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ON THE APPLICATION OF TRANSONIC SIMILARITY RULES

By John R. Spreiter

Ames Aeronautical Laboratory
Moffett Field, Calif.

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SUMMARY

The transonic aerodynamic characteristics of wings of finite span are discussed from the point of view of a unified small perturbation theory for subsonic, transonic, and supersonic flows about thin wings. This approach avoids certain ambiguities which appear if one studies transonic flows by means of equations derived under the more restrictive assumption that the local velocities are everywhere close to sonic velocity. The relation between the two methods of analysis of transonic flow is examined, the similarity rules and known solutions of transonic flow theory are reviewed, and the asymptotic behavior of the lift, drag, and pitching-moment characteristics of wings of large and small aspect ratio is discussed. It is shown that certain methods of data presentation are advantageous for the effective display of these characteristics.

INTRODUCTION

The small perturbation potential theory of transonic flow proposed apparently independently by Oswatitsch and Wieghardt, Busemann and Guderley, von Kármán (references 1 through 6), and others is now supplying a rapidly increasing fund of information regarding transonic flow about aerodynamic shapes. Solutions have been given recently for the flow around symmetrical nonlifting airfoils at both subsonic and supersonic speeds in papers by Guderley and Yoshihara, Vincenti and Wagoner, Cole, Trilling, Oswatitsch, Gullstrand (references 7 through 14), and others. In the application of these results to specific examples, two items of theoretical interest have been noted (see, in particular, references 8, 15, and 16): (a) The theoretical results appear to be applicable at Mach numbers far removed from 1 even though, in most cases, the results have been obtained from equations valid only in the immediate neighborhood of sonic speed. (b) In the application of theoretical results to specific examples at Mach numbers other than 1, it has been noted that certain ambiguities exist in the theoretical determination of aerodynamic quantities. It is one of the purposes of this report to investigate these two points. This is accomplished by examining transonic flow from the point of view of equations that are valid throughout the Mach number range rather than only in the neighborhood of sonic speed. Such an approach emphasizes the relation between the roles of linear theory and of nonlinear theory in the transonic range.

The similarity rules provided by the theory (references 5, 6, and 17 through 20) have also proved to be useful in the correlation and interpretation of experimental data. It is with the latter aspect of the transonic-flow problem that the present paper is primarily concerned. In this paper, the similarity rules and their application to the specific problem of concise presentation of lift, drag, and pitching-moment characteristics of wings are given in detail. The known solutions of two-dimensional transonic flow are reviewed and the asymptotic behavior of the aerodynamic characteristics of wings of large and small aspect ratios is examined. It is shown that certain methods of data presentation are advantageous for displaying these characteristics.

SYMBOLS

A	aspect ratio
$\tilde{A}$	$[(\gamma+1)(t/c)]^{1/3}A$
a	speed of sound
a_o	speed of sound in the free stream
a^*	critical speed of sound
b	wing semispan
C_D	drag coefficient
C_{D_o}	drag coefficient of symmetrical nonlifting wings
ΔC_D	contribution to drag coefficient due to lift
$\widetilde{(\Delta C_D/\alpha^2)}$	$[(\gamma+1)(t/c)]^{1/3}[\Delta C_D/\alpha^2]$
C_L	lift coefficient
$\widetilde{(C_L/\alpha)}$	$[(\gamma+1)(t/c)]^{1/3}[C_L/\alpha]$
C_m	pitching-moment coefficient
$\widetilde{(C_m/\alpha)}$	$[(\gamma+1)(t/c)]^{1/3}[C_m/\alpha]$
C_p	pressure coefficient

C.P.	center-of-pressure function
c	wing chord
c_{d_o}	section drag coefficient of symmetrical nonlifting airfoils
$\widetilde{c_{d_o}}$	$[(\gamma+1)^{1/3}/(t/c)^{5/3}]\ c_{d_o}$
Δc_d	contribution to section drag coefficient due to lift
$(\widetilde{\Delta c_d/\alpha^2})$	$[(\gamma+1)(t/c)]^{1/3}[\Delta c_d/\alpha^2]$
c_l	section lift coefficient
$(\widetilde{c_l/\alpha})$	$[(\gamma+1)(t/c)]^{1/3}[c_l/\alpha]$
D	drag function
D_o	drag function for symmetrical nonlifting wings
D_Δ	drag due to lift function
d_o	section drag function for symmetrical nonlifting airfoils
d_Δ	section drag due to lift function
L	lift function
l	section lift function
M	pitching-moment function
m	section pitching-moment function
M_o	free-stream Mach number
P	pressure function
s	stretching factors defined in equation (B7)
t	maximum thickness of wing
U_o	free-stream velocity
x,y,z	Cartesian coordinates where x extends in the direction of the free-stream velocity
$x_{c.p.}$	distance from wing leading edge to center of pressure
(Z/c)	ordinates of wing profiles in fractions of chord

α angle of attack

$\tilde{\alpha}$ $\alpha/(t/c)$

Γ $\gamma + 1$

γ ratio of specific heats, for air $\gamma = 1.4$

λ arbitrary constant

ξ_o $(M_o^2-1)/[(\gamma+1)(t/c)]^{2/3}$

τ ordinate-amplitude parameter

Φ velocity potential

φ perturbation velocity potential

Subscripts

l values given by linear theory

W conditions at the wing surface

FUNDAMENTAL CONCEPTS

Basic Equations

The quasi-linear partial differential equation satisfied by the velocity potential Φ of steady isentropic flow of a perfect inviscid gas can be expressed in the form

$$\left(1 - \frac{\Phi_x^2}{a^2}\right) \Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right) \Phi_{yy} + \left(1 - \frac{\Phi_z^2}{a^2}\right) \Phi_{zz} - 2 \frac{\Phi_x\Phi_y}{a^2} \Phi_{xy} -$$

$$2 \frac{\Phi_y\Phi_z}{a^2} \Phi_{yz} - 2 \frac{\Phi_z\Phi_x}{a^2} \Phi_{zx} = 0 \tag{1}$$

where the subscript notation is used to indicate differentiation and a is the local speed of sound given by the relation

$$a^2 = a_o^2 - \frac{\gamma-1}{2} \left(\Phi_x^2 + \Phi_y^2 + \Phi_z^2 - U_o^2\right) \tag{2}$$

In this latter equation U_o and a_o are, respectively, the velocity and speed of sound in the free stream and γ is the ratio of specific heats (for air $\gamma=1.4$).

Introducing the perturbation velocity potential φ, where

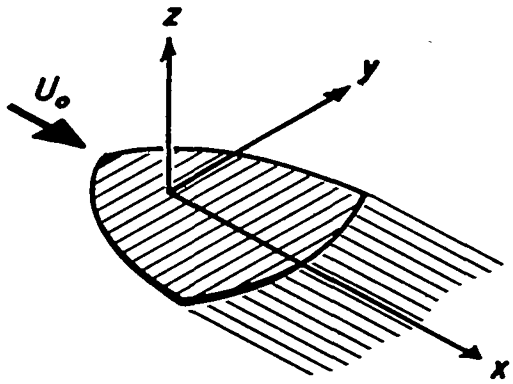
$$\varphi = -U_o x + \Phi \tag{3}$$

it is possible to express equation (1) in terms of the derivatives of φ as follows:

$$\left(1-M_o^2\right)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=\left\{\begin{array}{l}
\dfrac{\varphi_{xx}}{a_o^2}\left[(\gamma+1)U_o\varphi_x+\dfrac{\gamma+1}{2}\varphi_x^2+\dfrac{\gamma-1}{2}\left(\varphi_y^2+\varphi_z^2\right)\right]+ \\
\dfrac{\varphi_{yy}}{a_o^2}\left[(\gamma-1)U_o\varphi_x+\dfrac{\gamma-1}{2}\left(\varphi_x^2+\varphi_z^2\right)+\dfrac{\gamma+1}{2}\varphi_y^2\right]+ \\
\dfrac{\varphi_{zz}}{a_o^2}\left[(\gamma-1)U_o\varphi_x+\dfrac{\gamma-1}{2}\left(\varphi_x^2+\varphi_y^2\right)+\dfrac{\gamma+1}{2}\varphi_z^2\right]+ \\
2\dfrac{\varphi_{xy}}{a_o^2}\varphi_y\left[U_o+\varphi_x\right]+ \\
2\dfrac{\varphi_{xz}}{a_o^2}\varphi_z\left[U_o+\varphi_x\right]+ \\
2\dfrac{\varphi_{yz}}{a_o^2}\varphi_y\varphi_z
\end{array}\right\} \tag{4}$$

If it is assumed that all perturbation velocities and perturbation velocity gradients (represented by first and second derivatives, respectively, of φ) are small and that only the first-order terms in small quantities need be retained, equation (4) simplifies to the well-known Prandtl-Glauert equation of linear theory

$$\left(1-M_o^2\right)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0 \tag{5}$$



(a)

where the free-stream velocity is directed along the positive x axis as shown in sketch (a) and where M_o is the Mach number of the free stream. It is well-known that equation (5) leads to useful results in the study of subsonic and supersonic flows about thin wings and slender bodies but that it is incapable, in general, of treating transonic flows. The failure of linear theory in the transonic range is evidenced by the calculated value of φ_x growing to such magnitude that it can no longer be regarded as a small quantity when compared with U_o.

Second-order theory for thin wings would involve solution of the equation

$$\left(1-M_o^2\right)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=M_o^2\left[\frac{\gamma+1}{U_o}\varphi_x\,\varphi_{xx}+\frac{\gamma-1}{U_o}\varphi_x(\varphi_{yy}+\varphi_{zz})+\frac{2}{U_o}(\varphi_y\,\varphi_{xy}+\varphi_z\,\varphi_{xz})\right] \tag{6}$$

Actually, we are interested in retaining higher-order terms only to the extent that is necessary to allow the study of transonic flow. Examination of the known characteristics of transonic flow fields indicates that the first term on the right can often become of importance and should be retained. The remainder of the terms on the right can never become large for transonic flows about thin wings at small angles of attack and can be safely disregarded. Furthermore, since the right-hand side is merely an approximation to allow the treatment of transonic flows and rapidly diminishes in magnitude as M_o departs from unity, the equation may be further simplified without much, if any, loss in accuracy by setting $M_o = 1$ in the coefficient of the term on the right. (If one does not wish to make this added approximation, the results of this paper should be altered by replacing $\gamma + 1$ with $M_o^2(\gamma+1)$

wherever it occurs.) The simplified equation[1] is

$$\left(1-M_o^2\right) \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \frac{\gamma+1}{U_o} \varphi_x \varphi_{xx} \tag{7}$$

In addition to satisfying the differential equation, the perturbation potential must provide flows compatible with the following physical requirements: (a) The flow must be uniform far ahead of the wing and (b) the flow must be tangential to the wing surface. Therefore, the following boundary conditions are to be specified for the perturbation potential:

at $x = -\infty$

$$(\varphi_x)_o = (\varphi_y)_o = (\varphi_z)_o = 0 \tag{8}$$

at the wing surface, W

$$\frac{1}{U_o} (\varphi_z)_W = (\partial Z/\partial x) \tag{9}$$

where $(\partial Z/\partial x)$ refers to the local slope in the x direction of the wing surface. Furthermore, it is consistent with the assumption of small disturbances to satisfy the second boundary condition on the two sides of the xy plane rather than on the wing surface. Equation (9) is therefore replaced by

$$\frac{1}{U_o} (\varphi_z)_{z=o} = (\partial Z/\partial x) = \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \tag{10}$$

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[1] Although equation (7) is valid throughout the Mach number range, it is not the appropriate equation for the treatment of the "pseudo transonic" flow fields which are to be found around three-dimensional swept wings (consider, for instance, an infinite yawed wing) through a limited range of supersonic Mach numbers. Since these flows are characterized by shock waves standing essentially normal to the xy plane but oblique to the free-stream direction, equation (6) can be simplified only to the following:

$$\left(1-M_o^2\right) \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = M_o^2 \left(\frac{\gamma+1}{U_o} \varphi_x \varphi_{xx} + \frac{\gamma-1}{2} \varphi_x \varphi_{yy} + \frac{2}{U_o} \varphi_y \varphi_{xy}\right)$$

The discussion of these problems is outside the scope of the present report.

where the shape of the wing profile is given by

$$(Z/c) = \tau f(x/c,\ y/b) \tag{11}$$

where $f(x/c, y/b)$ represents the ordinate-distribution function and τ is an ordinate-amplitude parameter.[2] In order to obtain unique and physically important solutions, it is necessary to assume the Kutta condition that the flow leaves all subsonic trailing edges smoothly. To be complete, consideration should also be given to the conditions of transition through shock waves. This point has been discussed by Guderley (reference 4) and by Vincenti and Wagoner (reference 9). They indicated that (a) equation (7) is valid even if shock waves are present, and (b) the shock conditions agree with the similarity rules. In this report, the similarity rules are derived in appendix B and their compatibility with the shock relations is demonstrated in appendix C.

