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TWO-DIMENSIONAL STEADY NONVISCOUS AND VISCOUS COMPRESSIBLE FLOW THROUGH A SYSTEM OF EQUIDISTANT BLADES

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SUMMARY

This paper treats the two-dimensional flow of a compressible, nonviscous fluid through blade systems of equidistant spacing, of identical shape, and of straight-line arrangement of position, for which the names of grid, cascade, deflector, and lattice systems are in use.

The requirement of two-dimensional flow, if confined to a finite region, must be assumed to be realized by enclosing the flow between four frictionless walls, two of which are perpendicular to the span axis of the blades and two of which follow the inflow and outflow and coincide with the surface lines of the innermost and outermost blades of the system.

The treatment of this problem is carried out in two steps. First, it is assumed that the system of finitely spaced blades is replaced by a system of infinitesimally spaced blades, where the action of the blades is expressed by means of a continuous force field which is uniform in the direction of spacing. Then, in the second step, the transition to finite spacing is made by replacing the force field between the blades by the inertia and pressure terms which were omitted in the case of infinitesimal spacing.

A numerical example is worked out and represented graphically for a system of blades with a shape symmetric about their midpoint and which produce a 90° deflection of a uniform flow. The differential equation determining the change of the blade shape by finite spacing permits an infinite number of solutions. The solution in this paper leads to a line airfoil distorted very little from the shape for infinitesimal spacing.

Viscous flow through a grid system of equidistant, narrowly spaced blades is treated in an appendix by the introduction of a velocity, velocity gradient, pressure, and force field of uniformity across the blades. The method of treatment is a generalization of the method applied in the main text for nonviscous flow and in NACA TN 2493 for isentropic flow.

INTRODUCTION

Closely spaced airfoil blade systems in an incompressible or compressible fluid can, in a first approach, be represented by a continuous field of force intensity of either one-dimensional or circular two-dimensional symmetry. The arrangement of the blades is then either straight or circular; the blades may be either at rest or in steady motion.

Steady, nonviscous flow of an incompressible or compressible fluid through such a field can be analyzed by Euler's dynamic equations, the equation of state, the equation of change of state, the continuity equation, and the condition that the force field is of conservative character (i.e., without dissipation of energy).

The complete solution of this system of equations and the determination of those sets of streamlines which form the blade surfaces can be derived by integration. For a three-dimensional flow of given boundary conditions, it is best to prescribe the pressure function along the flow and one of the functions of a velocity component; while for a two-dimensional flow the prescription of the pressure function alone is sufficient for the simplest complete integration. In this paper, only the latter (two-dimensional) case is treated.

The transition from these solutions to finite spacing can be achieved by replacing the force field of infinitesimal spacing in the flow equations by those terms of Euler's dynamic equations which were missing in the first approach of infinitesimal spacing, where uniform symmetry of flow is assumed.

The analysis developed in this paper can be extended in two important directions. First, the method applied here for nonviscous flow can be adapted to viscous flow; second, it is possible to use the procedure in this report to predict the flow values through prescribed, two-dimensional blade systems. Viscous flow through a grid system of equidistant, narrowly spaced blades is treated in the appendix at the end of this paper.

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SYMBOLS

x,y Cartesian coordinates, where the y-axis connects the leading edges of blades

u,v velocity components in x- and y-directions, respectively

$\bar{f}$ impressed force intensity, per unit of volume

p pressure

ρ mass density

t time

T temperature

P enthalpy (pressure-density) relation $\left(\int \frac{dp}{\rho}\right)$

$\bar{k}$ impressed force intensity per unit of mass $\left(\bar{f}/\rho\right)$

R gas constant in technical units

$\overline{V}_R$ resultant velocity $\left(\left(u^2 + v^2\right)^{1/2}\right)$

γ ratio of specific heats

η nondimensional distance number $\left(\frac{y - y_o}{l}\right)$

l length of blade system in direction of x-axis

x_1 shift of a point of a streamline in direction of x-axis, caused by transition to finite spacing

$\xi_1 \equiv \frac{x_1}{l}$

$\xi_o \equiv \frac{x_o}{l}$

Subscripts:

in intake immediately at leading edge of blade system

ex exit

o flow functions in a blade system of infinitesimal spacing

SP finite spacing of blades

GENERAL ANALYSIS

Infinitesimal Spacing

Figure 1 shows a representation of a two-dimensional flow for an infinitesimally spaced curved blade system. The connecting straight line of the leading blade edges is chosen as the y-axis, and perpendicular to it, the x-axis. For infinitesimal spacing the pressure and inertia force discontinuities of the flow at each blade are replaced by a continuous field of force intensity ($\bar{k} = \bar{f}/\rho$) in a steady flow, which is two-dimensional in the dependent velocity components u and v and one-dimensional in x, for which

$$\frac{\partial}{\partial t} = 0$$

$$\frac{\partial}{\partial y} = 0$$

$$\frac{\partial}{dx} \neq 0$$

Force-field vector.- The field $\bar{k} = \bar{f}/\rho$ of force intensity per unit of mass in Cartesian coordinates is given by

$$k_x = \frac{f_x}{\rho}$$

$$k_y = \frac{f_y}{\rho}$$

where $\bar{f}$ is the force intensity per unit of volume.

Enthalpy function.- It is further assumed that the flow is nonviscous and isentropic (no heat input or loss) so that the density ρ is a known function of the pressure p. Because of this assumption, it becomes convenient to introduce the enthalpy $P = \int \frac{dp}{\rho}$.

This has the advantage of reducing in a simple manner the number of dependent variables.

The enthalpy function P, under the condition of isentropy, has in technical units the value

$$P = \int \frac{dp}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \tag{1}$$

which follows from the condition of change of state

$$\frac{p}{p_{in}} = \left(\frac{\rho}{\rho_{in}}\right)^{\gamma} \tag{2}$$

From the assumption of a perfect gas, the equation of state is

$$p = RT\rho g \tag{3}$$

The values of p and ρ can be expressed in terms of P, namely, by

$$\frac{p}{p_{in}} = \left(\frac{P}{P_{in}}\right)^{\frac{\gamma}{\gamma-1}} \tag{4}$$

and

$$\frac{\rho}{\rho_{in}} = \left(\frac{P}{P_{in}}\right)^{\frac{1}{\gamma-1}} \tag{5}$$

where the subscript in denotes, here and in the analysis to follow, values at the intake of the system. The flow values at the intake to the blade system, for this analysis of infinitesimal spacing, are assumed to be the same values as those in the free stream.

It follows also, from equations (2) and (3), that

$$P_{in} = \frac{\gamma}{\gamma - 1} \frac{p_{in}}{\rho_{in}} \tag{6}$$

Flow equations.- The dynamic equations of a two-dimensional flow between two blades are

$$\left.\begin{aligned} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0 \\ \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0 \end{aligned}\right\} \tag{7}$$

For infinitesimal spacing with independence of y for velocity and pressure and with the force field, the equations become

$$u \frac{du}{dx} + \frac{dP}{dx} = k_x \tag{8}$$

$$u \frac{dv}{dx} = k_y \tag{9}$$

Condition of nonviscous flow.- The condition of nonviscous flow requires that the force vector be perpendicular to the streamlines, so that

$$k_x u + k_y v = 0 \tag{10}$$

In terms of the streamline tangents (see fig. 1) it can be written in the form

$$\frac{v}{u} = - \frac{k_x}{k_y} = \frac{dy}{dx} \tag{11}$$

Continuity equation.- The system of equations is completed by the continuity equation of steady flow; namely,

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \tag{12}$$

The condition $\frac{\partial}{\partial y} = 0$, given above, simplifies equation (12) and leads to the following first integral:

$$\rho u = C = \rho_{in} u_{in} \tag{13}$$

Energy integral.- Flow equations (8) and (9) together with equation (10) furnish Bernoulli's integral; namely,

$$\frac{u^2 + v^2}{2} + P = \frac{{u_{in}}^2 + {v_{in}}^2}{2} + P_{in} \tag{14}$$

The velocities u and v can now, if equations (1), (2), (12), and (14) are observed, be expressed by the enthalpy P.

This can be done as follows. At first, the continuity equation (13) and the change-of-state relation (equation (2)) give

$$\frac{u}{u_{in}} = \left(\frac{P}{P_{in}}\right)^{\frac{1}{1-\gamma}} \tag{15}$$

Second, the y velocity component v follows from the energy integral (14) combined with equation (15) in the form

$$\left(\frac{v}{v_{in}}\right)^2 = 1 + \left(\frac{u_{in}}{v_{in}}\right)^2 + 2\frac{P_{in}}{{v_{in}}^2} - \frac{P}{P_{in}}\left[2\frac{P_{in}}{{v_{in}}^2} + \left(\frac{u_{in}}{v_{in}}\right)^2\left(\frac{P}{P_{in}}\right)^{\frac{1+\gamma}{1-\gamma}}\right] \tag{16}$$

It is necessary for the purpose of the analysis to give the expressions for the force field $\bar{k}$. These expressions are readily given terms of the function P. This is done by the insertion of the values of the velocity components u and v, given by equations (15) and (16), into the flow equations (8) and (9).

Using the abbreviation $\frac{d}{dx} \equiv {}'$, one obtains:

$$k_x = \left[\frac{{u_{in}}^2}{2}\left(\frac{P}{P_{in}}\right)^{\frac{2}{1-\gamma}} + P\right]'$$

or

$$k_x = \left[1 + \frac{1}{1-\gamma}\left(\frac{{u_{in}}^2}{P_{in}}\right)\left(\frac{P}{P_{in}}\right)^{\frac{1+\gamma}{1-\gamma}}\right]P' \tag{17}$$

and

$$k_y = -k_x\frac{u}{v} \tag{18}$$

From the results obtained in the preceding paragraphs, it can be seen that all the variables appearing in the complete setup of equations are expressible <u>in terms of the enthalpy function P</u> and the intake (entrance) values u_{in}, v_{in}, p_{in}, ρ_{in}, and P_{in}.

Compilation of expressions for infinitesimal spacing.- The following relations are used for calculating the variables for infinitesimal spacing, with $\gamma = 1.4$:

$$\frac{p}{\rho} = \frac{1}{3.5} P \tag{19}$$

$$\frac{p}{p_{in}} = \left(\frac{P}{P_{in}}\right)^{3.5} \tag{20}$$

$$\frac{\rho}{\rho_{in}} = \left(\frac{P}{P_{in}}\right)^{2.5} \tag{21}$$

$$\frac{u}{u_{in}} = \left(\frac{P}{P_{in}}\right)^{-2.5} \tag{22}$$

$$\left(\frac{v}{v_{in}}\right)^2 = 1 + \left(\frac{u_{in}}{v_{in}}\right)^2 + 2\frac{P_{in}}{{v_{in}}^2} - \frac{P}{P_{in}}\left[2\frac{P_{in}}{{v_{in}}^2} + \left(\frac{u_{in}}{v_{in}}\right)^2\left(\frac{P}{P_{in}}\right)^{-6}\right] \tag{23}$$

$$k_x = \left[1 - 2.5\frac{{u_{in}}^2}{P_{in}}\left(\frac{P}{P_{in}}\right)^{-6}\right]P' \tag{24}$$

$$k_y = -k_x\frac{u}{v} \tag{18}$$

$$\frac{dy}{dx} = \frac{v}{u} \tag{11}$$

The form of the relations above shows that the simplest calculation of all variables can be done with P as a basic chosen function. It follows therefore that, in the case of a continuous flow and force field (both constant in the direction of y), the velocity field, the streamlines, and the pressure and density distributions are determined by a free choice of the enthalpy function and of the intake values of the flow. This freedom of choice, however, will be somewhat restricted after the continuous symmetry of infinitesimal spacing is replaced by a discontinuous symmetry of finite blade spacing.

