NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2340

A SURVEY OF METHODS FOR DETERMINING STABILITY PARAMETERS OF AN AIRPLANE FROM DYNAMIC FLIGHT MEASUREMENTS

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SUMMARY

Various methods of reducing to stability parameter form the response to sinusoidal and transient disturbances are discussed, using the simplified longitudinal motion of an idealized airplane as an illustrative example. It is shown that there are basic limitations in the determination of some of the stability derivatives as compared with the transfer-function coefficients, which are certain combinations of stability derivatives directly related to the airplane response. Hence, most of the report is concerned with methods of determining transfer-function coefficients rather than stability derivatives.

It is shown how the method of least squares can be applied to give the desired parameters and also the ratio of their error to that of the basic data. The determination of these parameters and their corresponding error ratios is a nonlinear problem which it is shown can be solved by linearization using a first approximation to the unknown parameters. A number of methods of obtaining a good first approximation, which also involve a least squares procedure, are explained and illustrated in the numerical examples.

Although the examples are confined to a simplified case of longitudinal motion, the methods presented are applicable in general to other more complicated types of motion.

INTRODUCTION

The scarcity of reliable data on the stability characteristics of aircraft at transonic and supersonic speeds and the difficulties of obtaining this information from wind-tunnel tests (particularly with regard to dynamic parameters, such as the rotary damping derivatives), have accelerated interest in methods of obtaining such data from flight tests. Also the extensive use of automatic stabilization and control equipment and the uncertain and generally poorer dynamic-stability characteristics arising from the use of unconventional configurations necessitate comprehensive and refined measurements of the stability derivatives.

In the past, limited information has been derived from measurements of airplane characteristics in steady-straight and steady-turning flight. Recently, frequency-response measurements to sinusoidal control deflections have been employed to evaluate additional dynamic-stability parameters. The brief testing period available during flights of missiles and high-speed research airplanes has impelled consideration of transient-response flight-testing methods and associated instrumentation. Proper testing and analysis techniques should permit evaluation of all the stability derivatives, including the limited number which can be evaluated from steady-flight data.

The oscillation technique was discussed by Laitone, Cornell Aeronautical Laboratory, in an unpublished report on dynamic flight measurements in which the response of the airplane to sinusoidal elevator deflections was measured. It was shown that the lift-curve slope, elevator effectiveness, and damping and stiffness parameters associated with the short-period oscillation could be determined. A more detailed presentation of this method and an application to flight data are given in reference 1; similar theoretical and flight-test investigations for the lateral motions have also been reported. The method of analysis of reference 1 is only applicable to simple dynamical systems – that is, systems which mathematically are similar to one with a single degree of freedom, and is incapable of reducing the stability parameters to the basic stability derivative form.

The obvious advantage, from the standpoint of test simplicity and time, of using the response to a step elevator input instead of the frequency-response tests was soon realized, and the work reported in reference 2 shows how the step-response data can be converted into the frequency-response form. Subsequent work (reference 3) extended this method to the response to a pulse elevator input. Another method of analysis of response to an arbitrary elevator motion was suggested by Loring and Jonah of Chance-Vought Aircraft Company. This method does not require transference from the "time domain" to the "frequency domain" as is the case with references 1 through 3, and is referred to later in this report as the "derivative method." In this derivative method there appears for the first time the application of the method of least squares to obtain the most reasonable values of airplane parameters from redundant measurements.

Examination of the available literature indicates a lack of information concerning the general methods of analysis applicable to more complicated systems, for example, systems with more degrees of freedom or with higher-order derivatives. The purpose of the present report is to establish more general and rigorous methods for determining aerodynamic parameters from dynamic flight measurements. The following principal and basic problems are studied: the relation between the number and type of applied forcing functions and measured responses and the corresponding number and type of determinable aerodynamic parameters; various methods of converting flight data to a form suitable for determination of aerodynamic parameters; and the correct application of the method of least squares to compute the aerodynamic parameters.

Although the methods presented are applicable to more complicated systems, the examples in the present report are confined to the simplified longitudinal motions of an idealized airplane in order to facilitate computations.

NOTATION

General

α	angle of attack, radians
α_t	angle of attack of tail, radians
A,B	in- and out-of-phase components of oscillation
D	differential operator $\left(\frac{d}{dt}\right)$
δ	elevator deflection, radians
ϵ	downwash angle, radians
E	residual error in an equation
E_R	residual error in a real equation
E_I	residual error in an imaginary equation
g	acceleration due to gravity
θ	angle of pitch, radians
I_y	pitching moment of inertia, slug-feet squared
λ	roots of characteristic stability equation
L	lift force, pounds
L_α	$\frac{\partial L}{\partial \alpha}$
L_δ	$\frac{\partial L}{\partial \delta}$

L_t tail lift, pounds

L_V $\frac{\partial L}{\partial V}$

m mass of airplane, slugs

M pitching moment, foot-pounds

M_α $\frac{\partial M}{\partial \alpha}$

$M_{D\alpha}$ $\frac{\partial M}{\partial D\alpha}$

M_δ $\frac{\partial M}{\partial \delta}$

M_q $\frac{\partial M}{\partial q}$

M_V $\frac{\partial M}{\partial V}$

μ relative density parameter $\left(\frac{2m}{\rho S_w c} = \frac{2 \times \text{mass}}{\text{density} \times \text{wing area} \times \text{wing chord}}\right)$

n normal acceleration, g units

q angular pitching velocity, radians per second

R,φ amplitude and phase of complex number

S sum of weighted squares of residuals

t time, seconds

V velocity of airplane, feet per second

V_o trim velocity of airplane, feet per second

W weight of airplane, pounds

w_q weighting factor indicating accuracy of q measurement, etc.

x longitudinal distance between center of gravity and neutral point of airplane, feet

ω angular frequency, radians per second

Subscripts

c calculated

m measured

Transfer-Function Coefficients

b damping parameter $\left(\frac{-M_q - M_{D\alpha}}{I_y} + \frac{L_\alpha}{mV_o}\right)$

C_{1_α} $-\frac{L_\delta}{mV_o}$

C_{1_q} $\frac{M_\delta}{I_y} - \frac{L_\delta}{mV_o}\frac{M_{D\alpha}}{I_y}$

C_{o_α} $\frac{M_\delta}{I_y} + \frac{L_\delta}{mV_o}\frac{M_q}{I_y}$

C_{o_q} $\frac{M_\delta}{I_y}\frac{L_\alpha}{mV_o} - \frac{M_\alpha}{I_y}\frac{L_\delta}{mV_o}$

C_{o_n} $C_{o_q}\frac{V_o}{g}$

C_{1_n} $(C_{1_q} - C_{o_\alpha})\frac{V_o}{g}$

C_{2_n} — $C_{1_\alpha}\frac{V_o}{g}$

k stiffness parameter $\left(-\frac{M_\alpha}{I_y} - \frac{L_\alpha}{mV_o}\frac{M_q}{I_y}\right)$

MATHEMATICAL AND AERODYNAMIC PRELIMINARIES

In order to illustrate the types of dynamic parameters involved and the conditions under which they may be measured, the case of longitudinal motion of an airplane will be considered. In setting up the equations of motion, the following assumptions are made:

Basic Assumptions

1. Linear equations with constant coefficients

2. Measurement of forcing function not subject to error

3. Response measurements subject only to random errors

4. Constant airspeed and level flight

5. Aerodynamic lift equals $\alpha L_\alpha + \delta L_\delta$

6. Aerodynamic moment equals $\alpha M_\alpha + D\alpha\, M_{D\alpha} + q\, M_q + \delta M_\delta$

7. Rigid airplane

8. No unsteady lift effects except for downwash lag which introduces the derivative $M_{D\alpha}$. Only the first three of the above assumptions are necessary in order to apply the methods of this report; the others are made to simplify the numerical examples given later and because they do approximately describe the airplane motions in the maneuvers which are considered in this report.

Equations of Motion and Statement of "Inverse" Problem of Airplane Dynamics

Based on the above assumptions, the longitudinal equations of motion may be written

$$-\alpha L_\alpha - D\alpha\, mV_o + q\, mV_o = L \tag{1}$$

$$-\alpha M_\alpha - D\alpha\, M_{D\alpha} - q\, M_q + I_y\, Dq = M \tag{2}$$

The forcing functions, lift L and moment M, may be applied aerodynamically by deflecting an elevator, as discussed in a later section of the report, or nonaerodynamically as, for example, by releasing a bomb or firing a gun.

Most of the applications of the mathematical theory of dynamic behavior of airplanes have consisted of computations of airplane response characteristics when the stability derivatives such as L_α, M_α, and M_q are assumed known. The problem considered in the present report is the inverse; namely, given the airplane response in α, q, or n to a disturbance, to evaluate the stability derivatives of the airplane.

It will be proved later in this report (see section "Discussion") that the determination of moment derivatives from flight data is subject to certain basic theoretical limitations. There are certain combinations of moment derivatives, however, which determine the behavior of the airplane and which can be computed from the flight data. These combinations of derivatives are called "transfer coefficients" and are defined below. Most of this report is devoted to the determination of these transfer coefficients.

Transfer Functions for Control Deflection Input

For this case, in which $L=\delta L_\delta$ and $M=\delta M_\delta$, the operational solution of equations (1) and (2) is

$$\frac{\alpha}{\delta}=\frac{-\frac{L_\delta}{mV_o}D+\frac{M_\delta}{I_y}+\frac{L_\delta}{mV_o}\frac{M_q}{I_y}}{D^2+D\left(-\frac{M_q}{I_y}-\frac{M_{D\alpha}}{I_y}+\frac{L_\alpha}{mV_o}\right)-\frac{L_\alpha}{mV_o}\frac{M_q}{I_y}-\frac{M_\alpha}{I_y}} \tag{3}$$

$$=\frac{C_{1_\alpha}D+C_{0_\alpha}}{D^2+bD+k} \tag{4}$$

$$\frac{q}{\delta}=\frac{\left(\frac{M_\delta}{I_y}-\frac{L_\delta}{mV_o}\frac{M_{D\alpha}}{I_y}\right)D-\frac{M_\alpha}{I_y}\frac{L_\delta}{mV_o}+\frac{M_\delta}{I_y}\frac{L_\alpha}{mV_o}}{D^2+bD+k} \tag{5}$$

$$=\frac{C_{1_q}D+C_{0_q}}{D^2+bD+k} \tag{6}$$

Since the normal acceleration is often measured, it will be useful to give the solution

$$\frac{n}{\delta}=\frac{V_o}{g}\frac{q-D\alpha}{\delta}=\frac{V_o}{g}\frac{-C_{1_\alpha}D^2+(C_{1_q}-C_{0_\alpha})D+C_{0_q}}{D^2+bD+k}$$

$$=\frac{C_{2_n}D^2+C_{1_n}D+C_{0_n}}{D^2+bD+k} \tag{7}$$

Quantities such as C_{0_α}, C_{1_α}, C_{0_q}, C_{1_q}, b, and k will be called "transfer coefficients," because they determine the transfer function between a response and a disturbance.

It may be noted that there is a direct relation between the transfer function and frequency response. The sinusoidal response of the airplane to a sinusoidal disturbance of frequency ω is obtained by substituting $i\omega$ for D in the corresponding transfer function. The magnitude and phase angle of the complex number so obtained give the amplitude ratio and phase (lead) between the response and disturbance, which are of the same frequency.

COMPUTATION OF DYNAMIC PARAMETERS (TRANSFER COEFFICIENTS) FROM FLIGHT DATA

A more detailed study of the analysis procedures for reducing basic data obtained by various flight-test methods is now of interest. As previously noted, a dynamic flight test made for the purpose of measuring stability and control characteristics generally consists of the measurement of an input disturbance (such as control deflection) and the resulting airplane response in one or more degrees of freedom. The disturbance and response may be transient or may be sinusoidal motions, with time and angular frequency, respectively, as the independent variable. Equations of motion are then assumed, and the associated transfer coefficients and (to the extent it is possible) the stability derivatives are evaluated by a process usually referred to as "curve fitting." The minimum number of points required to fit a curve to the observed data is the same, of course, as the number of assumed transfer coefficients. A larger number of points ordinarily are obtained with either transient or sinusoidal tests, and the values of the stability parameters which will fit these redundant data with the least error should be determined. This is done by application of the principle of least squares, which also furnishes information about the accuracy of the parameter evaluations in terms of the accuracy of the basic data (error ratio).

Principle of Least Squares

This principle states that the most probable values of the unknown parameters are those for which the sum of the squares of the errors (differences between calculated and measured airplane motion, for instance) is a minimum. If only one airplane response is measured, for example, the transient pitching velocity q from t=0 to T, then the inverse problem of airplane dynamics is solved by finding values of b, k, C_{1_q} and C_{0_q} such that

$$S = \int_0^T (q_m - q_c)^2 \, dt$$

is a minimum.

If more than one airplane response is measured, for example, the transient responses in θ, q, and Dq from $t=0$ to T, then the values of the parameters b, k, C_{1_q} and C_{0_q} are to be determined such that

$$S = w_\theta \int_0^T (\theta_m - \theta_c)^2 \, dt + w_q \int_0^T (q_m - q_c)^2 \, dt + w_{Dq} \int_0^T (Dq_m - Dq_c)^2 \, dt$$

is a minimum. The weight w of a measurement is a number indicating the accuracy of that measurement (as regards random errors). More specifically, the weight is the reciprocal of the mean square error (normal distribution in the errors of measurement assumed, see reference 4).

If the frequency response (both amplitude and phase) is measured over a range of frequencies ω, then the parameters are to be determined by the condition that

$$S = \sum w_R (R_m - R_c)^2 + \sum w_\varphi (\varphi_m - \varphi_c)^2$$

is a minimum where the summation is taken over the frequencies at which the response is measured.