Upon solving the above boundary-value problem for the potential, one may determine the pressure coefficient by means of the formula

$$C_p = -\frac{2}{U_o}\varphi_x \tag{12}$$

Although Oswatitsch, Berndt, and Gullstrand (references 12, 13, 14, 19, and 20) have previously investigated transonic-flow phenomenon by means of equations derived by assuming that all velocities are small perturbations around the free-stream velocity U_o as is done above, most other workers have used equations derived under the more restrictive assumptions that all velocities are small perturbations around the critical velocity of sound a^*. In the latter scheme, the perturbation potential is defined by (see, for instance, reference 6 or 18)

$$\varphi' = -a^*x + \Phi \tag{13}$$

and the resulting differential equation for φ' is

$$\varphi'_{yy} + \varphi'_{zz} = \frac{\gamma+1}{a^*}\varphi'_x\,\varphi'_{xx} \tag{14}$$

The corresponding boundary conditions are specified as follows:

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[2]Note that, in general, a variation of τ represents a simultaneous change of the thickness ratio, camber, and angle of attack. In the special case of a nonlifting wing having symmetrical sections, τ is proportional to the thickness ratio; for inclined flat-plate wings of vanishing thickness, τ is proportional to the angle of attack.

at $x = -\infty$

$$(\varphi'_x)_o = U_o - a^* \approx -\frac{a^*}{\gamma+1}\left(1-M_o^2\right), \qquad (\varphi'_y)_o = (\varphi'_z)_o = 0 \tag{15}$$

at the wing surface

$$(\varphi'_z)_{z=o} \approx a^* \tau \frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \tag{16}$$

where the shape of the wing profile is still given by equation (11). The equation for the pressure coefficients is approximated similarly, thus,

$$C_p' \approx -\frac{2}{a^*}\left[\varphi'_x - (\varphi'_x)_o\right] \tag{17}$$

The two statements of the equations for transonic flow are clearly identical at a Mach number of unity. Although the derivation of the a^* equations requires that the free-stream Mach number be very close to unity, these equations have been used with good success by a number of authors to calculate the aerodynamic forces on airfoils at Mach numbers considerably removed from unity. In so doing, it has been suggested that it might be preferable to use more accurate relations for the pressure coefficient or the boundary conditions; for instance, it has been suggested that a^* be replaced with U_o in the equation for C_p'. This matter has been discussed at length in references 8, 15, and 16. Since no restriction requiring the Mach number to be near unity is made in the U_o analysis, it is informative to examine the relation between the results of the a^* and the U_o analyses. This is done in appendix A. It is found that the a^* analysis, if performed in a completely consistent manner using equations (13) through (17), yields values for C_p that are identical to those given by the more general U_o analysis. This somewhat paradoxical result is achieved through the action of a number of compensating effects and only applies to the pressures and the forces and moments derivable therefrom. It should be noted, in particular, that the values of the local velocities and Mach numbers provided by the a^* analysis for flows having free-stream Mach numbers other than unity are in error. Throughout the remainder of this report, the discussion will be based on the U_o analysis.

It is important to recognize that wing theory based on equation (7) is valid for all Mach numbers below the hypersonic range. At subsonic and supersonic speeds, equation (7) is of the same order of accuracy as the Prandtl-Glauert equation of linear theory (equation (5)) although more difficult to solve. At $M_o = 1$, equation (7) is identical with equation (14), now widely used in the study of transonic-flow problems.

On the other hand, there is no a priori method for determining whether or not a solution of the equations of linear theory will be valid in the transonic range. One can only decide by solving the problem under the assumptions of linear theory and then inspecting the magnitudes of the terms, particularly of φ_x, to see whether or not they can be regarded as small quantities. If the terms are sufficiently small, the linear-theory solution is presumed valid even though the Mach number may be near unity. Linearized-theory solutions have been obtained for a great number of practical wing problems and their behavior in the transonic range is now well known. To review briefly: For unswept wings of infinite span, linear theory indicates that the magnitude of φ_x on the surface of a given airfoil is proportional to $1/\sqrt{|1 - M_o^2|}$; consequently, φ_x approaches infinity as M_o approaches unity and the theory is clearly inapplicable. For wings of finite span, however, the perturbation velocities may be large or small at sonic velocity, depending on the particular problem as discussed in detail in reference 21. Specifically, for three-dimensional lifting surfaces of zero thickness, the velocities remain finite everywhere except at the leading edges, their magnitudes generally increasing with increasing aspect ratio and angle of attack. For wings of nonzero thickness, however, φ_x generally becomes large logarithmically as $1 - M_o^2$ approaches zero; consequently, linear theory is inapplicable within some Mach number range surrounding unity.

Summarizing, linear theory is applicable to lifting surfaces of small or moderate aspect ratio at all transonic speeds, but fails for wings of finite thickness within a range of Mach number surrounding unity. The range of inapplicability diminishes to zero as the aspect ratio, thickness ratio, and angle of attack of the wing tend to zero.

In treating transonic flows for which linear theory is applicable, it is often advantageous to consider the special case of sonic flow ($M_o = 1$) separately. Equation (5) for the perturbation potential then reduces to a particularly simple form

$$\varphi_{yy} + \varphi_{zz} = 0 \tag{18}$$

Solutions of equation (18), in conjunction with the boundary conditions given by equations (8) and (10), are identical to those of linear theory found by solving equation (5) and subsequently setting $M_o = 1$, but can be obtained with much less effort. Since, in addition, the results of this simple theory, now generally known as slender-wing theory, are also applicable to low-aspect-ratio lifting surfaces throughout the entire Mach number range,[3] a considerable number of solutions of slender-wing theory have been presented in the last few years. (See references 22, 23, and 24.) These results are, of course, applicable to flows at $M_o = 1$ to exactly the same extent as the results of linear theory.

Similiarity Rules

In reference 6, von Kármán derived similarity rules for the pressure distribution, lift, drag, and pitching moment of airfoils in transonic flow using equations (13) through (17). The same equations were used in reference 18 to determine the transonic similarity rules for wings of finite span. The corresponding similarity rules of linearized subsonic and supersonic wing theory were also derived and compared with the transonic similarity rules in the latter reference. It was shown that the similarity rules of linear theory contain an arbitrary parameter and can be expressed in many forms, one of which coincides with the similarity rules of transonic flow.

It follows from appendix A and is demonstrated in detail in appendix B that the similarity rules derived from equation (7) are identical to those previously given in references 6 and 18. The new derivation, however, possesses the advantage of being based on a single statement of the problem of wing theory that is uniformly valid at subsonic, transonic, and supersonic speeds. The similarity rules for C_p, C_L, C_m, and C_D were stated in reference 18 to be

$$C_p = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} P\left\{\frac{\sqrt{1-M_o^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_o^2}\,A; \frac{x}{c}, \frac{y}{b}\right\} \qquad (19)$$

$$C_L = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} L\left\{\frac{\sqrt{1-M_o^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_o^2}\,A\right\} \qquad (20)$$

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[3]This dual role of slender-wing theory stems from the two ways that one can reduce equation (5) to equation (18). One can neglect the term $(1 - M_o^2)\,\varphi_{xx}$ in comparison with φ_{yy} and φ_{zz} either because φ_{xx} is small, as may be the case with low-aspect-ratio wings or slender bodies, or because $(1 - M_o^2)$ is zero, provided φ_{xx} does not become very large in comparison with the other velocity gradients φ_{yy} and φ_{zz}. The application of this theory to slender wings antecedes the application to flows with sonic free-stream velocity, hence the name slender-wing theory.

$$C_m = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} M\left\{\frac{\sqrt{1-M_o^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_o^2}\,A\right\} \tag{21}$$

$$C_D = \frac{\tau^{5/3}}{(\gamma+1)^{1/3}} D\left\{\frac{\sqrt{1-M_o^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_o^2}\,A\right\} \tag{22}$$

where the geometry of related wings is given by equation (11):

$$(Z/c) = \tau f(x/c, y/b)$$

Equations (19) through (22) are functional equations. For example, equation (19) is to be interpreted as stating that the pressure coefficient C_p is equal to $\tau^{2/3}/(\gamma+1)^{1/3}$ times some function P of a number of specified parameters. The foregoing equations have been written for flows where $M_o \leq 1$. If $M_o \geq 1$, the radical $\sqrt{1-M_o^2}$ should be replaced with $\sqrt{M_o^2-1}$. The functions P, L, M, and D are different, however, for subsonic and supersonic flow. Consequently, subsonic flows may be related to other subsonic flows by the similarity rules, but not to supersonic flows, and conversely.

It is important to recognize that the similarity parameters may be combined or regrouped in any manner whatsoever, provided the same number of independent parameters is always retained. For instance, in much of what follows, it will be found desirable to use the square of $\sqrt{1-M_o^2}/[(\gamma+1)\tau]^{1/3}$ and to replace $\sqrt{1-M_o^2}\,A$ with a new parameter $[(\gamma+1)\tau]^{1/3}A$ obtained by dividing $\sqrt{1-M_o^2}\,A$ by $\sqrt{1-M_o^2}/[(\gamma+1)\tau]^{1/3}$. In terms of these parameters, the similarity rules are

$$C_p = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} P\left\{\frac{M_o^2-1}{[(\gamma+1)\tau]^{2/3}}, \quad [(\gamma+1)\tau]^{1/3}A;\ \frac{x}{c}, \frac{y}{b}\right\} \tag{23}$$

$$C_L = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} L\left\{\frac{M_o^2-1}{[(\gamma+1)\tau]^{2/3}}, \quad [(\gamma+1)\tau]^{1/3}A\right\} \tag{24}$$

$$C_m = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} M\left\{\frac{M_o^2-1}{[(\gamma+1)\tau]^{2/3}}, \quad [(\gamma+1)\tau]^{1/3}A\right\} \tag{25}$$

$$C_D = \frac{\tau^{5/3}}{(\gamma+1)^{1/3}} D \left\{ \frac{M_o^2-1}{[(\gamma+1)\tau]^{2/3}}, \quad [(\gamma+1)\tau]^{1/3} A \right\} \tag{26}$$

wherein the geometry of related wings is again given by equation (11).

Slope of Pressure Curve at $M_o = 1$

Liepmann and Bryson (reference 15) have recently determined the slope of the C_p versus M_o curve at $M_o = 1$ by means of the following simple and intuitive considerations. It is a well-known fact that, at slightly supersonic Mach numbers, the detached bow wave is far away from the airfoil and nearly normal. It is also well known that the Mach number downstream of a weak normal shock is as much below unity as the Mach number upstream is above unity. Consequently, the pressure or Mach number distribution on the airfoil should be independent of Mach number in the neighborhood of $M_o = 1$. The slope of the C_p versus M_o curve at $M_o = 1$ can then be found by simple and direct means and is the following for thin airfoils:

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1} = \frac{4}{\gamma + 1} \tag{27}$$

Vincenti and Wagoner (reference 8) have found by means of similar considerations that a more exact relation is given by

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1} = \frac{4}{\gamma + 1} - \frac{2}{\gamma + 1} \left(C_p\right)_{M_o=1} \tag{28}$$

Since the above derivation is based to a certain extent on physical reasoning which may be either more or less exact than small perturbation transonic theory, it is of interest from the present point of view to determine the equivalent result from the model of transonic flow provided by equation (7). The result, obtained by a similarity type of analysis and presented in detail in appendix D, is just that which would be found if one actually solved equation (7).

It is found that a solution for Mach numbers slightly less than unity satisfies the differential equation and the boundary conditions at the wing surface for flows with Mach numbers slightly greater than unity. However, the velocity perturbations do not go to zero infinitely far ahead of the wing, but go instead to the value corresponding to that associated with a normal shock wave. If it can be assumed that the bow wave approaches a normal shock wave standing infinitely far ahead of the wing at a sufficiently rapid rate as the free-stream Mach number approaches unity, the slope of the C_p versus M_o curve at $M_o = 1$ is as given by equation (27). Whether or not this is a permissible assumption for any given case still remains an unanswered question. Intuitive considerations suggest that the results are probably applicable to symmetrical airfoils at zero or infinitesimal angles of attack but not to airfoils at larger angles of attack or to wings of finite span.

APPLICATIONS

Fundamental Hypotheses and Principles

The remainder of this report is principally concerned with the deduction of the qualitative, and to some extent quantitative, characteristics of thin wings in transonic flow by means of simple logical considerations based primarily on the similarity rules together with the following hypotheses:

(a) Nonlinear theory based on equation (7) is applicable to all problems.

(b) Linear theory based on equation (5) is valid for all wings at Mach numbers either appreciably below or above unity.

(c) Linear theory is valid at all Mach numbers, except possibly very near unity, for wings of small aspect ratio.

(d) The differential pressures between the upper and lower surfaces of a wing having symmetrical airfoil sections are proportional to the angle of attack for at least a small range of angles about zero.

(e) The slope of C_p versus M_o at $M_o = 1$, defined by equation (27), is applicable at least to symmetrical airfoil sections at zero or infinitesimal angles of attack.

The consequences of the foregoing statements will be consistently pursued in the following sections in the discussion of the aerodynamic characteristics of airfoils and complete wings. Throughout, the analysis will be restricted to wings having symmetrical profiles. The decision to ignore the influence of camber is based not only on the desire for simplicity but on the fact that symmetrical airfoils appear experimentally to have superior aerodynamic characteristics in the transonic range. Whenever specific results are to be used to illustrate the statements, they will nearly always be for symmetrical-wedge or double-wedge profiles and for wings of triangular plan form. This choice is dictated by the present availability of theoretical results.