If, however, it is desired to calculate the variables for a flow through blade systems of prescribed shape, it is necessary to express the variables in terms of the streamline tangents. This "inverse procedure," though somewhat involved, should lead to valuable results, but will not be investigated in this report.

Equation of streamlines (for infinitesimal spacing).- The equation of the streamlines for an infinitesimally spaced blade system is derived from the streamline tangent equation (11), which is

$$\frac{dy}{dx} = \frac{v}{u} \tag{11}$$

From equation (11), the integration for y leads to

$$y = \int_0^x \frac{v}{u}\, dx + \text{Constant} \tag{25}$$

where the values of u and v in equation (25) are given by equations (15) and (16). In general, it is to expected that the integral of equation (25) will have to be evaluated graphically. A special example is given in the section, "An Application of the General Analysis."

Transition to Finite Blade Spacing

Set of series transforming velocities and enthalpy.- In the case of finite blade spacing, shown in figure 2, the condition $\frac{\partial}{\partial y} = 0$ is no longer valid and the force field k_x, k_y must disappear; furthermore the dynamic equations and the continuity equation must be complemented by the addition of the y derivatives in order to account for the inertia forces and the pressure gradients in the direction of y. In the analysis to follow the subscript o denotes the original values of infinitesimal spacing of all variables of the preceding sections. This transition to finite spacing must evidently change gradually the velocity components u and v and the enthalpy P to values different from the original values u_o and v_o of infinitesimal spacing.

The development is based on the assumption that a set of equidistant streamlines, as given by the continuous force field $\bar{k}$, of infinitesimal spacing is kept unchanged (fixed or frozen) in shape, velocity distribution, and pressure distribution (see fig. 2). The changed velocities and pressures on the other (free) streamlines will then be determined by a set of transforming series in powers of dimensionless distance numbers

$$\eta \equiv \frac{y - y_o}{l} \tag{26}$$

where l is the length of the blade system in the x-direction, $y_o(x)$ the original streamlines of infinitesimal spacing, and $y(x)$ a point on a streamline of finite spacing. Also

$$\eta_{SP} = \frac{y_{SP}}{l} \tag{27}$$

is the spacing number for the leading and trailing edges. This method of correcting series is justified by the compatibility of all resulting functions.

The series set is proposed in the form

$$\left.\begin{aligned} u &= u_o + \sum_{n=1}^{\infty} u_n \eta^n, \quad u_n = u_n(x) \\ v &= v_o + \sum_{n=1}^{\infty} v_n \eta^n, \quad v_n = v_n(x) \\ P &= P_o + \sum_{n=1}^{\infty} P_n \eta^n, \quad P_n = P_n(x) \end{aligned}\right\} \tag{28}$$

These series must satisfy the flow equations after they have been complemented by the addition of the y derivatives and stripped of the force field $\bar{k}$. The force field, however, appears indirectly after the original inertia terms (like $u_o \frac{\partial u_o}{\partial x}$, and so forth) are expressed by means of equations (8) and (9), with the use of the values in equations (24) and (18) of the original force field $\bar{k}$. The convergence of the series of equations (28) will be determined by means of the final numerical results.

The complete dynamic equations with the abbreviations $\frac{\partial}{\partial x} \equiv {}'$ and $\frac{\partial}{\partial y} \equiv {}^{\boldsymbol{\cdot}}$ are given by

$$\begin{aligned} uu' + vu^{\boldsymbol{\cdot}} + P' &= 0 \\ uv' + vv^{\boldsymbol{\cdot}} + P^{\boldsymbol{\cdot}} &= 0 \end{aligned} \tag{29}$$

The <u>complete</u> continuity equation, if ρ is replaced by equation (5), that is, by

$$\frac{\rho}{\rho_{in}} = \left(\frac{P}{P_{in}}\right)^{\frac{1}{\gamma-1}} \tag{5}$$

is

$$(\gamma - 1)P(u' + v^{\cdot}) + uP' + vP^{\cdot} = 0 \tag{30}$$

For abbreviation, one may write:

$$\left.\begin{aligned} u &= u_o + \sum_{n=1}^{n} u_n(x)\eta^n = u_o + U \\ v &= v_o + \sum_{n=1}^{n} v_n(x)\eta^n = v_o + V \\ P &= P_o + \sum_{n=1}^{n} P_n(x)\eta^n = P_o + \Pi \end{aligned}\right\} \tag{31}$$

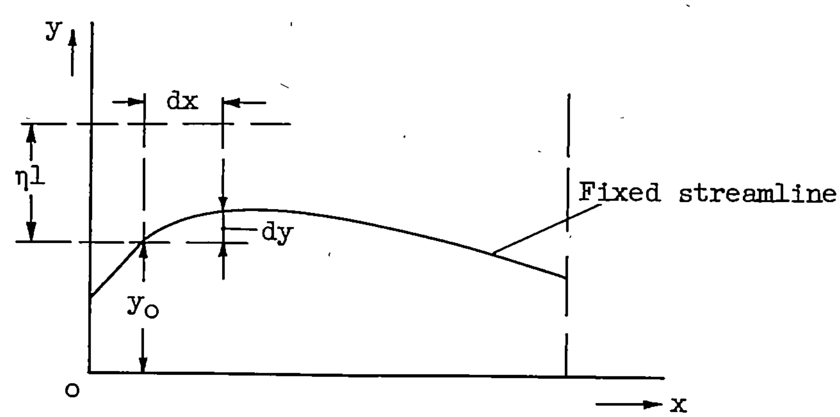
If u, v, and P of equations (29) and (30) are replaced by the right sides of equations (31) and if, moreover, equations (8), (9), and the continuity equation (12) are then subtracted from the corresponding equations (29) and (30), one obtains:

$$(\gamma - 1)\left[P_o(U' + V^{\cdot}) + \Pi u_o'\right] + u_o\Pi' + v_o\Pi^{\cdot} + UP_o' = 0 \tag{32}$$

$$u_oU' + Uu_o' + v_oU^{\cdot} + \Pi' = -k_x \tag{33}$$

$$u_oV' + Uv_o' + v_oV^{\cdot} + \Pi^{\cdot} = -k_y \tag{34}$$

For the expansion of equations (32), (33), and (34) the following method must be observed in regard to the partial derivatives of η.



According to the figure on the preceding page it is seen how the distance from the frozen streamline ηl changes with the partial changes of x and y.

If x changes while y remains constant, then the rate of change $\frac{\partial \eta l}{\partial x}$ is the same as the negative rate of change of the frozen streamline; hence,

$$\frac{\partial \eta l}{\partial x} = - \frac{dy_o}{dx} = - \frac{v_o}{u_o} \tag{35}$$

If y changes while x remains constant, the change of the distance ηl is the same as the change of the ordinate y; hence,

$$\frac{\partial \eta l}{\partial y} = 1 \tag{36}$$

so that

$$\left.\begin{aligned}
U' &= - \frac{v_o}{u_o l} \sum u_n n \eta^{n-1} + \sum u_n' \eta^n \\
U^{\cdot} &= \frac{1}{l} \sum u_n n \eta^{n-1} \\
V' &= - \frac{v_o}{u_o l} \sum v_n n \eta^{n-1} + \sum v_n' \eta^n \\
V^{\cdot} &= \frac{1}{l} \sum v_n n \eta^{n-1} \\
\Pi' &= - \frac{v_o}{u_o l} \sum P_n n \eta^{n-1} + \sum P_n' \eta^n \\
\Pi^{\cdot} &= \frac{1}{l} \sum P_n n \eta^{n-1}
\end{aligned}\right\} \tag{37}$$

It is important to note that in this section (of finite spacing) the notation of the abcissa x will be changed to x_o, in order to indicate that the original streamlines (except the "fixed" ones) will in general be distorted. Hence a point at x_o will move to a position $x = x_o + x_1$.

The recursion formulas for u_n and P_n are obtained by inserting relations (31) and (37) into equations (32), (33), and (34) as shown below.

The first equation (equation (32)) gives (see equation (11))

$$\frac{v_1}{u_1} = \frac{v_o}{u_o} = \frac{dy_o}{dx_o} \tag{38}$$

from which it is seen that the shape of the streamline is not changed by the first term of the series development and

$$2u_2\left(\frac{v_o}{u_o} - \frac{v_2}{u_2}\right) = l\left(u_1' + u_o'\frac{P_1}{P_o}\right) + l\frac{u_oP_1' + u_1P_o'}{(\gamma - 1)P_o} \tag{39}$$

$$(n + 1)u_{n+1}\left(\frac{v_o}{u_o} - \frac{v_{n+1}}{u_{n+1}}\right) = l\left(u_n' + u_o'\frac{P_n}{P_o}\right) + l\frac{u_oP_n' + u_nP_o'}{(\gamma - 1)P_o} \tag{40}$$

The second equation (equation (33)) furnishes the relations

$$P_1 = l\frac{u_o}{v_o}k_x = l\frac{u_o}{v_o}\left[1 - 2.5\frac{u_{in}^2}{P_{in}}\left(\frac{P_o}{P_{in}}\right)^{-6}\right]P_o' \tag{41}$$

$$P_2 = \frac{u_o l}{2v_o}\left(u_ou_1 + P_1\right)' \tag{42}$$

$$P_{n+1} = \frac{u_o l}{(n + 1)v_o}\left(u_ou_n + P_n\right)' \tag{43}$$

The third equation (equation (34)) becomes

$$P_1 = -lk_y \tag{44}$$

$$P_2 = -\frac{l}{2}\left(u_ov_1' + u_1v_o'\right) = -\frac{l}{2}\frac{u_o}{v_o}\left(v_ov_1\right)' \tag{45}$$

$$P_{n+1} = -\frac{l}{n + 1}\left(u_ov_n' + u_nv_o'\right) \tag{46}$$

Since the two values of P_1 must be equal, it is seen that

$$k_x u_o + k_y v_o = 0$$

which simply confirms equation (10), that is, the condition of frictionless flow.

The comparison of the two values of P_2, which must be equal, gives, after integration

$$u_o u_1 + v_o v_1 + P_1 = \text{Constant}$$

and, by means of equation (38),

$$u_1 = -u_o \frac{P_1 - \text{Constant}}{u_o^2 + v_o^2} \tag{47}$$

$$v_1 = -v_o \frac{P_1 - \text{Constant}}{u_o^2 + v_o^2} \tag{48}$$

$$P_2 = \frac{u_o}{2v_o}\left(\frac{P_1 v_o^2}{u_o^2 + v_o^2}\right)' = -\frac{l}{2}\frac{u_o}{v_o}\left(v_o v_1\right)' \tag{49}$$

The comparison of the two equal values of P_3 given by equations (43) and (46) requires that

$$\frac{u_o}{v_o}\left(u_o u_2 + P_2\right)' = -\left(u_o v_2' + v_o' u_2\right) \tag{50}$$

From equation (39) one has

$$u_2 v_o - v_2 u_o = \frac{l u_o}{2}\left(u_1' + u_o' \frac{P_1}{P_o}\right) + \frac{l u_o}{2}\frac{u_o P_1' + u_1 P_o'}{(\gamma - 1)P_o} \equiv f(P_o) \tag{51}$$

where the right side, expressible in terms of P_o, has been abbreviated to $f(P_o)$.