The problems above are nonlinear in the unknowns b, k, C_1, and C_0. The only practical method of solution is to linearize the problem and iterate from a first approximation. For the case in which only one quantity is subject to error (like the first case mentioned above) the methods of linearization and iteration are explained in reference 4, pages 214 and 84, respectively. The method of linearization is discussed briefly in the next section and in the appendix. The subsequent section (which constitutes the principal part of this report) deals with methods, most of which also involve least squares solutions, of obtaining a good first approximation.

The idea of determining the parameters from a transient response by linearization and iteration from a first approximation is due to Shinbrot of Ames Laboratory and is discussed and exploited more fully in reference 5.

Determination of the Parameters and Their Relative Accuracy by Linearization

To determine the parameters b, k, C_{1_q}, and C_{0_q} from q and δ measurements, it is first necessary to determine an initial approximation (b_0, k_0, $C_{1_{q_0}}$, and $C_{0_{q_0}}$) to the parameters (e.g., by one of the methods of the next section). Then linearization of the problem of determining the increments in the parameters which make S a minimum gives the equations:

$$\sum_{i=1}^{4} S_{ij}\, x_j = S_i \qquad i = 1, 2, 3, 4$$

where

$$S_{ij} = \frac{\partial^2 S}{\partial x_i \partial x_j}$$

and

$$S_i = -\frac{\partial S}{\partial x_i}$$

$$x_1 = b-b_0 \qquad x_2 = k-k_0 \qquad x_3 = C_{1_q}-C_{1_{q_0}} \qquad x_4 = C_{0_q}-C_{0_{q_0}}$$

The partial derivatives are to be computed at the values $b=b_0$, $k=k_0$, $C_{1_q}=C_{1_{q_0}}$ and $C_{0_q}=C_{0_{q_0}}$. Expressions for S_{ij} and S_i for the transient response and frequency response are given in the appendix.

Having obtained the increments x_i, it may be necessary to repeat the calculation using the corrected values of the parameters as a new first approximation and so on. This, of course, will not be necessary if a good first approximation is obtained. It appears to be worth while, therefore, to devote some effort to obtaining a good first approximation. In the next section it is shown how the method of least squares is used to find such an approximation.

The relative accuracy or weight (reciprocal of mean square error) of a computed parameter is obtained as shown in reference 4 from the following formula:

$$\frac{1}{w_{x_i}} = \frac{\text{cofactor of } S_{ii}}{|S_{ij}|}$$

A more useful indication of the accuracy of a computed parameter for some purposes is the ratio of percentage error in the computed parameter to the percentage error of the flight measurement. This ratio, which may be called the error ratio, is (for q measurements)

$$\frac{\text{percent error in } x_i}{\text{percent error in } q} = \text{error ratio of } x_i = \sqrt{\frac{w_q}{w_{x_i}}}\,\frac{q_{max}}{x_i}$$

$$= \sqrt{w_q}\,\frac{q_{max}}{x_i}\sqrt{\frac{\text{cofactor of } S_{ii}}{|S_{ij}|}}$$

where q_{max} is the full-scale reading of the q instrument, and w_q is the weight of the q measurements (reciprocal of mean square error).

Methods for Obtaining a First Approximation to the Parameters

Sinusoidal (frequency) response.- In this case the measurements consist of input and response amplitudes and phase shifts between the input and response in one or more degrees of freedom over a range of frequencies. For example, in the case of longitudinal motion with two degrees of freedom the transfer function between pitching velocity q and elevator angle δ has been shown to be:

$$\frac{q}{\delta} = \frac{C_{1_q}D+C_{0_q}}{D^2+bD+k}$$

If

$$\delta = \delta_0 \sin \omega t$$

and

$$\frac{q}{\delta_0} = R \sin (\omega t+\varphi) = A \sin \omega t + B \cos \omega t$$

then it can be shown that $A+iB$ is the value of the transfer function when $D=i\omega$. This gives the equation:

$$A(k-\omega^2) - Bb\omega + i[B(k-\omega^2)+Ab\omega] = C_{0_q} + C_{1_q}\,\omega i$$

Equating real and imaginary parts gives the two equations:

$$Ak - B\omega b - C_{o_q} = A\omega^2 \qquad \text{Real equation}$$

$$Bk + A\omega b - C_{1_q}\omega = B\omega^2 \qquad \text{Imaginary equation}$$

If the values of A and B are known at two frequencies, then it is possible to set up four simultaneous equations in the four unknowns b, k, C_{o_q}, and C_{1_q} as follows:

$$\begin{aligned}
A_1k - B_1\omega_1b - C_{o_q} &= A_1\omega_1^2 \\
A_2k - B_2\omega_2b - C_{o_q} &= A_2\omega_2^2 \\
B_1k + A_1\omega_1b - \omega_1C_{1_q} &= B_1\omega_1^2 \\
B_2k + A_2\omega_2b - \omega_2C_{1_q} &= B_2\omega_2^2
\end{aligned}$$

where A_1, A_2, B_1, B_2 are values of A and B at ω_1 and ω_2. The solution of these equations gives values of b, k, C_{1q}, and C_{oq} corresponding to a frequency response curve that agrees exactly with the measured curve at the two frequencies selected. This has been called the "method of selected points."

A better first approximation can be obtained at the expense of some additional labor by using data obtained at more than two frequencies. The method of least squares is used to calculate values of the parameters that make $\Sigma E_R^2 + \Sigma E_I^2$ a minimum where E_R and E_I are defined by:

$$Ak - B\omega b - C_{o_q} - A\omega^2 = E_R$$

$$Bk + A\omega b - C_{1_q}\omega - B\omega^2 = E_I$$

This leads to the following set of normal equations:

$$\begin{aligned}
&\Sigma(B^2\omega^2+A^2\omega^2)b + \Sigma B\omega C_{o_q} - \Sigma A\omega^2C_{1_q} = 0 \\
&\Sigma(B^2+A^2)k - \Sigma AC_{o_q} - \Sigma B\omega C_{1_q} = \Sigma(B^2\omega^2+A^2\omega^2) \\
&\Sigma B\omega b - \Sigma Ak + NC_{o_q} = -\Sigma A\omega^2 \\
&\Sigma A\omega^2b + \Sigma B\omega k - \Sigma\omega^2C_{1_q} = \Sigma B\omega^3
\end{aligned}$$

where N equals the number of frequencies and the summation is taken over all the frequencies at which data are obtained.

If the phase and amplitude of the normal acceleration response to sinusoidal elevator motion is measured, the coefficients of the transfer function

$$\frac{n}{\delta} = \frac{C_{2_n}D^2+C_{1_n}D+C_{0_n}}{D^2+bD+k}$$

can be determined in a similar manner.

To illustrate the computation of a first approximation by the method of least squares, the frequency response of an idealized airplane to sinusoidal elevator motion was calculated and the results were used to compute the transfer coefficients. The frequency-response points used in the computation are shown in figure 1. Table I gives a complete schedule of calculations of the transfer coefficients k, b, C_{1_q}, and C_{0_q} from the pitching-response data of figure 1.

Transient response.- Since it is much faster and easier to measure the transient response than the frequency response of a dynamical system, the analysis of transient responses to determine transfer coefficients is of great interest. A common type of transient function, the pulse, is shown in figure 2; some special types of input functions (impulse, step, and ramp) are shown in figure 3. Several methods of analysis, of varying degrees of generality, will be explained.

(1) Inspection of the transient.- If δ becomes constant after a brief transient period, b and k can be determined from the damping and period of the oscillations (assuming that the system is less than critically damped). If $T_{1/2}$ is the time for the free oscillations to damp to half amplitude and P is the period, then

$$b = \frac{1.386}{T_{1/2}} \quad \text{and} \quad k = \frac{0.48}{{T_{1/2}}^2} + \frac{39.48}{P^2}$$

If, in addition, the steady-state value of q and the steady-state value of δ are known, then their ratio q_∞/δ_∞ is equal to C_{0_q}/k. If the values of q_∞ and δ_∞ are zero (as for a pulse input) then

$$\frac{C_{0_q}}{k} = \frac{\int_0^\infty q \, dt}{\int_0^\infty \delta \, dt}$$

If the input is a step, then

$$C_{1_q} = \left(\frac{Dq}{\delta}\right)_{t=0}$$

Otherwise there is no way of determining C_{1_q} by inspection.

(2) Fourier transform.— The vector ratio of the Fourier integral of the output to the Fourier integral of the input gives the frequency response. That is, if

$$\frac{q}{\delta}(i\omega) = \frac{\int_0^\infty q(t)e^{-i\omega t}\,dt}{\int_0^\infty \delta(t)e^{-i\omega t}\,dt} = Re^{i\varphi}$$

then the response to $\delta = \sin \omega t$ is $q = R \sin (\omega t + \varphi)$. The application of the Fourier integral to the conversion of transient to frequency response is discussed in references 2 and 3.

If, as in the case of a step or ramp input, the slope of the output or input becomes essentially constant (and equal to Dq_T) when $t \geqq T$, the Fourier integral may be evaluated by the alternative substitutions

$$\int_0^\infty qe^{-i\omega t}\,dt = \frac{1}{i\omega}\int_0^{q_T} e^{-i\omega t}\,dq - \frac{1}{\omega^2}Dq_T\, e^{-i\omega T}$$

or

$$\int_0^\infty qe^{-i\omega t}\,dt = \int_0^T qe^{-i\omega t}\,dt + \frac{q_T e^{-i\omega T}}{i\omega} - \frac{Dq_T e^{-i\omega T}}{\omega^2}$$

where q_T is the value of q at $t = T$.

The transfer coefficients can, of course, be obtained from the frequency response by the methods of the previous section. An illustration of the application of the Fourier transform to compute frequency response from the step response of figure 4 is given in table II. The assumed transfer function is slightly different from that used in table I; hence a slightly different frequency response is obtained. The frequency response obtained as an intermediate step in the computation of transfer coefficients by this procedure is often itself of interest in problems of automatic stabilization.

The so-called "incomplete Fourier transforms" $\int_0^T qe^{-i\omega t}\,dt$ and $\int_0^T \delta e^{-i\omega t}\,dt$ can also be used to compute the parameters from a pulse which has not reached a steady state and for nonzero initial conditions. Assuming the transfer function

$$\frac{q}{\delta} = \frac{C_{1_q}D+C_{0_q}}{D^2+bD+k}$$

then it can be shown that

$$\left[(i\omega)^2+bi\omega+k\right]\int_0^T qe^{-i\omega t}\,dt + e^{-i\omega T}\left(Dq_T+bq_T+i\omega q_T\right) - Dq_o - bq_o - i\omega q_o$$

$$= \left(C_{1_q}i\omega+C_{0_q}\right)\int_0^T \delta e^{-i\omega t}\,dt + C_{1_q}\delta_T e^{-i\omega t} - C_{1_q}\delta_o$$

from which a real and an imaginary equation can be set up. These can be used to set up four simultaneous equations using computations at two frequencies, to get approximate values for b, k, C_{1_q}, and C_{0_q}. If more than two frequencies are involved, then the method of least squares can be used as in the case of frequency-response measurements.

(3) Derivative method.- Another method for computing the transfer coefficients from the transient response consists in using the measured values of a sufficient number of higher derivatives of input and response in the assumed transfer function. For example, if the assumed form of the transfer function is

$$\frac{q}{\delta} = \frac{D\theta}{\delta} = \frac{C_{1_q}D+C_{0_q}}{D^2+bD+k}$$

it is possible to determine the parameters from four sets of simultaneous measurements of θ, $D\theta$, $D^2\theta$, δ, and $\int_0^t \delta\,d\tau$ by fitting the equation

$$D^2\theta + bD\theta + k\theta - C_{1_q}\delta - C_{0_q}\int_0^t \delta d\tau = 0$$

If the subscript 1 denotes the value of a quantity at $t=t_1$ etc., then four simultaneous equations for the parameters b, k, C_{1_q}, and C_{0_q} can be set up as follows:

$$(D^2\theta)_1 + b(D\theta)_1 + k\theta_1 - C_{1q}\delta_1 - C_{0q}\int_0^{t_1} \delta d\tau = 0$$

$$(D^2\theta)_2 + b(D\theta)_2 + k\theta_2 - C_{1q}\delta_2 - C_{0q}\int_0^{t_2} \delta d\tau = 0$$

$$(D^2\theta)_3 + b(D\theta)_3 + k\theta_3 - C_{1q}\delta_3 - C_{0q}\int_0^{t_3} \delta d\tau = 0$$

$$(D^2\theta)_4 + b(D\theta)_4 + k\theta_4 - C_{1q}\delta_4 - C_{0q}\int_0^{t_4} \delta d\tau = 0$$

The parameters computed from the above equations can, of course, be used to compute a θ response for a δ input which passes through the four given values of δ and the integral of which corresponds to the four given values of $\int_0^t \delta d\tau$. However, the transient θ curve so obtained will not necessarily pass through the four given values of θ, but the above relation between $D^2\theta$, $D\theta$, θ, δ, and $\int_0^t \delta d\tau$ will be exactly satisfied at the four points.