The basic principle in the following analysis is to express the similarity rules in such forms that the lift, pitching-moment, and drag coefficients can be studied for limiting values of the parameters with no chance for ambiguity due to indeterminate forms. In this respect, the statement of the similarity rules provided by equations (23) through (26) will be found particularly useful.

The similarity rules thus formulated are totally equivalent to those given by equations (19) through (22) but possess three outstanding advantages:

(a) The indeterminacy at $M_o = 1$ resulting from two parameters simultaneously vanishing is removed.

(b) The squaring of the first parameter avoids the necessity of changing parameters as sonic speed is passed.

(c) The use of the parameter $[(\gamma+1)\tau]^{1/3}A$ rather than $\sqrt{1 - M_o^2}\,A$ aids in distinguishing the regimes in which linear theory is applicable in the transonic range from those in which nonlinear theory must be used. Thus as $[(\gamma+1)\tau]^{1/3}A$ approaches zero, linear theory is always applicable provided, in some cases, that M_o is not precisely equal to unity. On the other hand, as $[(\gamma+1)\tau]^{1/3}A$ becomes large, linear theory is not applicable in the transonic range and nonlinear theory must be used.

Pressure Drag of Symmetrical Nonlifting Wings

The similarity rule for the pressure drag coefficient of symmetrical nonlifting wings having profiles given by

$$(Z/c) = (t/c)\ f(x/c,\ y/b) \tag{29}$$

is obtained from equation (26) by identifying τ with t/c, the thickness ratio,

$$C_{D_o} = \frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}}\, D_o\left\{\frac{M_o^2-1}{[(\gamma+1)(t/c)]^{2/3}},\ [(\gamma+1)(t/c)]^{1/3}A\right\} = \frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}}\, D_o\ (\xi_o,\ \tilde{A}) \tag{30}$$

Therefore, drag results for symmetrical nonlifting wings should be presented by plotting the variation with ξ_o and $\tilde{A}$ of a generalized drag coefficient $\widetilde{C_{D_o}}$ defined as follows:

$$\widetilde{C_{D_o}} = \frac{(\gamma+1)^{1/3}}{(t/c)^{5/3}} C_{D_o} = D_o\ (\xi_o, \tilde{A}) \tag{31}$$

At Mach numbers sufficiently removed from unity for linear theory to apply, C_{D_o} must be independent of γ since γ does not appear in either the differential equation or boundary conditions of linear theory. Therefore,

$$\left(C_{D_o}\right)_l = \frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}} \left\{\frac{|M_o^2-1|}{[(\gamma+1)(t/c)]^{2/3}}\right\}^{-1/2} D_{o_l} \left\{\left[\frac{|M_o^2-1|}{[(\gamma+1)(t/c)]^{2/3}}\right]^{1/2} \times \right.$$

$$\left. [(\gamma+1)(t/c)]^{1/3} A\right\} = \frac{(t/c)^2}{\sqrt{|M_o^2-1|}} D_{o_l} \left(\sqrt{|M_o^2-1|}\ A\right) \tag{32}$$

where it should be recalled that D_{o_l} is a different function for subsonic and supersonic flow. Equation (32) is equivalent to the extended Prandtl-Glauert rule. For subsonic flow, D'Almbert's paradox requires that the drag be zero; therefore, for all wings,

$$\underset{M_o<1}{(C_{D_o})_l} = 0, \qquad \underset{\xi_o<0}{(\widetilde{C_{D_o}})_l} = 0 \tag{33}$$

For supersonic flow, wave drag exists which depends on $\sqrt{M_o^2 - 1}\ A$ as well as on the plan form and airfoil section. The general functional relation for the drag coefficient of a family of affinely related wings at zero angle of attack, as given by linear theory, is

$$\underset{M_o>1}{(C_{D_o})_l} = \frac{(t/c)^2}{\sqrt{M_o^2-1}} D_{o_l} \left(\sqrt{M_o^2-1}\ A\right), \qquad \underset{\xi_o>0}{(\widetilde{C_{D_o}})_l} = \xi_o^{-1/2} D_{o_l} (\xi_o^{1/2}\tilde{A}) \tag{34}$$

<u>Wings of infinite aspect ratio</u>.- For wings of infinite span (or airfoils), equation (32), representing the functional relation of linear theory for the drag coefficient, reduces to the following:

$$(c_{d_o})_{l \atop M_o>1} = \frac{(t/c)^2}{\sqrt{M_o^2-1}} \times \text{const.}, \qquad (\tilde{c}_{d_o})_{l \atop \xi_o>0} = \frac{\text{const.}}{\sqrt{\xi_o}} \tag{35}$$

where the value of the constant depends on the shape of the airfoil. Numerous experimental data show variations consistent with equation (35) at Mach numbers greater than about that of shock attachment. At Mach numbers closer to unity, however, the theoretical values provided by this equation are unreliable. It is evident from inspection of the results of linear theory itself that such a failure occurs, since the perturbation velocities assumed to be small in the derivation of the equations are found to become infinitely large as the Mach number approaches unity. It is apparent, therefore, that it is necessary to resort to nonlinear theory for the calculation of the drag of airfoils in the transonic speed range.

A similarity rule for the section drag coefficient of symmetrical nonlifting airfoils which is valid throughout the Mach number range may be obtained from equation (31) by setting $\tilde{A}$ equal to infinity.

$$\tilde{c}_{d_o} = \frac{(\gamma+1)^{1/3}}{(t/c)^{5/3}} c_{d_o} = \frac{(\gamma+1)^{1/3}}{(t/c)^{5/3}} (C_{D_o})_{\tilde{A}=\infty} = D_o\ (\xi_o, \infty) = d_o\ (\xi_o) \tag{36}$$

At a Mach number of unity, the similarity parameter ξ_o vanishes and the function $d_o\ (\xi_o)$ is a constant.

$$(\tilde{c}_{d_o})_{\xi_o=0} = d_o\ (0) = \text{const.}, \qquad (c_{d_o})_{M_o=1} = \frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}} \times \text{const.} \tag{37}$$

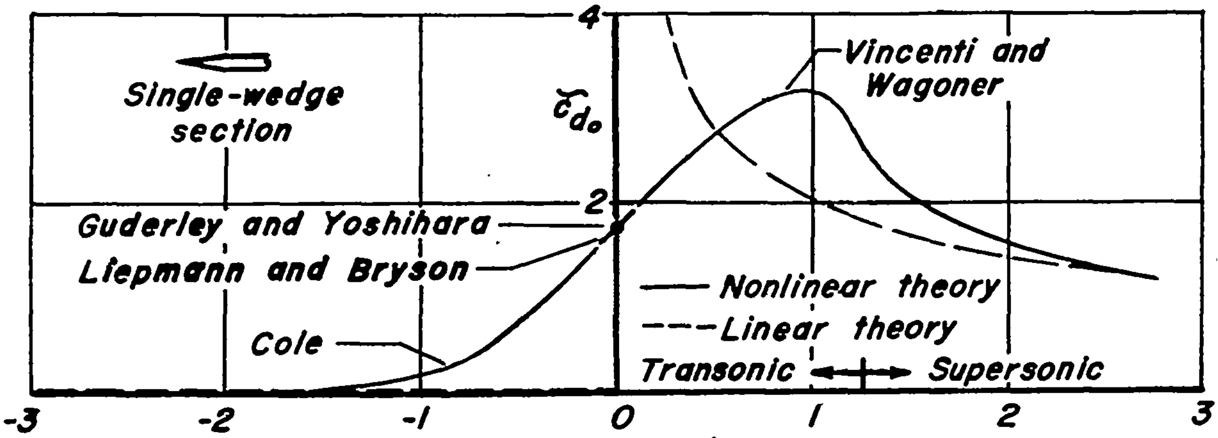
indicating that the section drag coefficients of affinely related airfoils are proportional to the five-thirds power of their thickness ratios. If hypothesis (e) is accepted, the variation of c_{d_o} with M_o at $M_o = 1$ is found to be zero for complete airfoils

$$\left(\frac{dc_{d_o}}{dM_o}\right)_{M_o=1} = \oint \left(\frac{dC_p}{dM_o}\right)_{M_o=1} \left(\frac{dZ}{dx}\right) \frac{dx}{c} = \frac{4}{\gamma+1} \oint \left(\frac{dZ}{dx}\right) \frac{dx}{c} = 0, \qquad \left(\frac{d\tilde{c}_{d_o}}{d\xi_o}\right)_{\xi_o=0} = 0 \tag{38}$$

Since calculations have been made of the drag in transonic flow of simple symmetrical sections at zero angle of attack, it is not necessary to speculate further regarding the variation of $\tilde{c}_{d_o}$ with ξ_o. At present, however, the profile for which the most complete information exists is not a complete airfoil but is a single-wedge section followed by a straight section extending infinitely far downstream. In accord with some of the original papers on this subject, the single-wedge section is considered as the front half of a symmetrical double-wedge airfoil having a chord c. The value of t/c in $\tilde{c}_{d_o}$ and ξ_o is defined accordingly. Solutions for this section obtained using the nonlinear small perturbation theory have been given for flows having subsonic, sonic, and supersonic free-stream velocities, respectively, by Cole (reference 10), Guderley and Yoshihara (reference 7), and Vincenti and Wagoner (references 8 and 9). The linear-theory solution for pure supersonic flows has been given by Ackeret (reference 25). The slope of the drag curve at $M_o = 1$ is no longer zero as indicated for the complete airfoil by equation (38) but takes on a positive value given originally by Liepmann and Bryson (reference 15) and readily derivable from equations (27) and (38).

$$\left(\frac{dc_{d_o}}{dM_o}\right)_{M_o=1} = \frac{4}{\gamma + 1}\frac{t}{c}, \qquad \left(\frac{d\tilde{c}_{d_o}}{d\xi_o}\right)_{\xi_o=0} = 2 \tag{39}$$

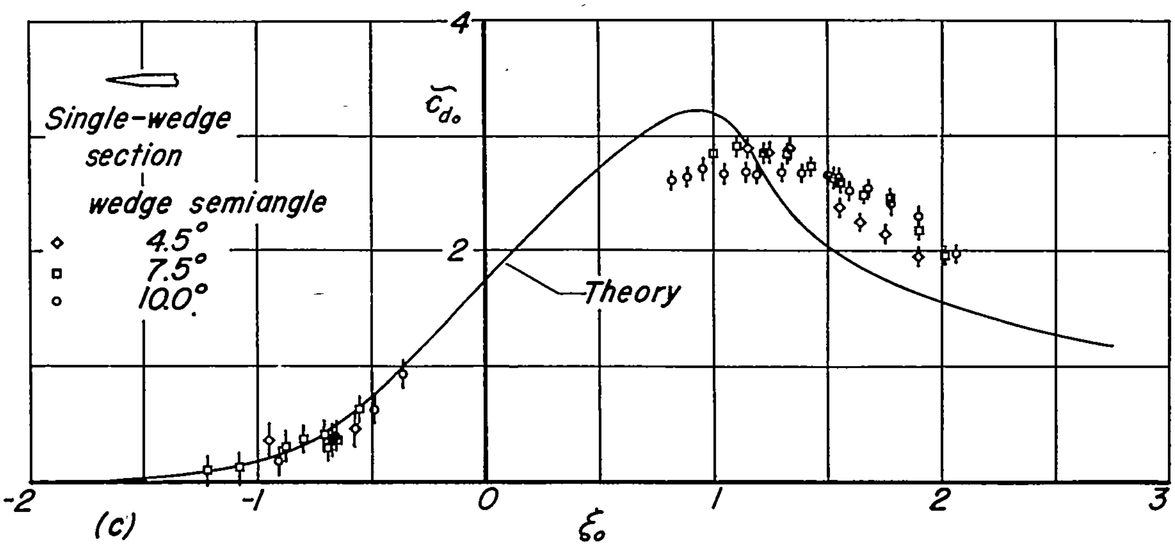
All of these results are combined on a single graph in sketch (b). It



(b)

may be seen that the preceding remarks concerning the relation of linear theory and nonlinear theory and the slope of the drag curve at a Mach number of unity are verified by this comparison.

