \varphi} + W_z\frac{\partial s}{\partial z}\right) = W_r\frac{\partial I}{\partial r} + \frac{W_u}{r}\frac{\partial I}{\partial \varphi} + W_z\frac{\partial I}{\partial z} \tag{28}$$

(2) The blade is so thin and the number of blades is so large or the pitch is so small compared with the blade chord that the terms containing the circumferential variation of the velocity components in equations (24) to (28) are much smaller than other terms in the same equations and can therefore be neglected. The circumferential variation of pressure or enthalpy is, however, preserved in equation (26) by the introduction of a blade force $\overline{F}$, which may be considered as either due to the pressure exerted by the blade surface on the circumferentially thin gas stream or as a circumferentially averaged blade force on the gas stream between two blades. This simplification is first introduced by Lorenz (reference 8) in order to follow the flow along a given surface. The physical concept involved is clarified by Stodola (reference 9). Ruden (reference 10) further proves, for incompressible flow, that the solution so obtained will give an average value in the circumferential direction for a finite number of blades, provided that the departure from the average value is small. For turbomachines having relatively thin blades of moderately high solidity, this solution can indeed be taken as that for a relative stream surface, which is about midway (based on mass flow) between two blades. Because the circumferential variation of pressure and density is considered in equation (26), it is better to refer to the present theory as a "through-flow" theory (following Ruden), or "large-number-of-thin-blades" theory, instead of "axially symmetric" or "infinite-number-of-blades" theory. (Differentiating velocity components obtained in this theory and combining them according to equation (16) does not give the true vorticity.)

With this second condition, equations (24) to (28) become

$$\frac{1}{r}\frac{\partial(\rho W_r r)}{\partial r} + \frac{\partial(\rho W_z)}{\partial z} = 0 \tag{29}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(h^{*\frac{1}{\bar{\gamma}-1}} e^{-\Delta s^*} W_r r\right) + \frac{\partial}{\partial z}\left(h^{*\frac{1}{\bar{\gamma}-1}} e^{-\Delta s^*} W_z\right) = 0 \tag{29a}$$

$$\begin{aligned} W_z\left(\frac{\partial W_r}{\partial z} - \frac{\partial W_z}{\partial r}\right) &= -\frac{\partial I}{\partial r} + \frac{W_u}{r}\frac{\partial(V_u r)}{\partial r} + T\frac{\partial s}{\partial r} + F_r \\ &= -\frac{\partial H}{\partial r} + \frac{V_u}{r}\frac{\partial(V_u r)}{\partial r} + T\frac{\partial s}{\partial r} + F_r \end{aligned} \tag{30}$$

$$\frac{D(V_u r)}{Dt} = W_r \frac{\partial(V_u r)}{\partial r} + W_z \frac{\partial(V_u r)}{\partial z} = F_u r \tag{31}$$

$$\begin{aligned} - W_r \left(\frac{\partial W_r}{\partial z} - \frac{\partial W_z}{\partial r} \right) &= - \frac{\partial I}{\partial z} + \frac{W_u}{r} \frac{\partial(V_u r)}{\partial z} + T \frac{\partial s}{\partial z} + F_z \\ &= - \frac{\partial H}{\partial z} + \frac{V_u}{r} \frac{\partial(V_u r)}{\partial z} + T \frac{\partial s}{\partial z} + F_z \end{aligned} \tag{32}$$

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and

$$Q = T \frac{Ds}{Dt} = T\left(W_r \frac{\partial s}{\partial r} + W_z \frac{\partial s}{\partial z}\right) \tag{33}$$

or

$$Q = \frac{DI}{Dt} = W_r \frac{\partial I}{\partial r} + W_z \frac{\partial I}{\partial z} \tag{33a}$$

The circumferential enthalpy or pressure gradient in equation (31) is replaced by the term F_u. The corresponding effects in the other two directions are represented by F_r and F_z in equations (30) and (32), respectively. The vector F is perpendicular to $\overline{W}$ or related to the shape of the blade or flow surface by the following equations:

$$F_r W_r + F_u W_u + F_z W_z = 0 \tag{34}$$

or

$$F_r \, dr + F_u r \, d\varphi + F_z \, dz = 0 \tag{34a}$$

Only six independent equations exist in the preceding equations: one continuity relation, three equations of motion, one energy equation, and one orthogonality relation between $\overline{F}$ and $\overline{W}$. For example, equation (33a) can be derived from the equations of motion, equation of energy (33), and equation (34). (See reference 4.)

DIRECT AND INVERSE PROBLEMS

In a direct problem, the blade surface or flow surface is considered to be given by the equation

$$S(r,\theta,z) = C \tag{35}$$

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and two relations among the force components are obtained by relating them to the partial derivatives of the function S:

$$\frac{F_r}{F_u} = \frac{\frac{\partial S}{\partial r}}{\frac{1}{r}\frac{\partial S}{\partial \theta}} \tag{36a}$$

$$\frac{F_z}{F_u} = \frac{\frac{\partial S}{\partial z}}{\frac{1}{r}\frac{\partial S}{\partial \theta}} \tag{36b}$$

Equations (29) to (34) and (36) therefore provide eight independent relations that completely define the direct problem involving the eight primary variables W_r, W_u, W_z, F_r, F_u, F_z, s, and I or H. (The heat transfer Q between the gas and the blade is only important in the case of cooled turbine blades and is considered to be given by cooling considerations.)

In the inverse problem, the blade surface is to be determined, which means that the two relations among the force components as given by equations (36) are not available. This unavailability does not mean, however, that the designer has freedom to prescribe the variation of two variables or two conditions among the variables throughout the blade region because, in order that the differential equation (34a) will lead to an integral blade or flow surface of the form of equation (35), the following necessary (and sufficient) condition of integrability must be satisfied (reference 11):

$$\overline{F}\cdot\nabla\times\overline{F} = 0 \tag{37}$$

which, for the present case, reduces to

$$\frac{\partial}{\partial r}\left(\frac{F_z}{F_u r}\right) = \frac{\partial}{\partial z}\left(\frac{F_r}{F_u r}\right) \tag{37a}$$

This condition of integrability was first pointed out by Bauersfeld as early as 1905 (reference 12), but is neglected in many recent investigations. In effect, it gives a restriction to the velocity variation that a designer can specify through the force terms in the motion equations (30) to (32).

Whereas in the direct problems two conditions are obtainable from the given surface, in the inverse problem one condition on the surface must be satisfied. Hence, the designer can specify, in the inverse problem, only one relation throughout the blade region, which can be a variation of one thermodynamic quantity of gas (V_u, V_z, p, or h), one relation on the blade surface (untwisted or radial blade element), or one relation among the gas properties (constant Mach number), and so forth. The variation prescribed should, of course, be reasonable so that the solution exists.

After this one relation is prescribed by the designer, the solution of the inverse problem is quite similar to the direct problem. Among all the equations to be satisfied, equations (29) to (34) are common to both problems. In addition to these six equations, equations (36) are available in the direct problem and equation (37a) and the one relation prescribed by the designer to be satisfied are available in inverse problem. If, in the design, a condition on the blade surface is specified such as the blade design in which all blade elements are radial, F_r is prescribed as zero by the designer, and the integrability condition equation (37a) leads to a simple relation between F_u and F_z. These same two relations are also directly given by equations (36a) and (36b), respectively, in the direct problem, and consequently the solution for the two problems is exactly the same. In other cases, however, the solution of the two problems is a little different.

PRINCIPAL EQUATION FOR TWO MAIN GROUPS OF DESIGN

In this and the following sections, a general method of solution for both the direct and the inverse problems will be described. From the preceding equations the through-flow considered herein is essentially described by the equation of continuity (29) and the equation of motion either in the radial direction (equation (30)), or in the axial direction (equation (32)). Except in the case of low-speed centrifugal impellers, it is always advantageous to use equation (30), because F_r is either zero in high-speed centrifugal or mixed-flow impellers or relatively small in axial-flow bladings. Also, either $\partial H/\partial r$ or $\partial I/\partial r$ is usually equal to zero, and $\partial(V_u r)/\partial r$ is often given. This choice is used hereinafter. If it is desirable to use equation (32), equations can be developed in a similar manner.