If it is desired to use more data (taken at more than four time instants) then the method of least squares may be used to compute values of b, k, C_{1q}, and C_{0q} such that $\sum_i E_i^2$ is a minimum

where

$$E_i = (D^2\theta)_i + b(D\theta)_i + k(\theta)_i - C_{1q}\delta_i - C_{0q}\int_0^{t_i} \delta\, d\tau$$

The normal equations are:

$$\sum_i (D\theta)_i^2 b + \sum_i (D\theta)_i(\theta)_i k - \sum_i (D\theta)_i(\delta)_i C_{1_q} - \sum_i (D\theta)_i \int_0^{t_i} \delta d\tau\, C_{o_q} = -\sum_i (D\theta)_i(D^2\theta)_i$$

$$\sum_i (D\theta)_i(\theta)_i b + \sum_i (\theta)_i^2 k - \sum_i (\theta)_i(\delta)_i C_{1_q} - \sum_i (\theta)_i \int_0^{t_i} \delta d\tau\, C_{o_q} = -\sum_i (\theta)_i (D^2\theta)_i$$

$$\sum_i (D\theta)_i(\delta)_i b + \sum_i (\theta)_i(\delta)_i k - \sum_i (\delta_i)^2 C_{1_q} - \sum_i (\delta)_i \int_0^{t_i} \delta d\tau\, C_{o_q} = -\sum_i (\delta)_i (D^2\theta)_i$$

$$\sum_i (D\theta)_i \int_0^{t_i} \delta d\tau\, b + \sum_i (\theta)_i \int_0^{t_i} \delta d\tau\, k - \sum_i (\delta_i) \int_0^{t_i} \delta d\tau\, C_{1_q} - \sum_i \left(\int_0^{t_i} \delta d\tau\right)^2 C_{o_q} = -\sum_i \int_0^{t_i} \delta d\tau\, (D^2\theta)_i$$

An example of the application of the derivative method to the determination of the parameters from the response θ, $D\theta$ and $D^2\theta$ to a step input in δ is shown in figure 5, and worked out in table III. For a step response the differential equation relating θ and δ is

$$D^2\theta + b\, D\theta + k\theta - C_{1_q} - C_{o_q} t = 0 \text{ or } E$$

and the normal equations resulting from the minimizing of $\Sigma\, E^2$ are

$$\sum_i (D\theta)_i^2 b + \sum_i (D\theta)_i(\theta)_i k - \sum_i (D\theta)_i C_{1_q} - \sum_i (D\theta)_i t_i C_{o_q} = -\sum_i (D\theta)_i(D^2\theta)_i$$

$$\sum_i (D\theta)_i(\theta)_i b + \sum_i (\theta)_i^2 k - \sum_i (\theta)_i C_{1_q} - \sum_i \theta_i t_i C_{o_q} = -\sum_i (D^2\theta)_i \theta_i$$

$$\sum_i (D\theta)_i b + \sum_i (\theta)_i k - N C_{1_q} - \sum_i t_i C_{o_q} = -\sum_i (D^2\theta)_i$$

$$\sum_i (D\theta)_i t b + \sum_i \theta_i t_i k - \sum_i t_i C_{1_q} - \sum_i t_i^2 C_{o_q} = -\sum_i (D^2\theta)_i t$$

where N is the number of time instants.

(4) <u>Prony's method</u>.— Prony's method (reference 6) for fitting a sum of exponentials to a number of equally spaced ordinates may be used to obtain the transfer coefficients from a step or impulse response. In the case of a step it can be assumed that q can be represented as a constant plus the sum of ν exponentials

$$q = q_\infty + A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \ldots A_\nu e^{\lambda_\nu t}$$

and the values of $q(t)$ at n equidistant instants Δ seconds apart will satisfy a difference equation of the form

$$q_{(m+\nu)\Delta} + a_\nu q_{(m+\nu-1)\Delta} + \ldots a_1 q_{m\Delta} = 0$$

where q_Δ, $q_{2\Delta}$, . . . , $q_{m\Delta}$ indicate the value of $q-q_\infty$ at $t=\Delta$, 2Δ, . . . , $m\Delta$, respectively. The following set of $n-\nu$ equations may be written:

$$q_{(\nu+1)\Delta} + a_\nu q_{\nu\Delta} + \ldots a_1 q_\Delta = 0$$

$$q_{(\nu+2)\Delta} + a_\nu q_{(\nu+1)\Delta} + \ldots a_1 q_{2\Delta} = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$q_{n\Delta} + a_\nu q_{(n-1)\Delta} + \ldots a_1 q_{(n-\nu)\Delta} = 0$$

If the value of q_∞ is known, (i.e., if a practically steady state has been reached) then the above equations can be solved for the coefficients a_i by the method of least squares (if $n > 2\nu$). If the value of q_∞ is unknown (because a steady state has not been reached at the end of the record), then the difference equation may be written, calling

$$q = q_m \quad \text{at} \quad t = m\Delta$$

$$q_{m+\nu} + a_\nu q_{m+\nu-1} + \ldots a_1 q_m = (a_1 + \ldots a_\nu + 1) q_\infty = -a_{\nu+1}$$

This leads to a similar set of difference equations with the additional unknown $a_{\nu+1}$. These also can be solved by the method of least squares (if $n > 2\nu$) for the coefficients a_i, from which, of course, q_∞ is determined. The least squares solution for the coefficients a_i is determined by minimizing $\sum_1^{n-\nu} E_m^2$ where

$$E_m = q_{m+\nu} + a_\nu \; q_{m+\nu-1} + \ldots a_1 q_m + a_{\nu+1}$$

The values of a_i are then used to form an algebraic equation

$$x^\nu + a_\nu \, x^{\nu-1} + \ldots a_1 = 0$$

the roots of which are related to λ (the characteristic roots) by

$$\lambda = \frac{\log x}{\Delta}$$

Knowing the ν values of λ the set of n equations

$$q_\Delta = A_1 \, e^{\lambda_1 \Delta} + A_2 \, e^{\lambda_2 \Delta} + \ldots A_\nu \, e^{\lambda_\nu \Delta}$$

$$q_{2\Delta} = A_1 \, e^{2\lambda_1 \Delta} + A_2 \, e^{2\lambda_2 \Delta} + \ldots A_\nu \, e^{2\lambda_\nu \Delta}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_{n\Delta} = A_1 \, e^{n\lambda_1 \Delta} + A_2 \, e^{n\lambda_2 \Delta} + \ldots A_\nu \, e^{n\lambda_\nu \Delta}$$

can be set up, and the method of least squares is used a second time to determine the unknown coefficients A_i. The transfer function is now completely determined. To express it in terms of the differential operator D, each term $e^{\lambda t}$ is replaced by $D/D-\lambda$, its Laplace transform multiplied by D, and the resulting fractions are added, that is,

$$\begin{aligned} q &= q_\infty + \frac{A_1 D}{D-\lambda_1} + \frac{A_2 D}{D-\lambda_2} + \ldots \frac{A_\nu D}{D-\lambda_\nu} \\ &= \frac{C_\nu \, D^\nu + C_{\nu-1} \, D^{\nu-1} + \ldots C_0}{D^\nu + B_{\nu-1} \, D^{\nu-1} + \ldots B_0} \end{aligned}$$

where the coefficients B_i are functions of the λ roots, whereas the coefficients C_i are functions of the coefficients A_i and of the λ roots.

A numerical example illustrating the application of Prony's method to the response to a step input is shown in tables IV(a) and IV(b). This example is based on the computed response shown in figure 4. The form of transfer function for the pitch response to elevator deflection is the same as that assumed previously, that is,

$$\frac{q}{\delta} = \frac{C_{1_q} D + C_{0_q}}{D^2 + bD + k}$$

The values of q are tabulated at equal intervals of time Δ of 0.1 second, from $t=0$ to $t=1.0$ second, and the solution is carried out on the assumption that the steady-state value of q is unknown.

The values of b and k are determined in table IV(a); the values of C_{1_q} and C_{0_q} are determined in table IV(b). The small value obtained for the coefficient of D^2 (0.47) (rather than the zero theoretical value) is due to imperfect fit.

DISCUSSION

Comparison of Fourier Transform Method, Derivative Method, and Prony's Method of Obtaining a First Approximation

The Fourier transform method appears to be very convenient, especially if a harmonic analyzer is available. The derivative method does not appear to be practical unless the necessary higher derivatives are measured during the flight test. Although Prony's method is limited to the case where the response is a sum of exponentials (the step response is included), it can always be used with a pulse input by fitting the free oscillations following the pulse. This determines the characteristic roots of the system, or the transfer coefficients in the denominator of the transfer function. It is shown in reference 5 that once the transfer coefficients of the denominator have been determined, the transfer coefficients of the numerator of the transfer function can be computed by a linear calculation. In the case of the response due to a step input, the complete transfer function can be determined, as shown in the example, but this is a special case.

Determination of Lift Derivatives

It can be shown that, given a complete set of transfer coefficients (e.g., those corresponding to q/δ and α/δ), the lift derivative L_α is given by:

$$\frac{L_\alpha}{mV_o} = \frac{b\ C_{1_\alpha}\ C_{o_q} - C_{o_\alpha}\ C_{o_q} - k\ C_{1_\alpha}\ C_{1_q}}{C_{1_\alpha}\ C_{o_q} - C_{o_\alpha}\ C_{1_q}}$$

If L_δ is negligible this reduces to $\frac{L_\alpha}{mV_o} = \frac{C_{o_q}}{C_{1_q}}$.

The value of L_α can also be determined from dynamic measurements of a single response at two center-of-gravity positions as follows:

$$k = -\frac{L_\alpha}{mV_o}\frac{M_q}{I_y} - \frac{M_\alpha}{I_y}$$

Differentiating this with respect to x (assuming I_y and M_q do not vary appreciably with x) gives

$$\frac{dk}{dx} = \frac{L_\alpha}{I_y}$$

from which

$$L_\alpha = I_y \frac{dk}{dx}$$

The above methods of determining L_α can be used no matter how large the value of L_δ is. If L_δ is small enough, the value of L_α can be determined from static-stability measurements or from steady-turn measurements. (See subsequent section entitled "Relation Between Static and Dynamic Tests.")

Basic Limitations in Determination of Moment Derivatives

Various methods of computing transfer coefficients and lift derivatives are mentioned above; the computation of moment derivatives is not explained. For the two degree of freedom case discussed above, the unique determination of the damping moment derivatives from the airplane response to control deflections is impossible, as will now be shown.

If α, $D\alpha$, q, δ, and Dq are known at four different instants of a transient disturbance, the moment equation can be used to form four simultaneous linear equations for the determination of the four unknowns, M_α, $M_{D\alpha}$, M_q and M_δ. The coefficients of these unknowns are the values of α, $D\alpha$, q and δ at these instants. Unfortunately, these four values of α, $D\alpha$, q and δ are linearly dependent according to the lift equation, which states that

$$\alpha L_\alpha + D\alpha\, mV_o + \delta L_\delta - q\, mV_o = 0$$

Therefore, the four simultaneous equations formed to determine the moment derivatives are linearly dependent and, as shown in reference 7, cannot be solved uniquely for the unknowns. If the value of any one of the moment derivatives is known from other sources, then the values of the other three may be uniquely determined from the flight measurements. Otherwise, only combinations of the moment derivatives such as the following can be evaluated from the flight data:

$$M_{D\alpha} + M_q$$

$$mV_o\, M_\alpha + M_q\, L_\alpha$$

$$mV_o\, M_\delta + M_q\, L_\delta$$

In practice, this indeterminacy prevents only the separation of the damping derivatives in the first of the above expressions, because they are of the same order of magnitude, at least for most configurations.

The second of the above expressions often may be considered as practically equal to $mV_o\, M_\alpha$, for sufficiently high static margins, altitude, and wing loading. This can be more readily shown by taking the ratio of mV_oM_α to M_qL_α which can be shown to be

$$\frac{mV_oM_\alpha}{M_qL_\alpha} = \frac{2\mu x}{Cm_{\frac{qc}{2V}}}$$

where

$$\mu = \frac{2m}{\rho S_w c} = 26.1\,\frac{W}{S_w}\left(\frac{\rho}{\rho_o}\right)\frac{1}{c}$$

$$x = \text{static margin}$$

For $W/S_W = 100$, $c = 2$, $\rho/\rho_o = 0.53$ (20,000 feet altitude), $x = 0.2$ of M.A.C., and $Cm_{\frac{qc}{2V}} = -15$ (conventional configuration), the ratio is approximately 20. This means that the error in M_α involved in neglecting the M_q term in the expression $mV_oM_\alpha + M_qL_\alpha$ is 5 percent.

The last of the above expressions, namely $mV_oM_\delta + M_qL_\delta$, also can be assumed equal to mV_oM_δ to about 5 percent for conventional plan forms, so that M_δ may be known to about this accuracy.

If the tail length l defined as the distance from the airplane center of gravity to the center of pressure due to δ is known, then M_δ also can be found from L_δ by the simple relation:

$$M_\delta = -lL_\delta$$

This additional relation between the derivatives M_δ and L_δ makes possible, in principle, the unique determination of all the derivatives; however, in practice, the uncertainty (especially for unconventional configurations) in the value of l and the low accuracy of determining L_δ (for conventional configurations where L_δ is almost negligible) make the accuracy of the determination of the individual derivatives by this method very much lower than that for the transfer coefficients or combinations of derivatives.

The indeterminacy of the moment derivatives from control-response data persists, in general, even when changes in forward speed are taken into account. Measurements of airplane response in the three longitudinal degrees of freedom (e.g., V, n, and q) to a known control deflection will yield values of

$$L_V\ M_\alpha - M_V\ L_\alpha$$

$$L_V\ M_{D\alpha} - M_V\ mV_o$$

and so forth.

The derivative M_V, a measure of the effect of airspeed changes alone on pitching moment, is caused primarily by power and Mach number effects. In cases where this derivative can be assumed negligible, measurements involving forward speed permit evaluation of all derivatives. It is interesting to note that a similar type of indeterminacy exists in the lateral motions if product-of-inertia terms are present.

Information Obtained From Tail Load Measurements

If the loads on the tail of a conventional-type airplane are measured during a maneuver, the value of downwash slope ϵ_α may be inferred, provided the following assumption is made:

$$L_t = \alpha L_{t_{\alpha_t}} (1 - \epsilon_\alpha) + D\alpha\, L_{t_{\alpha_t}} \epsilon_\alpha \frac{l}{V_o} + q\, L_{t_{\alpha_t}} \frac{l}{V_o} + \delta L_\delta$$

By measuring the time histories of α (or n), δ, q and L_t, and knowing l and V_o, the values of $L_{t_{\alpha_t}}$, ϵ_α, and L_δ may be determined using the same numerical methods outlined above.