If the value of u_2 of equation (39) is inserted into equations (43) and (46), equation (50) becomes

$$\left(u_o u_2\right)' + \frac{P_2'}{l} = -\left(v_2 v_o\right)' = \frac{v_o}{u_o} f(P_o) \tag{52}$$

Observing that P_2/l is also a known function of P_o one can introduce the abbreviation

$$-\left[P_2' + \frac{v_o'}{u_o}f(P_o)\right] \equiv g(P_o) \tag{53}$$

and can write

$$\left(u_o u_2 + v_o v_2\right)' \equiv g(P_o) \tag{54}$$

which by integration becomes

$$\left(u_o u_2 + v_o v_2\right) - \int g(P_o)\ dx_o + c \equiv h(P_o) \tag{55}$$

If equation (51) is also used, namely,

$$v_o u_2 - u_o v_2 \equiv f(P_o)$$

or, dimensionless,

$$\frac{v_o u_2 - u_o v_2}{u_o v_o} = \frac{f(P_o)}{u_o v_o} \tag{56}$$

then the velocity components of the second term of the power-series terms assume the values

$$u_2 = \frac{u_o h + v_o f}{u_o^2 + v_o^2} \tag{57}$$

$$v_2 = \frac{-u_o f + v_o h}{u_o^2 + v_o^2} \tag{58}$$

The results show that all enthalpy (also pressure) functions up to the series terms of factor $\eta^2 = \frac{(y - y_o)^2}{l}$ are expressed as functions of the enthalpy function $P_o(x_o)$. It will be seen in the following section that the choice of this function has to satisfy the conditions of regularity of the derivatives and the requirements for continuous flow into and out of the blade system.

Intake and exit boundary conditions.- The continuous flow, produced by a continuous force field, is evidently also continuous at the intake and exit. After the transition to finite blade spacing, the flow inside the blade system becomes discontinuous across a solid blade. Since it must be assured that outside the blade system the flow remains continuous, the condition of continuity of pressure and velocity must be imposed at the leading and trailing edges of the finitely spaced blades. These conditions are automatically satisfied for the even powers of n in the transition terms $\left(P_n\eta^n,\ u_n\eta^n,\text{ and }\ v_n\eta^n\right)$. It must therefore be only required that u_1, u_3, v_1, v_3, P_1, P_3, and so forth are zero at $x = 0$ and $x = l$.

In the case treated in the section "An Application of the General Analysis," the above conditions (as seen below) are accomplished up to the second series term P_1, by requiring that the enthalpy function of the continuous force field, according to equation (41), satisfies the condition

$$\left(\frac{dP_o}{dx_o}\right)_{\substack{x=0\\x=l}} = \left(k_x\right)_{\substack{x=0\\x=l}} = 0 \qquad (59)$$

It is also necessary to cancel the arbitrary constant in equations (47) and (48). The equations then show that the condition $P_o' = 0$ at $x = 0$ and l satisfies also the continuity condition for the inflow and outflow velocities.

Blade shape.- From equation (38) it is seen, as mentioned above, that the first series term of the transition to finite spacing keeps unchanged the shape of the original streamlines of infinitesimal spacing, which, of course, means that, at least in the first approximation, the blade shape is not changed for any spacing.

The second series terms u_2 and v_2 change the shape of the streamlines and of the blade according to equations (57) and (58) by means of the relation between the velocity components $u_o + u_1\eta + u_2\eta^2$ and $v_o + v_1\eta + u_2\eta^2$ as follows:

$$\frac{dy}{dx} = \frac{v_o + v_1\eta + v_1\eta^2}{u_o + u_1\eta + u_2\eta^2}$$

Equations (38) and (11), if only terms up to the factor η^2 are kept, lead to

$$\frac{dy}{dx} = \frac{dy_o}{dx_o}\left[1 + \eta^2\left(\frac{v_2}{v_o} - \frac{u_2}{u_o}\right)\right] \tag{60}$$

The coordinates x and y of a free streamline, particularly the one which shall become part of a blade surface, will in general differ from the coordinate x_o and y_o of the frozen streamline. Therefore, the left side of equation (60) must be developed by means of the relations

$$x = x_o + x_1$$

and

$$y = y_o + \eta l$$

With these relations, together with definition (51) for $\frac{v_2}{v_o} - \frac{u_2}{u_o}$, equation (60) becomes

$$\frac{dy}{dx} = \frac{dy_o}{dx_o}\,\frac{1 + l\,\frac{d\eta}{dy_o}}{1 + \frac{dx_1}{dx_o}} = \frac{dy_o}{dx_o}\left[1 - \eta^2\,\frac{f(P_o)}{u_o v_o}\right] \tag{61}$$

If the numerator and denominator of equation (61) are multiplied by $\left(1 - \frac{dx_1}{dx_o}\right)$ and second and higher powers of $\frac{dx_1}{dx_o}$ are neglected (an assumption which must be borne out by the results of the analysis), then equation (61) is transformed as follows:

$$\frac{d\eta}{dx_o} + \eta^2\frac{f(P_o)}{l u_o{}^2} = \frac{dx_1}{dx_o}\left(\frac{v_o}{l u_o} + \frac{d\eta}{dx_o}\right) \tag{62}$$

If now it is further anticipated that $\frac{d\eta}{dx_o}\,\frac{dx_1}{dx_o}$ is small and of higher order, one obtains from equation (62) the equation

$$\frac{d\eta}{dx_o} + \eta^2\,\frac{f(P_o)}{l u_o{}^2} = \frac{dx_1}{dx_o}\,\frac{v_o}{l u_o} \tag{63}$$

This neglect of $\frac{d\eta}{dx_0} \frac{dx_1}{dx_0}$ must be justified by the choice of the magnitude of η_{SP} and by the numerical results.

In equation (63) which is the only one controlling the blade shape of finite spacing η_{SP}, there are the two variables η and x_1; it is therefore permissible, for instance, to choose one variable and solve for the other by integration of equation (63). After both of these values have been determined, then the final blade shape is developed as shown schematically in figure 2, and discussed in the paragraph below.

The construction of the airfoil shape is as follows.- For a given value of the abscissa $\frac{x_0 + x_1}{l}$, the corresponding values of $\eta_{\pm}$, following from equation (63), are measured up and down from a pair of fixed equidistant streamlines with the spacing η_{SP}. In general, the values of $\eta_{\pm}$ do not have to be equal to each other; this fact permits a large variation in the design of blade shapes. One other pertinent general fact to be observed is that the values of $\eta_{\pm}$ and $x_0 + x_1$ must be determined with the requirement that the airfoil will be closed at the leading and trailing edges. The boundary conditions for the integral of equation (63) must therefore require that the upper and lower airfoil surfaces intersect at the leading and trailing edges.

There are two simple integrals obtainable from equation (63) of which solutions will now be derived and discussed.

If x_1 for a first possibility is chosen to be equal to zero; that is, if the points on the blade are required not to change their x_0 abscissa, then the solution of equation (63) becomes

$$\eta = \frac{l}{\int_0^{x_0} \frac{f(P_0)}{u_0^2} dx_0 + \text{Constant}} \tag{64}$$

It is seen from equation (64) that the variation of η with x_0 depends upon the integral quantity $\int_0^{x_0} \frac{f(P_0)}{u_0^2} dx_0$ and that this variation will be the same for η_+ as for η_-. In order that the airfoil, derived from $\eta_{\pm}$, will be closed at the leading and trailing edges, it must be required from equation (64) that $\eta_+ + \eta_- = \eta_{SP}$ at the intake and exit of the blade system. This requirement is satisfied at the intake by setting the constant in equation (64) equal to zero. At the exit, however, $\eta_+ = \eta_-$ only if the integral $\int_0^l \frac{f(P_o)}{u_o^2} dx_o$ is zero taken over the limits of $x_o = 0$ to $x_o = l$. In practice, this requirement will be very difficult to accomplish because of the complexity of that integral. However, a number of numerical calculations with a P_o function, chosen symmetrically with respect to the midchord point, have shown that the value of this integral in equation (64), between the limits of $x_o = 0$ to $x_o = l$, is quite small. Hence, the airfoils derived from those calculations can, for all practical purposes, be considered as line airfoils (airfoils of zero thickness). The method to obtain airfoils of finite thickness is discussed later on in this section.

A second possibility of a simple solution of equation (63) is to prescribe $\eta_+ = \eta_{in,+}$ = Constant and $\eta_- = \eta_{in,-}$ = Constant, where η must be one-half of the blade spacing η_{SP}. Then the solution of equation (63), following from $\frac{dx_1}{dx_o} = \eta^2 \frac{f(P_o)}{u_o v_o}$, where $\frac{f(P_o)}{u_o v_o} \equiv f(x_o)$, becomes

$$x_1 = \eta_{in}^2 \int_0^{x_o} \frac{f(P_o)}{u_o v_o} dx_o + \text{Constant} \tag{65}$$

If the dimensionless functions $\frac{x_1}{l} = \xi_1$ and $\frac{x_o}{l} = \xi_o$ are introduced and it is required that the intake (entrance) edge of the blade remain in place, then equation (65) can be written in the following form:

$$\xi_1 = \eta_{in}^2 \int_0^{\xi_o} \frac{f(P_o)}{u_o v_o} d\xi_o \tag{66}$$

Only line airfoils can be derived from equation (65). Furthermore, as stated above, there is a special condition which must be required to obtain these airfoils; namely, that the value of $\eta_{in,\pm}$ must be one-half of the grid spacing η_{SP}, in order that x_1 be the same coming from both the fixed streamlines.

It is possible (and has been done in the following section) to integrate equations (64) and (65) graphically by means of the known functions y_o, u_o, v_o, u_2, and v_2, which follow from the choice of the pressure function $P_o(x_o)$ (equations (11), (22), (23), (57), and (58), respectively). It may be remarked that up to the first and second transition terms the inflow and outflow remain unchanged (u and v are continuous outside the blade system). This is so since $u_1\eta$ and $v_1\eta$ are zero and $u_2{\eta_+}^2 = u_2{\eta_-}^2$ and $v_2{\eta_+}^2 = v_2{\eta_-}^2$ at the entrance and exit. The transition to the series term with $u_2\eta^2$ and $v_2\eta^2$ yields an outside flow without discontinuities, but which is a periodic function of y, with period η_{SP} near $x = 0$ and $x = l$.

The method for airfoils of finite thickness which are closed at the leading and trailing edges must use the differential equation (63) in a more general manner. In this equation both η and x are dependent variables, so that any choice of a function is permitted for one of these variables in order to find the other; or, if it is more convenient, one of the variables can be prescribed as a function of the other (e.g., $x_1 = f(\eta)$). Such a choice, with a sufficient number of constants, will give an airfoil of finite thickness which will be closed at the leading and trailing edges.

AN APPLICATION OF THE GENERAL ANALYSIS

The design of a system of blades, deflecting a ducted uniform flow about a prescribed flow angle, is explained in the following sections.