The combination of equations (29) and (30) is carried out by the use of a stream function, which is defined as follows and satisfies the continuity equation (29):

$$\frac{\partial \psi}{\partial r} = r\rho^* W_z = rh^{*\frac{1}{\gamma-1}} e^{-\Delta s^*} W_z \tag{38a}$$

$$\frac{\partial \psi}{\partial z} = -r\rho^* W_r = -rh^{*\frac{1}{\gamma-1}} e^{-\Delta s^*} W_r \tag{38b}$$

From equation (37),

$$\left.\begin{aligned} rh^{*\frac{1}{\gamma-1}} e^{-\Delta s^*} \frac{\partial W_z}{\partial r} &= \frac{\partial^2 \psi}{\partial r^2} + \left(-\frac{1}{r} - \frac{1}{a^2}\frac{\partial h}{\partial r} + \frac{\partial s^*}{\partial r}\right)\frac{\partial \psi}{\partial r} \\ -rh^{*\frac{1}{\gamma-1}} e^{-\Delta s^*} \frac{\partial W_r}{\partial z} &= \frac{\partial^2 \psi}{\partial z^2} + \left(-\frac{1}{a^2}\frac{\partial h}{\partial z} + \frac{\partial s^*}{\partial z}\right)\frac{\partial \psi}{\partial z} \end{aligned}\right\} \tag{39}$$

The succeeding development is a little different for the two main groups of designs to be considered. In the first group, the variation of the angular momentum of the gas about the axis of rotation is prescribed by the designer; that is,

$$V_u r = f_1(r,z) \tag{40}$$

is given. Among this group are the free-vortex type in which f_1 is just a function of z, the more general solid-body-rotation type, the symmetrical-velocity-diagram type, and others. In the second group of designs, the following relation between tangential and axial velocity is prescribed by the designer:

$$\mu = \tan \beta = \frac{W_u}{W_z} = f_2(r,z) \tag{41}$$

Among this group are the common blade design for high-speed centrifugal- and mixed-flow impellers in which all blade elements are radial with $\mu = r\, f_3(z)$, the less general design with $\mu = \tan \beta = f_4(z)$, which gives a practically untwisted blade and is most suitable for cooled turbine rotor, and others. The principal equation will now be obtained in a form most convenient for these two main groups of designs.

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Group of Designs in which Equation (40) is Specified

From equations (9) and (38),

$$h = I - \frac{W_u^{\ 2}}{2} + \frac{\omega^2 r^2}{2} - \frac{1}{2} r^{-2} h^{*\frac{2}{\overline{\gamma}-1}} e^{2\Delta s^*} \left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] \tag{42}$$

Differentiating with respect to r and z yields

$$\left.\begin{aligned}
\frac{\partial h}{\partial r} &= \frac{a^2}{a^2-W_m^{\ 2}}\left[\frac{\partial I}{\partial r} - W_u \frac{\partial W_u}{\partial r} + \omega^2 r - W_m^{\ 2}\left(-\frac{1}{r} + \frac{\partial s^*}{\partial r}\right) - \right. \\
&\qquad \left. r^{-1} h^{*-\frac{1}{\overline{\gamma}-1}} e^{\Delta s^*}\left(W_z \frac{\partial^2\psi}{\partial r^2} - W_r \frac{\partial^2\psi}{\partial r\,\partial z}\right)\right] \\
\frac{\partial h}{\partial z} &= \frac{a^2}{a^2-W_m^{\ 2}}\left[\frac{\partial I}{\partial z} - W_u \frac{\partial W_u}{\partial z} - W_m^{\ 2} \frac{\partial s^*}{\partial z} - \right. \\
&\qquad \left. r^{-1} h^{*\frac{1}{\overline{\gamma}-1}} e^{\Delta s^*}\left(W_z \frac{\partial^2\psi}{\partial r\,\partial z} - W_r \frac{\partial^2\psi}{\partial z^2}\right)\right]
\end{aligned}\right\} \tag{43}$$

Substituting equation (43) into equation (39) and adding yield

$$(a^2-W_m^{\ 2}) r h^{*\frac{1}{\overline{\gamma}-1}} e^{-\Delta s^*}\left(\frac{\partial W_z}{\partial r} - \frac{\partial W_r}{\partial z}\right) = (a^2-W_r^{\ 2}) \frac{\partial^2\psi}{\partial r^2} - 2W_r W_z \frac{\partial^2\psi}{\partial r\,\partial z} +$$

$$(a^2-W_z^{\ 2}) \frac{\partial^2\psi}{\partial z^2} + \left(-\frac{a^2}{r} - \frac{\partial I}{\partial r} + W_u \frac{\partial W_u}{\partial r} - \omega^2 r + a^2 \frac{\partial s^*}{\partial r}\right) \frac{\partial\psi}{\partial r} +$$

$$\left(-\frac{\partial I}{\partial z} + W_u \frac{\partial W_u}{\partial z} + a^2 \frac{\partial s^*}{\partial z}\right) \frac{\partial\psi}{\partial z}$$

$$= 0 \tag{44}$$

Substituting equation (44) into equation (30) and dividing by a^2 yield the following principal equation:

$$\left(1 - \frac{W_r^2}{a^2}\right)\frac{\partial^2\psi}{\partial r^2} - 2\,\frac{W_rW_z}{a^2}\,\frac{\partial^2\psi}{\partial r\,\partial z} + \left(1 - \frac{W_z^2}{a^2}\right)\frac{\partial^2\psi}{\partial z^2} + L_1\,\frac{\partial\psi}{\partial r} + N_1\,\frac{\partial\psi}{\partial z} = 0 \qquad (45)$$

where

$$L_1 = -\frac{1}{r} + \frac{1}{a^2}\left[-\frac{\partial I}{\partial r} + W_u\,\frac{\partial W_u}{\partial r} - \omega^2 r + a^2\,\frac{\partial s^*}{\partial r} + \frac{a^2-W_m^2}{W_z^2}\left(-\frac{\partial I}{\partial r} + \frac{W_u}{r}\,\frac{\partial(V_ur)}{\partial r} + F_r + T\,\frac{\partial s}{\partial r}\right)\right]$$

$$N_1 = \frac{1}{a^2}\left(-\frac{\partial I}{\partial z} + W_u\,\frac{\partial W_u}{\partial z} + a^2\,\frac{\partial s^*}{\partial z}\right)$$

With the variation of V_u or W_u prescribed by the designer, the meridional velocity components are determined by the principal equation (45). The other equations are used to determine various terms involved in the coefficients L_1 and N_1. From the coefficients of the second derivatives, the principal equation (45) is hyperbolic when the meridional velocity $W_m = \sqrt{W_r^2 + W_z^2}$ is greater than the speed of sound, and elliptic when the meridional velocity is less than the speed of sound. For the hyperbolic case, the method of characteristics can be used and will be discussed later. For the elliptic case, it is convenient to put the principal equation in a slightly different form, as follows: From equation (38),

$$r\rho^*\,\frac{\partial W_z}{\partial r} = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\,\frac{\partial\psi}{\partial r} - \frac{\partial(\log_e\,\rho^*)}{\partial r}\,\frac{\partial\psi}{\partial r} \qquad r\rho^*\,\frac{\partial W_r}{\partial z} = \frac{\partial^2\psi}{\partial z^2} - \frac{\partial(\log_e\,\rho^*)}{\partial z}\,\frac{\partial\psi}{\partial z} \qquad (46)$$

Substituting equation (46) into equation (30) results in

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\,\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} - \left[\frac{\partial\psi}{\partial r}\,\frac{\partial(\log_e\,\rho^*)}{\partial r} + \frac{\partial\psi}{\partial z}\,\frac{\partial(\log_e\,\rho^*)}{\partial z}\right] + \frac{(r\rho^*)^2}{\frac{\partial\psi}{\partial r}}\left[\frac{W_u}{r}\,\frac{\partial(V_ur)}{\partial r} - \frac{\partial I}{\partial r} + T\,\frac{\partial s}{\partial r} + F_r\right] = 0 \qquad (47)$$

In this form, all terms except the first three are taken as constant during the successive improvement of ψ values throughout the whole region in the numerical solution so that the coefficients in the difference equations of ψ's will not vary during successive cycles.