Use of Nonaerodynamic Forcing Functions

It has been shown above that the indeterminacy that exists when the response to a control deflection is measured would be overcome if the location of the center of pressure of the lift due to control deflection were known. This immediately suggests a testing technique in which the force is applied at a known position in the aircraft by, for example, dropping a bomb or firing a gun or jet normal to the longitudinal axis.

If the known applied lift L acts at a known distance l' behind the center of gravity, then the applied moment M is equal to $-l'L$ and the equations of motion are

$$-\alpha L_\alpha - D\alpha\, mV_o + q\, mV_o = L$$

$$-\alpha M_\alpha - D\alpha\, M_{D\alpha} - q\, M_q + Dq\, I_y = -l'L$$

Solving these equations for α and q gives

$$\frac{\alpha}{L} = \frac{-\dfrac{1}{MV_o} D + \dfrac{M_q - l'mV_o}{I_y mV_o}}{D^2 + b\,D + k}$$

$$\frac{q}{L} = \frac{-\dfrac{M_{D\alpha} + l'mV_o}{I_y mV_o} D - \dfrac{M_\alpha + l'L_\alpha}{I_y mV_o}}{D^2 + b\,D + k}$$

where b and k are the same as before. The above transfer coefficients (determined by the methods of the earlier part of this report) can obviously be broken down immediately to give M_q and $M_{D\alpha}$. The determination of L_α and M_α from the transfer coefficients is also feasible, although somewhat less straightforward.

If the bomb in the previous case is replaced by a gun firing vertically, the same method of analysis applies. If the impulse supplied by the gun is I (= mass of shell times muzzle velocity), the equations of motion are:

$$-\alpha L_\alpha - mV_o\, D\alpha + mV_o\, q = ID$$

$$-\alpha M_\alpha - M_{D\alpha}\, D\alpha - M_q\, q + I_y\, Dq = I\, l'D$$

where l' is now the distance between gun and center of gravity. It is not necessary to know the value of the impulse I in order to determine the derivatives – a fact which is of importance since the impulse is not as easy to measure as the weight of the bomb in the previous example.

Relation Between Static and Dynamic Tests

In static or maneuvering stability tests, the rate of change of trim with control deflection is measured. This is, of course, the ratio of the constants in the transfer function. For example, the longitudinal maneuvering stability is determined by the normal acceleration per degree elevator deflection in a steady turn which is

$$\left(\frac{n}{\delta}\right)_{D=0} = \frac{C_{o_n}}{k} = \frac{V_o(M_\alpha L_\delta - M_\delta L_\alpha)}{g(mV_o M_\alpha + M_q L_\alpha)}$$

The variation of this with center-of-gravity position is

$$\frac{d}{dx}\left(\frac{n}{\delta}\right) = -\left(\frac{n}{\delta}\right)\frac{dk/dx}{k}$$

The ratio

$$\frac{n/\delta}{\frac{d}{dx}(n/\delta)} = -\frac{k}{dk/dx}$$

is the distance between the center of gravity and maneuver point (i.e., the point where $k=0$).

If the angle-of-attack change (as well as the normal acceleration change) in a steady turn is measured, we get

$$\left(\frac{\alpha}{\delta}\right)_{D=0} = \frac{M_\delta \, mV_o + M_q \, L_\delta}{-M_q \, L_\alpha - M_\alpha \, mV_o} = \frac{C_{o_\alpha}}{k}$$

The variation with center-of-gravity position is

$$\frac{d}{dx}\left(\frac{\alpha}{\delta}\right) = \frac{k\left(\frac{dC_{o_\alpha}}{dx}\right) - C_{o_\alpha}\left(\frac{dk}{dx}\right)}{k^2}$$

The ratio

$$\frac{\alpha/\delta}{\frac{d}{dx}(\alpha/\delta)}$$

does not accurately give the maneuver point unless L_δ is negligible. If L_δ is not negligible, as for a tailless airplane, it is necessary to measure δ/α at a number of center-of-gravity positions to determine the position for which δ/α is zero, in order to find the maneuver point. Thus, normal acceleration measurements enable a somewhat more convenient maneuver point determination than angle-of-attack measurements. The ratio of n/δ to α/δ gives

$$\frac{n}{\alpha} = -\frac{V_o}{g} \frac{M_\alpha L_\delta - M_\delta \, L_\alpha}{mV_o \, M_\delta + M_q \, L_\delta}$$

If L_δ is negligible, this equals L_α/mg. If L_δ is not negligible, it is not possible to determine L_α from steady-turn measurements.

In static longitudinal-stability tests the values of V/δ and its variation with center-of-gravity position similarly determine the neutral-point position. It can be shown that

$$\frac{V}{\delta} = \frac{M_\delta \, L_\alpha - M_\alpha L_\delta}{(2M_\alpha W/V_o) - M_V \, L_\alpha}$$

and

$$\frac{V/\delta}{\frac{d}{dx}(V/\delta)} = \frac{2M_{\alpha} W - M_V L_{\alpha} V_o}{2 W L_{\alpha}}$$

which is equal to the distance between the neutral point and the center of gravity. The measurement of α/δ gives

$$\frac{\alpha}{\delta} = \frac{mV_o M_{\delta} + M_q L_{\delta}}{-L_{\alpha} M_q - mV_o M_{\alpha}}$$

Just as in the case of steady turns, the value of

$$\frac{\alpha/\delta}{d/dx\ (\alpha/\delta)}$$

does not quite give the neutral-point position unless L_{δ} is negligible. Measurement of V/δ and α/δ gives

$$\frac{V}{\alpha} = \frac{L_{\alpha} M_{\delta} - M_{\alpha} L_{\delta}}{L_{\delta} M_V - M_{\delta} 2W/V_o}$$

If L_{δ} is negligible, the above expression reduces to

$$\frac{-L_{\alpha} V_o}{2W}$$

but no static or steady-turn measurement at a single center-of-gravity position can determine L_{α} or L_{δ} if L_{δ} is appreciable compared with L_{α}.

Aerodynamic Lag

The lag between aerodynamic forces and moments and angle of attack, control deflection, or pitching velocity is variously labeled as:

Unsteady lift
Indicial lift
Nonstationary or unstationary lift
Oscillating airfoil effect
Wagner effect

The assumption was made above that the aerodynamic forces are represented by

$$L = \alpha L_\alpha + \delta L_\delta$$

$$M = \alpha M_\alpha + D\alpha\, M_{D\alpha} + \delta M_\delta + q M_q$$

These equations indicate that the lift due to a sudden change in control deflection, angle of attack, or pitching deflection, and the pitching moment due to control deflection or pitching velocity develop instantaneously. The lag in pitching moment due to angle of attack, supposedly represented by $M_{D\alpha}$, is obviously an approximation to the relation between angle of attack and pitching moment due to the complex transient downwash disturbance at the tail. Actually the lift due to sudden changes in angle of attack, etc., lags somewhat behind the angle of attack at first. The same effect can be expressed in terms of the frequency response of lift due to sinusoidal variation in angle of attack and, therefore, is also referred to as the "oscillating airfoil" effect.

The aerodynamic lag effect can be expressed in a number of ways, namely,

(1) The expressions for L and M given above require additional derivative terms on both sides of the equation to take account of the aerodynamic lag. The correct expression for L, for example, would be of the form:

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots a_1 D + 1)L =$$

$$L_\alpha(1 + b_1 D + \ldots b_m D^m)\alpha + L_\delta(1 + c_1 D + \ldots C_r D^r)\delta$$

where m and r are less than n.

(2) The relation between an aerodynamic force and airplane motion in a particular degree of freedom can be expressed as an "aerodynamic transfer function" which can be described in terms of an "indicial lift response" or an "oscillating lift response." The indicial lift response is the variation of lift with time for step function displacements in the particular degree of freedom. The oscillating lift response is the amplitude and phase of the lift with respect to displacement for sinusoidal motion in the particular degree of freedom as a function of frequency.

The two methods of expressing aerodynamic lag are, of course, related by the fact that the expression given under (1) is the rational fraction approximation to the "operational equivalent" of the indicial lift response mentioned under (2). This means that the indicial lift response or oscillating lift response as calculated from (1) would be the same as that defined in (2).

It is theoretically impossible (in the most general case) to determine these unsteady lift functions from the response to control deflection alone, as pointed out above for the determination of moment derivatives. Their importance in airplane maneuvers of the type discussed in this report is believed to be small, however, except possibly in the transonic speed range.

Suggestions for Future Work

It would be of great interest to compute the error ratios of the parameters involved in the longitudinal and lateral dynamic airplane responses, for various forcing functions and airplane configurations. This would show what accuracy can be expected from various flight-testing techniques. The problem of taking account of errors in measurement of the forcing function (e.g., control angle deflection) remains to be solved.

In applying the methods of this report it is necessary to assume the form of the transfer function in advance. Further research is desirable on a method of solution which would yield the form of the transfer function as well as the values of the transfer coefficients.

CONCLUDING REMARKS

A mathematical study of the determination of dynamic airplane parameters from flight measurements showed, as would be expected, that the determinability and accuracy of the parameters depend on many factors, such as type of input disturbance, the amount of response data, and the method of analysis. The determination of stability derivatives as distinct from transfer-function coefficients requires, in general, a more complicated flight-testing technique. In the case of longitudinal motions, for example, measurement of the response of the airplane to elevator motions theoretically will not enable derivatives to be uniquely determined; instead, the values of linear combinations of pairs are obtained. Practically, the static derivatives can be determined with reasonable accuracy in many cases. Measurement of the response to a known force, as applied for example by a bomb drop or gun recoil, yields additional data from which all the derivatives can be determined.

Application of these analysis methods and principles should assist in developing testing and analysis techniques to satisfy the requirements of various specific research programs. Most previous work concentrated on sinusoidal control-deflection inputs or converted data from the response to step or pulse inputs to frequency-response form for further analysis. Conversion of the transient response to the frequency response is not the only way to analyze the data to determine dynamic parameters. Methods such as Prony's and the "derivative method" are also available. Prony's method, in particular, appears to be a simple and direct method of determining the characteristic roots from free-oscillation data, and is also applicable to the response to a step input.

In the determination of the dynamic parameters from flight data a rational least-squares procedure should be used to give the best fit to redundant data. Application of this procedure also gives an evaluation of the accuracy of a parameter, relative to that of the basic flight data.

The methods of analysis presented in this report are applicable to more complicated problems than the ones used for illustration. The inclusion of additional degrees of freedom, and of additional higher-order derivatives to take account of unsteady lift effects is possible by the methods of this report, and remains a field for future work.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., Dec. 19, 1950.

APPENDIX

Details of Calculation of Parameters From First Approximation

In the case where one transient response is measured (say q) the equations for the increments in the parameters become:

$$\sum_{j=1}^{4} S_{ij}\, x_j = S_i \qquad i = 1, 2, 3, 4$$

$$S_{ij} = \int_0^T \left(\frac{\partial q_c}{\partial x_i}\right)\left(\frac{\partial q_c}{\partial x_j}\right)dt \qquad S_i = -\int_0^T (q_m - q_c)\,\frac{\partial q_c}{\partial x_i}\,dt$$

where

$$q_c = \frac{C_{1q_o}\lambda_1 + C_{oq_o}}{\lambda_1 - \lambda_2}\, e^{\lambda_1 t}\int_0^t \delta(\tau) e^{-\lambda_1 \tau} d\tau + \frac{C_{1q_o}\lambda_2 + C_{oq_o}}{\lambda_2 - \lambda_1}\, e^{\lambda_2 t}\int_0^t \delta(\tau) e^{-\lambda_2 \tau} d\tau$$

λ_1 and λ_2 are the roots of $\lambda^2 + b_o\lambda + k_o = 0$

and

$$x_1 = b - b_o \qquad x_2 = k - k_o \qquad x_3 = C_{1_q} - C_{1_{q_o}} \qquad x_4 = C_{o_q} - C_{o_{q_o}}$$

where $\left\{ b_o,\ k_o,\ C_{1_{q_o}},\ C_{o_{q_o}} \right\}$ are the first approximation

If λ_1 and λ_2, the conjugate complex roots, are equal to $l \pm il'$, then[1]

$$q_c = \frac{e^{lt}}{l'}\left\{\left(C_{1_{q_o}} l'\cos l't + C_{1_{q_o}} l \sin l't + C_{o_{q_o}} \sin l't\right)\int_0^t \delta(\tau) e^{-l\tau}\cos l'\tau\, d\tau - \right.$$

---

[1] The symbols l and l' used in the appendix should not be confused with those in the text.

$$\left(C_{1_{q_0}} l \cos l't + C_{o_{q_0}} \cos l't - C_{1_{q_0}} l' \sin l't\right) \int_0^t \delta(\tau) e^{-l\tau} \sin l'\tau d\tau \Bigg\}$$

In order to obtain the required derivatives $\frac{\partial q_c}{\partial b}$ and $\frac{\partial q_c}{\partial k}$, it is first necessary to obtain the derivatives $\frac{\partial q_c}{\partial \lambda_1}$ and $\frac{\partial q_c}{\partial \lambda_2}$ and then use the expressions

$$\frac{\partial q_c}{\partial b} = \frac{\partial q_c}{\partial \lambda_1} \frac{\lambda_1}{\lambda_2 - \lambda_1} + \frac{\partial q_c}{\partial \lambda_2} \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

$$\frac{\partial q_c}{\partial k} = \frac{\partial q_c}{\partial \lambda_1} \frac{1}{\lambda_2 - \lambda_1} + \frac{\partial q_c}{\partial \lambda_2} \frac{1}{\lambda_1 - \lambda_2}$$