Computations for an infinitesimally spaced blade system.- A technically important example of a two-dimensional blade system is the case of a system of airfoil vanes deflecting a uniform flow about an angle of prescribed magnitude.

This case, is, for instance, realized by the deflector system in a wind tunnel with a deflection angle of 90°. Such a case will be treated in this second section.

It was shown in the general analysis that the enthalpy function determines the functions of all other variables. In this example an enthalpy function P_o is chosen originally symmetric about the midpoint $x_o = l/2$ of the chord. It follows then from equations (15) and (16) that the velocity function v_o is antisymmetric about the same point $x_o = l/2$. Hence the original streamline shape, following from $\frac{dy_o}{dx_o} = \frac{v_o}{u_o}$, will also have a symmetrical shape about the half-chord point.

The development of the series terms of finite spacing leads to a restriction in the choice of the function P_o. This restriction is necessary to prevent a singularity of the function P_1 at the point $x_o = l/2$ (where in this example $v_o = 0$); namely, that $P_o' = 0$ and $P_o'' = 0$ at this point. This is seen from formula (41) for P_1 and its first derivative given as follows:

$$P_1 = l\,\frac{u_o}{v_o}\left[1 - 2.5\,\frac{{u_{in}}^2}{P_{in}}\left(\frac{P_o}{P_{in}}\right)^{-6}\right]P_o' \tag{41}$$

and

$$\begin{aligned} P_1' = {} & -l\,\frac{u_o v_o'}{{v_o}^2}\left[1 - 2.5\,\frac{{u_{in}}^2}{P_{in}}\left(\frac{P_o}{P_{in}}\right)^{-6.0}\right]P_o' + \\ & l\,\frac{u_o'}{v_o}\left[1 - 2.5\,\frac{{u_{in}}^2}{P_{in}}\left(\frac{P_o}{P_{in}}\right)^{-6.0}\right]P_o' + \\ & \frac{l u_o}{v_o}\left[1 - 2.5\,\frac{{u_{in}}^2}{P_{in}}\left(\frac{P_o}{P_{in}}\right)^{-6.0}\right]P_o'' \end{aligned}$$

It is deduced from an inspection of these two formulas that, in order to avoid infinite values of P_1 and P_1' where $v_o = 0$, it becomes necessary to require that there $P_o' = P_o'' = 0$. This fact is also valid for nonsymmetric streamlines at the point where the streamline tangents are parallel to the x-axis.

A summary of the requirements for the choice of the basic enthalpy function in the continuous force field of infinitisimal spacing is now as follows:

(a) The condition for equal pressure at inflow and exit is

$$P_o = P_{in} \quad \text{at} \quad \xi_o \equiv \frac{x_o}{l} = 0 \quad \text{and at} \quad \xi_o = 1$$

(b) The conditions for equal inflow and outflow (P_1, u_1, and $v_1 = 0$ and the conditions (59)) and for avoidance of a singularity of P_1 at $\xi_o = 0.5$ are

$$P_o' = 0 \quad \text{at} \quad \xi_o = 0,\ \ 0.5,\ \text{and}\ \ 1$$

(c) The condition of preventing a singularity of P_1' at $\xi_o = 0.5$ is

$$P_o'' = 0 \quad \text{at} \quad \xi_o = 0.5 \quad \text{(see explanation above)}$$

The following polynomial satisfies the above conditions for P_o:

$$P_o = P_{in} + P^*f(\xi_o) \tag{67}$$

where

$$f(\xi_o) = \xi_o^2(\xi_o - 1)^2(\xi_o^2 - \xi_o + \frac{3}{8}) \tag{68}$$

The value of P^* is determined by the fact that $v_o = 0$ at the location $\xi_o = 0.5$. This is done in the following manner.

From the energy integral (14) one has, where $v_o = 0$,

$$u_o^2 + 2P_o = u_{in}^2 + v_{in}^2 + 2P_{in}$$

Replacing P_o in this energy relation by its value from equation (67), it follows that

$$u_o^2 + 2P^*f(\xi_o) = u_{in}^2 + v_{in}^2$$

or

$$2P^*f(\xi_o) = \left(u_{in}^2 + v_{in}^2\right) - u_o^2$$

or

$$2P^*f(\xi_o) = \left\{\left[1 + \left(\frac{v_{in}}{u_{in}}\right)^2\right] - \left(\frac{u_o}{u_{in}}\right)^2\right\}u_{in}^2 \tag{69}$$

If now it is required that the flow deflection be rectangular (see fig. 3) then the intake and exit angles must be equal and of value 45°. This requirement is expressed by

$$\left.\begin{aligned} u_{in} &= v_{in} \\ u_{ex} &= v_{ex} \end{aligned}\right\} \tag{70}$$

Equation (69) with these values from equation (70) and $f(\xi_o) = f(0.5)$ then becomes

$$P^* = \frac{u_{in}^2}{2f(0.5)}\left[2 - \left(\frac{u_o}{u_{in}}\right)^2\right] \tag{71}$$

The value of $f(0.5)$ follows from equation (68); namely,

$$f(\xi_o) = f(0.5) = \frac{1}{128} = 7.8 \times 10^{-3}$$

Furthermore, according to equation (22),

$$\frac{u_o}{u_{in}} = \left(\frac{P_o}{P_{in}}\right)^{-2.5} \tag{72}$$

which, if P_o is replaced by its value from equation (67), leads to

$$\left(\frac{u_o}{u_{in}}\right)^2 = \left[1 + \left(7.8 \times 10^{-3}\right)\frac{P^*}{P_{in}}\right]^{-5} \tag{73}$$

The formula desired for P^* now follows by replacing $\left(\frac{u_o}{u_{in}}\right)$ in equation (71) by its value given in equation (73); this formula for P^* is

$$2\left[\left(7.8 \times 10^{-3}\right)\frac{P^*}{u_{in}^2} - 1\right] + \left[1 + \left(7.8 \times 10^{-3}\right)\frac{P^*}{P_{in}}\right]^{-5.0} = 0 \tag{74}$$

Equation (74) is solved graphically with the condition that P^* is positive, since $P_o = \frac{\gamma}{\gamma - 1}\frac{p_o}{l_o}$ must be greater than zero all along a streamline.

The following data are chosen for the numerical calculations:

$$u_{in} = 600 \text{ feet per second} = v_{in}$$

$$P_{in} = \frac{\gamma}{\gamma - 1}\frac{p_{in}}{\rho_{in}} = \frac{\gamma}{\gamma - 1} RgT_{in} = 3.087 \times 10^6 \text{ (ft/sec)}^2$$

$$p_{in} = 2.097 \times 10^3 \text{ pounds per square foot}$$

$$\rho_{in} = 2.378 \times 10^{-3} \text{ slugs per cubic foot}$$

$$\gamma = 1.4$$

$$R = 53.3 \text{ feet per degree}$$

$$T_{in} = 514^\circ \text{ F absolute}$$

With these data, the solution for P^* from equation (74) is found to be

$$P^* = 30.123 \times 10^6 \ (\text{ft/sec})^2$$

so that

$$\frac{P^*}{P_{in}} = 9.758$$

On the basis of this result, together with equations (67) and (4), the enthalpy P_o and the pressure distribution along the streamlines for infinitesimal spacing are calculated and presented in table I. This enthalpy function P_o is plotted in figure 4.

After the enthalpy distribution is known, all other flow variables and the streamline shape for infinitesimal spacing are ready for computation. The necessary formulas for these computations are tabulated in the section entitled "Compilation of expressions for infinitesimal spacing" under General Analysis. Tabulated values of P_o, u_o and v_o, and y_o are given in tables I, II, and III, respectively.

Computation of shape of finitely spaced blades.- After several test computations of blade shapes by means of formulas (64) and (65), it was found to be preferable to use formula (65) or (66); that is, to assume the distance numbers $\eta = \dfrac{y - y_o}{l}$ = Constant. It follows then that all streamlines, except the fixed streamlines, are distorted only in the direction of the (dimensionless) chord abscissa ξ, so that, according to equation (66),

$$\frac{x_1}{l} = \xi_1 = \eta_{in}^{\ 2} \int_0^{\xi_o} \frac{f(P_o)}{u_o v_o} \, d\xi_o \tag{66}$$

By means of equations (22) and (23) and the data chosen above, the functions $u_o = u_o(\xi_o)$ and $v_o = v_o(\xi_o)$ were computed and the results are shown in table II. It is, furthermore, necessary to compute the values of $\frac{f(P_o)}{u_o v_o}$, which is obtained from formula (51). The values of P_1 and u_1 (with the constant equal to zero) are given by formulas (41) and (47).

Table III shows the numerical values of the function $\frac{f(P_o)}{u_o v_o}$, the integrals $\int_0^{\xi_o} \frac{f(P_o)}{u_o v_o} d\xi_o$, and the specific distortion $\xi_1 = \eta_{in}^{\ 2} \int_0^{\xi_o} \frac{f(P_o)}{u_o v_o} d\xi_o$, where $\eta_{in} = \frac{1}{2} \eta_{SP} = 0.1$, so that $\eta_{SP} = 0.2$. The choice of this value of 0.1 means that it is assumed that the distance between the blades is one-fifth of the chord.

The shape of the blade in comparison with the original streamline, as derived for the case of infinitesimal spacing, is shown in figure 3 in the scale 1:1, with the assumption of a chord l equal to 12 inches. It is seen that the chord is increased 1/2 percent corresponding to $\frac{1}{16.67}$ inch and that the form becomes slightly fuller at the intake and flatter at the exit. In general, however, the blade shape is very nearly the same as the streamline shape of infinitesimal spacing.

The entrance and exit angles of flow are slightly changed, as can be seen from equation (60), written as follows:

$$\tan\left(\varphi_o + \epsilon\right) = \tan\left(\varphi_o\right)\left[1 + \eta^2\left(\frac{v_2}{v_o} - \frac{u_2}{u_o}\right)\right] \tag{75}$$

where $\tan \varphi_o = \pm 1$, and $\tan \epsilon$ is the deviation caused by the transition to higher-order terms. If powers of $\tan \epsilon$ higher than 1 are neglected, then it follows from equation (75) that

$$\tan \epsilon = \frac{\tan \varphi_o}{1 + \tan^2\varphi_o} \eta^2\left(\frac{v_2}{v_o} - \frac{u_2}{u_o}\right) = \pm\frac{1}{2} \eta^2\left(\frac{v_2}{v_o} - \frac{u_2}{u_o}\right) \tag{76}$$

The data of the numerical example above furnish the value $\epsilon = \pm 1.3^o$, showing that a very small change results from the required deflection, after the transition was made to finite spacing.

Computation of flow variables for finitely spaced blades.- By means of the velocity and enthalpy formulas given for u_1, u_2, v_1, v_2, and P_1 and P_2 (equations (47), (57), (48), (58), and (41) and (49), respectively), the values of the flow variables u, v, P, and p for the finitely spaced blade system are found. The results of the calculations, based on the above formulas, are presented in table IV. In figure 5 the resultant velocity-ratio distributions are presented along the suction (upper) and pressure (lower) surfaces of the blades; namely,

$$\frac{V_R}{V_{R,in}} = \left(\frac{u^2 + v^2}{u_{in}^2 + v_{in}^2}\right)^{1/2} = \left[\frac{\left(u_o + u_1\eta + u_2\eta^2\right)^2 + \left(v_o + v_1\eta + v_2\eta^2\right)^2}{u_{in}^2 + v_{in}^2}\right]^{1/2}$$

where η is positive for the pressure surface and negative for the suction surface. The resultant velocity curves in figure 5 are plotted against $x = x_o + x_1$, and the curves are seen to be very nearly symmetrical with respect to the chord midpoint. It is also observed from figure 5 that the resultant velocities for the intake and exit stations, on both surfaces, are equal to each other; this equality satisfies the previously discussed conditions of assuring continuous flow into and out of the blade system.

