NATIONAL ADVISORY COMMITTEE
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TECHNICAL NOTE 3049

AN ANALYTICAL INVESTIGATION OF THE EFFECT

OF THE RATE OF INCREASE OF TURBULENT KINETIC ENERGY IN

THE STREAM DIRECTION ON THE DEVELOPMENT OF TURBULENT

BOUNDARY LAYERS IN ADVERSE PRESSURE GRADIENTS

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SUMMARY

A general integral form of the boundary-layer equation which includes the Reynolds normal-stress term was derived. From this general form, two special equations - namely, the modified momentum equation and the modified kinetic-energy equation - are obtained. These modified equations include the effect of the Reynolds normal stress in the stream direction.

The parameters for the dissipation of the mean-flow kinetic energy by the shearing stress are suggested by the use of the Fediaevsky analysis for the shearing stress across turbulent boundary layers and by the use of a friction formula that makes the surface shear depend on the shape of the velocity profile, as well as on the Reynolds numbers. The parameters obtained in this way are found to be the same as those previously assumed by other investigators.

The parameters for the Reynolds normal stress are suggested by assuming that a relation between the local fluctuating axial velocity and the local mean velocity that is useful in wind tunnels can be used to determine some of the parameters upon which the Reynolds normal stress may depend. A test of these parameters by a limited amount of data shows no better correlation than previously obtained by other investigators who used only one of the three parameters obtained in this analysis. This lack of correlation may be caused by incorrect parameters or insufficient data.

INTRODUCTION

The importance of the problem of predicting the behavior of the turbulent boundary layer in regions of adverse pressure gradients has led to

a number of investigations during the past. Some of the more represen-
tative papers are those of Buri (ref. 1), Gruschwitz (ref. 2), whose
main effort was directed toward finding a single shape parameter against
which the velocity profiles could be correlated, and Von Doenhoff and
Tetervin (ref. 3), who found that Gruschwitz's suggestion of a single-
parameter family of velocity profiles appeared to be essentially correct
and who developed an empirical equation for the prediction of the develop-
ment of the turbulent boundary layer. None of these investigations, how-
ever, considered the effect of the rate of increase in the stream direc-
tion of the Reynolds normal-stress term.

When the wall shear was computed from experimental measurements of
the boundary-layer velocity profiles by using the Von Kármán momentum
equation (ref. 4), the wall shear stress was found to increase toward
separation. Wall-shear-stress measurements made with a heat-transfer
instrument (ref. 5), however, indicated that the wall shear stress
diminished toward separation. This seeming inconsistency of the wall
shear stress was clarified somewhat in a paper by Wallis (ref. 6), who
suggested that the usual procedure of neglecting the effect of the rate
of increase of the Reynolds normal-stress term would give misleading
results in the computing of the behavior of turbulent boundary layers
near separation. Neglect of this effect would result in computed values
giving too low a value of momentum-thickness gradient near separation.

The first work toward checking the suggestion of reference 6 was
done by Bidwell (ref. 7), who analytically arrived at a modified momentum
equation which included the effect of all the Reynolds stress terms.
The data of Schubauer and Klebanoff (ref. 8) were used in evaluating the
additional effects, with the result that only the effect of the rate of
increase of the Reynolds normal-stress term gave an important contribution.

Another work toward clarifying the problem was done by Rubert and
Persh (ref. 9). These investigators found from a number of measurements
of mean-flow terms that the values of momentum-thickness gradient near
separation computed from the Von Kármán momentum equation (ref. 10) did
not check with their experimental values. Realizing the limitations of
the Von Kármán momentum equation, these investigators summed the mean-
flow terms and the wall shear stress and then subtracted this sum from
the experimentally measured momentum-thickness gradient. The residue
was assumed to be due entirely to the rate of increase in the stream
direction of the Reynolds normal-stress term. In addition, they inves-
tigated the kinetic-energy expression of reference 11. The residue for
their modified expression, however, was found to be negligible.

The purpose of this paper is to provide a derivation of the general
integral form of the boundary equation without neglecting the Reynolds
normal-stress term. Two special cases of this equation are given explic-
itly. They are the momentum equation and the kinetic-energy equation.

Both expressions include the Reynolds normal-stress term. The present work also suggests parameters which influence the dissipation of the mean-flow kinetic energy by the shearing stress and by the Reynolds normal stress.

This paper, with modifications as to some details, was submitted to the University of Virginia in the form of a thesis in partial fulfillment of the requirements of the degree of Master of Science of Aeronautical Engineering.