Equation (45) or (47) is then the principal equation for this group of designs to be used for a meridional velocity greater or less than the velocity of sound, respectively. The process involved in solving this principal equation, together with other equations in the inverse and direct problems, is as follows:

In the inverse problem, V_u or W_u is given by equation (40). Equation (31) is first used to compute F_u. The energy equation (33) is then used to determine the variation of s along the streamline. The variation of I along the streamline is obtained from equation (33a). Equation (32) is used to compute F_z and F_r is then obtained by integrating equation (36a) along a constant r line:

$$F_r = F_u r \int_{z_o}^{z} \frac{\partial}{\partial r}\left(\frac{F_z}{F_u r}\right) d\zeta \tag{48}$$

where $F_r = 0$ at $z = z_o$. The solution is then carried downstream by equation (45) in the hyperbolic case, whereas successive sets of improved values of ψ are obtained throughout the region in the elliptic case. The quantities W_r and W_z are then computed from equation (38).

In the direct problem, equations (36a) and (36b) are given. It is most convenient to obtain W_u from equation (34) as follows:

$$W_u = -\left(\frac{F_r}{F_u} W_r + \frac{F_z}{F_u} W_z\right) \tag{49}$$

The quantity F_u is then computed from equation (31) and F_z and F_r are obtained from equations (36). Equation (33) is used to determine the variation of s along the streamline and equation (32) to determine the variation of I or H. Equation (30) is then used to solve for ψ as before.

Group of Designs in which Equation (41) is Specified

For this group of designs, it is necessary to combine W_u into W_z according to equation (41) as follows: Substituting equation (41) into equation (30) yields

$$(1+\mu^2)\frac{\partial W_z}{\partial r} - \frac{\partial W_r}{\partial z} + \mu\left(\frac{\mu}{r} + \frac{\partial \mu}{\partial r}\right)W_z + 2\omega\mu + \frac{1}{W_z}\left(-\frac{\partial I}{\partial r} + T\frac{\partial s}{\partial r} + F_r\right) = 0 \tag{50}$$

Instead of equation (42),

$$h = I + \frac{\omega^2 r^2}{2} - \frac{1}{2} r^{-2} h^{*\,-\frac{2}{\gamma-1}} e^{2\Delta s^*}\left[(1+\mu^2)\left(\frac{\partial \psi}{\partial r}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2\right] \tag{51}$$

Differentiating with respect to r and z, combining with equation (39), and substituting into equation (50) give the following form of the principal equation:

$$(1+\mu^2)\left(1 - \frac{W_r^2}{a^2}\right)\frac{\partial^2 \psi}{\partial r^2} - 2(1+\mu^2)\frac{W_r W_z}{a^2}\frac{\partial^2 \psi}{\partial r\,\partial z} + \left(1 - \frac{W_u^2 + W_z^2}{a^2}\right)\frac{\partial^2 \psi}{\partial z^2} +$$

$$L_2\frac{\partial \psi}{\partial r} + N_2\frac{\partial \psi}{\partial z} = 0 \tag{52}$$

where

$$L_2 = (1+\mu^2)\left[-\frac{1}{r} + \frac{\partial s^*}{\partial r} - \frac{1}{a^2}\left(\frac{\partial I}{\partial r} + \omega^2 r - W_z^2 - \mu\frac{\partial \mu}{\partial r}\right)\right] +$$

$$\frac{a^2 - W^2}{a^2}\mu\left(\frac{\mu}{r} + \frac{\partial \mu}{\partial r}\right) + \frac{a^2 - W^2}{a^2 W_z^2}\left(-\frac{\partial I}{\partial r} + T\frac{\partial s}{\partial r} + F_r + 2\omega W_u\right)$$

$$N_2 = \frac{\partial s^*}{\partial z} - \frac{1}{a^2}\frac{\partial I}{\partial z} + \frac{W_z^2}{a^2}\mu\frac{\partial \mu}{\partial z}$$

Equation (52) becomes hyperbolic when the relative velocity is supersonic, and elliptic when the relative velocity is subsonic. For subsonic velocity, a more convenient form of this equation for computation is obtained by substituting equation (46) into (49):

$$(1+\mu^2)\frac{\partial^2\psi}{\partial r^2} - \left(\frac{1}{r} - \mu\,\frac{\partial\mu}{\partial r}\right)\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{\rho}\left[(1+\mu^2)\frac{\partial\psi}{\partial r}\,\frac{\partial\rho}{\partial r} + \frac{\partial\psi}{\partial z}\,\frac{\partial\rho}{\partial z}\right] +$$

$$2\mu\omega r\rho^* + \frac{r^2\rho^{*2}}{\frac{\partial\psi}{\partial r}}\left(-\frac{\partial I}{\partial r} + T\,\frac{\partial s}{\partial r} + F_r\right) = 0 \tag{53}$$

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The process involved in solving the principal equation simultaneously with other equations in the direct and inverse problems is essentially the same as that given for the previous group of designs. The only difference is that W_u is now computed from equation (41) and, in the elliptic case, is to be reevaluated after successive improvements of ψ. In the design where radial blade elements are employed, the computation is considerably shortened with F_r equal to zero, which shows the advantage of using the cylindrical coordinate system for this problem. (In the customary method employing a coordinate system along the streamline and normal to it, the blade force along the normal is not zero.) In this design, the process for the direct and the inverse problem is exactly the same.

The use of $\mu = f_3(z)$ in the design will lead to a blade close to the untwisted type if the hub-tip-radius ratio is not too small. In the direct problem with such a blade given, this relation may be used or the flow may be more accurately obtained by using the equations given for the first group.

The different character of the principal equation for the two groups of designs considered is interesting. The character of the principal equation depends on the variation prescribed in the design or considered as given in the direct problem. This fact may be utilized to solve some flow problems in which the flow in some region is slightly supersonic. When a tangential-velocity variation is given, the equation for the whole region may be elliptic.