The values of resultant velocities along the suction surface, on the forward and rearward 25-percent-chord portions of the blades, become greater than the intake velocity, while along the middle 50 percent of this surface the velocities are lower than the intake value. The maximum and minimum velocities on this suction surface are approximately 10 percent greater and 38 percent lower, respectively, than the intake velocity. Along the pressure surface, the velocity variation, also almost symmetrical about the midchord point, has values which are always lower than the intake value; the minimum value is approximately 45 percent lower than the intake velocity.

The distribution of the pressure ratios p/p_{in} for the suction and pressure surfaces was computed from the calculated enthalpy values P, given in table IV. In terms of the enthalpy for finite spacing, the pressure relation (20) becomes

$$\frac{p}{p_{in}} = \left(\frac{P}{P_{in}}\right)^{3.5} = \left(\frac{P_o + P_1\eta + P_2\eta^2}{P_{in}}\right)^{3.5}$$

Values of p/p_{in} are plotted against $x_o + x_1$ in figure 3 perpendicular to the airfoil surface. It may be noted here that the pressure curves on the suction (convex) and pressure (concave) surfaces are symmetrical with respect to the midchord point. Furthermore, the pressure jump from the upper to the lower surface is small, so that the resulting airfoil lift coefficient will also be small. In the direction of flow, approximately 30 percent of the convex surface has a small pressure decrease, while the variation along the concave surface is everywhere greater than the ambient value. By a different assumption of the original pressure function, however, it is possible to attain a much greater region of suction pressure on the convex surface.

It is of interest to note that the original assumption of an enthalpy function P_o which is symmetrical about the midchord point leads for finite spacing to very nearly symmetrical variations, not only for the blade shape, but also for the flow variables.

In order to judge the convergence of the series for the velocity and enthalpy (pressure) variations with ξ_o, table IV is presented. From this table it is seen that the convergence is everywhere good, with the exception of the values of $v_2\eta^2 = -23.76$ feet per second against $v_1\eta = -7.0$ feet per second for $\xi_o = 0.4$ and 0.6. The reason for the poor convergence at this point seems to be related to the assumed enthalpy variation of infinitesimal spacing and could have been avoided by a further refinement in the blade design.

CONCLUDING REMARKS

A method of obtaining two-dimensional systems of equidistant blades has been presented. Such blade systems will produce a prescribed deflection in an originally uniform stream of a compressible nonviscous fluid. The essential features of the method are that it is assumed, in a first step, that the blade system is replaceable by a system of infinitesimally spaced blades, while, in a second step, the transition is made to finitely spaced blades.

The analysis shows that two types of finitely spaced airfoils can be derived, namely, line airfoils (zero thickness) or airfoils of finite thickness. The actual choice of one or the other type depends upon a solution which is multivalued because of two dependent variables in an ordinary differential equation (of only one independent variable). The technical requirements of the problem must also be considered for a decision.

A numerical example is given of the procedure for the design of blades, deflecting a uniform nonviscous ducted flow (Mach number equal to 0.72) about a prescribed angle of 90°. The example can be interpreted as the theory of a system of guide vanes in the corner of a wind tunnel.

It is found that the desired deflection can be achieved by a system of line (zero-thickness) airfoils which are nearly symmetrical about the chord midpoint. The shape of the line airfoils for finitely spaced blades is, for this example, found to be very nearly the same as the shape of the streamlines of infinitesimal spacing. The pressure jump across a line airfoil in the system is found to be small, so that the resulting airfoil lift coefficient would also be small. In the direction of flow, approximately 30 percent of the upper surface has a pressure suction, while the pressure variation along the lower surface is everywhere greater than the ambient value. With a different assumption, however, of the initial pressure function, it would be possible to have on the upper airfoil surface, if desired, a much greater region of suction pressure.

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APPENDIX

TWO-DIMENSIONAL, VISCOUS, LAMINAR, STEADY COMPRESSIBLE FLOW THROUGH A SYSTEM OF EQUIDISTANT NARROWLY SPACED BLADES

Viscous flow through a grid system of equidistant, narrowly spaced blades is treated in this appendix by the introduction of a velocity, velocity gradient, pressure, and force field of uniformity across the blades. This is accomplished by integration of the continuity and the Navier-Stokes equations in the direction y across the blades, in order to establish the uniformity of the field in this direction, but retaining the variation in the direction x perpendicular to y. In this manner the actually concentrated boundary forces of the blades are transformed to a uniform force field. From the thermodynamic energy equation, applied to the heat produced by viscosity, an integral is derived for the pressure function in terms of a freely choosable streamline (blade) curve and one of the velocity components.

The method of treatment applied in this appendix is a generalization of the method applied in the body of the paper and in a preceding paper (reference 1) for isentropic flow.

An extension for viscous flow of the method was necessary because of the difficulty of the boundary conditions of zero velocity and of the more complicated problem of transformation of mechanical into heat energy. The extension consists of the following points:

(a) An explicit method of averages in the direction y connecting corresponding points of the blade system

(b) A determination of the heat generated by viscosity and from it the derivation of the pressure function

(c) A free choice of the streamline (blade) equation and of one of the velocity-component functions

The blade system considered is shown in figures 6 and 7, arranged in line and staggered, respectively.

Analysis

The analysis will be based on the following assumptions.

(1) The actual pressure and velocity field is replaced by a field of average pressure, average velocity, and average velocity gradient which is uniform in the y-direction but variable in the x-direction.

(2) The original velocity distribution across the blade pairs is assumed of parabolic character analogous to the Poiseuille flow,[1] while the original pressure and density distribution is assumed to be of little change across and nearly linear. These assumptions for the average equations give certain numerical factors of an appropriate order of magnitude, but are open for any change by refinement of the theory or comparison with experience.[1]

(3) The set of concentrated force layers of the blade surfaces is replaced by a force field uniform in the direction y (across the blades), which represents in the average the action of the blades.

(4) The values of the coefficients μ of viscosity and of $\nu = \frac{\mu}{\rho}$ of kinematic viscosity will, in view of the slight changes of temperature, be considered as constant.

(5) The conduction of heat in view of the high velocity of flow will be neglected.

The following abbreviations are used:

$$\frac{\partial}{\partial x} \equiv {}'$$

$$\frac{\partial}{\partial y} \equiv {}^{\cdot}$$

$$\int_{y_o}^{y_o+h} \frac{(\ )dy}{h} \equiv (\bar{\ }) \equiv (\text{Average})$$

where $(\bar{\ })$ denotes an average value and h the spacing distance in the y-direction.

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[1]One important refinement should work with the experimental fact that, between narrowly spaced walls, the boundary layer is thin at the inflow and becomes parabolic toward the exit.

The continuity equation and the Navier-Stokes equations for viscous two-dimensional flow between a pair of blades may be at first stated in their complete forms; namely,

$$(\rho u)' + (\rho v)^{\bullet} = 0 \tag{A1}$$

$$uu' + v\dot{u} + P' - \frac{1}{3}\nu(u' + \dot{v})' - \nu(u'' + \ddot{u}) = 0 \tag{A2}$$

$$uv' + v\dot{v} + P^{\bullet} - \frac{1}{3}\nu(u' + \dot{v})^{\bullet} - \nu(v'' + \ddot{v}) = 0 \tag{A3}$$

The continuity equation (A1), to be averaged along the y-axis at any fixed cross section x, must be transformed as follows:

$$\int_0^1 (\rho u)' d\eta + \int_0^1 (\rho v)^{\bullet} d\eta = 0$$

where $\eta = \frac{y - y_0}{h}$, $d\eta = h^{-1}\,dy$, and h is the distance between blades.

The second integral is zero because the velocity v is zero at the limits because of viscosity.

The result is therefore

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\int_0^1 (\rho u)\, d\eta = 0 \\ &\overline{\rho u} = \text{Constant} = m \end{aligned}\right\} \tag{A4}$$

The method of averages in the y-direction for the terms in the dynamic equations (A2) and (A3) is done as follows.

It is assumed that the distribution of the velocity components u and v in all cross sections along the y-axis is, as said before, of parobolic character in order to satisfy the condition of zero velocity at the limits (analogously to the Poiseuille flow). This is expressed by

$$\left.\begin{array}{ll}
u = 4u_{max}\eta(1 - \eta) & v = 4v_{max}\eta(1 - \eta) \\[2ex]
\bar{u} = \frac{2}{3}\, u_{max} & \bar{v} = \frac{2}{3}\, v_{max} \\[2ex]
\overline{u'} = \frac{2}{3}\, u_{max}' & \overline{v'} = \frac{2}{3}\, v_{max}' \\[2ex]
\overline{\dot{u}} = 0 & \overline{\dot{v}} = 0 \\[2ex]
\overline{u''} = \frac{2}{3}\, u_{max}'' & \overline{v''} = \frac{2}{3}\, v_{max}'' \\[2ex]
\overline{\ddot{u}} = -8u_{max}h^{-2} = -12\bar{u}h^{-2} & \overline{\ddot{v}} = -12\bar{v}h^{-2} \\[2ex]
\overline{uu'} = \frac{6}{5}\,\bar{u}\,\overline{u'} & \overline{vv'} = \frac{6}{5}\,\bar{v}\,\overline{v'} \\[2ex]
\overline{uv'} = \frac{6}{5}\,\bar{u}\,\overline{v'} & \overline{vu'} = \frac{6}{5}\,\bar{v}\,\overline{u'} \\[2ex]
\overline{v\dot{u}} = 0 & \overline{v\dot{v}} = 0 \\[2ex]
\overline{\dot{u}'} = 0 & \overline{\dot{v}'} = 0
\end{array}\right\} \tag{A5}$$

In the dynamic equations (A2) and (A3) the forces acting on the flow do not appear explicitly, but only indirectly in the boundary conditions. In the averaged equations these concentrated boundary forces of a pair of blades will be uniformly spread over the distance h between the blades. Their values (functions of x) are finally found by the values of the variables $\bar{u}$, $\bar{v}$, and $\bar{P}$.