SYMBOLS

A_1 Fediaevsky parameter, $\dfrac{\delta}{\tau_0} \dfrac{d\bar{p}}{dx}$

f velocity ratio, $\bar{u}/U$

g shear-stress ratio, τ/τ_0

H boundary-layer shape parameter, δ^*/θ

$$I = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1}\left(1 - f^{m+1}\right)\left(\int_0^y \frac{\partial f}{\partial H}\, dy\right) dy$$

$$I_1 = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1}\left(1 - f^{m+1}\right)\left[\int_0^y \frac{\partial(1 - f)}{\partial x}\, dy\right] dy$$

$$J = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1}\left(1 - f^{m+1}\right)\left[\int_0^y (1 - f)\,dy\right] dy$$

K ratio of kinetic-energy thickness to momentum thickness,

$$\frac{1}{\theta} \int_0^\delta (1 - f^2)f\, dy$$

$$K' = \frac{dK}{dH}$$

$$L = \frac{1}{\theta^{n+1}} \int_0^\delta f\left(1 - f^{m-1}\right)y^n dy$$

l contraction ratio

$$M = \frac{1}{\theta^{n+1}} \int_0^\delta y^n \left(1 - f^{m+1}\right) dy$$

m exponent of $\bar{u}$ in derivation of general integral equation

$$N = \frac{1}{\theta^{n+1}} \int_0^\delta y^n \left(1 - f^{m+1}\right) f \, dy$$

$$N' = \frac{dN}{dH}$$

n exponent of y in derivation of general integral equation

$\bar{p}$ mean pressure

q dynamic pressure outside boundary layer, $\frac{1}{2} \rho U^2$

R_δ boundary-layer Reynolds number based on boundary-layer thickness, $U\delta/\nu$

R_θ boundary-layer Reynolds number based on momentum thickness, $U\theta/\nu$

r power-profile exponent

U mean velocity just outside boundary layer

$\bar{u}$ mean velocity in boundary layer in x-direction

u' fluctuating velocity in x-direction

$\overline{u'}$ time average of u'

$\bar{v}$ mean velocity in boundary layer in y-direction

v' fluctuating velocity in y-direction

$\overline{v'}$ time average of v'

x distance along surface

y distance normal to surface

δ boundary-layer thickness

$\delta*$ boundary-layer displacement thickness, $\displaystyle\int_0^\delta (1 - f)\,dy$

$\eta = \dfrac{y}{\theta}$

θ boundary-layer momentum thickness, $\displaystyle\int_0^\delta (1 - f)f\,dy$

μ absolute value of dynamic viscosity

ν kinematic viscosity of air

ξ power-profile parameter, y/δ

ρ mean density of air

τ total shear stress

τ_0 wall shear stress

$\dfrac{\tau_R}{2q}$ turbulent-normal-stress coefficient, $\dfrac{1}{U^2}\displaystyle\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})\,dy$

ANALYSIS

Derivation of Modified Integral Momentum and Kinetic-Energy

Equations by the Method of Tetervin and Lin

The analysis described in this section differs from the method of Tetervin and Lin (ref. 12) in three aspects. First of all, the effect of suction (or blowing) velocity is not considered; consequently, all the v_0 terms in the analysis of Tetervin and Lin are omitted. Second, this analysis is for two-dimensional flow only. Although the momentum equation for two-dimensional flow is not the same as for flow over a body of revolution, it can be shown that the kinetic-energy equation is the same. Third, and most important, this analysis includes the effect of the rate of increase of the Reynolds normal stress in the stream direction. As a result, the Von Kármán momentum equation is modified to include an additional stress term and the kinetic-energy

equation is modified to include two Reynolds stress terms which differ in sign.

If the Reynolds stress term $(-\rho\overline{u'u'})$ is included in the boundary-layer equation for steady two-dimensional incompressible flow, this equation becomes

$$\rho\left(\overline{u}\,\frac{\partial\overline{u}}{\partial x} + \overline{v}\,\frac{\partial\overline{u}}{\partial y}\right) = -\frac{d\overline{p}}{dx} + \frac{\partial}{\partial y}\left(\mu\,\frac{\partial\overline{u}}{\partial y} - \rho\overline{u'v'}\right) - \frac{\partial}{\partial x}(\rho\overline{u'u'}) \qquad (1)$$

where $\dfrac{\partial}{\partial y}(-\rho\overline{u'v'})$ is the Reynolds shear-stress term and $\dfrac{\partial}{\partial x}(\rho\overline{u'u'})$ is the term due to the increasing Reynolds normal stress. It is thus seen that the effect of an axially increasing or decreasing Reynolds normal stress is such as to produce an additional stress. The general integral form of the boundary-layer equation of the analysis of reference 12, when modified as indicated in equation (1), becomes