GENERAL METHOD OF SOLVING PRINCIPAL EQUATION

Elliptic Case

For convenience of discussion, the fundamental equation (47) or (53) of the two groups of designs can be written in the following general form:

$$\frac{\partial\psi}{\partial r}\left[J\,\frac{\partial^2\psi}{\partial r^2} - \frac{K}{r}\,\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{\rho}\left(J\,\frac{\partial\psi}{\partial r}\,\frac{\partial\rho}{\partial r} + \frac{\partial\psi}{\partial z}\,\frac{\partial\rho}{\partial z}\right) + L\,\rho^*\right] - r^2\rho^{*2}M = 0 \qquad (54)$$

The equation is nonlinear even for incompressible flow. The equation may be more conveniently rewritten in a linear form as

$$J\,\frac{\partial^2\psi}{\partial r^2} - \frac{K}{r}\,\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = N \qquad (55)$$

where

$$N = \frac{1}{\rho}\left(J\,\frac{\partial\psi}{\partial r}\,\frac{\partial\rho}{\partial r} + \frac{\partial\psi}{\partial z}\,\frac{\partial\rho}{\partial z}\right) - L\,\rho^* + \frac{r^2\rho^{*2}}{\frac{\partial\psi}{\partial r}}\,M \qquad (56)$$

and is evaluated from an approximate solution at the start of the calculation and from the ψ and ρ values obtained in the previous cycle during the calculation. For simple boundary shapes and simple functions of J and K, it may be possible to find a Green's function $G(y,z,\eta,\zeta)$ with its proper properties so that the solution of the problem can be written in the following form:

$$\psi(r,z) = \iint G(r,z,\eta,\zeta)\;N(\eta,\zeta)\;d\eta\;d\zeta \qquad (57)$$

For example, for flow with the design in which the tangential velocity is prescribed, the principal equation takes the form

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\,\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = N \qquad (58)$$

If the boundary walls are cylindrical surfaces, the total-enthalpy and entropy are uniform, and the tangential and the radial velocity both vanish at the inlet and the exit, the radial variation of ψ at the inlet and the exit are the same and can be subtracted from ψ, which results in

$$\frac{\partial^2(\psi-\psi_i)}{\partial r^2} - \frac{1}{r}\,\frac{\partial(\psi-\psi_i)}{\partial r} + \frac{\partial^2(\psi-\psi_i)}{\partial z^2} = N \qquad (59)$$

which can be written as

$$\frac{\partial^2}{\partial r^2}\left(\frac{\psi-\psi_i}{r}\right) + \frac{\partial}{\partial r}\left(\frac{\psi-\psi_i}{r^2}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\psi-\psi_i}{r}\right) = \frac{N}{r} \tag{60}$$

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The quantity $\left(\frac{\psi-\psi_i}{r}\right)$ is zero on the boundary and the corresponding Green's function is available from a similar equation in reference 3. When the Green's function of reference 3 is used, the solution for ψ is

$$\psi = \psi_i + r \iint G(r,z,\eta,\zeta)\ \frac{N}{\eta}\ (\eta,\zeta)\ d\eta\ d\zeta \tag{61}$$

If G is tabulated at several values of r, $\psi-\psi_i$ can be conveniently obtained by a numerical double integration process on a punch-card machine. For curved boundary walls in the meridional plane, it is necessary, in this method, to use the technique of conformal transformation to render the given boundary shape into a rectangular one. Inasmuch as this process involves a numerical solution of the Laplace equation with the given boundary shape, it is found better to solve directly the given equation (55) with the given shape by the numerical method. Furthermore, this solution will be the only choice in the general case where J and K are not equal to 1 or the boundary condition is more general, which makes the task of obtaining the proper Green's function a very difficult one, if not impossible.

In order to solve directly the given equation (55), a general numerical differentiation formula for first and second derivatives with function value given at unequally spaced grid points using second- and higher-order polynomial representation is required to give conveniently and accurately the finite-difference expressions at the grid point near the curved boundary, which is done as follows: When the value of any quantity q is known corresponding to a number of unequally spaced values of the independent variable x, denoted by x_0, x, . . . x_n, the variation of q with respect to x is most conveniently expressed by a Lagrangian polynomial of the n^{th} degree:

$$q(x) = \sum_{j=0}^{n} \frac{\Pi_{n+1}(x)}{x-x_j}\ \frac{q^j}{\Pi'_{n+1}(x_j)} + {}_n^0R \tag{62}$$

where

$$\Pi_{n+1}(x) = (x-x_0)(x-x_1)\ .\ .\ .\ (x-x_n) \tag{63}$$

$$\Pi'_{n+1}(x_j) = (x_j-x_0)(x_j-x_1)\ .\ .\ .\ (x_j-x_{j-1})(x_j-x_{j+1})\ .\ .\ .\ (x_j-x_n) \tag{64}$$

and

$${}^0_nR = \frac{\Pi_{n+1}(x)}{(n+1)!}\, q^{(n+1)}(\nu) \tag{65}$$

where ν lies between x_0 and x_n. The successive derivatives of q with respect to x at any point x can be expressed as (reference 13)

$$D^1 q = \sum_{j=0}^{n}\left[\frac{q^j}{\Pi'_{n+1}(x_j)}\,\frac{\Pi_{n+1}(x)}{x-x_j}\sum_{\substack{k=0\\k\neq j}}^{n}\frac{1}{x-x_k}\right] + {}^1_nR \tag{66}$$

$$D^2 q = 2!\sum_{j=0}^{n}\left[\frac{q^j}{\Pi'_{n+1}(x_j)}\,\frac{\Pi_{n+1}(x)}{x-x_j}\sum_{\substack{k=0\\k\neq j}}^{n}\left(\frac{1}{x-x_k}\sum_{\substack{l=k+1\\l\neq j}}^{n}\frac{1}{x-x_l}\right)\right] + {}^2_nR \tag{67}$$

and so forth, with

$${}^1_nR = \frac{\Pi'_{n+1}(x)}{(n+1)!}\, q^{(n+1)}(\nu) + \frac{\Pi_{n+1}(x)}{(n+1)!}\, q^{(n+2)}(\nu)\,\frac{d\nu}{dx} + \ .\ .\ . \tag{68}$$

$${}^2_nR = \frac{\Pi''_{n+1}(x)}{(n+1)!}\, q^{(n+1)}(\nu) + 2\,\frac{\Pi'_{n+1}(x)}{(n+1)!}\, q^{(n+2)}(\nu)\,\frac{d\nu}{dx} + \ .\ .\ . \tag{69}$$

The summation operation is very easily performed when x is a grid point, because most of the products vanish. At these points, it is convenient to write

$$(D^m q)_{x=x_i} = \sum_{j=0}^{n} {}^m_nB^i_j\, q^j + {}^m_nR^i \tag{70}$$

The differentiation coefficients B and the coefficients of the derivative in the first or second remainder term have been explicitly expressed in reference 13 in terms of the spacings between the successive grid points for general nonuniform spacing throughout and for the special case near a tapered or curved boundary where only the first or last spacing is different from the others, using polynomials of second, third, and fourth order. For the special case, these coefficients have also been computed for difference ratios from 0.1 to 1.29 in intervals of 0.01 of the distance between the boundary and the nearest point and the other spacings and are given in reference 13. For spacing lying between tabulated intervals, the interpolation coefficients given in reference 14 can be used to obtain the required values of B.

In the present fluid-flow problems, it is necessary to cover a large region in order to reach the boundary condition that is always given at stations far upstream and downstream of the blade row. In order to reduce the labor of computation, it is desirable to determine if the number of grid points required for a given accuracy can be reduced by using an order of polynomial higher than the customary second order. A study of the remainder terms (reference 13) and actual experience in the present problems show that, in most cases, the use of fourth-order polynomial will reduce the necessary number of grid points to less than one-quarter of that required by the second-order polynomial. In setting up the grid pattern, it is always desirable to map the flow region in such a manner that the distance between the boundary and a point next to it is not too small compared with the other distances, because the differentiation coefficient becomes very sensitive to the small ratio. If the small ratio cannot be avoided, it is best not to include these points in the calculation.

When the grid pattern and the order of polynomial representation have been selected, the coefficients B at each point can be obtained from the table given in reference 13. Then the differential equation (55) at any grid point whose ψ value is ψ^i (fig. 1) is replaced by the following algebraic equation:

$$\sum_{j=0}^{n} \left(J^i \, {}_n^2B_j^i - \frac{K^i}{r^i} \, {}_n^1B_j^i\right)\psi^j + \sum_{k=0}^{n} {}_n^2B_k^i \, \psi^k - N^i = 0 \tag{71}$$

where ψ^j and ψ^k denote the values of ψ along constant z and constant r lines, respectively.