If now these forces (called k_x and k_y) and the expressions (A5) above are inserted in the dynamic equations, they assume the following form:

$$\frac{3}{5}\left(\bar{u}^2\right)' + \overline{P}' - \nu\,\frac{4}{3}\,\bar{u}'' - 12h^{-2}\bar{u} = k_x \tag{A6}$$

$$\frac{6}{5}\,\bar{u}\,\overline{v'} + \overline{\dot{P}} - \nu\left(\bar{v}'' - \frac{4}{3}\,12h^{-2}\bar{v}\right) = k_y \tag{A7}$$

The Bernoulli equation is obtained if equation (A6) is multiplied by $\overline{u}$ and equation (A7) by $\overline{v}$ and added. Moreover it will be observed that

$$\overline{u}\ \overline{u}'' = \left(\frac{\overline{u}^2}{2}\right)'' - (\overline{u}')^2$$

$$\overline{v}\ \overline{v}'' = \left(\frac{\overline{v}^2}{2}\right)'' - (\overline{v}')^2$$

according to relations (A5),

$$\left(\frac{2}{3}\,\overline{u}^2 + \frac{1}{2}\,\overline{v}^2\right)'' = 0$$

and, following from symmetry,

$$\overline{P}' = \int_0^1 \frac{\partial P}{\partial \eta}\,dy = P_1 - P_o = 0$$

Then the result is

$$\frac{D}{Dt}\left[\frac{3}{5}(\overline{u}^2 + \overline{v}^2) + \overline{P} - \nu\left(\frac{2}{3}\overline{u}^2 + \frac{1}{2}\overline{v}^2\right)'\right] +$$
$$\nu\left[\frac{4}{3}(\overline{u}')^2 + (\overline{v}')^2 + 12h^{-2}\left(\overline{u}^2 + \frac{4}{3}\overline{v}^2\right)\right] = k_x\overline{u} + k_y\overline{v} \qquad \text{(A8)}$$

The first term on the left side of equation (A8) represents the total mechanical energy (per unit of mass) which travels with the flow without change. The second term represents the loss of mechanical energy that is the amount of heat energy (per unit of mass) generated by the viscous stresses and also traveling with the steady flow. This substantial derivative denoted by $\frac{DQ}{Dt}$ will now be expressed by the rise of temperature T and the work of pressure $P\,\frac{DV}{Dt}$, where $V = \rho^{-1}$. The equation following from this relation is given by

$$\frac{DQ}{Dt} - p\frac{D\rho^{-1}}{Dt} = c_v\frac{DT}{Dt} \qquad \text{(A9)}$$

This equation signifies that the heat generation plus the work of the pressure $\left(-p\frac{D\rho^{-1}}{Dt}\text{ is positive}\right)$ raises the temperature, a work to which also the mechanical energy contributes, without however generating heat.

The development of the equation above leads to

$$\nu\left[\frac{4}{3}(u')^2 + (v')^2 + 12h^{-2}\left(u^2 + \frac{4}{3}v^2\right)\right] = u\left(\frac{p}{\rho}\right)'(\gamma - 1) + up(\rho^{-1})' \qquad \text{(A10)}$$

where, for the sake of simplicity, the average bars are left out. This will also be done in the next equations.

The average density $\bar{\rho}$ must now be expressed in terms of the average velocity $\bar{u}$ by means of the continuity equation

$$\overline{\rho u} = m = \text{Constant} \qquad \text{(A11)}$$

A certain difficulty must be overcome to apply this equation, since there is in principle a difference between $\overline{\rho u}$ and $\bar{\rho}\,\bar{u}$. However one can show that the difference is very small if ρ varies little from one blade to the other.

Assume, for instance,

$$\rho = \rho_{max} - \rho_1\, 4\eta(1 - \eta), \quad \rho_1 \ll \rho_{max}$$

$$u = u_{max} 4\eta(1 - \eta)$$

Then,

$$\overline{\rho u} = \int_0^1 d\eta \left[\rho_{max} - \rho_1 4\eta(1 - \eta)\right]\left[u_{max} 4\eta(1 - \eta)\right]$$

$$= \frac{2}{3}\, u_{max}\left(\rho_{max} - \rho_1 \frac{4}{5}\right)$$

$$\bar{\rho}\,\bar{u} = \left(\rho_{max} - \rho_1 \frac{2}{3}\right) u_{max} \frac{2}{3}$$

The difference consists only in the factors 0.8 and 0.667 of the small value ρ_1.

It seems therefore permissible to derive from equation (A11)

$$\overline{\rho} = \frac{m}{\overline{u}}$$

With this relation equation (A10) becomes

$$p' + p\gamma(u^2/2)'u^{-2} = m(\gamma - 1)u^{-2}\,\nu\left[\frac{4}{3}(u')^2 + (v')^2 + 12H^{-2}\left(u^2 + \frac{4}{3}v^2\right)\right] \quad \text{(A12)}$$

This differential equation does not present any difficulties, particularly not after the functions u and v of x are chosen.

They can be freely chosen, since the four equations (equations (A1), (A2), (A3), and (A12)) have the six variables ρ, u, v, p, k_x, and k_y. The following simple example starting from a prescribed streamline surface may be given as follows.

The streamline surface in figure 8 is defined by the equation

$$y_o/H = \frac{297}{14}\left[\xi^3 - 3\xi^2 + 2\xi - \frac{9}{22}\left(\xi^4 - 5\xi^2 + 4\xi\right)\right] \quad \text{(A13)}$$

where y_o is the ordinate function of the curve, $H = y_{o,max}$, $\xi = x/l$, and l is the length of the chord (see fig. 8). It is correct as stated above to choose either the function $v(x)$ or $u(x)$.

A practical assumption is

$$u = \text{Constant} = u_{in} \quad \text{(A14)}$$

and with it there follows

$$v = u_{in}\frac{dy_o}{dx} = u_{in}\frac{d\left(y_o/H\right)}{d(x/l)}\frac{H}{l}, \qquad \xi = x/l \quad \text{(A15)}$$

and consequently

$$v_{in} = u_{in}\left[\frac{d\left(y_o/H\right)}{d\xi}\right]_{\xi=0}\frac{H}{l} = u_{in}\frac{54}{7}\frac{H}{l} \quad \text{(A16)}$$

where the last term is computed by means of equation (A13). Dividing equation (A15) by equation (A16) one obtains

$$v = \frac{d(y_o H)}{d\xi} v_{in} \frac{7}{54}$$

and, in consequence,

$$v = v_{in} \frac{11}{4} \left[3\xi^2 - 6\xi + 2 - \frac{9}{22}\left(4\xi^3 - 10\xi + 4\right)\right] \tag{A17}$$

This example shows that the velocities, the force field $\overline{k}$, and the pressure function can be derived from a prescribed shape of the streamline surface.

With the values of u and v from equations (A14) and (A17) of this example, the differential equation (A12) can be solved by a simple quadrature, in order to obtain $\overline{p}$ as a function of x.

REFERENCE

1. Reissner, H. J., and Meyerhoff, L.: Analysis of an Axial Compressor Stage with Infinitesimal and Finite Blade Spacing. NACA TN 2493, 1951.

TABLE I. - DISTRIBUTION OF ENTHALPY AND PRESSURE

FOR INFINITESIMALLY SPACED BLADES

$$\left[\frac{P_o}{P_{in}} = 1 + \frac{P^*}{P_{in}} f(\xi_o);\ \text{see equation (68)}\right]$$

ξ_o	$\frac{P^*}{P_{in}} f(\xi_o)$	$\frac{P_o}{P_{in}}$	$\frac{p_o}{p_{in}}$
0	0	1.0	1.0
.1	.022526	1.0225	1.0810
.2	.053708	1.0537	1.2009
.3	.071004	1.0710	1.2713
.4	.075878	1.0759	1.2918
.5	.076235	1.0762	1.2931
.6	.075878	1.0759	1.2918
.7	.071004	1.0710	1.2713
.8	.053708	1.0537	1.2009
.9	.022526	1.0225	1.0810
1.0	0	1.0	1.0

TABLE II. - DISTRIBUTION OF VELOCITY COMPONENTS

FOR INFINITESIMALLY SPACED BLADES

ξ_o	u_o (ft/sec)	v_o (ft/sec)
0.0	600	600
.1	567.53	568.83
.2	526.45	333.54
.3	505.45	161.74
.4	499.72	40.988
.5	499.36	0
.6	499.72	-40.988
.7	505.45	-161.74
.8	526.45	-333.54
.9	567.53	-568.83
1.0	600.00	-600.00

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TABLE III. - SHAPE OF STREAMLINES y_o OF INFINITESIMALLY SPACED BLADES AND NONDIMENSIONAL SHIFT ξ_1 OF BLADE CHORD DUE TO TRANSITION TO FINITE SPACING

ξ_o	$\frac{f(P_o)}{l^2 u_o v_o}$	$\frac{1}{l^2}\int_0^{\xi_o} \frac{f(P_o)}{u_o v_o}\,d\xi_o$	$\xi_1 = \frac{\eta i n^2}{l}\int_0^{\xi_o} \frac{f(P_o)}{u_o v_o}\,d\xi_o$	y_o (in.)
0	-4.6497	0	0	0
.1	-2.6129	-.34503	-.00345	1.138
.2	-3.0214	-.68805	-.00688	2.056
.3	7.7160	-.75492	-.00755	2.628
.4	66.636	.08899	.000890	2.867
.5		.28157	.00282	2.915
.6	66.636	.47414	.00474	2.867
.7	7.7160	1.3181	.0132	2.628
.8	-3.0214	1.2512	.0125	2.056
.9	-2.6129	.9082	.00908	1.138
1.0	-4.6497	.56314	.00563	0

TABLE IV. - DISTRIBUTION OF VELOCITY COMPONENTS, ENTHALPY, AND PRESSURE FOR UPPER AND LOWER SURFACES OF FINITELY SPACED BLADES

(1)	(2)	(3)	(4)	(5)	(6)
ξ_o	u_o	$\pm u_1\eta$ (a)	$u_2\eta^2$	u_{upper}	u_{lower}
0	600	0	-47.41	552.6	552.6
.1	567.5	±84.49	-1.16	650.8	481.9
.2	526.5	±131.5	28.98	687.0	424.0
.3	505.5	±137.4	60.27	703.2	428.4
.4	499.7	±85.35	43.31	628.4	457.7
.5	499.4	0	0	499.4	499.4
.6	499.7	±85.35	43.31	628.4	457.7
.7	505.5	±137.4	60.27	703.2	428.4
.8	526.5	±131.5	28.98	687.0	424.0
.9	567.5	±84.49	-1.16	650.8	481.9
1.0	600	0	-47.41	552.6	552.6

(1)	(2)	(3)	(4)	(5)	(6)
ξ_o	v_o	$v_1\eta$ (a)	$v_2\eta^2$	v_{upper}	v_{lower}
0	600	0	-19.51	580.5	580.5
.1	508.8	±75.75	12.25	596.8	445.3
.2	333.5	±83.29	28.84	445.6	279.1
.3	161.7	±43.97	6.80	212.5	124.5
.4	40.99	±7.00	-23.76	24.2	10.2
.5	0	0	0	0	0
.6	-40.99	∓7.00	23.76	-24.2	10.2
.7	-161.7	∓43.97	-6.80	-212.5	124.5
.8	-333.5	∓83.29	-28.84	-445.6	279.1
.9	-508.8	∓75.75	-12.24	-596.8	445.3
1.0	-600	0	19.51	-580.5	580.5

[a]The ± sign indicates the sign to be used for the upper and lower surface, respectively. The value of η is -0.1 for upper surface and 0.1 for lower surface. See figure 2.