$$(n + 1)N\,\frac{d\theta}{dx} + \theta\left(\frac{dN}{dx} - nI_1\right) + \frac{\theta}{U}\,\frac{dU}{dx}\left[N(m + 2) - n(J - M) - L(m + 1)\right] =$$

$$-(m + 1)\,\frac{1}{\rho U^2}\left[\int_0^\delta \eta^n_f^m\,\frac{\partial\tau}{\partial y}\,dy - \int_0^\delta \eta^n_f^m\,\frac{\partial}{\partial x}(\rho\overline{u'u'})dy\right] \qquad (2)$$

where τ is the total shear stress.

The general integral equation (2) was derived by multiplying equation (1) by $\overline{u}^m$ and y^n (where m and n are arbitrary integers) and integrating from $y = 0$ to $y = \delta$. This equation contains the Von Kármán integral momentum and the integral kinetic-energy equations, since these equations are derived from the boundary-layer equation by taking appropriate values of m and n. The Von Kármán momentum equation is derived from the boundary-layer equation when it is integrated with respect to y. Thus, for $m = 0$ and $y = 0$, equation (2) should reduce to the modified momentum equation.

For $m = n = 0$, equation (2) becomes

$$N\,\frac{d\theta}{dx} + \theta\,\frac{dN}{dx} + \frac{\theta}{U}\,\frac{dU}{dx}(2N - L) = -\frac{1}{\rho U^2}\int_0^\delta \frac{\partial\tau}{\partial y}\,dy + \frac{1}{\rho U^2}\int_0^\delta \frac{\partial}{\partial x}(\rho\overline{u'u'})dy \qquad (3)$$

since for $n = 0$,

$$J\theta = M\theta = I_1\theta = 0$$

Also, for $m = n = 0$,

$$N\theta = \int_0^\delta (1 - f)f \, dy$$

and

$$L\theta = \int_0^\delta (f - 1)dy$$

Thus, it follows that

$$N = 1$$

$$L = -H$$

and equation (3) becomes

$$\frac{d\theta}{dx} + \frac{\theta}{U}\frac{dU}{dx}(H + 2) = \frac{\tau_o}{\rho U^2} + \frac{1}{U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy \tag{4}$$

which is the modified Von Kármán momentum equation. Comparison of equation (4) with the well-known Von Kármán momentum equation indicates that the term

$$\frac{1}{U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy \tag{5}$$

is the contribution due to the rate of increase of the Reynolds normal stress in the stream direction.

If the value of $d\theta/dx$ of equation (4) is substituted into equation (3), and the further approximation is made that the velocity profiles form a single-parameter family of curves $f = f(\eta, H)$ where

$$\eta = \frac{y}{\theta}$$

$$H = \frac{\delta^*}{\theta}$$

equation (3) becomes, after lengthy manipulation,

$$\theta \frac{dH}{dx}\left(\frac{dN}{dH} + nI\right) = \frac{\theta}{U}\frac{dU}{dx}\left[-n(J - M)(H + 1) + L(m + 1) + N(H - m)\right] +$$

$$\frac{\tau_0}{\rho U^2}\left[n(J - M) - N - (m + 1)\int_0^\delta \eta^n f^m \frac{\partial g}{\partial y}\,dy\right] +$$

$$\frac{1}{U^2}\left\{\left[n(J - M) - N\right]\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy + \right.$$

$$\left. (m + 1)\int_0^\delta \eta^n f^m \frac{\partial}{\partial x}(\overline{u'u'})dy\right\} \tag{6}$$

The kinetic-energy equation is obtained by letting $m = 1$ and $n = 0$. For $n = 0$, the integrals I, J, and M have finite integrands. When $m = 1$ and $n = 0$, $L = 0$ and

$$N = \frac{1}{\theta}\int_0^\delta f\left(1 - f^2\right)dy = K$$

Then

$$\frac{dN}{dH} = N' = K' = \frac{dK}{dH}$$

For these conditions, equation (6) reduces to

$$\theta \frac{dH}{dx} = \frac{\theta}{U}\frac{dU}{dx}\left[\frac{K(H - 1)}{K'}\right] - \frac{\tau_0}{\rho U^2}\left(\frac{-K + 2\int_0^\delta f\frac{\partial g}{\partial y}\,dy}{K'}\right) -$$

$$\frac{1}{U^2}\left[\frac{K\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy - 2\int_0^\delta f\frac{\partial}{\partial x}(\overline{u'u'})dy}{K'}\right] \tag{7}$$

which is the modified kinetic-energy equation.