The values of ψ along the hub and the casing walls can be arbitrarily chosen, with the difference proportional to the mass flow between them. At the first station on the left 1-1 and at the last station 2-2 on the right (fig. 2), however, the ψ values are unknown. The boundary condition at stations 1-1 and 2-2 is, usually, that the flow is parallel to the bounding walls. When the hub and casing walls are horizontal at the inlet and exit of the machine and the stations chosen are far enough out, the value ψ^b required at a point δ_z distance away from the ψ^a point is also equal to ψ^a. Whether or not the inlet and the exit stations are chosen far enough out will be indicated by the variation of ψ's along these stations obtained in the solution.

After equation (71) is obtained at every interior grid point, a number of methods can be used to solve the set of simultaneous algebraic equations. For hand computation, the relaxation method developed by Southwell has proved to be superior to others for this type of equation (references 12 and 15 to 18). If the fourth-order polynomial representation is decided, the calculation can be carried out in two steps by using only the five main coefficients in early stages and then the residuals are recomputed and relaxed by all nine coefficients (reference 13).

If a high-speed, large-scale, digital computing machine is available, the set of equations can be solved either by an indirect or direct method. In the indirect method, the straight iterative method of Liebmann is used, wherein ψ^i is solved at each point from its surrounding values according to equation (71) and the process is repeated until the change at any point is no longer significant. This method is simplest to set up, but is slowest. A better method is to set the machine to perform a simple relaxation process by computing the residual at each point and relaxing according to a fixed relation with respect to the amount of residual just found.

A much quicker machine method, especially when a number of solutions with, for example, different inlet Mach numbers are to be found with a given geometrical shape of the problem, is the direct method that solves the set of simultaneous equations by a matrix process. The details of a matrix method, which fully utilizes the great number of zero elements of the original coefficient matrix, is given in reference 13.

Hyperbolic Case

In the hyperbolic case, the problem consists in solving ψ from the following principal equation, written in a general form for the two groups of designs:

$$J \frac{\partial^2 \psi}{\partial z^2} + 2K \frac{\partial^2 \psi}{\partial z \, \partial r} + L \frac{\partial^2 \psi}{\partial r^2} + M \frac{\partial \psi}{\partial z} + N \frac{\partial \psi}{\partial r} = 0 \tag{72}$$

with the initial condition that ψ and its normal derivative are given on a curve that is not a characteristic line.

From equation (72), the equation of the characteristic line is

$$J \left(\frac{dr}{dz}\right)^2 - 2K \left(\frac{dr}{dz}\right) + L = 0 \tag{73}$$

The slopes of the characteristic lines in the r,z-plane are

$$\lambda_1 = \left(\frac{dr}{dz}\right)_1 = \frac{K}{J} - \frac{1}{J}\sqrt{K^2 - JL} \tag{74a}$$

$$\lambda_2 = \left(\frac{dr}{dz}\right)_2 = \frac{K}{J} + \frac{1}{J}\sqrt{K^2 - JL} \tag{74b}$$

When the point on the r,z-plane moves along the characteristic curve l, corresponding to a small change dz in z, the change in r is $dr = \lambda_1 \, dz$. Because of these two small changes, the change of any quantity q is

$$dq = \frac{dq}{dz} dz = \frac{\partial q}{\partial z} dz + \frac{\partial q}{\partial r} \lambda_1 \, dz \tag{75}$$

or

$$\frac{dq}{dz} = \frac{\partial q}{\partial z} + \lambda_1 \frac{\partial q}{\partial r} \tag{76}$$

Hence, along l_1,

$$\frac{d}{dz}\left(\frac{\partial \psi}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\right) + \lambda_1 \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial z}\right) = \frac{\partial^2 \psi}{\partial z^2} + \lambda_1 \frac{\partial^2 \psi}{\partial z \, \partial r} \tag{77}$$

$$\frac{d}{dz}\left(\frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial r}\right) + \lambda_1 \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial r}\right) = \frac{\partial^2 \psi}{\partial z \, \partial r} + \lambda_1 \frac{\partial^2 \psi}{\partial r^2} \tag{78}$$

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From equatio) and (78),

$$\frac{\partial^2\psi}{\partial z\,\partial r} = \frac{d}{dz}\left(\frac{\partial\psi}{\partial r}\right) - \lambda_1 \frac{\partial^2\psi}{\partial r^2} \tag{79}$$

$$\frac{\partial^2\psi}{\partial z^2} = \frac{d}{dz}\left(\frac{\partial\psi}{dz}\right) - \lambda_1 \frac{d}{dz}\left(\frac{\partial\psi}{\partial r}\right) + \lambda_1{}^2 \frac{\partial^2\psi}{\partial r^2} \tag{80}$$

Substituting equations (79) and (80) into equation (72) yields

$$J \frac{d}{dz}\left(\frac{\partial\psi}{\partial z}\right) + (2K-J\lambda_1) \frac{d}{dz}\left(\frac{\partial\psi}{\partial r}\right) + (J\lambda_1{}^2-2K\lambda_1+L)\frac{\partial^2\psi}{\partial r^2} + M \frac{\partial\psi}{\partial z} + N \frac{\partial\psi}{\partial r} = 0 \tag{81}$$

By virtue of equations (74a) and (74b), equation (81) becomes

$$\frac{d}{dz}\left(\frac{\partial\psi}{\partial z}\right) + \lambda_2 \frac{d}{dz}\left(\frac{\partial\psi}{\partial r}\right) + \frac{M}{J}\frac{\partial\psi}{\partial z} + \frac{N}{J}\frac{\partial\psi}{\partial r} = 0 \tag{82}$$

Similarly, along the second characteristic line l_2,

$$\frac{d}{dz}\left(\frac{\partial\psi}{\partial z}\right) + \lambda_1 \frac{d}{dz}\left(\frac{\partial\psi}{\partial r}\right) + \frac{M}{J}\frac{\partial\psi}{\partial z} + \frac{N}{J}\frac{\partial\psi}{\partial r} = 0 \tag{83}$$

Starting from two points a and b a short distance apart on the curve where the initial condition is given, equations (74a) and (74b) give the tangent to the characteristic curves at these two points and equations (82) and (83) give the new value of $\partial\psi/\partial z$ and $\partial\psi/\partial r$ at the point of intersection c of the two tangent lines. The auxiliary equations corresponding to the particular problem are then used to determine other pertinent quantities at the point c. This process is to be carried step by step downstream. The method is the same as for ordinary two-dimensional rotational flow. (For details of calculation, see reference 19.)

APPLICATION TO TYPICAL DESIGNS

The following sections include a brief discussion of the manner in which the fundamental and auxiliary equations reduce to particular forms for several typical designs. In actual computations, it is always desirable to render all quantities dimensionless. A convenient system is to divide r or z, W or V, ρ, T, s, H, or I,

F, ψ, and ω by r_t, U_t, $\rho_{T,i}$, U_t^2/R, R, U_t^2, U_t^2/r_t, $r_t^2U_t$, and U_t/r_t, respectively. These dimensionless values are used in the following equations.