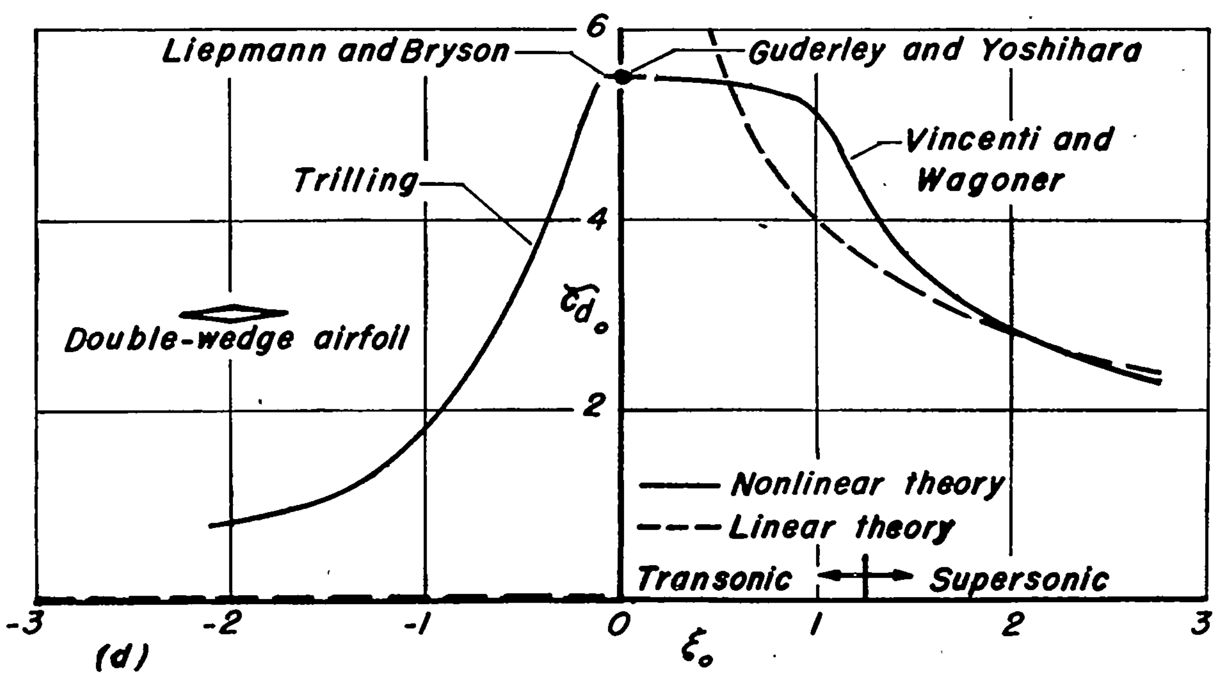
An indication of the accuracy of the theory is provided in sketch (c), which shows the theoretical curve of sketch (b) together with corresponding results obtained from wind-tunnel experiments by Liepmann and Bryson (references 15 and 16). The vertical lines through



(c)

the experimental points indicate estimated accuracy of the data. Three single-wedge models having semiapex angles of 4.5°, 7.5°, and 10° were used in the tests. This figure illustrates the degree to which the similarity rules are able to reduce data from a family of profiles having different thickness ratios to essentially a single curve.

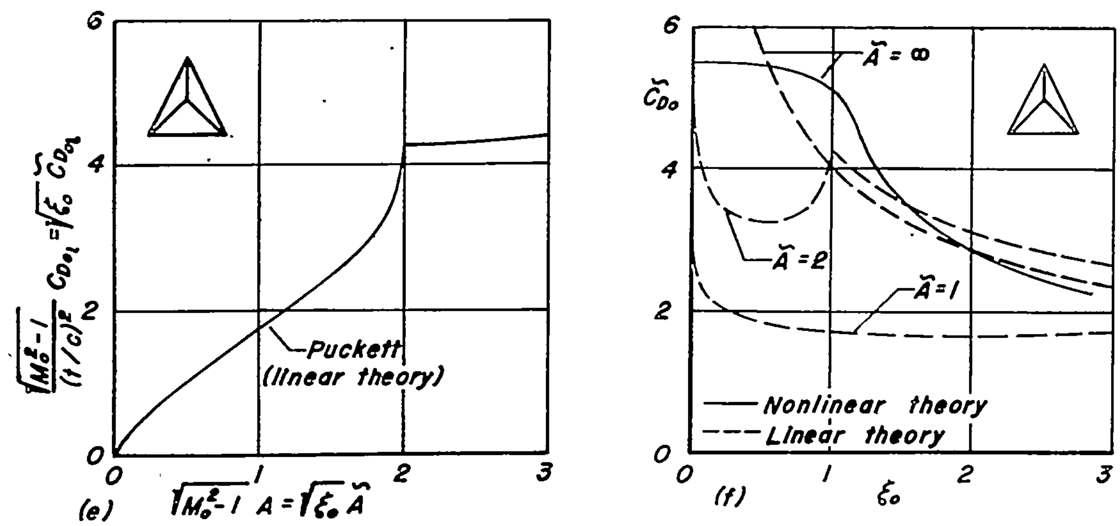
The double-wedge airfoil has also been treated theoretically throughout the entire Mach number range. Solutions for $M_o < 1$ have been given by Trilling (reference 11), for $M_o = 1$ by Guderley and Yoshihara (reference 7), and for $M_o > 1$ by Vincenti and Wagoner (references 8 and 9). The results of their calculations are shown in sketch (d).



(d)

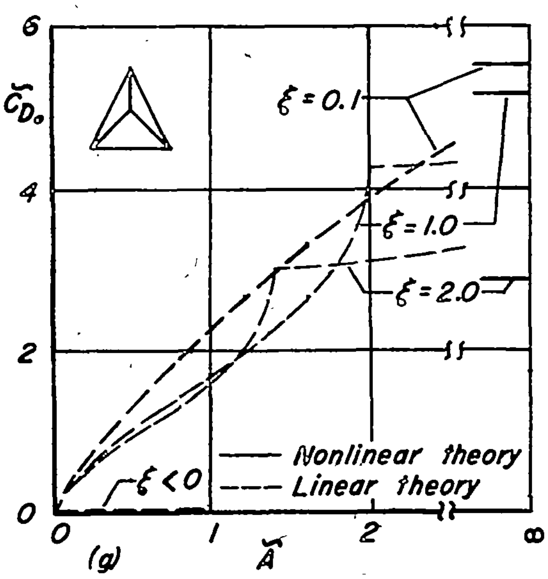
For $M_0 < 1$, Oswatitsch has given approximate solutions for the pressure distribution on symmetrical biconvex airfoils (references 12 and 13) and for the pressure distribution and drag on NACA four-digit symmetrical airfoils (reference 13). This work has recently been extended to several NACA 6-series airfoils by Gullstrand (reference 14). Their drag results are generally similar to those indicated at corresponding Mach numbers for the double-wedge section in sketch (d).

Wings of finite aspect ratio.- The similarity rules for wings of finite aspect ratio are given by equations (31) and (32). Although no essential simplification of the rules occurs for wings of small aspect ratio, the range of applicability of linear theory increases as the aspect ratio decreases. This point can be illustrated by considering the results provided by linear theory in a specific case. A good example to select for this purpose is that of the drag in supersonic flow of a triangular wing with symmetrical double-wedge airfoils. (See reference 26.) This particular choice was made for the following reasons: (a) Solutions are known for all supersonic Mach numbers; (b) the double-wedge airfoil discussed in the preceding sections corresponds to the limiting case of the wing of very great aspect ratio. The drag results provided for this wing by linear theory are presented in sketch (e).



The results of sketch (e) are presented in a different manner in sketch (f) wherein $\widetilde{C_{D_0}}$ is plotted as a function of ξ_0 for various values of $\tilde{A}$ as suggested by equation (31). For purposes of comparison, the curves for the drag of airfoils ($\tilde{A}=\infty$) computed by both linear and nonlinear theory are also included on the graph. As noted in the preceding section on airfoils, comparison of the results of linear theory with those of nonlinear theory for wings of $\tilde{A} = \infty$ shows that good agreement exists for larger ξ_0, but that at smaller ξ_0, the values predicted by linear theory become too large. For wings of finite $\tilde{A}$, only the results of linear theory are available. They also exhibit the trend of indicating infinite drag as ξ_o approaches zero; however, the range of ξ_o in which the value of $\tilde{C}_{D_o}$ is excessive is much less than is the case for $\tilde{A} = \infty$. In general, $\tilde{C}_{D_o}$ of wings of small aspect ratio diminishes with decreasing aspect ratio.

The drag results of sketch (e) are presented in still another form in sketch (g) in which is plotted the variation of $\tilde{C}_{D_o}$ with $\tilde{A}$ for various values of ξ_o. The principal merit of this method of plotting is that it aids in distinguishing the region where nonlinear theory must be used from that where linear theory may be useful. Thus the two-dimensional nonlinear theory results appear on the right of the graph corresponding to large $\tilde{A}$, whereas the three-dimensional linear theory results appear on the left for small $\tilde{A}$. The filling in of the remainder of the graph requires either the solution of the equations of three-dimensional nonlinear wing theory or the use of experimental data. It should again be noted that the present drag considerations apply only to the pressure drag. Before plotting experimental results in the manner indicated, it is necessary to first subtract the friction drag.



Lift

Equation (24) indicates that the similarity rule for the lift coefficient of wings having profiles given by

$$(Z/c) = \tau f(x/c, y/b)$$

is

$$C_L = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} L\left\{\frac{M_o^2-1}{[(\gamma+1)\tau]^{2/3}}, \quad [(\gamma+1)\tau]^{1/3} A\right\}$$

This equation has been the source of some confusion due to the multiple role that τ plays in determining the thickness ratio, camber, and angle of attack. A more explicit statement of the similarity rule that is useful for all wings having symmetrical profiles of nonzero thickness is given by the following pair of equations:

$$(Z/c) = (t/c)\left[f(x/c,\ y/b) - \left(\frac{\alpha}{t/c}\right)\left(\frac{x}{c}\right)\right] \tag{40}$$

$$C_L = \frac{(t/c)^{2/3}}{(\gamma+1)^{1/3}}\ L'\left\{\frac{M_0{}^2 - 1}{[(\gamma+1)(t/c)]^{2/3}},\ [(\gamma+1)(t/c)]^{1/3} A,\ \frac{\alpha}{t/c}\right\} = \frac{(t/c)^{2/3}}{(\gamma+1)^{1/3}}\ L'\ (\xi_0,\ \tilde{A},\ \tilde{\alpha}) \tag{41}$$

where the primes serve as a reminder that L and L' are different functions of the parameters indicated. If hypothesis (d) is accepted, C_L varies linearly with α for at least small angles of attack. It is advantageous, therefore, to consider the lift ratio C_L/α rather than C_L alone, thereby minimizing the influence of $\tilde{\alpha}$.

$$\frac{C_L}{\alpha} = \frac{1}{\alpha}\frac{(t/c)^{2/3}}{(\gamma+1)^{1/3}}\ L'(\xi_0,\ \tilde{A},\ \tilde{\alpha}) = \frac{1}{[(\gamma+1)(t/c)]^{1/3}}\ L''(\xi_0,\ \tilde{A},\ \tilde{\alpha}) \tag{42}$$

Therefore, lift results may be presented by plotting the variation with ξ_0, $\tilde{A}$, and $\tilde{\alpha}$ of a generalized lift ratio $\widetilde{C_L/\alpha}$ defined as follows (the primes on L have been omitted for simplicity):

$$\widetilde{C_L/\alpha} = [(\gamma+1)(t/c)]^{1/3}\ (C_L/\alpha) = L\ (\xi_0,\ \tilde{A},\ \tilde{\alpha}) \tag{43}$$

Equation (43) shows that $\widetilde{C_L/\alpha}$ depends upon three parameters, one more than the number which can readily be treated on a simple plot. Simplification can be gained, of course, by holding one of the parameters constant for an entire graph. Results so presented are particularly interesting for $\xi_0 = 0$, $(M_0=1)$; $\tilde{A}=\infty$, $(A=\infty)$; and $\tilde{\alpha}=0$, $(\alpha=0)$. The latter scheme is especially good since experiments indicate that lift curves of wings are often relatively straight lines at all Mach numbers. The values of $\widetilde{C_L/\alpha}$ at $\tilde{\alpha} = 0$ might, therefore, be expected to be good indications of the actual values for other $\tilde{\alpha}$. The appropriate similarity rule may then be written

$$(\widetilde{C_L/\alpha})_{\tilde{\alpha}=0} = L\ (\xi_0,\ \tilde{A},\ 0) = L_0\ (\xi_0,\ \tilde{A}) \tag{44}$$

For cases where linear theory applies (hypotheses (b) and (c)), two statements can be made immediately which provide further information about C_L/α: (a) $(C_L/\alpha)_l$ must be independent of γ, and (b) $(C_L/\alpha)_l$ must be independent of $\tilde{\alpha}$ by virtue of the superposition principle of linear theory. Therefore, following the procedure used in equation (32) gives

$$\left(\frac{C_L}{\alpha}\right)_l = \frac{1}{\sqrt{|M_0^2-1|}} L_l\left(\sqrt{|M_0^2-1|}\,A\right), \qquad \left(\widetilde{\frac{C_L}{\alpha}}\right)_l = \frac{1}{\sqrt{|\xi_0|}} L_l\left(\sqrt{|\xi_0|}\,\tilde{A}\right) \tag{45}$$

where again L_l is a different function for subsonic and supersonic flow.

Wings of infinite aspect ratio.- For wings of infinite aspect ratio, the functional relation of linear theory for the lift ratio given by equation (45) reduces to

$$\left(\frac{c_l}{\alpha}\right)_l = \frac{\text{const.}}{\sqrt{|M_0^2-1|}}, \qquad \left(\widetilde{\frac{c_l}{\alpha}}\right)_l = \frac{\text{const.}}{\sqrt{|\xi_0|}} \tag{46}$$

Solutions of the equations of linear theory show that the value of the constant is 2π for subsonic flow and four for supersonic flow. Examination of these results indicates that they are valid at Mach numbers appreciably less than or greater than unity, but are invalid for Mach numbers near 1.