The values of the partial derivatives are:

$$\frac{\partial q_c}{\partial \lambda_1} = \frac{C_{1_0}\lambda_2 + C_{o_0}}{(\lambda_1 - \lambda_2)^2} \left[e^{\lambda_2 t} \int_0^t \delta(\tau) e^{-\lambda_2 \tau} d\tau - e^{\lambda_1 t} \int_0^t \delta(\tau) e^{-\lambda_1 \tau} d\tau \right]$$

$\frac{\partial q_c}{\partial \lambda_2}$ = expression obtained by interchanging λ_1 and λ_2 in above

$$\frac{\partial q_c}{\partial C_{1_q}} = \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \int_0^t \delta(\tau) e^{-\lambda_1 \tau} d\tau + \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \int_0^t \delta(\tau) e^{-\lambda_2 \tau} d\tau$$

$$\frac{\partial q_c}{\partial C_{o_q}} = \frac{e^{\lambda_1 t} \int_0^t \delta(\tau) e^{-\lambda_1 \tau} d\tau}{\lambda_1 - \lambda_2} + \frac{e^{\lambda_2 t} \int_0^t \delta(\tau) e^{-\lambda_2 \tau} d\tau}{\lambda_2 - \lambda_1}$$

Where θ, q, and Dq are measured in a transcient response, the linearized equations for the increments become:

$$\sum_{j=1}^{4} S_{ij} x_j = S_i \qquad i = 1, 2, 3, 4$$

$$S_{ij}=w_\theta \int_0^T \left(\frac{\partial\theta_c}{\partial x_i}\right)\left(\frac{\partial\theta_c}{\partial x_j}\right)dt+w_q\int_0^T\left(\frac{\partial q_c}{\partial x_i}\right)\left(\frac{\partial q_c}{\partial x_j}\right)dt+w_{Dq}\int_0^T\left(\frac{\partial Dq_c}{\partial x_i}\right)\left(\frac{\partial Dq_c}{\partial x_j}\right)dt$$

$$-S_i=w_\theta \int_0^T (\theta_m-\theta_c)\frac{\partial\theta_c}{\partial x_i}dt+w_q\int_0^T(q_m-q_c)\frac{dq_c}{\partial x_i}dt+w_{Dq}\int_0^T(Dq_m-Dq_c)\frac{\partial Dq_c}{\partial x_i}dt$$

$x_1 = \Delta b \qquad x_2 = \Delta k \qquad x_3 = \Delta C_{1_q} \qquad x_4 = \Delta C_{o_q}$

where

$$\theta_c = \frac{C_{o_{q_o}}}{k}\int_0^t \delta(\tau)d\tau + \frac{C_{1_{q_o}}\lambda_1 + C_{o_{q_o}}}{\lambda_1(\lambda_1-\lambda_2)}e^{\lambda_1 t}\int_0^t \delta(\tau)e^{-\lambda_1\tau}d\tau + \frac{C_{1_{q_o}}\lambda_2 + C_{o_{q_o}}}{\lambda_2(\lambda_2-\lambda_1)}$$
$$e^{\lambda_2 t}\int_0^t \delta(\tau)e^{-\lambda_2\tau}d\tau$$

$$q_c = \frac{C_{1_{q_o}}\lambda_1 + C_{o_{q_o}}}{\lambda_1-\lambda_2}e^{\lambda_1 t}\int_0^t \delta(\tau)e^{-\lambda_1\tau}d\tau + \frac{C_{1_{q_o}}\lambda_2 + C_{o_{q_o}}}{\lambda_2-\lambda_1}e^{\lambda_2 t}\int_0^t \delta(\tau)e^{-\lambda_2\tau}d\tau$$

$$Dq_c = C_{1_{q_o}}\delta(t) + \frac{\left(C_{1_{q_o}}\lambda_1+C_{o_{q_o}}\right)\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_1 t}\int_0^t \delta(\tau)e^{-\lambda_1\tau}d\tau + \frac{\left(C_{1_{q_o}}\lambda_2 + C_{o_{q_o}}\right)\lambda_2}{\lambda_2-\lambda_1}$$
$$e^{\lambda_2 t}\int_0^t \delta(\tau)e^{-\lambda_2\tau}d\tau$$

or if λ_1 and λ_2 are the conjugate complex roots equal to $l \pm il'$ then

$$\theta_c=\frac{e^{lt}}{l'k}\left[\left(-C_{o_{q_o}}l'\cos l't + C_{o_{q_o}}l\sin l't + C_{1_{q_o}}k\sin l't\right)C + \left(-C_{1_{q_o}}k\cos l't - C_{o_{q_o}}l\cos l't - C_{o_{q_o}}l'\sin l't\right)D\right] +$$
$$\frac{C_{o_{q_o}}}{k}\int_0^t \delta(\tau)\,d\tau$$

$$q_c = \frac{e^{lt}}{l'}\left[\left(C_{1_{q_o}} l' \cos l't + C_{1_{q_o}} l \sin l't + C_{o_{q_o}} \sin l't\right) C - \left(C_{1_{q_o}} l \cos l't + C_{o_{q_o}} \cos l't - C_{1_{q_o}} l' \sin l't\right) D\right]$$

$$Dq_c = \frac{e^{lt}}{l'}\left\{\left[2 C_{1_{q_o}} l l' \cos l't + C_{o_{q_o}} l' \cos l't + \left(C_{1_{q_o}} l^2 + C_{o_{q_o}} l - C_{1_{q_o}} l'^2\right) \sin l't\right] C + \left[\left(2 C_{1_{q_o}} l l' + C_{o_{q_o}} l'\right) \sin l' t + \left(C_{1_{q_o}} l'^2 - C_{o_{q_o}} l - C_{1_{q_o}} l^2\right) \cos l't\right] D\right\} + C_{1_{q_o}} \delta(t)$$

where

$$C = \int_0^t \delta(\tau) e^{-l\tau} \cos l'\tau \, d\tau \quad \text{and} \quad D = \int_0^t \delta(\tau) e^{-l\tau} \sin l'\tau \, d\tau$$

If the frequency response $q = R \sin(\omega t + \varphi)$ to $\delta = \sin \omega t$ is measured, the equations for the increments in b, k, C_{1_q}, and C_{o_q} over the first approximation are denoted by:

$$\sum_{j=1}^{4} S_{ij} x_j = S_i \qquad i = 1, 2, 3, 4$$

where

$$S_{ij} = \omega_R \sum \left(\frac{\partial R_c}{\partial x_i}\right)\left(\frac{\partial R_c}{\partial x_j}\right) + \omega_\varphi \sum \left(\frac{\partial \varphi}{\partial x_i}\right)\left(\frac{\partial \varphi}{\partial x_j}\right)$$

$$-S_i = \omega_R \sum (R_m - R_c) \frac{\partial R_c}{\partial x_i} + \omega_\varphi \sum (\varphi_m - \varphi_c) \frac{\partial \varphi_c}{\partial x_i}$$

where the summation is to be taken over the frequencies.

$$R_c = \sqrt{\frac{{C_{o_{q_o}}}^2 \, {C_{1_{q_o}}}^2 \, \omega^2}{(k_o - \omega^2)^2 + {b_o}^2 \omega^2}} \quad \text{and} \quad \tan \varphi_c = \frac{\dfrac{C_{1_{q_o}} \omega}{C_{o_{q_o}}} - \dfrac{b_o \omega}{k_o - \omega^2}}{1 + \dfrac{C_{1_{q_o}} b \, \omega^2}{C_{o_{q_o}} (k_o - \omega^2)}}$$

and

$$\frac{\partial R_c}{\partial b} = \frac{\left(-C_{o_{q_o}}{}^2 - C_{1_{q_o}}{}^2 \omega^2\right) \omega^2 b}{R_c \left[\left(k_o - \omega^2\right)^2 + b_o{}^2 \omega^2\right]^2}$$

$$\frac{\partial R_c}{\partial k} = \frac{\left(-C_{o_q}{}^2 - C_{1_q}{}^2 \omega^2\right)\left(k_o - \omega^2\right)}{R_c \left[(k_o - \omega^2)^2 + b_o{}^2 \omega^2\right]^2}$$

$$\frac{\partial R_c}{\partial C_o} = \frac{C_{o_{q_o}}}{R_c \left[(k_o - \omega^2)^2 + b_o{}^2 \omega^2\right]}$$

$$\frac{\partial R_c}{\partial C_1} = \frac{\omega^2 C_{1_{q_o}}}{R_c \left[(k_o - \omega^2)^2 + b_o{}^2 \omega^2\right]}$$

$$\frac{\partial \varphi_c}{\partial b} = \cos^2 \varphi_c \frac{\left(C_{o_{q_o}}{}^2 \omega^3 - C_{o_{q_o}}{}^2 k\omega - C_{1_{q_o}}{}^2 k\omega^3 + C_{1_{q_o}}{}^2 \omega^5\right)}{\left(C_{o_{q_o}} k_o - C_{o_{q_o}} \omega^2 + C_{1_{q_o}} b_o \omega^2\right)^2}$$

$$\frac{\partial \varphi_c}{\partial k} = \cos^2 \varphi_c \frac{\left(C_{o_{q_o}}{}^2 b_o \omega^3 + C_{o_{q_o}}{}^2 b_o \omega\right)}{\left(C_{o_{q_o}} k_o - C_{o_{q_o}} \omega^2 + C_{1_{q_o}} b_o \omega^2\right)^2}$$

$$\frac{\partial \varphi_c}{\partial C_o} = \cos^2 \varphi_c \frac{\left(C_{1_{q_o}} k_o \omega^3 - C_{1_{q_o}} b_o{}^2 \omega^3 - C_{1_{q_o}} k_o{}^2 \omega - C_{1_{q_o}} \omega^5 + C_{1_{q_o}} k_o \omega^3\right)}{\left(C_{o_{q_o}} k_o - C_{o_{q_o}} \omega^2 + C_{1_{q_o}} b_o \omega^2\right)^2}$$

$$\frac{\partial \varphi_c}{\partial C_1} = \cos^2 \varphi_c \frac{\left(C_{o_{q_o}} k_o{}^2 \omega - 2 C_{o_{q_o}} k_o \omega^3 + C_{o_{q_o}} \omega^5 + C_{o_{q_o}} b_o{}^2 \omega^3\right)}{\left(C_{o_{q_o}} k_o - C_{o_{q_o}} \omega^2 + C_{1_{q_o}} b_o \omega^2\right)^2}$$

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TABLE I.— LEAST SQUARES CALCULATION OF FIRST APPROXIMATION TO TRANSFER COEFFICIENTS FROM FREQUENCY RESPONSE IN PITCHING

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Row	ω	R	φ (deg)	$\sin\varphi$	$\cos\varphi$	$R \sin\varphi = B$	$R \cos\varphi = A$	A^2	$AB\omega$	$B\omega$	$B^2\omega^2$	ω^2	$A^2\omega^2$	$A\omega^2$	$B\omega^3$	B^2
1	1	8.8587	184.8	-0.06662	-0.99778	-0.5902	- 8.8390	78.1279	5.2168	-0.5902	0.34834	1	78.1279	-8.8390	-0.5902	0.34834
2	2	10.0211	183.4	-.05878	-.99827	-.5890	-10.0038	100.0760	11.7845	-1.1780	1.38768	4	400.304	-40.0152	-4.712	.34692
3	3	11.3560	177.8	.03847	-.99926	.4369	-11.3476	128.7680	-14.8733	1.3107	1.71793	9	1158.912	-102.1284	11.7963	.19088
4	4	12.2791	168.7	.19543	-.98072	2.3997	-12.0424	145.0194	-115.5926	9.5988	92.1370	16	2320.31	-192.6784	153.5808	5.75856
5	5	12.4966	158.4	.36748	-.93003	4.5923	-11.6222	135.0755	-266.8631	22.964	527.230	25	3376.888	-290.555	574.0375	21.08922
6	6	12.0825	148.8	.51818	-.85527	6.2609	-10.3338	106.7874	-388.1933	32.5654	1411.16	36	3844.346	-372.0168	1352.354	39.19887
7	7	11.3058	140.6	.63500	-.77251	7.1792	- 8.7338	76.2793	-438.9119	50.2544	2525.50	49	3737.686	-427.9562	2462.47	51.54091
8	8	10.4232	133.9	.72007	-.69390	7.5054	- 7.2327	52.3119	-434.2745	60.0432	3605.19	64	3347.962	- 462.8928	3842.76	56.33103
9	9	9.5589	128.6	.78119	-.62429	7.4673	- 5.9675	35.6111	-401.0500	67.2057	4516.61	81	2884.499	- 483.3675	5443.662	55.76057
10	10	8.7671	124.4	.82551	-.56439	7.2373	- 4.9481	24.4837	-358.1088	72.3730	5237.85	100	2448.37	- 494.810	7237.30	52.3785
11		Σ					-91.0709	882.5402	-2400.8662	314.5445	17919.13	385	23597.40	-2875.259	21072.66	282.9438

Real Equation of Condition

$$B\omega b - Ak + C_{o_q} + A\omega^2 = E_R$$

Imaginary Equation of Condition

$$A\omega b + Bk - \omega C_{1_q} - B\omega^2 = E_I$$

Normal Equations Obtained by Minimizing $\Sigma E_R^2 + \Sigma E_I^2$.

$$\begin{aligned}
&[\Sigma 12 + \Sigma 14]b && + \Sigma 11\, C_{o_q} - \Sigma 15\, C_{1_q} = 0 \\
&[\Sigma 17 + \Sigma 9]k && - \Sigma 8\, C_{o_q} - \Sigma 11\, C_{1_q} = \Sigma 12 + \Sigma 14 \\
&\Sigma 11\, b - \Sigma 8\, k && + 10.\ C_{o_q} = -\Sigma 15 \\
&\Sigma 15\, b + \Sigma 11\, k && - \Sigma 13\, C_{1_q} = \Sigma 16
\end{aligned}$$

$$\begin{aligned}
41516.53\, b &+ 314.5445\, C_{o_q} + 2875.259\, C_{1_q} = 0 \\
1165.484\, k &+ 91.0709\, C_{o_q} - 314.5445\, C_{1_q} = 41516.53 \\
314.5445\, b + 91.0709\, k &+ 10.\ C_{o_q} = 2875.259 \\
-2875.259\, b + 314.5445\, k &- 385.\ C_{1_q} = 21072.66
\end{aligned}$$