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Free-Vortex Design

In the free-vortex design, the variation of V_ur is prescribed as a function of z only. With a free-vortex flow, the total enthalpy at any point z is simply related to the inlet value by

$$H_j^*-H_i^* = \int_{t_i}^{t_j} \frac{Q}{U_t^2}\,dt + \omega^*\left[(V_u^*r^*)_j-(V_u^*r^*)_i\right] \tag{84}$$

where $(V_u^*r^*)_i$ is a constant. If H_i is uniform with respect to r and $\partial Q/\partial r$ is zero, $\partial H/\partial r$ will be zero everywhere; but $\partial H/\partial z$ is not equal to zero in the rotor, whereas $\partial I/\partial z$ is zero and it is therefore convenient to use the system of equations involving I. The principal equation is then

$$\frac{\partial^2\psi^*}{\partial r^{*2}} - \frac{1}{r^*}\frac{\partial\psi^*}{\partial r^*} + \frac{\partial^2\psi^*}{\partial z^{*2}} - \frac{1}{\rho^*}\left(\frac{\partial\psi^*}{\partial r^*}\frac{\partial\rho^*}{\partial r^*} + \frac{\partial\psi^*}{\partial z^*}\frac{\partial\rho^*}{\partial z^*}\right) + \frac{r^{*2}\rho^{*2}}{\frac{\partial\psi^*}{\partial r^*}}\left(F_r^* + T^*\frac{\partial s^*}{\partial r^*}\right) = 0 \tag{85}$$

The auxiliary equations are

$$F_u^*r^* = \frac{1}{r^*\rho^*}\frac{\partial\psi^*}{\partial r^*}\frac{\partial(V_u^*r^*)}{\partial r^*} \tag{86}$$

$$F_z^* = -\frac{W_u^*}{r^*}\frac{\partial(V_u^*r^*)}{\partial z^*} - T^*\frac{\partial s^*}{\partial z^*} - \frac{1}{r^{*2}\rho^{*2}}\frac{\partial\psi^*}{\partial z^*}$$
$$\left[\frac{\partial^2\psi^*}{\partial r^{*2}} - \frac{1}{r^*}\frac{\partial\psi^*}{\partial r^*} + \frac{\partial^2\psi^*}{\partial z^{*2}} - \frac{1}{\rho^*}\left(\frac{\partial\psi^*}{\partial r^*}\frac{\partial\rho^*}{\partial r^*} + \frac{\partial\psi^*}{\partial z^*}\frac{\partial\rho^*}{\partial z^*}\right)\right] \tag{87}$$

$$F_r^* = F_u^*r^*\int_{z_0^*}^{z^*}\frac{\partial}{\partial r^*}\left(\frac{F_z^*}{F_u^*r}\right)d\zeta \tag{88}$$

$$\rho^{*2} = \left[\frac{I^*}{H_i^*} + \frac{(\omega^* r^*)^2}{2H_i^*} - \frac{W_u^{*2}}{2H_i^*} - \frac{\left(\frac{\partial \psi^*}{\partial r^*}\right)^2 + \left(\frac{\partial \psi^*}{\partial z^*}\right)^2}{2r^{*2}\rho^{*2}H_i^*} \right]^{\frac{2}{\gamma-1}} e^{-2\Delta s^*} \tag{89}$$

The computation can be started with an assumed value of ψ conforming to the boundary shapes. In the early stage, it is advantageous to omit equations (87) and (88) and to use the following approximate formula of F_r, which is obtained by solving the two equations by assuming $\partial\psi/\partial r$ constant and neglecting small terms:

$$F_r^* = \frac{2}{r^{*3}} \frac{\partial (V_u^* r^*)}{\partial z^*} \int_{z_0^*}^{z^*} V_u^* r^* \, d\zeta \tag{90}$$

Density can be obtained from the ψ derivatives as follows: First, equation (89) is written as $\sigma = [1 - \varphi/\sigma]^{\frac{2}{\gamma-1}}$, where

$$\sigma = \rho^{*2} e^{2\Delta s^*} \left[\frac{I^*}{H_i^*} + \frac{\omega^{*2} r^{*2}}{2H_i^*} - \frac{W_u^{*2}}{2H_i} \right]^{-\frac{2}{\gamma-1}}$$

$$\varphi = \left[\left(\frac{\partial \psi^*}{\partial r^*}\right)^2 + \left(\frac{\partial \psi^*}{\partial z^*}\right)^2 \right] e^{2\Delta s^*} \left(2r^{*2}H_i^*\right)^{-1} \left(\frac{I^*}{H^*} + \frac{\omega^{*2} r^{*2}}{2H_i^*} - \frac{W_u^{*2}}{2H_i^*} \right)^{-\frac{\gamma+1}{\gamma-1}}$$

Second, either φ is computed for a number of values of σ and a curve is plotted, or σ is obtained for a number of values of φ through iteration of the preceding equation and a table obtained for equal intervals in φ. In either case, after the ψ derivatives are obtained, φ is computed and σ or ρ is then obtained from either the curve or the table.

Design Based on Symmetrical Velocity Diagram at All Radii

Generalized for compressible flow, the symmetrical velocity diagram at all radii is defined as follows (reference 4): (See fig. 2 for station notation.)

$$\frac{d(V_u^*r^*)_b}{dr_b^*} + \frac{d(V_u^*r^*)_e}{dr_b^*} = 2r_b^* \tag{91}$$

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If it is desirable to maintain a constant total state along the radius, the variation of $V_u r$ with z is such that the same amount of work is done along all streamlines:

$$(V_u^*r^*)_j - (V_u^*r^*)_b = f(z_j^*) \tag{92}$$

Hence,

$$\frac{\partial(V_u^*r^*)_j}{\partial r_b^*} - \frac{\partial(V_u^*r^*)_b}{\partial r_b^*} = 0 \tag{93}$$

Combining equations (91) and (93) yields

$$\frac{\partial(V_u^*r^*)_j}{\partial r_b^*} = r_b^* \tag{94}$$

Hence,

$$(V_u^*r^*)_b = \frac{r_b^{*2}}{2} - \frac{(V_u^*r^*)_{e,t} - (V_u^*r^*)_{b,t}}{2} \tag{95}$$

$$(V_u^*r^*)_e = \frac{r_b^{*2}}{2} + \frac{(V_u^*r^*)_{e,t} - (V_u^*r^*)_{b,t}}{2} \tag{96}$$

$$(V_u^*r^*)_j = (V_u^*r^*)_b + \left[(V_u^*r^*)_{j,t} - (V_u^*r^*)_{b,t}\right] \tag{97}$$

In this design, the quantity $\partial I/\partial r$ is not zero in the rotor, but with constant work input, $\partial H/\partial r$ is zero. Using the system of equations involving H is therefore convenient:

$$\frac{\partial^2\psi^*}{\partial r^{*2}} - \frac{1}{r^*}\frac{\partial\psi^*}{\partial r^*} + \frac{\partial^2\psi^*}{\partial z^{*2}} - \frac{1}{\rho^*}\left(\frac{\partial\psi^*}{\partial r^*}\frac{\partial\rho^*}{\partial r^*} + \frac{\partial\psi^*}{\partial z^*}\frac{\partial\rho^*}{\partial z^*}\right) +$$

$$\frac{r^{*2}\rho^{*2}}{\dfrac{\partial\psi^*}{\partial r^*}}\left[\frac{V_u^*}{r^*}\frac{\partial(V_u^*r^*)}{\partial r^*} + F_r^* + T^*\frac{\partial s^*}{\partial r^*}\right] = 0 \tag{98}$$

$$F_u^*r^* = \frac{1}{r^*\rho^*}\left[-\frac{\partial\psi^*}{\partial z^*}\frac{\partial(V_u^*r^*)}{\partial r^*} + \frac{\partial\psi^*}{\partial r^*}\frac{\partial(V_u^*r^*)}{\partial z^*}\right] \tag{99}$$

$$F_z^* = -\frac{W_u^*}{r^*}\frac{\partial(V_u^* r^*)}{\partial z^*} - T^*\frac{\partial s^*}{\partial r^*} - \omega^*\frac{\dfrac{\partial\psi^*}{\partial z^*}}{\dfrac{\partial\psi^*}{\partial r^*}}\frac{\partial(V_u^* r^*)}{\partial r^*} - \frac{1}{r^{*2}\rho^{*2}}\frac{\partial\psi^*}{\partial z^*}$$

$$\left[\frac{\partial^2\psi^*}{\partial r^{*2}} - \frac{1}{r^*}\frac{\partial\psi^*}{\partial r^*} + \frac{\partial^2\psi^*}{\partial z^{*2}} - \frac{1}{\rho^*}\left(\frac{\partial\psi^*}{\partial r^*}\frac{\partial\rho^*}{\partial r^*} + \frac{\partial\psi^*}{\partial z^*}\frac{\partial\rho^*}{\partial z^*}\right)\right] \tag{100}$$

$$F_r^* = F_u^* r^* \int_{z_0^*}^{z} \frac{\partial}{\partial r^*}\left(\frac{F_z^*}{F_u^* r^*}\right) d\zeta \tag{101}$$

$$\rho^{*2} = \left[\frac{H^*}{H_i^*} - \frac{(V_u^* r^*)^2}{2r^{*2}H_i^*} - \frac{\left(\dfrac{\partial\psi^*}{\partial r^*}\right)^2 + \left(\dfrac{\partial\psi^*}{\partial z^*}\right)^2}{2r^{*2}\rho^{*2}H_i^*}\right]^{\frac{2}{\gamma-1}} e^{-2\Delta s^*} \tag{102}$$