A similarity rule for the section lift coefficients of a family of affinely related symmetrical airfoils which is valid throughout the Mach number range may be obtained from equation (43) by setting $\tilde{A} = \infty$.

$$(\widetilde{c_l/\alpha}) = L(\xi_0, \infty, \tilde{\alpha}) = l(\xi_0, \tilde{\alpha}) \tag{47}$$

At a Mach number of unity, ξ_0 is identically zero and the expression for the lift ratio becomes

$$(\widetilde{c_l/\alpha})_{\xi_0=0} = l(0, \tilde{\alpha}), \qquad (c_l/\alpha)_{M_0=1} = \frac{1}{[(\gamma+1)(t/c)]^{1/3}}\, l\left(0, \frac{\alpha}{t/c}\right) \tag{48}$$

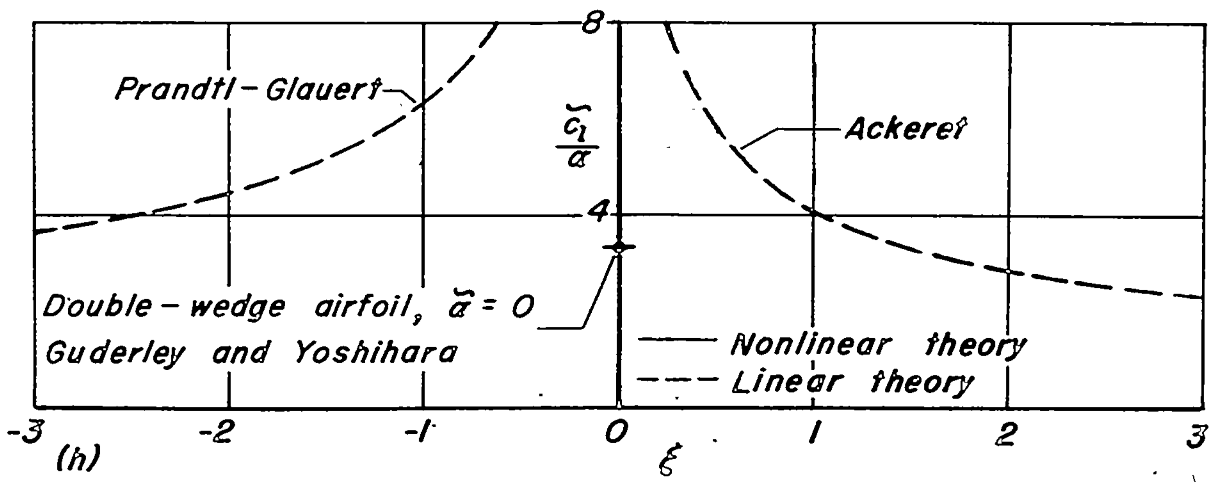
Equation (48), when considered together with hypothesis (d), indicates that at sonic speed the lift-curve slope at zero angle of attack of airfoils of a single family varies inversely as the cube root of the thickness ratio. Note that as the thickness ratio goes to zero, the value of the lift-curve slope at zero angle of attack is indicated to become infinite just as is indicated by equation (46) to be the case according to linear theory. On the other hand, for very large $\tilde{\alpha}$, it is plausible that the thickness ratio does not have any effect on c_l; therefore c_l is proportional to the two-thirds power of α for large $\tilde{\alpha}$. If hypothesis (e) is acceptable, that is, if it can be assumed that the detached bow wave in front of a symmetrical airfoil at an infinitesimal angle of attack is a normal shock wave so that $(dC_p/dM_o)_{M_o=1}$ is given by equation (27), the lift-curve slope at zero angle of attack is stationary with Mach number at $M_o = 1$, since

$$\left[\frac{d}{dM_o}\left(\frac{c_l}{\alpha}\right)_{\alpha=0}\right]_{M_o=1} = \oint \left(\frac{dC_p}{dM_o}\right)_{M_o=1} \frac{dx}{c} = \frac{4}{\gamma+1}\oint \frac{dx}{c} = 0, \quad \left[\frac{d}{d\xi_o}\left(\widetilde{\frac{c_l}{\alpha}}\right)_{\tilde{\alpha}=0}\right]_{\xi_o=0} = 0 \tag{49}$$

At present, the only available solution of the nonlinear equations for transonic flow about lifting airfoils is that of Guderley and Yoshihara (reference 27) for the case of sonic flow about a symmetrical double-wedge airfoil at an infinitesimal angle of attack. They found that

$$\left(\widetilde{c_l/\alpha}\right)_{\substack{\xi_o=0\\ \tilde{\alpha}=0}} = 3.32 \tag{50}$$

The foregoing results are summarized in graphical form in sketch (h).



(h)

<u>Wings of vanishing aspect ratio</u>.- Two well-known results of linear theory are that the lift-curve slopes of wings of finite aspect ratio remain finite throughout the entire Mach number range and that the

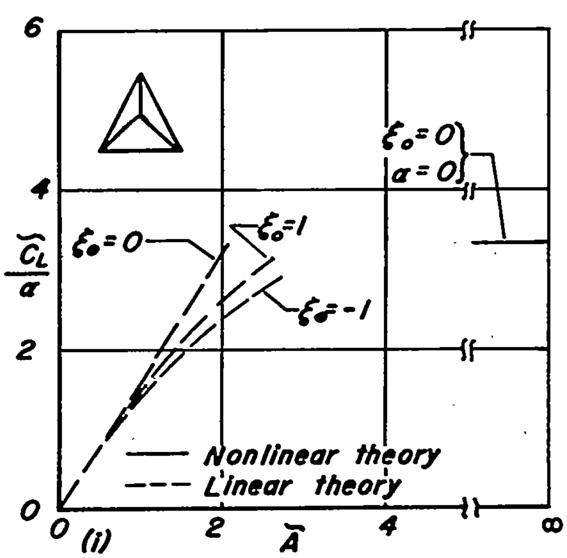
lift-curve slopes of wings of vanishing aspect ratio are independent of Mach number. Therefore, equation (45) implies that $(\widetilde{C_L/\alpha})_l$ must be proportional to $\tilde{A}$ either for wings of vanishing $\tilde{A}$ in any flow, or for any wing in a flow of vanishing ξ_o:

$$(\widetilde{C_L/\alpha})_{l\,\tilde{A}\to 0} = (\widetilde{C_L/\alpha})_{l\,\xi_o=0} = \tilde{A} \times \text{const.} \tag{51}$$

The value of the constant must be determined for each plan form by actually solving the equations of linear theory. For wings having trailing edges which possess no cutouts extending forward of the most forward station of maximum span, that is, triangular, rectangular, elliptical, etc., as well as certain swept-back wings, the value of the constant is $\pi/2$. (References 21, 22, 23, and 24 should be consulted for further discussion of this point as well as for the values of the constant for wings having cutouts in the trailing edge which violate the above stated condition.)

It is seen from equation (51) that the lift-curve slopes of wings in sonic flow decrease continuously in magnitude as the aspect ratio diminishes towards zero. Since, in addition, the lift-curve slope given by linear theory has its maximum value at $M_o = 1$, it is conjectured that the lift results of linear theory are a good approximation to those of nonlinear theory not only for all wings at Mach numbers far from unity but also for all Mach numbers for wings of sufficiently small aspect ratio.

Wings of finite aspect ratio.- At the present time no solutions of the nonlinear theory are available for wings of finite aspect ratio. However, from the remarks of the preceding paragraphs, it is apparent that a curve representing the variation of $\widetilde{C_L/\alpha}$ with $\tilde{A}$ for constant ξ_o and $\tilde{\alpha}$ would have the following asymptotic properties: $\widetilde{C_L/\alpha}$ would increase linearly with $\tilde{A}$ for small $\tilde{A}$ and be independent of $\tilde{A}$ for large $\tilde{A}$. In order to give a better idea of the numerical values to be expected, a set of typical results of this type are shown in sketch (i) for wings having triangular plan forms and symmetrical double-wedge airfoil sections. The supersonic results are those of Stewart, Brown (references 28 and 29), and others. The subsonic results are those calculated by De Young and Harper (reference 30) using Weissinger's modified lifting line theory.



(i)

An interesting result of the foregoing remarks concerns the influence of thickness ratio on the lift-curve slope at zero angle of attack of wings in sonic flow. For wings of large aspect ratio, the lift-curve slope is inversely proportional to the cube root of the thickness ratio. For wings of small aspect ratio, the lift-curve slope is independent of the thickness ratio.

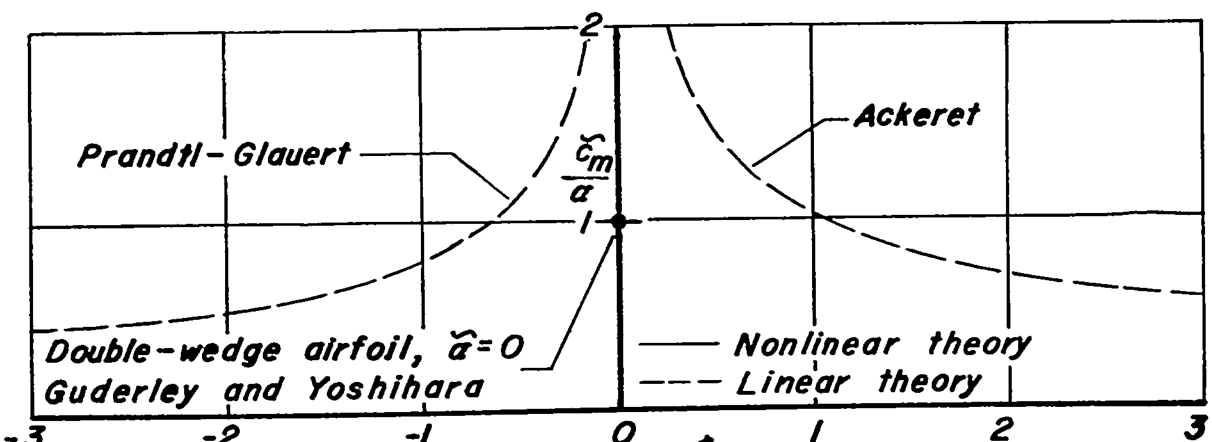
Pitching Moment

The remarks on the pitching-moment characteristics of wings follow in a manner exactly analogous to those just stated for the lift characteristics. The corresponding statements for the pitching-moment coefficient C_m may be obtained by simply replacing C_L with C_m and L with M. Thus, the similarity rules for C_m corresponding to equations (43) and (45) are the following, respectively:

$$(\widetilde{C_m/\alpha}) = [(\gamma+1)(t/c)]^{1/3}\,(C_m/\alpha) = M(\xi_o, \tilde{A}, \tilde{\alpha}) \tag{52}$$

$$\left(C_m/\alpha\right)_l = \frac{1}{\sqrt{|M_o^2-1|}}\, M_l\left(\sqrt{|M_o^2-1|}\,A\right), \quad (\widetilde{C_m/\alpha})_l = \frac{1}{\sqrt{|\xi_o|}}\, M_l\left(\sqrt{|\xi_o|}\,\tilde{A}\right) \tag{53}$$

where once more M and M_l are different functions for subsonic and supersonic flow. The only difference between the discussion of C_m and C_L is that the values of the constants of equations (46), (50), and (51) are, of course, different. Graphs of theoretical



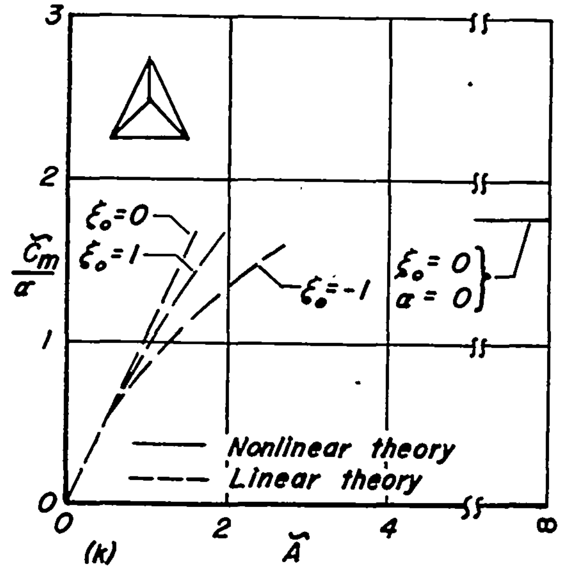
(j)

pitching-moment characteristics for airfoils and for triangular wings corresponding to the lift results of sketches (h) and (i) are shown in sketches (j) and (k). The moment axis is taken to be through the most forward point of the wing.