$b = 8.048$ $\quad k = 28.616$ $\quad C_{o_q} = -226.2$ $\quad C_{1_q} = -91.46$

TABLE II.– APPLICATION OF FOURIER TRANSFORM TO CONVERT THE RESPONSE TO A STEP INPUT INTO THE FREQUENCY RESPONSE

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Row	t	q/δ	2t	3t	4t	5t	6t	7t	8t	9t	10t	sin (4)	sin (5)	sin (6)	sin (7)	sin (8)	sin (9)	sin (10)	sin (11)	sin (12)	cos (4)	cos (5)	cos (6)	cos (7)	cos (8)	cos (9)	cos (10)	cos (11)	cos (12)
1	0	−0.08	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	.1	−7.05	.2	.3	.4	.5	.6	.7	.8	.9	1.0	.19867	.29552	.38942	.47943	.56462	.64422	.71736	.78333	.84147	.98007	.95534	.92106	.87758	.82534	.76484	.69671	.62161	.54030
3	.2	−10.42	.4	.6	.8	1.0	1.2	1.4	1.6	1.8	2.0	.38942	.56462	.71736	.84147	.93204	.98545	.99957	.97385	.90930	.92106	.82534	.69671	.54030	.36236	.16997	−.02920	−.22720	−.41615
4	.3	−11.54	.6	.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0	.56462	.78333	.93204	.99749	.97385	.86321	.67546	.42738	.14112	.82534	.62161	.36236	.07074	−.22720	−.50485	−.73739	−.90407	−.98999
5	.4	−11.48	.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	.71736	.93204	.99957	.90930	.67546	.33499	−.05837	−.44252	−.75680	.69671	.36236	−.02920	−.41615	−.73739	−.94222	−.99829	−.89676	−.65364
6	.5	−10.88	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	.84147	.99749	.90930	.59847	.14112	−.35078	−.75680	−.97753	−.95892	.54030	.07074	−.41615	−.80114	−.98999	−.93646	−.65364	−.21080	.28366
7	.6	−10.24	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4	6.0	.93204	.97385	.67546	.14112	−.44252	−.87158	−.99616	−.77276	−.27942	.36236	−.22720	−.73739	−.98999	−.89676	−.49026	.08750	.63469	.96017
8	.7	−9.66	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3	7.0	.98545	.86321	.33499	−.35078	−.87158	−.98245	−.63127	.01681	.65699	.16997	−.50485	−.94222	−.93646	−.49026	.18651	.77557	.99986	.75390
9	.8	−9.18	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2	8.0	.99957	.67546	−.05837	−.75680	−.99616	−.63127	.11655	.79367	.98936	−.02920	−.73739	−.99829	−.65364	.08750	.77557	.99318	.60835	−.14550
10	.9	−8.90	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1	9.0	.97385	.42738	−.44252	−.97753	−.77276	.01681	.79367	.96989	.41212	−.22720	−.90407	−.89676	−.21080	.63469	.99986	.60835	−.24354	−.91113
11	1.0	−8.79	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	.90930	.14112	−.75680	−.95892	−.27942	.65699	.98936	.41212	−.54402	−.41615	−.98999	−.65364	.28366	.96017	.75390	−.14550	−.91113	−.83907
12	1.1	−8.70	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9	11.0	.80850	−.15774	−.95160	−.70554	.31154	.98817	.58492	−.45754	−.99999	−.58850	−.98748	−.30733	.70867	.95023	.15337	−.81109	−.88919	.00426
13	1.2	−8.71	2.4	3.6	4.8	6.0	7.2	8.4	9.6	10.8	12.0	.67546	−.44252	−.99616	−.27942	.79367	.85460	−.17433	−.98093	−.53657	−.73739	−.89676	.08750	.96017	.60835	−.51929	−.98469	−.19433	.84385
14	1.3	−8.73	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7	13.0	.51550	−.68777	−.92581	.21512	.99854	.31910	−.82783	−.76198	.42017	−.85689	−.72593	.37798	.97659	.05396	−.94772	−.56098	.64760	.90745
15	1.4	−8.77	2.8	4.2	5.6	7.0	8.4	9.8	11.2	12.6	14.0	.33499	−.87158	−.63127	.65699	.85460	−.36648	−.97918	.03362	.99061	−.94222	−.49026	.77557	.75390	−.51929	−.93043	.20300	.99943	.13674
16	1.5	−8.79	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5	15.0	.14112	−.97753	−.27942	.93800	.41212	−.87970	−.53657	.80379	.65029	−.98999	−.21080	.96017	.34664	−.91113	−.47554	.84385	.59492	−.75969
17	1.6	−8.77	3.2	4.8	6.4	8.0	9.6	11.2	12.8	14.4	16.0	−.05837	−.99616	.11655	.98936	−.17433	−.97918	.23151	.96566	−.28790	−.99829	.08750	.99318	−.14550	−.98469	.20300	.97283	−.25982	−.95766

NACA

TABLE II. – CONCLUDED

1	2	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
Row	t	(3)×(13)	(3)×(14)	(3)×(15)	(3)×(16)	(3)×(17)	(3)×(18)	(3)×(19)	(3)×(20)	(3)×(21)	(3)×(22)	(3)×(23)	(3)×(24)	(3)×(25)	(3)×(26)	(3)×(27)	(3)×(28)	(3)×(29)	(3)×(30)	Simpson's factor
1	0	0	0	0	0	0	0	0	0	0	-0.08000	-0.08000	-0.08000	-0.08000	-0.08000	-0.08000	-0.08000	-0.08000	-0.08000	1
2	.1	-1.40062	-2.08342	-2.74541	-3.37998	-3.98057	-4.54175	-5.05739	-5.52248	-5.93236	-6.90949	-6.73515	-6.49347	-6.18694	-5.81865	-5.39212	-4.91181	-4.38235	-3.80912	4
3	.2	-4.05776	-5.88334	-7.47489	-8.76816	-9.71186	-10.26839	-10.41552	-10.14752	-9.47491	-9.59745	-8.60004	-7.25972	-5.62993	-3.77579	-1.77109	.30426	2.36742	4.33628	2
4	.3	-6.51571	-9.03963	-10.75574	-11.51103	-11.23823	-9.96144	-7.79481	-4.93197	-1.62852	-9.52442	-7.17338	-4.18163	-.81634	2.62189	5.82597	8.50948	10.43297	11.42448	4
5	.4	-8.23529	-10.69982	-11.47506	-10.43876	-7.75428	-3.84569	.67009	5.08013	8.68806	-7.99823	-4.15989	.33522	4.77740	8.46524	10.81669	11.46037	10.29480	7.50379	2
6	.5	-9.15519	-10.85269	-9.89318	-6.51135	-1.53539	3.81649	8.23398	10.63553	10.43305	-5.87846	-.76965	4.52771	8.71640	10.77109	10.18868	7.11160	2.29350	-3.08622	4
7	.6	-9.54409	-9.97222	-6.91671	-1.44507	4.53140	8.92498	10.20068	7.91306	2.86126	-3.71057	2.32653	7.55087	10.13745	9.18282	5.02026	-.89600	-6.49923	-9.83214	2
8	.7	-9.51945	-8.33861	-3.23600	3.38853	8.41946	9.49047	6.09807	-.16238	-6.34652	-1.64191	4.87685	9.10184	9.04620	4.73591	-1.80169	-7.49201	-9.65865	-7.28267	4
9	.8	-9.17605	-6.20072	.53584	6.94742	9.14475	5.79506	-1.06993	-7.28589	-9.08232	.26806	6.76924	9.16430	6.00042	-.80325	-7.11973	-9.11739	-5.58465	1.33569	2
10	.9	-8.66727	-3.80368	3.93843	8.70002	6.87756	-.14961	-7.06366	-8.63202	-3.66787	2.02208	8.04622	7.98116	1.87612	-5.64874	-8.89875	-5.41432	2.16751	8.10906	4
11	1.0	-7.99275	-1.24044	6.65227	8.42891	2.45610	-5.77494	-8.69647	-3.62253	4.78194	3.65796	8.70201	5.74550	-2.49337	-8.43989	-6.62678	1.27895	8.00883	7.37543	2
12	1.1	-7.03395	1.37233	8.27892	6.13820	-2.71040	-8.59708	-5.08880	3.98060	8.69991	5.11995	8.59108	2.67377	-6.16543	-8.26700	-1.33432	7.05648	7.73595	-.03706	4
13	1.2	-5.88326	3.85435	8.67655	2.43375	-6.91287	-7.44357	1.51841	8.54390	4.67352	6.42267	7.81078	-.76213	-8.36308	-5.29873	4.52302	8.57665	1.69261	-7.34993	2
14	1.3	-4.50032	6.00423	8.08232	-1.87800	-8.71725	-2.78574	7.22696	6.65209	-3.66808	7.48065	6.33737	-3.29977	-8.52563	-.47107	8.27360	4.89736	-5.65355	-7.92204	4
15	1.4	-2.93786	7.64376	5.53624	-5.76180	-7.49484	3.21403	8.58741	-.29485	-8.68765	8.26327	4.29958	-6.80175	-6.61170	4.55417	8.15987	-1.78031	-8.76500	-1.19921	2
16	1.5	-1.24044	8.59249	2.45610	-8.24502	-3.62253	7.13256	4.71645	-7.06531	-5.71605	8.70201	1.85293	-8.43989	-3.04697	8.00883	4.18000	-7.41744	-5.22935	6.67768	4
17	1.6	.51190	8.73632	-1.02214	-8.67669	1.52887	8.58741	-2.03034	-8.46884	-2.52488	8.75500	-.7673	-8.71019	1.27604	8.63573	-1.78031	-8.53172	2.27862	8.39868	1
18	Simpson Summation	-287.27402	-108.85646	-25.45190	-79.07855	-95.98573	-32.59403	4.64220	-28.28000	-41.26108	.76806	93.55412	14.63327	-23.57994	40.05311	68.30965	20.39870	-3.94770	28.95594	
19	$\frac{1}{30}$ × row 18	-9.57580	-3.62855	-.84840	-2.63595	-3.19952	-1.08647	.15474	-.94267	-1.37537	.02560	3.11847	.48778	-.78600	1.33510	2.27699	.67996	-.13159	.96520	
20	ω	2	3	4	5	6	7	8	9	10	2	3	4	5	6	7	8	9	10	
21	ω × row 19	-19.15160	-10.88562	-3.39360	-13.17975	-19.19712	-7.60529	1.23792	-8.48403	-13.75370	.05120	9.35541	1.95112	-3.93000	8.01060	15.93893	5.43968	-1.18431	9.65200	
22	cosine 1.6ω	-.99829	.08750	.99318	-.14550	-.98469	.20300	.97824	-.25978	-.95767	-.05837	-.99616	.11655	.98936	-.17433	-.97918	.23151	.96566	-.28790	sin 1.6ω
23	8.77 × row 22	-8.75500	.76738	8.71019	-1.27604	-8.63573	1.78031	8.57916	-2.27827	-8.39877	.51190	-8.73632	1.02214	8.67669	-1.52887	-8.58741	2.03034	8.46884	-2.52488	8.77 × row 22
		← In Phase Component, A →									← Out of Phase Component, B →									
	row 21–row 23	-10.3966	-11.6530	-12.103	-11.9037	-10.5614	-9.3856	-7.3412	-6.2058	-5.3549	-.4607	.6190	2.9733	4.7467	6.4817	7.3515	7.4700	7.2845	7.1271	

Simpson Summation $= \sum_{t=0}^{1.6}$ (no. from column __) × (corresponding Simpson's factor from column 49)

By Simpson's rule the integral of column __ is given by $\frac{1}{30}$ times the Simpson's Summation.