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TABLE IV. - DISTRIBUTION OF VELOCITY COMPONENTS, ENTHALPY, AND PRESSURE FOR UPPER AND LOWER SURFACES OF FINITELY SPACED BLADES - CONCLUDED

ξ_o	P_o	$\pm P_1\eta$ (a)	$P_2\eta^2$	P_{upper}	P_{lower}
0	3.087×10^{-6}	0×10^{-6}	0.040×10^{-6}	3.127×10^{-6}	3.127×10^{-6}
.1	3.157	.0865	-.0043	3.066	3.239
.2	3.253	.0970	-.0214	3.135	3.329
.3	3.306	.0766	-.0277	3.202	3.355
.4	3.321	.0429	-.0192	3.259	3.344
.5	3.322	0	0	3.322	3.322
.6	3.321	.0429	-.0192	3.259	3.344
.7	3.306	.0766	-.0277	3.202	3.355
.8	3.253	.0970	-.0214	3.135	3.329
.9	3.157	.0865	-.0043	3.066	3.239
1.0	3.087	0	.0401	3.127	3.127

ξ_o	$\left(\frac{p}{p_{in}}\right)_{upper}$	$\left(\frac{p}{p_{in}}\right)_{lower}$
0	1.0	1.0
.1	.9336	1.128
.2	1.008	1.245
.3	1.086	1.279
.4	1.155	1.266
.5	1.236	1.236
.6	1.155	1.266
.7	1.086	1.279
.8	1.008	1.245
.9	.9336	1.128
1.0	1.0	1.0

[a]The ± sign indicates the sign to be used for the upper and lower surface, respectively. The value of η is -0.1 for upper surface and 0.1 for lower surface. See figure 2.

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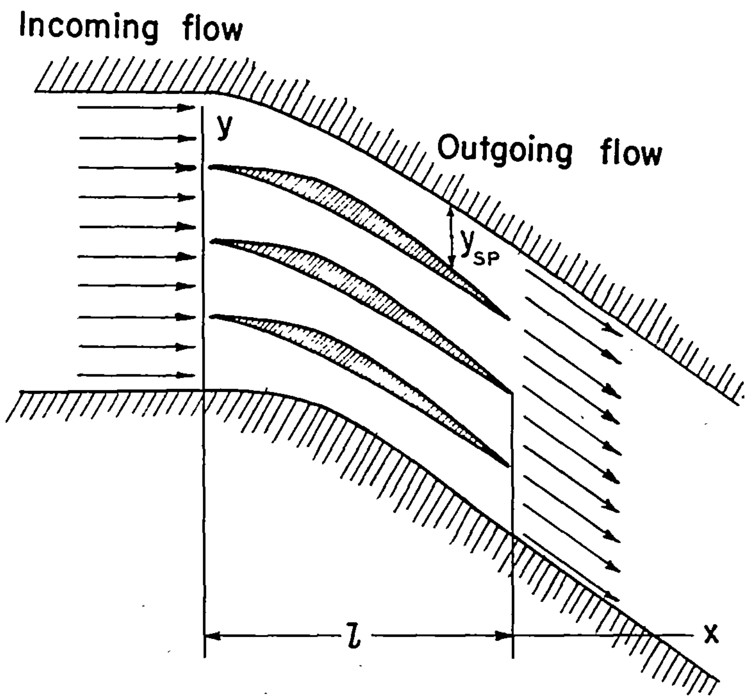
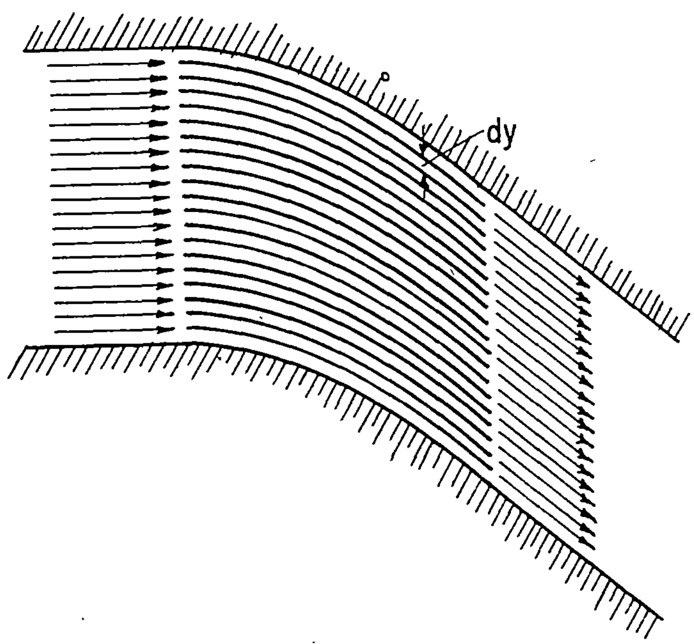


(a) Two-dimensional flow through a finitely spaced deflecting blade system.



(b) System of infinitesimally spaced streamlines replacing in a first approximation the flow shown in sketch above.

Figure 1.- Infinitesimally and finitely spaced blade systems.

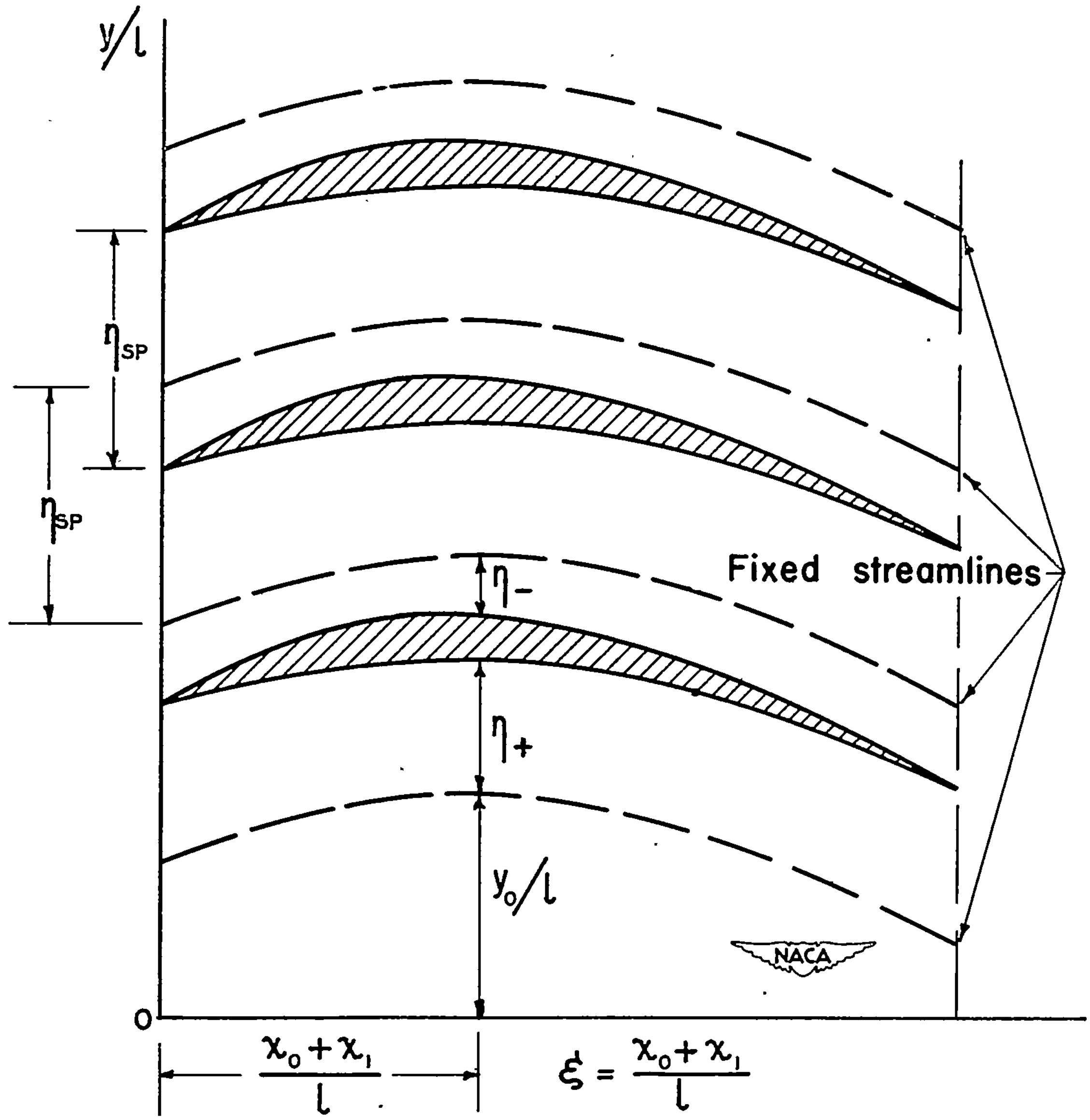


Figure 2.- General representation of blade surfaces for finite spacing.

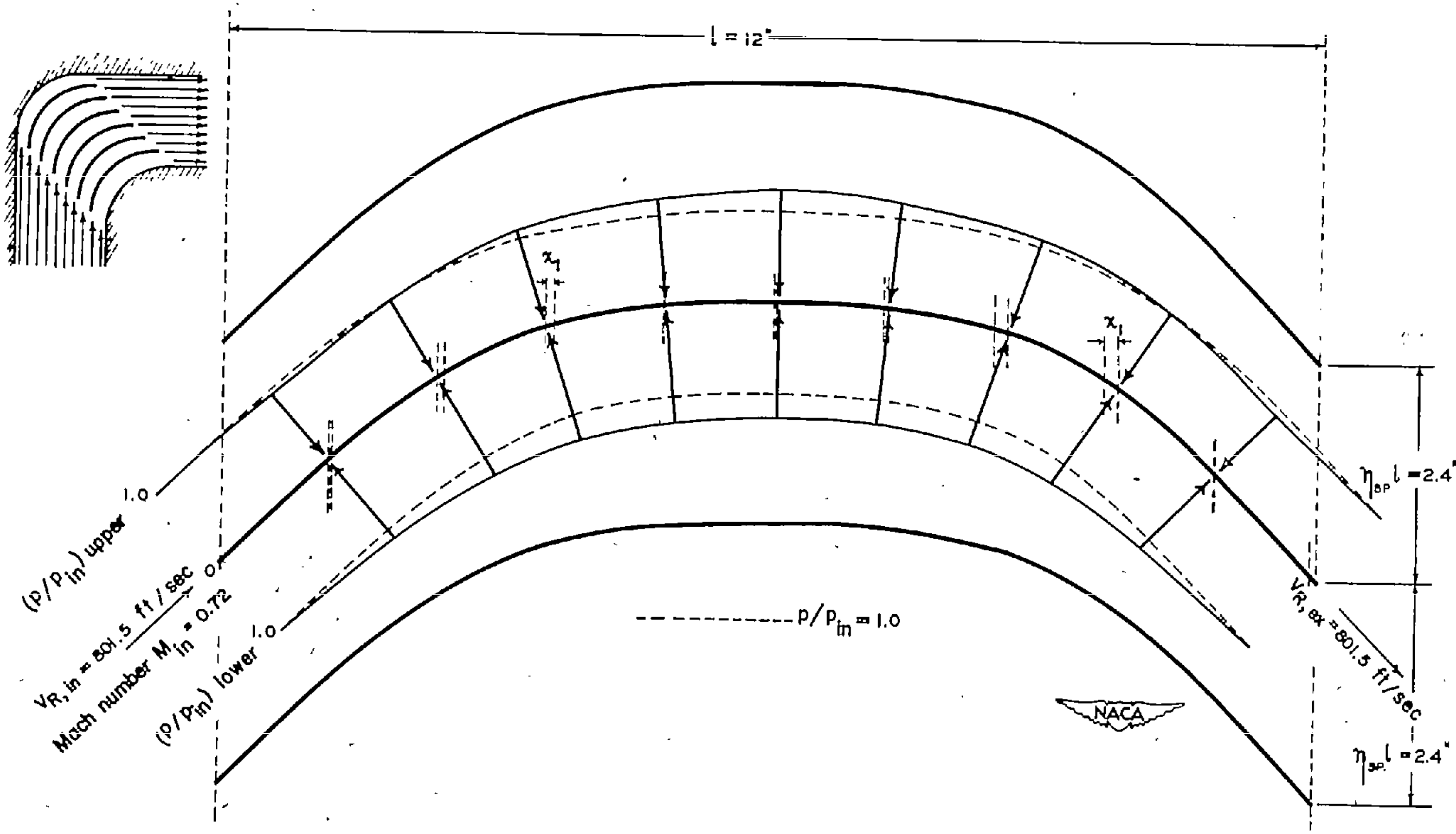


Figure 3.- Pressure distribution on a 90° deflector blade in a compressible fluid. $V_{R,in}$, velocity at airfoil leading edge; x_1, deformation shift. Pressure at airfoil leading edge P_{in}, 2.307×10^3; undisturbed free stream pressure, 2.097×10^3; undisturbed free stream velocity, 800 feet per second.