Sometimes it is desired to present pitching-moment characteristics of wings in terms of center-of-pressure position rather than pitching-moment coefficient. Since the center-of-pressure position can be expressed in terms of C_m and C_L by

$$\frac{x_{c.p.}}{c} = -\frac{C_m}{C_L} \tag{54}$$

the resulting expression for the center-of-pressure position found through application of equations (43) and (52) is



$$\frac{x_{c.p.}}{c} = -\frac{M(\xi_o, \tilde{A}, \tilde{\alpha})}{L(\xi_o, \tilde{A}, \tilde{\alpha})} = \text{C. P.}\ (\xi_o,\ \tilde{A},\ \tilde{\alpha}) \tag{55}$$

The corresponding relation for linear theory is

$$\left(\frac{x_{c.p.}}{c}\right)_l = \text{C.P.}_l\left(\sqrt{|M_o^2 - 1|}\ A\right) = \text{C.P.}_l\left(\sqrt{|\xi_o|}\ \tilde{A}\right) \tag{56}$$

Pressure Drag Due to Lift

The similarity rule for the pressure drag of inclined wings having symmetrical airfoils is indicated by equation (26) to be the following if the dependence on α and t/c is written in the same manner as in equations (40) and (41):

$$C_D = \frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}}\ D\ (\xi_o,\ \tilde{A},\ \tilde{\alpha}) \tag{57}$$

The portion of the drag due to lift is therefore

$$\left.\begin{aligned}\Delta C_D = C_D - C_{D_0} = \frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}}\left[D(\xi_o, \tilde{A}, \tilde{\alpha}) - D(\xi_o, \tilde{A}, 0)\right] = \\ \frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}} D_\Delta(\xi_o, \tilde{A}, \tilde{\alpha})\end{aligned}\right\} \tag{58}$$

Since C_D varies, in most cases, with the square of α for at least a small range of α surrounding zero (hypothesis (d)), it is advantageous to consider the drag-rise ratio $\Delta C_D/\alpha^2$ rather than ΔC_D alone, thus,

$$\frac{\Delta C_D}{\alpha^2} = \frac{1}{\alpha^2}\frac{(t/c)^{5/3}}{(\gamma+1)^{1/3}} D_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) = \frac{1}{[(\gamma+1)(t/c)]^{1/3}} D'_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) \tag{59}$$

Therefore, drag-due-to-lift data should be presented by plotting the variation with ξ_o, $\tilde{A}$, and $\tilde{\alpha}$ of a generalized drag-rise ratio $\widetilde{\Delta C_D/\alpha^2}$ defined as follows (the prime on D_Δ being dropped for simplicity):

$$(\widetilde{\Delta C_D/\alpha^2}) = [(\gamma+1)(t/c)]^{1/3}[\Delta C_D/\alpha^2] = D_\Delta(\xi_o, \tilde{A}, \tilde{\alpha}) \tag{60}$$

The actual presentation of the results of this three-parameter system may be accomplished as described in the section on the lift of wings. Of particular interest is the simplification resulting from presenting only the values found at $\alpha = 0$. The simplified similarity rule is then

$$(\widetilde{\Delta C_D/\alpha^2})_{\tilde{\alpha}=0} = D_\Delta(\xi_o, \tilde{A}, 0) \tag{61}$$

For cases where linear theory applies, the following results hold:

$$\left(\frac{\widetilde{\Delta C_D}}{\alpha^2}\right)_l = \frac{1}{\sqrt{|M_o^2-1|}} D_{\Delta l}\left(\sqrt{|M_o^2-1|}\,A\right), \quad \left(\frac{\widetilde{\Delta C_D}}{\alpha^2}\right)_l = \frac{1}{\sqrt{|\xi_o|}} D_{\Delta l}\left(\sqrt{|\xi_o|}\,\tilde{A}\right) \tag{62}$$

<u>Wings of infinite aspect ratio</u>.- For wings of infinite aspect ratio, the functional relation of linear theory for the drag due to lift, equation (62), reduces to

$$\left(\frac{\Delta c_d}{\alpha^2}\right)_l = \frac{\text{const.}}{\sqrt{|M_o^2-1|}}, \quad \left(\frac{\widetilde{\Delta c_d}}{\alpha^2}\right)_l = \frac{\text{const.}}{\sqrt{|\xi_o|}} \tag{63}$$

Solutions of the equations of linear theory show that the value of the constant is zero for subsonic flow and four for supersonic flow about any symmetrical airfoil. These results are valid at Mach numbers appreciably less than or greater than unity but are invalid for Mach numbers near 1.

A similarity rule for the drag due to lift of a family of affinely related symmetrical airfoils that is valid throughout the Mach number range may be obtained from equation (60) by setting $\tilde{A} = \infty$.

$$\left(\frac{\overset{\smile}{\Delta c_d}}{\alpha^2}\right) = D_\Delta(\xi_o, \infty, \tilde{\alpha}) = d_\Delta\ (\xi_o, \tilde{\alpha}) \tag{64}$$

At a Mach number of unity, equation (64), for the drag due to lift, reduces to the following:

$$\left(\frac{\Delta c_d}{\alpha^2}\right)_{\xi_o=0} = d_\Delta\ (0,\tilde{\alpha}), \qquad \left(\frac{\overset{\smile}{\Delta c_d}}{\alpha^2}\right)_{M_o=1} = \frac{1}{[(\gamma+1)(t/c)]^{1/3}} d_\Delta \left(0, \frac{\alpha}{t/c}\right) \tag{65}$$

Equation (65), together with hypothesis (d), indicates that, at sonic speed, the drag-rise ratio $\Delta c_d/\alpha^2$ of airfoils of a single family varies inversely as the cube root of the thickness ratio. For very large values of $\tilde{\alpha}$, the thickness ratio cannot have any effect on Δc_d; therefore, Δc_d is proportional to the five-thirds power of the angle of attack. If hypothesis (e) is accepted, c_d/α^2 at infinitesimal angles of attack is stationary with Mach number at $M_o = 1$, that is,

$$\left[\frac{d}{dM_o}\left(\frac{c_d}{\alpha^2}\right)\right]_{\substack{\alpha=0\\ M_o=1}} = 0, \qquad \left[\frac{d}{d\xi_o}\left(\frac{\overset{\smile}{c_d}}{\alpha^2}\right)\right]_{\substack{\tilde{\alpha}=0\\ \xi_o=0}} = 0 \tag{66}$$

Wings of vanishing aspect ratio.- Since the drag due to lift calculated by means of linear theory remains finite throughout the Mach number range, it is assumed, as in the preceding sections on lift and pitching moment, that linear theory is capable of describing the drag-due-to-lift characteristics of wings of vanishing aspect ratio at all Mach numbers. Therefore the following relations stemming from equation (62) hold:

$$\left(\frac{\overset{\smile}{\Delta C_D}}{\alpha^2}\right)_{l,\ \tilde{A}\to 0} = \left(\frac{\overset{\smile}{\Delta C_D}}{\alpha^2}\right)_{l,\ \xi_o=0} = \tilde{A} \times \text{const.} \tag{67}$$

Solutions of the equations of linear theory show that the value of the constant is $\pi/4$ for all wings of small $\sqrt{|M_o^2 - 1|}\,A$ whose trailing edges possess no cutouts extending forward of the most forward station of maximum span (i.e., triangular, rectangular, elliptical wings, etc.). The quoted value of the constant corresponds to the development of the full "leading-edge force." It is known that this force is oftentimes not completely realized due to a local separation and subsequent reattachment of the flow around the leading edge. If the leading-edge force is nonexistent, the corresponding value for the constant is $\pi/2$.

Wings of finite aspect ratio.- At the present time no drag-due-to-lift results have been obtained from the nonlinear theory for wings of either finite or infinite aspect ratio. The foregoing remarks, however, are sufficient to determine that a curve representing the variation of $\widetilde{\Delta C_D/\alpha^2}$ with $\tilde{A}$ for constant ξ_o and $\tilde{\alpha}$ would increase linearly with $\tilde{A}$ for small $\tilde{A}$ (unless the degree of attainment of the leading-edge force also depends on $\tilde{A}$) and become independent of $\tilde{A}$ for large $\tilde{A}$. The resulting curve would presumably have the same general appearance as that shown in sketch (i) for $\widetilde{C_L/\alpha}$.

It may sometimes be desired to present drag-due-to-lift results in terms of $\Delta C_D/C_L^2$ or $\Delta C_D/\alpha C_L$ rather than $\Delta C_D/\alpha^2$. The similarity rules for these quantities can be quickly deduced from the foregoing results.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., March 6, 1952

APPENDIX A

RELATION BETWEEN U_o AND a^* STATEMENTS OF THE TRANSONIC-FLOW EQUATIONS

Equations (3), (7), (8), (10), and (12) were presented in the text as being applicable to the study of transonic, as well as subsonic and supersonic, flow about thin wings. These equations, repeated below as equations (A1) through (A5), will be referred to as the U_o statement of the problem since the perturbation velocities are taken about the free-stream velocity U_o. The perturbation potential φ is defined by

$$\varphi = -U_o x + \Phi \tag{A1}$$

The differential equation is

$$(1 - M_o^2)\,\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = \frac{\gamma + 1}{U_o}\,\varphi_x\,\varphi_{xx} \tag{A2}$$

The boundary conditions are

at $x = -\infty$

$$(\varphi_x)_o = (\varphi_y)_o = (\varphi_z)_o = 0 \tag{A3}$$

at the wing surface

$$(\varphi_z)_{z=0} = U_o\left(\frac{\partial z}{\partial x}\right) \tag{A4}$$

The pressure coefficient is given by

$$C_p = -\frac{2}{U_o}\,(\varphi_x) \tag{A5}$$

In the a^* statement of the transonic-flow equations (equations (13) through (17) in the text), it is assumed that all velocities are only slightly different from the critical speed of sound a^*. The perturbation potential is defined by

$$\varphi' = -a^* x + \Phi \tag{A6}$$

and the resulting differential equation is

$$\varphi'_{yy} + \varphi'_{zz} = \frac{\gamma + 1}{a^*}\,\varphi'_x\,\varphi'_{xx} \tag{A7}$$

If the perturbation analysis is carried out in a completely consistent manner the boundary conditions are:

at $x = -\infty$

$$(\varphi'_x)_o = (M_o^2 - 1)\,\frac{a^*}{\gamma + 1}, \qquad (\varphi'_y)_o = (\varphi'_z)_o = 0 \tag{A8}$$

at the wing surface

$$(\varphi'_z)_{z=0} = a^*\left(\frac{\partial Z}{\partial x}\right) \tag{A9}$$

and the pressure coefficient is given by

$$C_p{}' = -\frac{2}{a^*}\left[\varphi'_x - (\varphi'_x)_o\right] \tag{A10}$$

The relation between the U_o and the a^* statements can be determined directly in the following manner. The differential equations for φ' and φ will be the same if

$$\frac{\gamma + 1}{a^*}\,\frac{\partial \varphi'}{\partial x} = \frac{\gamma + 1}{U_o}\,\frac{\partial \varphi}{\partial x} - (1 - M_o^2)$$

or if

$$\varphi' = \frac{a^*}{U_o}\,\varphi - \frac{a^*\,(1 - M_o^2)}{\gamma + 1}\,x \tag{A11}$$

The boundary conditions for φ' corresponding to those stated for φ in equations (A3) and (A4) are

at $x = -\infty$

$$\left.\begin{aligned}(\varphi'_x)_o &= \frac{a^*}{U_o}\,(\varphi_x)_o - \frac{a^*\,(1 - M_o^2)}{\gamma + 1} = -\frac{a^*\,(1 - M_o^2)}{\gamma + 1}\\ (\varphi'_y)_o &= \frac{a^*}{U_o}\,(\varphi_y)_o = 0, \qquad (\varphi'_z)_o = \frac{a^*}{U_o}\,(\varphi_z)_o = 0\end{aligned}\right\} \tag{A12}$$

at the wing surface

$$(\varphi'_z)_{z=0} = \frac{a^*}{U_o}\,(\varphi_z)_{z=0} = a^*\left(\frac{\partial Z}{\partial x}\right) \tag{A13}$$

Finally the pressure coefficient is given by

$$\left.\begin{aligned}C_p{}' = C_p &= -\frac{2}{U_o}\,\varphi_x = -\frac{2}{U_o}\left[\frac{U_o}{a^*}\,\varphi'_x + \frac{a^*\,(1 - M_o^2)}{\gamma + 1}\right] = \\ &-\frac{2}{a^*}\left[\varphi'_x - (\varphi'_x)_o\right]\end{aligned}\right\} \tag{A14}$$

Comparison of the above equations with those given previously for the completely consistent a^* analysis reveals their identity. As far as obtaining the values of C_p is concerned, therefore, the a^* statement of the problem may be regarded as a transformation of the more general U_o statement. The results so obtained are consequently valid throughout the Mach number range. As is evident from equation (A11), however, the local velocities, and consequently the local Mach numbers, found in the a^* analysis are only correct when the free-stream Mach number is unity.