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TABLE III.— LEAST-SQUARES DETERMINATION OF FIRST APPROXIMATION TO TRANSFER COEFFICIENTS FROM THE RESPONSE TO A UNIT STEP INPUT USING DERIVATIVE METHOD

1	2	3	4	5	6	7	8	9	10	11	12	13	14
	← Given Data →												
Row	t	θ	$D\theta$	$D^2\theta$	(4)^2	(4) × (3)	(4) × (2)	(4) × (5)	(3)^2	(3) × (2)	(3) × (5)	(2)^2	(2) × (5)
1	0	0	−0.08	−80.0	0.0064	0	0	6.400	0	0	0	0	0
2	.1	−.432	−7.05	−46.8	49.7025	3.04560	−.705	329.940	.186624	−.0432	20.2176	.01	−4.68
3	.2	−1.296	−10.42	−20.0	108.5764	13.50432	−2.084	208.400	1.679616	−.2592	25.9200	.04	−4.00
4	.3	−2.456	−11.54	−4.5	133.1716	28.34224	−3.462	51.930	6.031936	−.7368	11.0520	.09	−1.35
5	.4	−3.640	−11.48	4.0	131.7904	41.78720	−4.592	−45.920	13.249600	−1.4560	−14.5600	.16	1.60
6	.5	−4.736	−10.88	7.4	118.3744	51.52768	−5.440	−80.512	22.429696	−2.3680	−35.0464	.25	3.70
7	.6	−5.760	−10.24	6.9	104.8576	58.98240	−6.144	−70.656	33.177600	−3.4560	−39.7440	.36	4.14
8	.7	−6.750	−9.66	5.0	93.3156	65.20500	−6.762	−48.300	45.562500	−4.7250	−33.7500	.49	3.50
9	.8	−7.690	−9.18	3.5	84.2724	70.59420	−7.344	−32.130	59.136100	−6.1520	−26.9150	.64	2.80
10	.9	−8.600	−8.90	2.2	79.2100	76.54000	−8.010	−19.580	73.960000	−7.7400	−18.9200	.81	1.98
11	1.0	−9.500	−8.79	1.0	77.2641	83.50500	−8.790	−8.790	90.250000	−9.5000	−9.5000	1.00	1.00
Σ	5.5	−50.86	−98.22	−131.3	980.5414	493.03364	−53.333	290.782	345.663672	−36.4362	−121.2458	3.85	8.69

Equation of Condition: $D^2\theta + bD\theta + k\theta - C_{1_q} - C_{o_q}t = E$

Normal Equations Obtained by Minimizing ΣE^2

$$\Sigma(6)\, b + \Sigma(7)\, k - \Sigma(4)\, C_{1_q} - \Sigma(8)\, C_{o_q} = -\Sigma(9)$$

$$\Sigma(7)\, b + \Sigma(10)\, k - \Sigma(3)\, C_{1_q} - \Sigma(11)\, C_{o_q} = -\Sigma(12)$$

$$-\Sigma(4)\, b - \Sigma(3)\, k + 11\, C_{1_q} + \Sigma(2)\, C_{o_q} = +\Sigma(5)$$

$$-\Sigma(8)\, b - \Sigma(11)\, k + \Sigma(2)\, C_{1_q} + \Sigma(13)\, C_{o_q} = +\Sigma(14)$$

$$980.5414\, b + 493.03364\, k + 98.22 C_{1_q} + 53.333 C_{o_q} = -290.782$$

$$493.03364 b + 345.663672 k + 50.86 C_{1_q} + 36.4362 C_{o_q} = 121.2458$$

$$98.22\, b + 50.86\, k + 11\, C_{1_q} + 5.5\, C_{o_q} = -131.30$$

$$53.333\, b + 36.4362\, k + 5.5\, C_{1_q} + 3.85\, C_{o_q} = 8.69$$

$$b = 7.941 \qquad k = 29.22 \qquad C_{1_q} = -90.31 \qquad C_{o_q} = -255.3$$

TABLE IV.– APPLICATION OF PRONY'S METHOD TO THE RESPONSE TO A STEP INPUT.

(a) Computation of b and k.

1	2	3	4	5	6	7	8	9	10
m	t	q_m	q_{m+1}	q_{m+2}	③^2	③ × ④	③ × ⑤	④^2	④ × ⑤
1	0	–0.08	–7.05	–10.42	0.0064	0.5640	0.8336	49.7025	73.4610
2	.1	–7.05	–10.42	–11.54	49.7025	73.4610	81.3570	108.5764	120.2468
3	.2	–10.42	–11.54	–11.48	108.5764	120.2468	119.6216	133.1716	132.4792
4	.3	–11.54	–11.48	–10.88	133.1716	132.4792	125.5552	131.7904	124.9024
5	.4	–11.48	–10.88	–10.24	131.7904	124.9024	117.5552	118.3744	111.4112
6	.5	–10.88	–10.24	–9.66	118.3744	111.4112	105.1008	104.8576	98.9184
7	.6	–10.24	–9.66	–9.18	104.8576	98.9184	94.0032	93.3156	88.6788
8	.7	–9.66	–9.18	–8.90	93.3156	88.6788	85.9740	84.2724	81.7020
9	.8	–9.18	–8.90	–8.79	84.2724	81.7020	80.6922	79.2100	78.2310
10	.9	–8.90	–8.79						
11	1.0	–8.79							
12	$\sum_{t=0}^{1.0}$	–80.53	–89.35	–91.09	824.0673	832.3638	810.6928	903.2709	910.0308

Equation of Condition: $a_1 q_m + a_2 q_{m+1} + a_3 + q_{m+2} = E$

Normal Equations Obtained by Minimizing ΣE^2

$$\Sigma ⑥ a_1 + \Sigma ⑦ a_2 + \Sigma ③ a_3 = -\Sigma ⑧ \qquad 824.0673\, a_1 + 832.3638\, a_2 - 80.53\, a_3 = -810.6928$$

$$\Sigma ⑦ a_1 + \Sigma ⑨ a_2 + \Sigma ④ a_3 = -\Sigma ⑩ \qquad 832.3638\, a_1 + 903.2709\, a_2 - 89.35\, a_3 = -910.0308$$

$$\Sigma ③ a_1 + \Sigma ④ a_2 + \Sigma ⑨ a_3 = -\Sigma ⑤ \qquad -80.53\, a_1 - 89.35\, a_2 + 9\, a_3 = 91.09$$

$$a_1 = 0.432127 \qquad a_2 = -1.227683 \qquad a_3 = 1.799523$$

Let $x = e^{0.1\lambda}$ then $x^2 + a_2 x + a_1 = 0$

Solution for $x = x_1 \pm ix_2 = 0.61382 \pm 0.235214i = Re^{\pm i\theta} = 0.657364\, e^{\pm 0.36558i}$

$$b = -(\lambda_1 + \lambda_2) = -\frac{\log Re^{i\theta} + \log Re^{-i\theta}}{0.1} = -\frac{\log R^2}{0.1} = -\frac{\log a_1}{0.1} = \frac{\log\left(\frac{1}{0.432127}\right)}{0.1} = 8.3904$$

$$k = \lambda_1\lambda_2 = \frac{(\log Re^{i\theta})(\log Re^{-i\theta})}{0.1} = \frac{(\log R)^2 + \theta^2}{0.01} = \frac{(\log 0.65736)^2 + (0.36558)^2}{0.01} = +30.9643$$

Equation for $\lambda = l + il'$ $\qquad \lambda^2 + b\lambda + k = 0$

$$\lambda = -4.1949 \pm 3.6618i \quad \text{then} \quad l = -4.1949$$

$$l' = 3.6618$$

In addition

$$q_\infty = -\frac{a_3}{a_1 + a_2 + 1} = -8.802034$$

TABLE IV.– CONCLUDED

(b) Computation of C_{1_q} and C_{O_q}.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
	t	$q - q_\infty = q - 8.80$	lt	e^{-lt}	③ × ⑤	$l't$	sin ⑦	cos ⑦	⑧^2	⑨^2	⑧ × ⑨	⑧ × ⑥	⑨ × ⑥
1	0	8.72	0	1.00000	8.72	0	0	1.00000	0	1.00000	0	0	8.72
2	.1	1.75	−.41949	1.52120	2.66210	.36618	.35804	.93371	.128193	.871814	.334306	.953138	2.485629
3	.2	−1.62	−.83898	2.31405	−3.74876	.73236	.66861	.74361	.447039	.552956	.497185	−2.506458	−2.787615
4	.3	−2.74	−1.25847	3.52014	−9.64518	1.09854	.89053	.45492	.793044	.206952	.405120	−8.589322	−4.387785
5	.4	−2.68	−1.67796	5.35484	−14.35097	1.46472	.99437	.10592	.988772	.011219	.105324	−14.270174	−1.520055
6	.5	−2.08	−2.09745	8.14578	−16.94322	1.83090	.96638	−.25713	.933890	.066116	−.248485	−16.373589	4.356610
7	.6	−1.44	−2.51694	12.39137	−17.84357	2.19708	.81025	−.58609	.656505	.343501	−.474879	−14.457753	10.457938
8	.7	−.86	−2.93643	18.84033	−16.20268	2.56326	.54669	−.83734	.298870	.701138	−.457765	−8.857843	13.567152
9	.8	−.38	−3.35592	28.64560	−10.88533	2.92944	.21064	−.97756	.044369	.955624	−.205913	−2.292886	10.641063
10	.9	−.10	−3.77541	43.59751	−4.39575	3.29562	−.15333	−.98817	.023510	.976480	.151516	.674000	4.343748
11	1.0	.01	−4.19490	66.35373	.66354	3.66180	−.49705	−.86772	.247059	.752938	.431300	−.329813	−.575767
12	Σ								4.561251	6.438738	.537709	−66.050700	45.300918

Equation of Condition $\quad q - q_\infty - e^{lt}(M \sin l't + N \cos l't) = E$

Normal Equations Obtained by Minimizing ΣE^2

$$\Sigma ⑩ M + \Sigma ⑫ N = \Sigma ⑬$$

$$\Sigma ⑫ M + \Sigma ⑪ N = \Sigma ⑭$$

$4.561251\ M + 0.537709N = -66.0507$

$0.537709\ M + 6.438738N = 45.300918$

$M = -15.4625 \qquad N = 8.3270$

Response Equation $\quad q = -8.802 + e^{-4.195t}(-15.46 \sin 3.662t + 8.327 \cos 3.662t)$

Laplace Transform of $e^{-4.195t} \sin 3.662t = \dfrac{3.662}{p^2 + bp + k}$

Laplace Transform of $e^{-4.195t} \cos 3.662t = \dfrac{p + 4.195}{p^2 + bp + k}$

Transfer Function of $\dfrac{q}{\delta} = -8.802 - 15.46\dfrac{3.662D}{D^2 + 8.39D + 30.96} + 8.327\dfrac{D(D + 4.195)}{D^2 + 8.39D + 30.96}$

$$= \frac{-0.47D^2 - 91.54D - 258.4}{D^2 + 8.39D + 30.96}$$

$C_{1_q} = -91.54 \qquad C_{O_q} = -258.4$

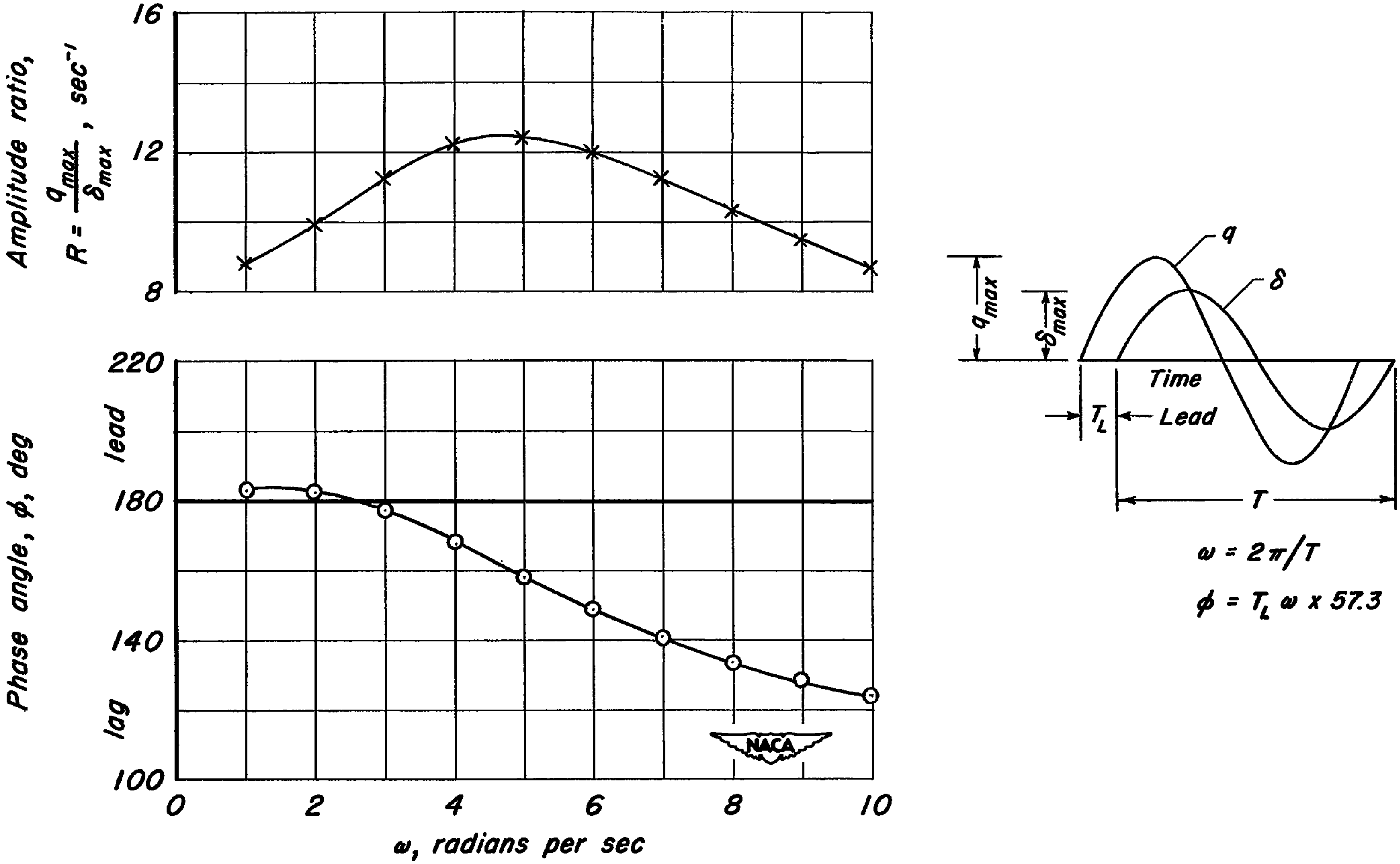

Figure 1.- Frequency response of pitching velocity of example airplane to sinusoidal elevator oscillations.