With only the additional complication that the value of V_u in each cycle is determined from a knowledge of the streamline in the previous cycle, the solution of this problem is obtained in the same manner as in the previous case. For a multistage compressor, it is important to account for the effect of loss on density rise by including the $e^{-\Delta s^*}$ factor in equation (102). The increase of entropy can be estimated by a knowledge of the polytropic efficiency (reference 4).

A nonvortex-type velocity diagram quite similar to the preceding one is the solid-body rotation design that has a tangential velocity varying linearly with radius in front of the rotor; that is,

$$(V_u^* r^*)_b = C r_b^{*2} \tag{103}$$

$$\frac{\partial(V_u^* r^*)_b}{\partial r_b^*} = 2C r_b^* \tag{104}$$

If total enthalpy is to be constant along the radius through the rotor, equations (92) and (93) also apply. Hence,

$$(V_u^* r^*)_j = C r_b^{*2} + \left[(V_u^* r^*)_{j,t} - (V_u^* r^*)_{b,t}\right] \tag{105}$$

and

$$(V_u^* r^*)_e = C r_b^{*2} + \left[(V_u^* r^*)_{e,t} - (V_u^* r^*)_{b,t}\right] \tag{106}$$

Comparison of these equations with those in the previous design is interesting. In the previous design the change of whirl through the rotor is distributed evenly in the whirl in front of and behind the rotor, whereas in the present design it is completely put into the whirl behind the rotor. Except for this difference, the calculation is quite similar to that given in the previous design.

Designs Involving Untwisted Rotor Blades

Untwisted rotor blades are desirable because of simplicity in manufacturing, and seem to be the most practical design for cooled-turbine rotor blades. They can be efficiently used if the stator blades are designed to fit them. If the blade is not too long, it can be designed on the basis that $\mu = W_u/W_z$ is a function of z only. The principal equation then takes the following form:

$$(1+\mu^2)\frac{\partial^2\psi^*}{\partial r^{*2}} - \frac{1}{r^*}\frac{\partial\psi^*}{\partial r^*} + \frac{\partial^2\psi^*}{\partial z^{*2}} - \frac{1}{\rho^*}\left[(1+\mu^2)\frac{\partial\psi^*}{\partial r^*}\frac{\partial\rho^*}{\partial r^*} + \frac{\partial\psi^*}{\partial z^*}\frac{\partial\rho^*}{\partial z^*}\right] -$$

$$2\mu\omega^* r^* \rho^* + \frac{r^{*2}\rho^{*2}}{\frac{\partial\psi^*}{\partial r^*}}\left(F_r^* - \frac{\partial I^*}{\partial r^*} + T^*\frac{\partial s^*}{\partial r^*}\right) = 0 \tag{107}$$

Radial- and Mixed-Flow Impeller with All Radial Blade Elements

The speed of rotation of the rotor can be increased by having all blade elements radial. With $F_r = 0$, the integrability equation (37a) gives the result that $F_z/F_u r$ is a function of z only. Thus, when

$$\frac{F_z^*}{F_u^*} = r^* f_3(z^*) \tag{108}$$

equation (41) becomes

$$\mu = \tan\beta = \frac{W_u^*}{W_z^*} = -\frac{F_z^*}{F_u^*} = -r^* f_3(z^*) \tag{109}$$

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The principal ion is then

$$\left[1+(r^* f_3)^2\right]\frac{\partial^2\psi^*}{\partial r^{*2}} - \frac{1+(r^* f_3)^2}{r^*}\frac{\partial\psi^*}{\partial r^*} + \frac{\partial^2\psi^*}{\partial z^{*2}} -$$

$$\frac{1}{\rho^*}\left[(1+r^{*2} f_3{}^2)\frac{\partial\psi^*}{\partial r^*}\frac{\partial\rho^*}{\partial r^*} + \frac{\partial\psi^*}{\partial z^*}\frac{\partial\rho^*}{\partial z^*}\right] - 2\mu\omega^* r^*\rho^* - \frac{r^{*2}\rho^{*2}}{\frac{\partial\psi^*}{\partial r^*}}\left(\frac{\partial I^*}{\partial r^*} - T^*\frac{\partial s^*}{\partial r^*}\right) = 0 \tag{110}$$

With $F_r = 0$, use of the three auxiliary equations to compute F_r is unnecessary.

Equation (110) is further simplified with $\partial I/\partial r$ equal to zero if the inlet flow is of the free-vortex type or has no whirl.

Simple Approximate Correction for Blade Thickness

If the blade is not quite thin, it is desirable to add a simple approximate correction factor b in the definition of stream functions of equation (38) as follows:

$$b\frac{\partial\psi}{\partial r} = r\rho^* W_z = rh^{*\frac{1}{\gamma-1}}e^{-\Delta s^*}W_z \tag{111a}$$

$$b\frac{\partial\psi}{\partial z} = -r\rho^* W_r = -rh^{*\frac{1}{\gamma-1}}e^{-\Delta s^*}W_r \tag{111b}$$

A good conception of this thickness correction factor can be obtained by analyzing the effect of blade thickness on the specific mass flow along the mean streamline (based on mass flow) in two-dimensional cascades. Yet unpublished calculations made for a typical subsonic turbine cascade and two supersonic compressor cascades show that the specific mass flow ρW_z on the mean streamline is about 4 and 10 percent higher than that given by one-dimensional calculations corresponding to the same reduction in channel area for the subsonic and supersonic cascades, respectively. The influence of the blade thickness also extends a short distance upstream and downstream of the blades. The shape of the mean streamline is also seen to follow approximately the mean channel line, (but with lower curvature). When this correction factor b is used in equations (111), all the equations previously obtained should be modified by replacing ρ by ρ/b.

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CONCLUDING REMARKS

Equations of motion and energy for unsteady, three-dimensional flow of a nonviscous fluid are expressed in terms of gas quantities most convenient for analyzing flow in turbomachines. Entropy change due to heat transfer in a cooled turbine and due to strong shock wave in supersonic flow can be taken into calculation.

The general equations are simplified according to the standard assumption for steady through-flow calculation in turbomachines having thin blades of high solidity. The problem is completely defined in the direct problem with blade shape given; whereas in the inverse or design problem, with the inclusion of the integrability condition for the blade surface, either one flow variable or one relation among several variables can be prescribed by the designer.

Through the use of the stream function, the continuity equation and the equation of motion in the radial direction are combined to form a principal equation for the present problem. The principal equation contains some terms that are either prescribed or to be determined by other equations defining the problem. Two forms of the principal equation are obtained for the two main groups of current compressor and turbine designs in which either the variation of tangential velocity or the variation of the ratio of tangential to axial velocity throughout the blade region are given. When the tangential velocity is given, the principal equation is elliptic or hyperbolic depending on whether the meridional velocity is subsonic or supersonic. When a relation between the tangential and axial velocity is given, the principal equation becomes hyperbolic when the relative velocity is supersonic.