APPENDIX B

DERIVATION OF SIMILARITY RULES

The basic equations of linear theory and of nonlinear theory of transonic flow may be summarized as follows. The differential equations are:

$$(1 - M_o^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \begin{cases} 0 & \text{Linear} \quad \text{(B1)} \\ \dfrac{\gamma + 1}{U_o}\dfrac{\partial\varphi}{\partial x}\dfrac{\partial^2\varphi}{\partial x^2} & \text{Nonlinear} \quad \text{(B2)} \end{cases}$$

The boundary conditions are:

at $x = -\infty$

$$\left(\frac{\partial\varphi}{\partial x}\right)_o = \left(\frac{\partial\varphi}{\partial y}\right)_o = \left(\frac{\partial\varphi}{\partial z}\right)_o = 0 \tag{B3}$$

at the wing surface

$$\frac{1}{U_o}\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = \tau\frac{\partial}{\partial(x/c)} f\left(\frac{x}{c}, \frac{y}{b}\right) \tag{B4}$$

where the geometry of the wing is given by

$$(Z/c) = \tau\ f(x/c,\ y/b) \tag{B5}$$

The pressure coefficient is given by

$$C_p = -\frac{2}{U_o}\frac{\partial\varphi}{\partial x} \tag{B6}$$

If the differential equations are now transformed into a system with primed quantities and the proportionality or stretching factors are denoted by s with appropriate subscripts such that

$$\left.\begin{aligned} &x' = s_x x, \quad y' = s_y y, \quad z' = s_z z, \quad \varphi' = s_\varphi \varphi, \quad U_o' = s_U U_o \\ &\sqrt{(1 - M_o^2)'} = s_\beta\sqrt{1 - M_o^2}\ , \qquad (\gamma + 1)' = \Gamma' = s_\Gamma \Gamma = s_\Gamma(\gamma + 1) \end{aligned}\right\} \tag{B7}$$

equations (B1) and (B2) become

$$\frac{s_x^2}{s_\beta^2 s_\varphi}(1-M_o^2)'\frac{\partial^2\varphi'}{\partial x'^2}+\frac{s_y^2}{s_\varphi}\frac{\partial^2\varphi'}{\partial y'^2}+\frac{s_z^2}{s_\varphi}\frac{\partial^2\varphi'}{\partial z'^2}=\begin{cases}0 & \text{Linear} \quad \text{(B8)}\\[2ex] \dfrac{s_U s_x^3}{s_\Gamma s_\varphi^2}\dfrac{(\gamma+1)'}{U_o'}\dfrac{\partial\varphi'}{\partial x'}\dfrac{\partial^2\varphi'}{\partial x'^2} & \text{Nonlinear} \quad \text{(B9)}\end{cases}$$

Similarity is assured if φ' satisfies the same differential equation and boundary conditions as φ. In order for the two flows to be the same, therefore, the following relations must hold:

$$\frac{s_y s_\beta}{s_x}=1,\quad \frac{s_y}{s_z}=1,\quad \frac{s_\varphi}{s_x s_U}=\begin{cases}\lambda/\lambda' & \text{Linear} \quad \text{(B10)}\\[2ex] s_\beta^{\,2}/s_\Gamma & \text{Nonlinear} \quad \text{(B11)}\end{cases}$$

where, for linear theory, λ/λ' is an arbitrary constant which can be equated to s^2_β/s_Γ if desired.

An immediate consequence of this transformation is that the wing plan forms undergo an affine transformation such that the aspect ratios of wings in similar flow fields are related, according to both linear and nonlinear theory, by

$$A' = \frac{s_y}{s_x}A = \frac{1}{s_\beta}A = \sqrt{\frac{1-M_o^2}{(1-M_o^2)'}}\,A \tag{B12}$$

or by

$$\sqrt{(1-M_o^2)'}\,A' = \sqrt{1-M_o^2}\,A \tag{B13}$$

Since φ' is proportional to φ, the boundary conditions at $x = -\infty$ are automatically satisfied. The boundary conditions at the wing may be given in either of two forms:

$$\left(\frac{\partial\varphi'}{\partial z'}\right)_{z'=0} = \frac{s_\varphi}{s_z}\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = \frac{s_\varphi}{s_z}U_o\tau\frac{\partial}{\partial(x/c)}f\left(\frac{x}{c},\frac{y}{b}\right) \tag{B14}$$

$$\left(\frac{\partial\varphi'}{\partial z'}\right)_{z'=0} = U_o'\tau'\frac{\partial}{\partial(x'/c')}f'\left(\frac{x'}{c'},\frac{y'}{b'}\right) = s_U U_o\,\tau'\frac{\partial}{\partial(x'/c')}f'\left(\frac{x'}{c'},\frac{y'}{b'}\right) \tag{B15}$$

whence, if the two wings have the same ordinate-distribution functions, that is, if $f'(x'/c', y'/b') = f(x/c, y/b)$, the ordinate-amplitude parameters are related as follows:

$$\tau' = \frac{s_\varphi}{s_U s_z}\,\tau = \frac{s_\varphi s_\beta}{s_U s_x}\,\tau = \begin{cases} \dfrac{\lambda}{\lambda'}\sqrt{\dfrac{(1-M_0^2)'}{1-M_0^2}}\,\tau & \text{Linear} \quad \text{(B16)} \\[2ex] \dfrac{s_\beta^3}{s_\Gamma}\,\tau = \left[\dfrac{(1-M_0^2)'}{1-M_0^2}\right]^{3/2}\left[\dfrac{\gamma+1}{(\gamma+1)'}\right]\tau & \text{Nonlinear} \quad \text{(B17)} \end{cases}$$

or as

$$\frac{\sqrt{(1-M_0^2)'}}{\lambda'\tau'} = \frac{\sqrt{1-M_0^2}}{\lambda\tau} \qquad \text{Linear} \quad \text{(B18)}$$

$$\frac{\sqrt{(1-M_0^2)'}}{[(\gamma+1)'\tau']^{1/3}} = \frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}} \qquad \text{Nonlinear(B19)}$$

The relation between the pressure coefficients at corresponding points on the wing surface is given by

$$C_p' = -\frac{2}{U_0'}\left(\frac{\partial\varphi'}{\partial x'}\right)_{z'=0} = \frac{s_\varphi}{s_U s_x}\left(-\frac{2}{U_0}\frac{\partial\varphi}{\partial x}\right)_{z=c} =$$

$$\frac{s_\varphi}{s_U s_x}\,C_p = \begin{cases} \dfrac{\lambda}{\lambda'}\,C_p & \text{Linear} \quad \text{(B20)} \\[2ex] \dfrac{s_\beta^2}{s_\Gamma}\,C_p = \left(\dfrac{\tau'}{\tau}\right)^{2/3}\left[\dfrac{\gamma+1}{(\gamma+1)'}\right]^{1/3} C_p & \text{Nonlinear} \quad \text{(B21)} \end{cases}$$

or more completely by

$$C_p'\left[\frac{\sqrt{(1-M_0^2)'}}{\lambda'\tau'},\ \sqrt{(1-M_0^2)'}\ A';\ \frac{x'}{c'},\ \frac{y'}{b'}\right] =$$

$$\frac{\lambda}{\lambda'}\,C_p\left(\frac{\sqrt{1-M_0^2}}{\lambda\tau},\sqrt{1-M_0^2}\,A;\ \frac{x}{c},\ \frac{y}{b}\right) \qquad \text{Linear} \quad \text{(B22)}$$

$$C_p'\left\{\frac{\sqrt{(1-M_0^2)'}}{[(\gamma+1)'\tau']^{1/3}},\sqrt{(1-M_0^2)'}\,A;\frac{x'}{c'},\ \frac{y'}{b'}\right\} =$$

$$\left[\frac{\gamma+1}{(\gamma+1)'}\right]^{1/3}\left(\frac{\tau'}{\tau}\right)^{2/3} C_p\left\{\frac{\sqrt{1-M_0^2}}{[(\gamma+1)\tau]^{1/3}},\sqrt{1-M_0^2}\ A;\ \frac{x}{c},\ \frac{y}{b}\right\} \qquad \text{Nonlinear(B23)}$$

The foregoing relationships may be summarized in the following statement: The similarity rule for the pressure coefficients on a family of wings having their geometry given by

$$(Z/c) = \tau f(x/c,\ y/b) \qquad \text{(B24)}$$

is

$$C_{p_l} = \frac{1}{\lambda} P_l \left(\frac{\sqrt{1-M_o{}^2}}{\lambda\tau}, \sqrt{1-M_o{}^2}A; \ \frac{x}{c}, \frac{y}{b} \right) \qquad \text{Linear} \qquad (B25)$$

$$C_p = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} P \left\{ \frac{\sqrt{1-M_o{}^2}}{[(\gamma+1)\tau]^{1/3}}, \sqrt{1-M_o{}^2}A; \ \frac{x}{c}, \frac{y}{b} \right\} \ \text{Nonlinear} \ (B26)$$

The similarity rules for the lift, pitching-moment, and drag coefficients given by linear theory are

$$C_{L_l} = \frac{1}{\lambda} L_l \left(\frac{\sqrt{1-M_o{}^2}}{\lambda\tau}, \sqrt{1-M_o{}^2} \ A \right) \qquad \text{Linear} \qquad (B27)$$

$$C_{m_l} = \frac{1}{\lambda} M_l \left(\frac{\sqrt{1-M_o{}^2}}{\lambda\tau}, \sqrt{1-M_o{}^2} \ A \right) \qquad \text{Linear} \qquad (B28)$$

$$C_{D_l} = \frac{\tau}{\lambda} D_l \left(\frac{\sqrt{1-M_o{}^2}}{\lambda\tau}, \sqrt{1-M_o{}^2} \ A \right) \qquad \text{Linear} \qquad (B29)$$

The corresponding similarity rules given by nonlinear theory are

$$C_L = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} L \left\{ \frac{\sqrt{1-M_o{}^2}}{[(\gamma+1)\tau]^{1/3}}, \ \sqrt{1-M_o{}^2} \ A \right\} \qquad \text{Nonlinear} \ (B30)$$

$$C_m = \frac{\tau^{2/3}}{(\gamma+1)^{1/3}} M \left\{ \frac{\sqrt{1-M_o{}^2}}{[(\gamma+1)\tau]^{1/3}}, \ \sqrt{1-M_o{}^2} \ A \right\} \qquad \text{Nonlinear} \ (B31)$$

$$C_D = \frac{\tau^{5/3}}{(\gamma+1)^{1/3}} D \left\{ \frac{\sqrt{1-M_o{}^2}}{[(\gamma+1)\tau]^{1/3}}, \ \sqrt{1-M_o{}^2} \ A \right\} \qquad \text{Nonlinear} \ (B32)$$

It should be noted that the foregoing equations have been written for subsonic flow where $M_o \leq 1$. If $M_o \geq 1$, the radical $\sqrt{1-M_o{}^2}$ should be replaced with $\sqrt{M_o{}^2 - 1}$.

In the linear-theory analysis, λ has remained a completely arbitrary coefficient to be selected as best suits the particular problem at hand. For instance, the compressible-flow relationships between two wings having identical pressure distributions are found by setting $\lambda = 1$. If, on the other hand, λ is set equal to $\sqrt{1-M_o{}^2}$, the thickness ratio, camber, and angle of attack of related wings are identical. The greatest simplification of the similarity rules of linear theory occurs when λ is set equal to $\sqrt{1-M_o{}^2}/\tau$ since the number of parameters necessary to show the results of linear theory is thereby decreased by one. This degree of arbitrariness in the similarity rules for linear theory is in contrast to the case for nonlinear transonic theory in which no undetermined coefficient like λ is to be found.

For the present purpose of gaining a better understanding of transonic flows, the most advantageous choice for λ is

$$\lambda = \frac{(\gamma + 1)^{1/3}}{\tau^{2/3}} \tag{B33}$$

because then the similarity rules for linear theory assume forms identical to those for nonlinear transonic theory. This is important since it implies that solutions of linear theory and of nonlinear transonic theory can be expressed as functions of the same parameters and hence can both be plotted on a single graph in terms of one set of parameters. The two theories would, of course, yield two distinct curves on such a plot. The curve for linear theory would be accepted as valid for purely subsonic and purely supersonic flows but may or may not be valid in the transonic range, as discussed previously. The curve for nonlinear transonic theory is valid not only for transonic flows, but for subsonic and supersonic small perturbation flows as well. As is evident from the derivation of the basic equations, however, the results of the nonlinear transonic theory should be considered to be of only equal accuracy to those of linear theory in the definitely subsonic and supersonic regimes, despite the fact that the solutions are much more difficult to obtain.

APPENDIX C

SHOCK-WAVE RELATIONS

Similarity rules were derived in appendix B through consideration of the differential equation for the perturbation velocity potential φ. Since the transonic equation involves the existence of a velocity potential, the derived similarity rules might be assumed limited to regions of flow lying between discontinuities or shock waves. It will be shown in the following, however, that the same basic parameters govern the transition through weak normal or oblique shocks so that the similarity rules can be used to relate flows containing shock waves.