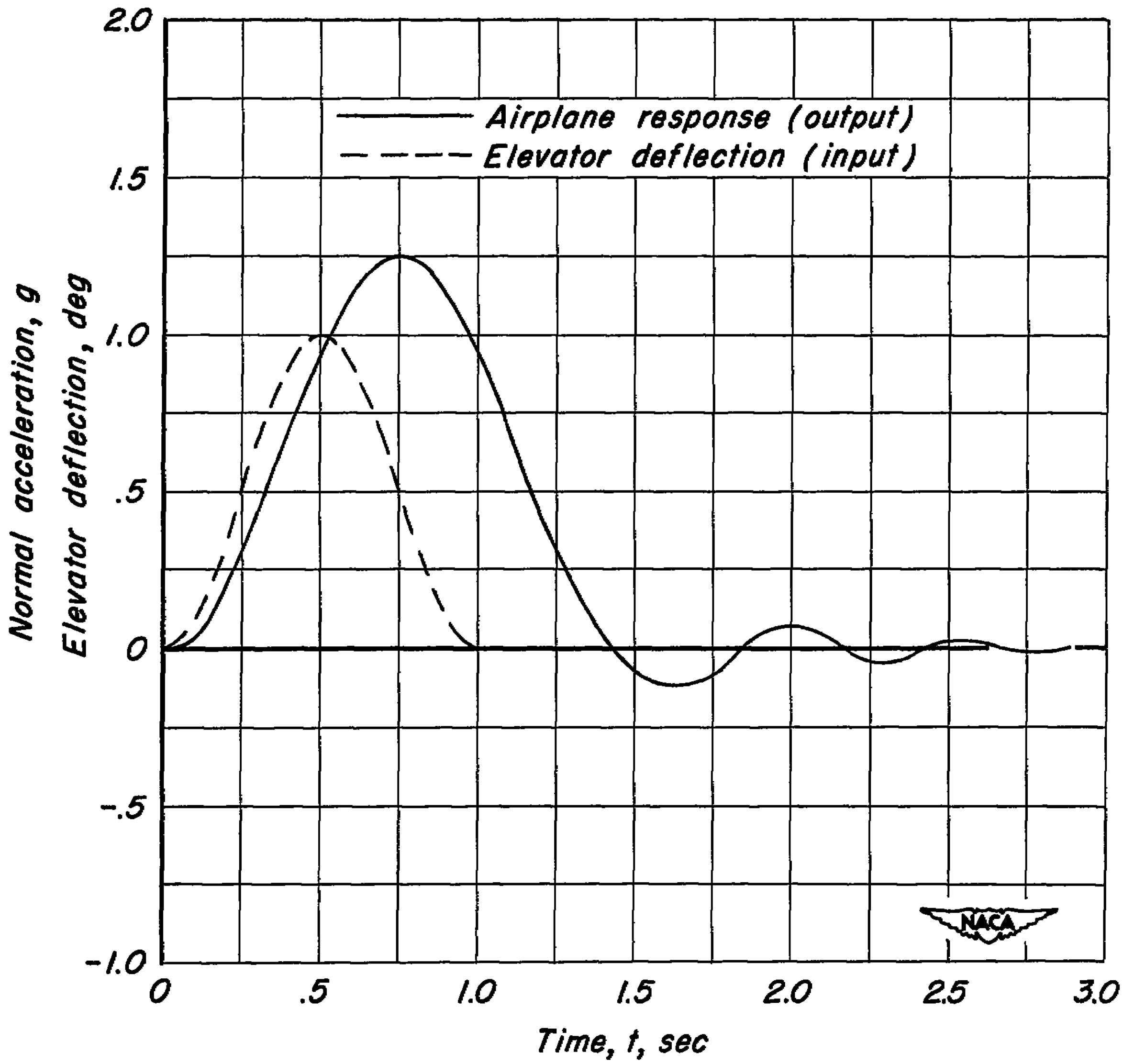

Figure 2.- Illustration of transient response to elevator pulse.

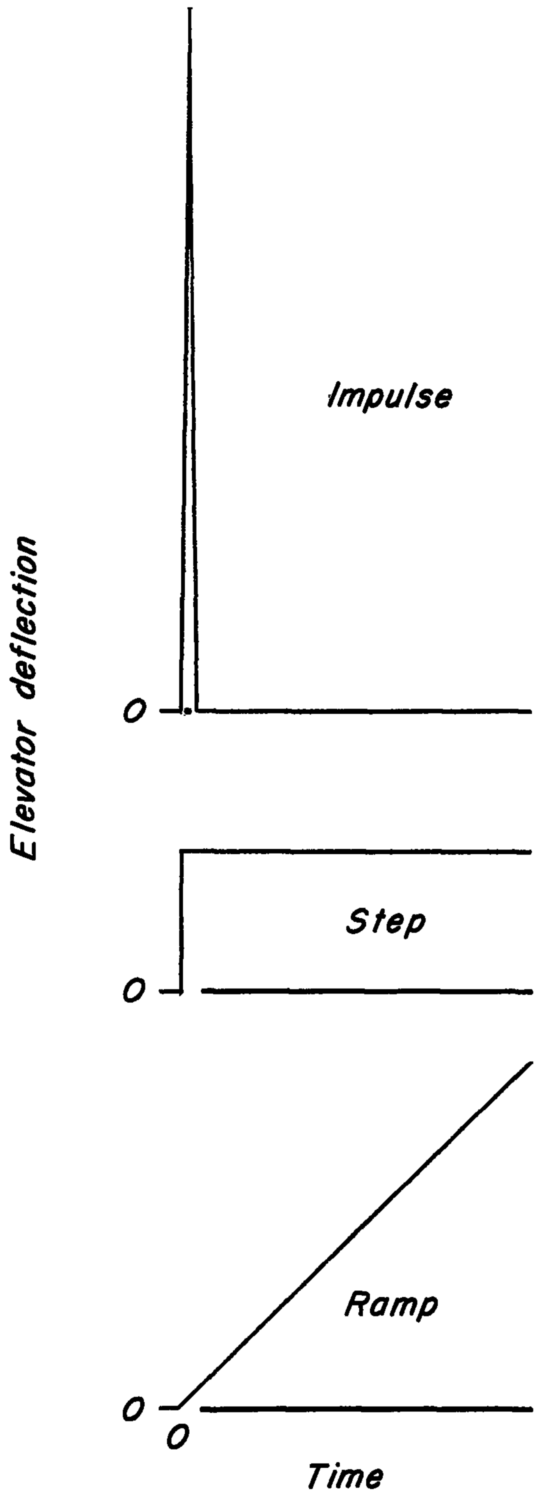

Figure 3.— Illustrative impulse, step, and ramp functions.

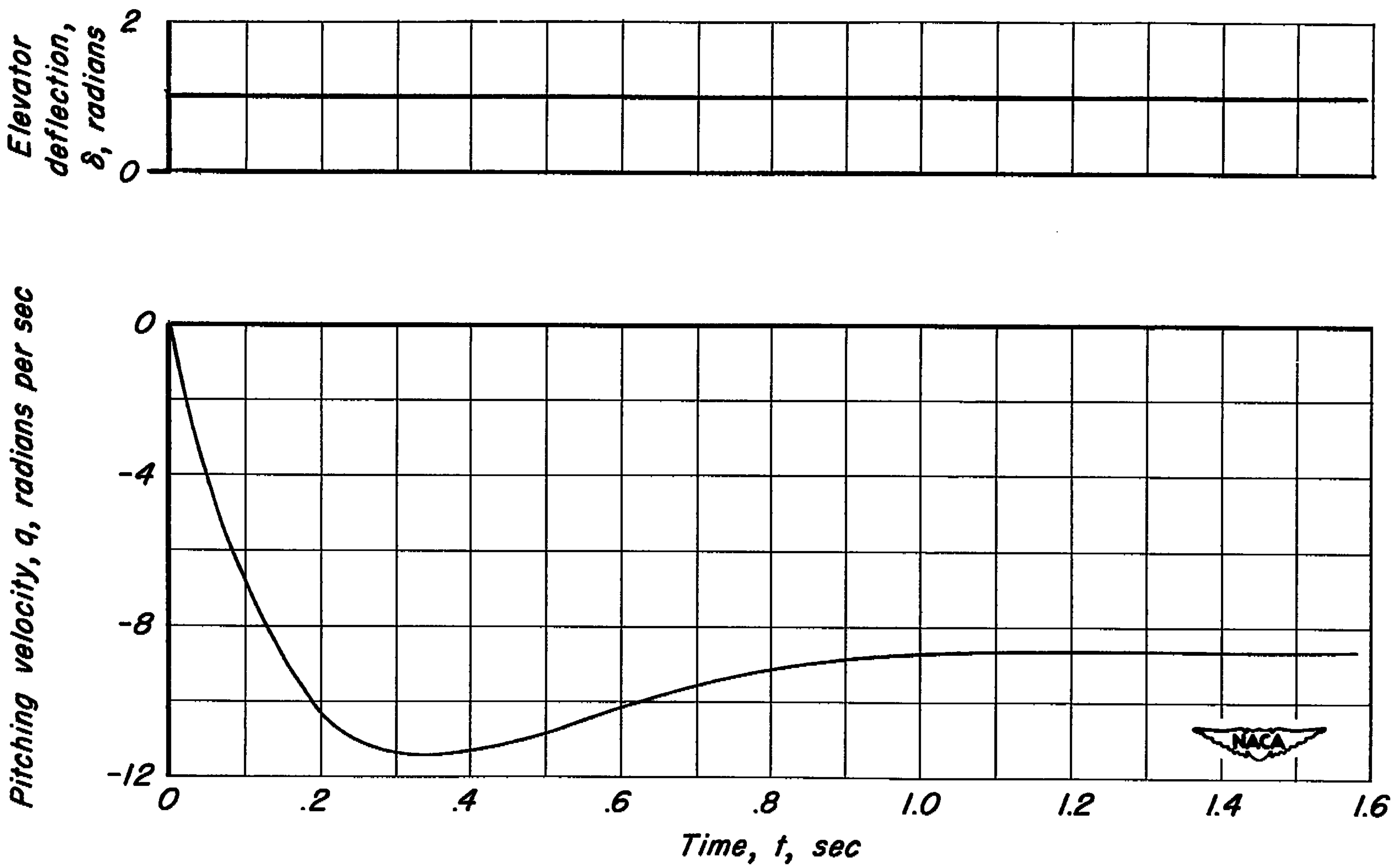

Figure 4.– Typical aircraft pitching velocity response to step elevator input.

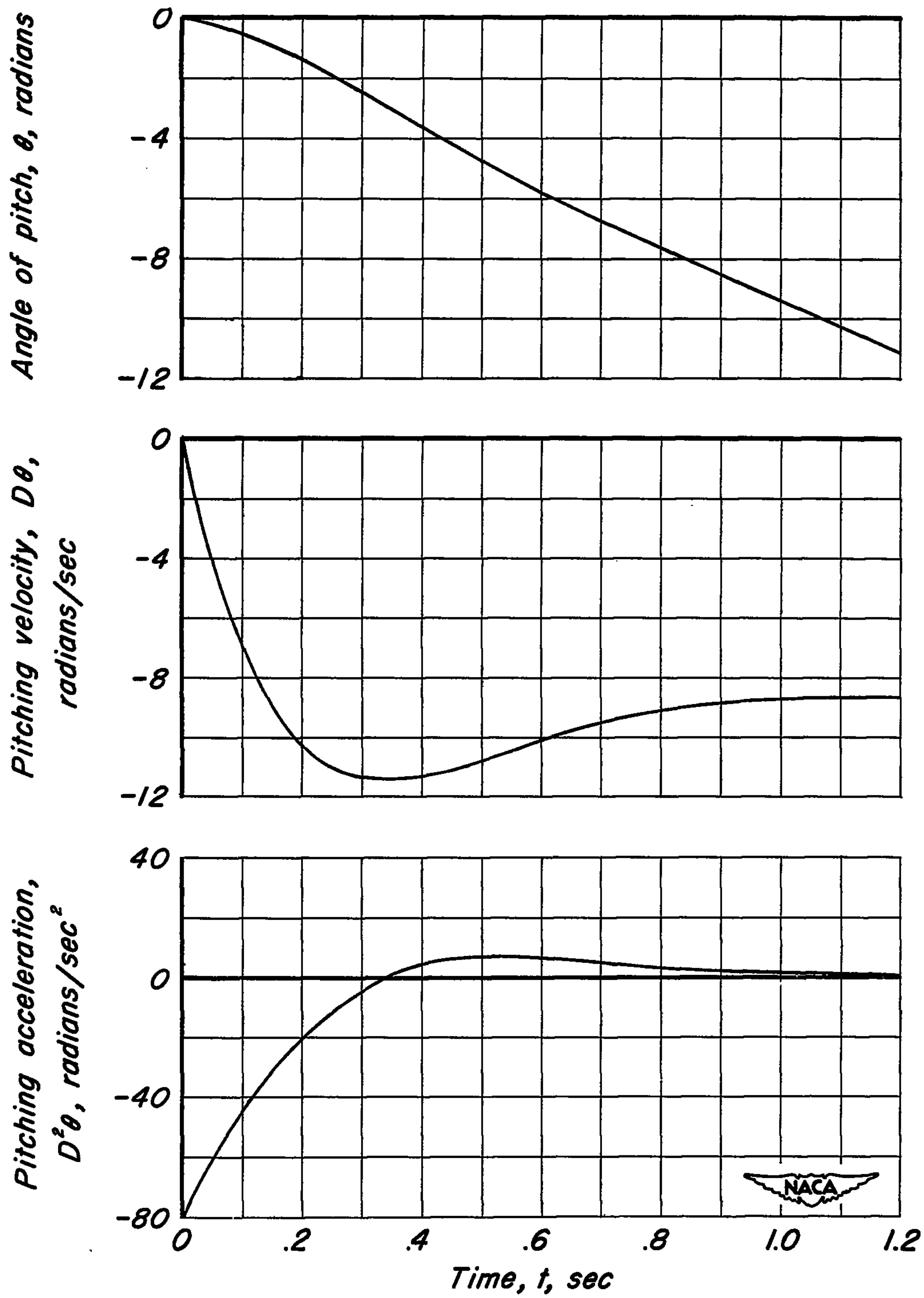

Figure 5.– The airplane response in angle of pitch, pitching velocity, and pitching acceleration to a unit step elevator input.