A general method of solution for both the elliptic and the hyperbolic cases is outlined. Specific applications of the theory to several common types of compressor and turbine employing free-vortex, symmetrical-velocity-diagram, solid-rotation, nontwisted-blade, and radial-blade-element designs are discussed. A simple correction factor for blade-thickness effect is also suggested.

Lewis Flight Propulsion Laboratory,
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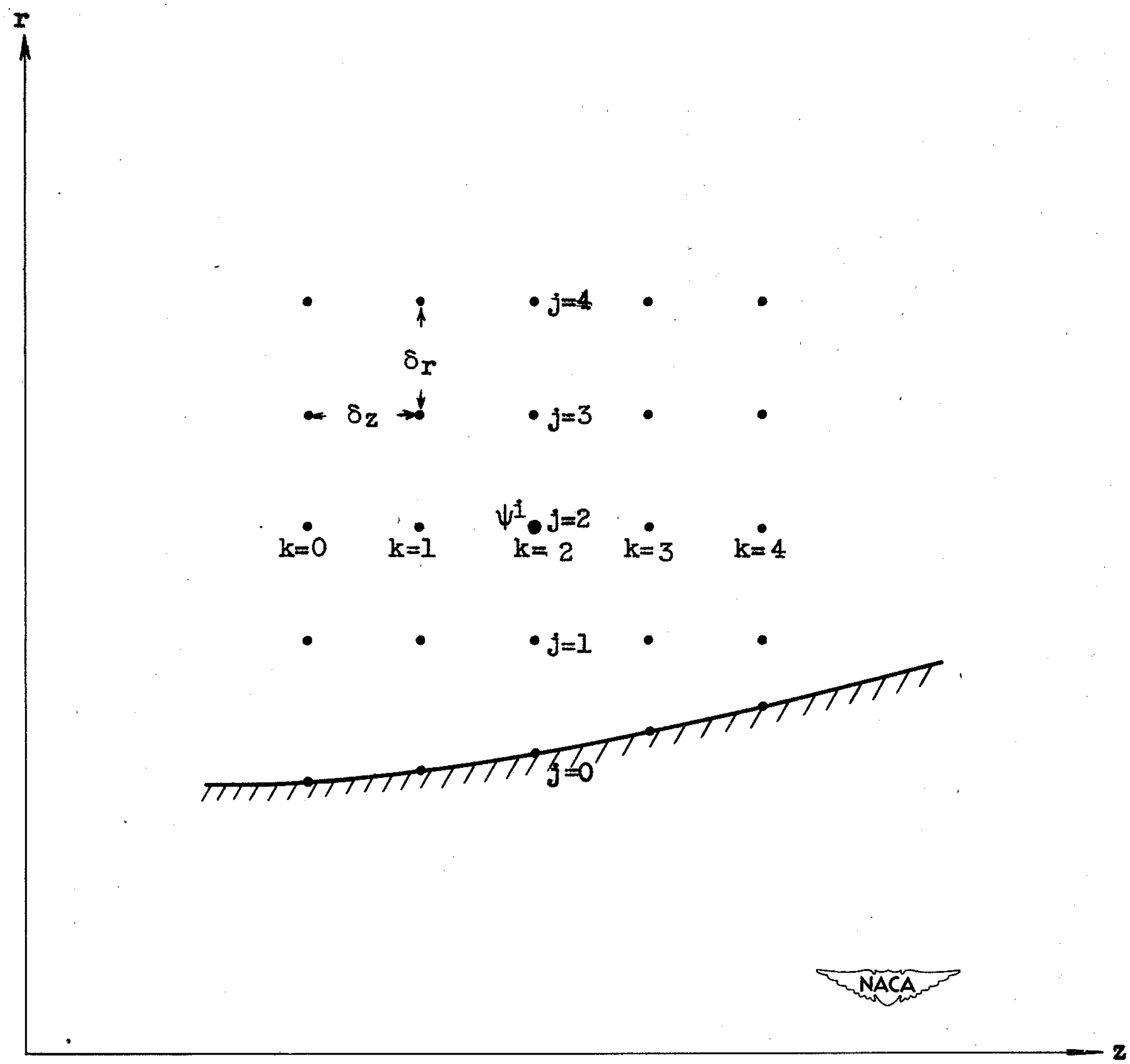

Figure 1. - Grid pattern.

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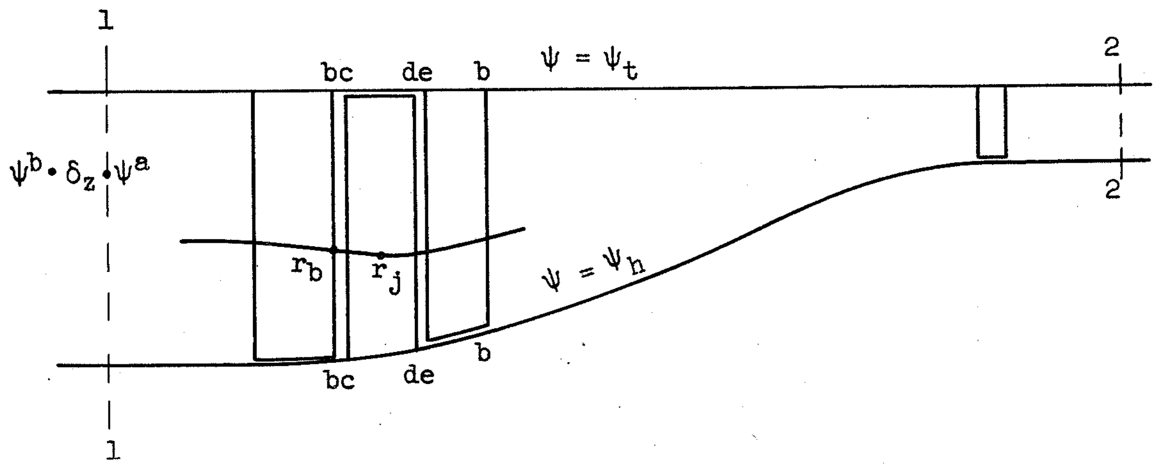

Figure 2. - Boundary conditions and station notation.

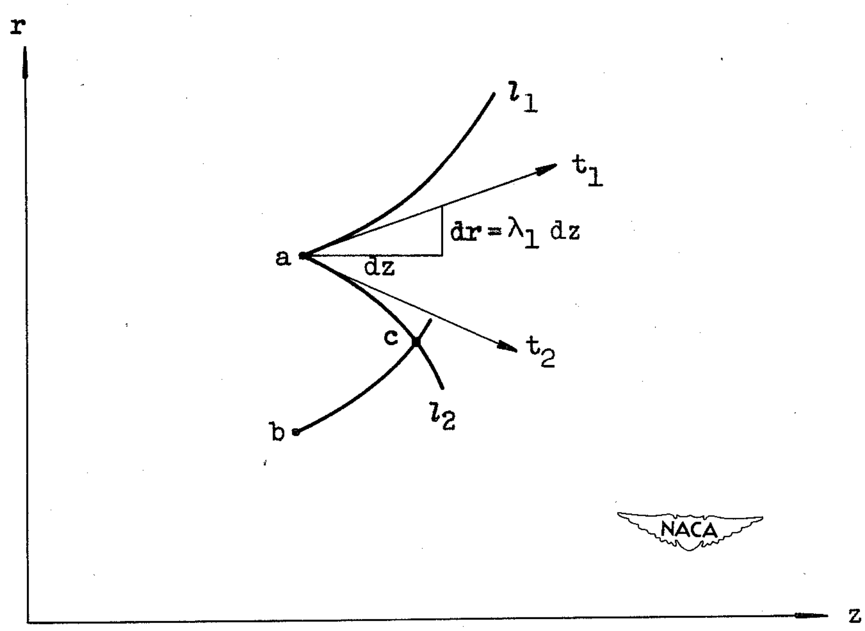

Figure 3. - Characteristic curves.