If the velocity immediately before the shock is designated by $\bar{U}_1$ and the velocity components immediately behind the shock extending in directions parallel and perpendicular to $\bar{U}_1$ are designated, respecrespectively, by $\bar{U}_2$ and $\sqrt{\bar{V}_2{}^2 + \bar{W}_2{}^2}$, the classical equation for the shock polar provides that

$$\bar{V}_2{}^2 + \bar{W}_2{}^2 = \left(\bar{U}_1-\bar{U}_2\right)^2 \frac{\bar{U}_1\bar{U}_2 - a^{*2}}{\frac{2}{\gamma+1}\bar{U}_1{}^2 - \bar{U}_1\bar{U}_2 + a^{*2}} \tag{C1}$$

Except for the important case of the bow wave in supersonic flow, $\bar{U}_1$ is not generally alined with the direction of the x axis, but is inclined a small angle. With the resolution into components parallel to the axes of the coordinate system and upon carrying out a small perturbation analysis analogous to that performed in the derivation of equation (7), equation (C1) provides the following relation between the velocity components (potential gradients) immediately fore and aft of the shock:

$$\left(1-M_o{}^2\right)\left(\varphi_{x_2}-\varphi_{x_1}\right)^2 + \left(\varphi_{y_2}-\varphi_{y_1}\right)^2 + \left(\varphi_{z_2}-\varphi_{z_1}\right)^2 =$$

$$\frac{\gamma+1}{U_o}\left(\frac{\varphi_{x_2}+\varphi_{x_1}}{2}\right)\left(\varphi_{x_2}-\varphi_{x_1}\right)^2 \tag{C2}$$

This equation corresponds to the shock-polar curve for weak shock waves inclined at any angle between that of normal shock waves and that of the Mach lines.

The striking correspondence of equations (C2) and (7) make it almost self evident that the shock relations satisfy the same similarity rules as the differential equation for the perturbation velocity potential. This can be verified quickly by transforming equation (C2) into a system with primed quantities related to the original quantities by equation (B7), thus

$$\frac{s_x^2}{s_\beta^2 s_\varphi^2}\left(1-M_o^2\right)'\left(\varphi'_{x'_2}-\varphi'_{x'_1}\right)^2+\frac{s_y^2}{s_\varphi^2}\left(\varphi'_{y'_2}-\varphi'_{y'_1}\right)^2+\frac{s_z^2}{s_\varphi^2}\left(\varphi'_{z'_2}-\varphi'_{z'_1}\right)^2=$$

$$\frac{s_U s_x^3}{s_\Gamma s_\varphi^3}\frac{(\gamma+1)'}{U_o'}\left(\frac{\varphi'_{x'_2}+\varphi'_{x'_1}}{2}\right)\left(\varphi'_{x'_2}-\varphi'_{x'_1}\right)^2 \tag{C3}$$

In order to assure that flows through shocks in the primed and unprimed systems are similar, the following relations must hold:

$$\frac{s_y s_\beta}{s_x}=1,\qquad \frac{s_y}{s_z}=1,\qquad \frac{s_\varphi s_\Gamma}{s_x s_U s_\beta^2}=1 \tag{C4}$$

These are identical to the requirements already specified in equation (B11) for similarity of the flows in the shock-free regions. Therefore it follows that the similarity rules developed in appendix B on the basis of the potential equation are, in fact, valid for flows containing shock waves.

APPENDIX D

SLOPE OF PRESSURE CURVE AT $M_o = 1$

Consider a transonic flow problem in which the free-stream Mach number is slightly less than unity ($M_o^- = 1 - \epsilon$) where ϵ is a small positive quantity. Since the object is only to determine the value of a slope as ϵ approaches zero, there is no loss in accuracy introduced by retaining only the leading powers of ϵ. To this order, the differential equation for φ^- is

$$2\epsilon\, \varphi_{xx}^- + \varphi_{yy}^- + \varphi_{zz}^- = \frac{\gamma+1}{a_o(1-\epsilon)}\, \varphi_x^-\, \varphi_{xx}^- \tag{D1}$$

The boundary conditions are:

at $x = -\infty$

$$(\varphi_x^-)_o = (\varphi_y^-)_o = (\varphi_z^-)_o = 0 \tag{D2}$$

at the wing surface

$$\frac{1}{a_o(1-\epsilon)}(\varphi_z^-)_{z=o} = \tau^- \frac{\partial}{\partial(x/c)} f^-\left(\frac{x}{c}, \frac{y}{b}\right) \tag{D3}$$

The pressure coefficient is given by

$$C_p^- = -\frac{2}{a_o(1-\epsilon)}\, (\varphi_x^-)_{z=o} \tag{D4}$$

If the Mach number of the flow is now increased to $M_o^+ = 1 + \epsilon$, keeping a_o constant, the differential equation for φ^+ is

$$-2\epsilon\, \varphi_{xx}^+ + \varphi_{yy}^+ + \varphi_{zz}^+ = \frac{\gamma+1}{a_o(1+\epsilon)}\, \varphi_x^+\, \varphi_{xx}^+ \tag{D5}$$

hence φ^+ and φ^- both satisfy the same equation provided

$$\frac{\gamma+1}{a_o(1+\epsilon)}\, \varphi_x^+ + 2\epsilon = \frac{\gamma+1}{a_o(1-\epsilon)}\, \varphi_x^- - 2\epsilon \tag{D6}$$

or if

$$\varphi^+ = \frac{1+\epsilon}{1-\epsilon}\, \varphi^- - \frac{4\epsilon a_o}{\gamma+1}\, x + \text{const.} \tag{D7}$$

Consideration of the boundary conditions at the wing shows that the wing geometry is preserved by the new potential since

$$\tau^+ \frac{\partial}{\partial(x/c)} f^+\left(\frac{x}{c}, \frac{y}{b}\right) = \frac{1}{(1+\epsilon)a_o} (\varphi^+_x)_{z=o} =$$

$$\frac{1}{(1-\epsilon)a_o} (\varphi^-_z)_{z=o} = \tau^- \frac{\partial}{\partial(x/c)} f^-\left(\frac{x}{c}, \frac{y}{b}\right) \tag{D8}$$

At $x = -\infty$, however, an alteration of the boundary conditions is observed since the velocity perturbations no longer vanish but take on a uniform value given by

$$(\varphi^+_x)_o = -\frac{4\epsilon a_o}{\gamma+1} = -\frac{4(M_o^+-1)a_o}{\gamma+1}, \qquad (\varphi^+_y)_o = (\varphi^+_z)_o = 0 \tag{D9}$$

It can be shown that such a velocity perburbation is just that which would exist if there were a normal shock wave of infinite lateral extent standing infinitely far ahead of the wing. (See equation (C2).) If it can be assumed that such a shock is actually present, the pressure coefficient in slightly supersonic flow is related to that in slightly subsonic flow in the following manner:

$$C_p^+ = -\frac{2}{a_o(1+\epsilon)}(\varphi^+_x)_{z=o} = -\frac{2}{a_o(1-\epsilon)}(\varphi^-_x)_{z=o} + \frac{8\epsilon}{\gamma+1} = C_p^- + \frac{8\epsilon}{\gamma+1} \tag{D10}$$

The slope of the pressure curve at $M_o = 1$ is therefore

$$\left(\frac{dC_p}{dM_o}\right)_{M_o=1} = \lim_{\epsilon \to o} \frac{C_p^+ - C_p^-}{2\epsilon} = \frac{4}{\gamma+1} \tag{D11}$$

the value originally given by Liepmann and Bryson (reference 15).

It is to be emphasized that it is necessary to assume the presence of the velocity field at $x = -\infty$ given by equation (D9), due presumably to a normal shock wave, in both the present derivation and those of references 8 and 15. Whether or not this is a permissible assumption for any given case still remains an unanswered question. Intuitive considerations suggest that the results are probably applicable to symmetrical airfoils at zero or infinitesimal angles of attack but not to airfoils at larger angles of attack or to wings of finite span.

REFERENCES

1. Oswatitsch, K., and Wieghardt, K.: Theoretische Untersuchungen über stationäre Potential Strömungen und Grenzschichten bei hohen Geschwindigkeiten. Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht S 13/1d, 1942, S. 7-24. (Also available as NACA TM 1189, 1948)

2. Busemann, A., and Guderley, K. Gottfried: The Problem of Drag at High Subsonic Speeds. British Ministry of Aircraft Production, Volkenrode, Rept. and Trans. No. 184, March 1947.

3. Guderley, K. Gottfried: Considerations of the Structure of Mixed Subsonic - Supersonic Flow Patterns. Tech. Rept. F-TR-2168-ND, AAF, Air Materiel Command, Wright Field, 1947.

4. Guderley, K. Gottfried: On the Transition from a Transonic Potential Flow to a Flow with Shocks. Tech. Rept. F-TR-2160-ND, AAF, Air Materiel Command, Wright Field, August 1947.

5. von Kármán, Theodore: Supersonic Aerodynamics - Principles and Applications. Jour. Aero. Sci., vol. 14, no. 7, July 1947, pp. 373-409.

6. von Kármán, Theodore: The Similarity Law of Transonic Flow. Jour. Math. and Physics, vol. XXVI, no. 3, Oct. 1947, pp. 182-190.

7. Guderley, Gottfried, and Yoshihara, Hideo: The Flow Over a Wedge Profile at Mach Number 1. Jour. Aero. Sci., vol. 17, no. 11, Nov. 1950, pp. 723-735.

8. Vincenti, Walter G., and Wagoner, Cleo B.: Transonic Flow Past a Wedge Profile With Detached Bow Wave - General Analytical Method and Final Calculated Results. NACA TN 2339, 1951.

9. Vincenti, Walter G., and Wagoner, Cleo B.: Transonic Flow Past a Wedge Profile With Detached Bow Wave - Details of Analysis. NACA TN 2588, 1951.

10. Cole, Julian D.: Drag of Finite Wedge at High Subsonic Speeds. Jour. Math. and Physics, vol. 30, no. 2, July 1951, pp. 79-93.

11. Trilling, Leon: Transonic Flow Past a Wedge at Zero Angle of Attack. Paper presented at American Physical Society annual meeting, New York, Jan. 1952.

12. Oswatitsch, K.: Die Geschwindigkeitsverteilung bie lokalen Überschallgebieten an flachen Profilen. Zeitschrift für Angewandte Mathematik und Mechanik, Bd. 30, Nr. 1/2, Jan./Feb. 1950, S.17-24.

13. Oswatitsch, K.: Die Geschwindigkeitsverteilung an symmetrischen Profilen beim Auftreten lokalen Uberschallgebiete. Acta Physica Austriaca, Bd. 4, Nr. 2-3, Dec. 1950, S.228-271.

14. Gullstrand, Tore R.: The Flow Over Symmetrical Aerofoils Without Incidence in the Lower Transonic Range. KTH Aero TN 20, Royal Institute of Technology, Stockholm, Sweden, 1951.

15. Liepmann, H. W., and Bryson, A. E. Jr.: Transonic Flow Past Wedge Sections. Jour. Aero. Sci., vol. 17, no. 12, Dec. 1950, pp. 745-755.

16. Bryson, Arthur Earl, Jr.: An Experimental Investigation of Transonic Flow Past Two-Dimensional Wedge and Circular-Arc Sections Using a Mach-Zehnder Interferometer. NACA TN 2560, 1951.

17. Kaplan, Carl: On Similarity Rules for Transonic Flows. NACA TN 1527, 1948.

18. Spreiter, John R.: Similarity Laws for Transonic Flow About Wings of Finite Span. NACA TN 2273, 1951.

19. Berndt, S. B.: Similarity Laws for Transonic Flow Around Wings of Finite Aspect Ratio. KTH Aero TN 14, Royal Inst. Tech., Stockholm, Sweden, 1950.

20. Oswatitsch, K., and Berndt, S. B.: Aerodynamic Similarity at Axisymmetric Transonic Flow Around Slender Bodies. KTH Aero TN 15, Royal Inst. Tech., Stockholm, Sweden, 1950.

21. Heaslet, Max. A., Lomax, Harvard, and Spreiter, John R.: Linearized Compressible-Flow Theory for Sonic Flight Speeds. NACA Rep. 956, 1950. (Formerly NACA TN 1824)

22. Jones, Robert T.: Properties of Low-Aspect Ratio Pointed Wings at Speeds Below and Above the Speed of Sound. NACA Rep. 835, 1946. (Formerly NACA TN 1032)

23. Mangler, K. W.: Calculation of the Pressure Distribution over a Wing at Sonic Speeds. RAE Rep. No. Aero. 2439 (British), Sept. 1951.

24. Lomax, Harvard and Heaslet, Max. A.: Linearized Lifting-Surface Theory for Swept-Back Wings with Slender Plan Forms. NACA TN 1992, 1949.

25. Ackeret, J.: Luftkäfte auf Flügel, die mit grösstererals Schallgeschwindigkeit bewegt werden: Zeitschrift für Flugtechik und Motorluftschiffahrt, Feb. 14, 1925, S.72-74. (Also as NACA TM 317, 1925)

26. Puckett, Allen E.: Supersonic Wave Drag of Thin Airfoils. Jour. Aero. Sci., vol. 13, no. 9, Sept. 1946, pp. 475-484.

27. Guderley, Gottfried, and Yoshihara, Hideo: Unsymmetrical Flow Patterns at Mach Number One. Air Force, Air Materiel Command, Tech. Rep. 6683.

28. Stewart, H. J.: The Lift of a Delta Wing at Supersonic Speeds. Quart. App. Math., vol. IV, no. 3, Oct. 1946, pp. 246-254.

29. Brown, Clinton E.: Theoretical Lift and Drag of Thin Triangular Wings at Supersonic Speeds. NACA Rep. No. 839, 1946. (Formerly NACA TN 1183)

30. De Young, John, and Harper, Charles W.: Theoretical Symmetric Span Loading at Subsonic Speeds for Wings Having Arbitrary Plan Form. NACA Rep. No. 921, 1948.

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