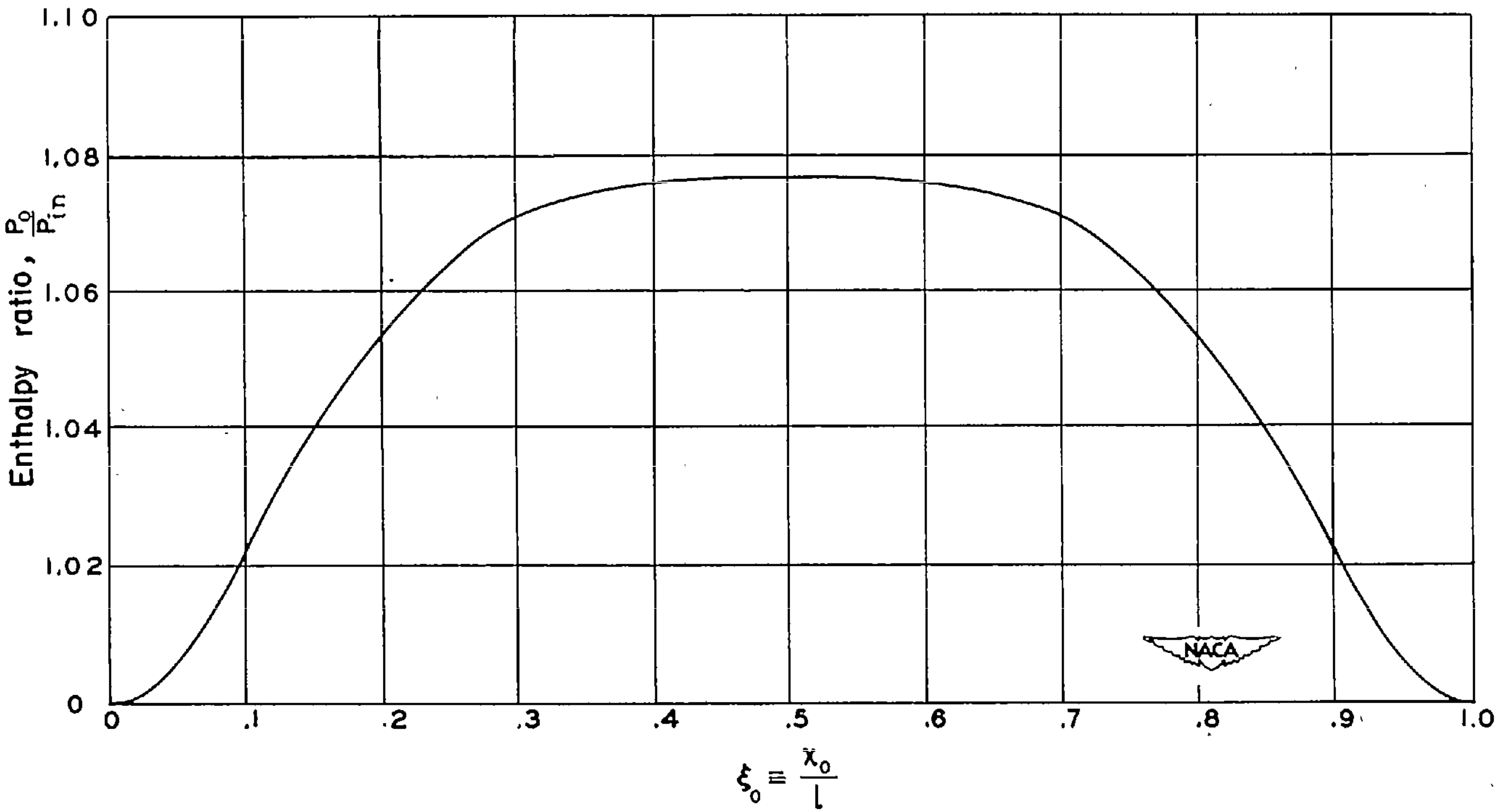


Figure 4.- Enthalpy-ratio distribution along a streamline for infinitesimal spacing. $P_{in} = \gamma \frac{p_{in}}{\rho_{in}} = 3.087 \times 10^6 \text{ (ft/sec)}^2$; $\frac{p_o}{p_{in}} = \left(\frac{P_o}{P_{in}}\right)^{3.5}$.

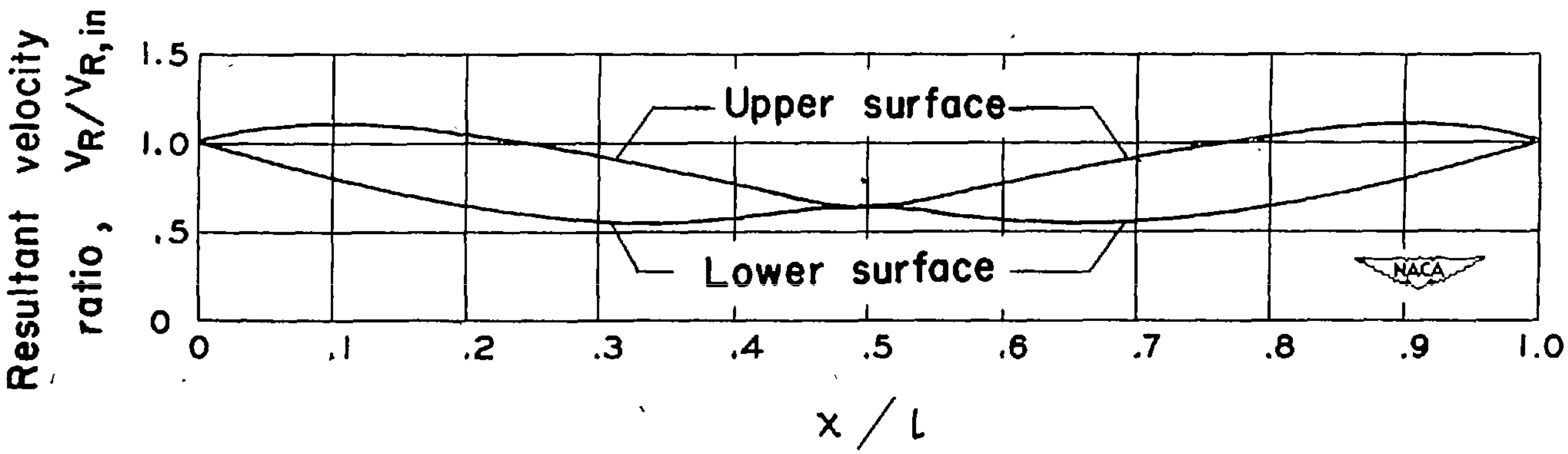


Figure 5.- Distribution of resultant velocity on a 90° deflector blade in a compressible fluid.

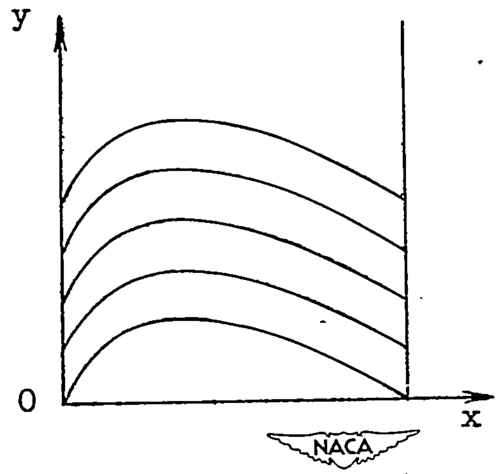


Figure 6.- Blade system arranged in line.

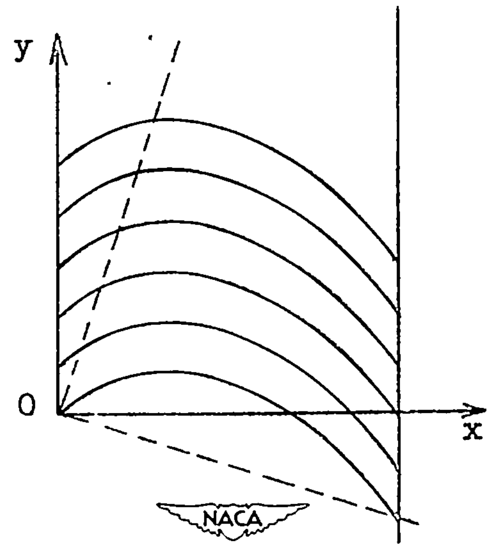


Figure 7.- Blade system staggered.

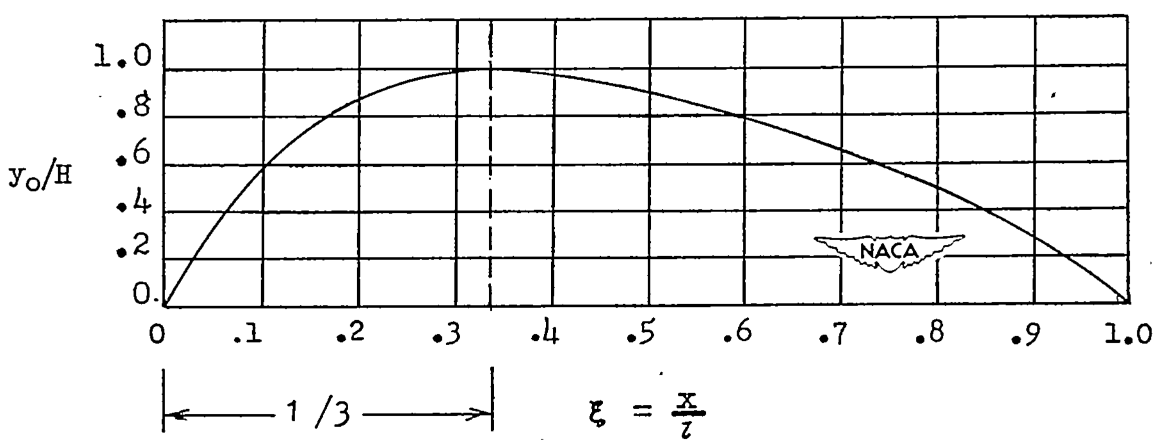


Figure 8.- Plot of y_o/H against ξ.

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