Equation (7) shows that the contribution of the rate of increase of the Reynolds normal stress in the streamwise direction results in two terms which differ in sign.

In reference 9, power profiles were assumed in the kinetic-energy equation without obtaining any residue term. Equation (7) was then introduced in appendix C of reference 9 and power profiles were again assumed but, for the data available, no pertinent values for the sum of the last two terms of equation (7) were obtained.

Determination of the Streamwise Variation

of the u' Fluctuation

The integrals

$$\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})\,dy$$

and

$$\int_0^\delta f \frac{\partial}{\partial x}(\overline{u'u'})\,dy$$

are contained in the modified momentum and kinetic-energy equations. If it were possible to predict exactly the manner in which u' varies with x, the above integrals could be obtained either by numerical or graphical integration. Since no exact expression exists, the integrals must be evaluated by approximate methods. One such method involves the determination of the parameters upon which u' depends. This method requires some suitable approximation of the relation between u' and x. The work of G. I. Taylor (ref. 13) suggests such a relationship.

If the mean flow in a converging entry is assumed to be irrotational, the disturbance or fluctuation can be regarded as due to vorticity. The effect of the contraction is to elongate filaments of particles lying in the direction of the mean flow and to contract filaments lying perpendicular to the mean flow. It is shown in detail in reference 13 that, if the vortices are very much elongated in the direction of the stream, the contraction of the disturbances is in inverse proportion to the contraction of the mean flow. For a different arrangement

or elongation of these vortices the proportionality constant varies from
1.0 to 0.5. Thus, if the mean flow increases from $\bar{u}$ to $l\bar{u}$, the fluc-
tuation decreases from u' to u'/l or $u'/2l$, depending on the vor-
tices predominant in the flow. This concept gives fair results for the
relation between u' and $\bar{u}$ for the flow in a contracting channel like
the entrance cone of a wind tunnel.

It is realized that this concept pertains to a flow where the
effects of viscous and turbulence stresses are small. In the present
work a relation that can be used in a boundary layer is needed. A test
of Taylor's concept was made from the data of reference 8. The rela-
tion is

$$u' = u_r'\bar{u}_r\, \frac{1}{\bar{u}} \tag{8}$$

or, in nondimensional form,

$$\frac{u'}{U} = \left(\frac{u'}{U}\right)_r\left(\frac{\bar{u}}{U}\right)_r \frac{U}{\bar{u}} \tag{8a}$$

where the subscript r denotes reference quantities which are constant
for any particular streamline. The procedure for calculating the axially
fluctuating terms consists in determining the quantities for a reference
station. Equation (8) is then used to calculate u' along a streamline
for various values of x, when $\bar{u}$ is known as a function of x.

The results obtained by using equation (8a), where the reference
quantities were evaluated only at $x = 17.5$ feet, are given in table I
and shown plotted in figure 1. The results indicate that equation (8a),
insofar as the flow in a boundary layer is concerned, is considerably
affected by the viscous and turbulence stresses, which are large in this
case. In figure 2, for which the calculations are given in table II,
two computed curves are shown for the station at $x = 22.5$ feet. The
17.5-foot station was used as the reference to evaluate the lower curve
and the 20.0-foot station was used as the reference to evaluate the
upper curve. Table II and figure 2 indicate that the results are in
fair agreement when the reference station is taken just ahead of the
station for which the computations are made. These results indicate
that the Taylor relation may be sufficiently accurate to suggest param-
eters upon which the terms involving the effect of the rate of increase
of the Reynolds normal stress depend.

Determination of the Parameters for the Term

$$\frac{1}{U^2} \int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy$$

The purpose of this section is to determine parameters upon which the term (eq. (5)) involving the effect of the rate of increase of the Reynolds normal stress may depend. The results, of course, are not based on a rigorous analysis since, as indicated in reference 13, the exact vortex patterns and the effects of viscosity and diffusion cannot be determined.

If the Taylor relationship is assumed to be sufficiently accurate to apply to the flow in a boundary layer, equation (8) may be squared to obtain

$$\overline{u'^2} = \overline{u_r'^2}\bar{u}_r^2 \frac{1}{\bar{u}^2} \tag{9}$$

Since $\overline{u'^2}$ at the outer edge of the boundary layer is assumed to be zero, the expression for equation (5) is

$$\frac{1}{U^2} \int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy = \frac{1}{U^2} \frac{d}{dx}\left(\int_0^\delta \overline{u'^2}dy \right)$$

or, upon use of equation (9),

$$\frac{1}{U^2} \int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy = \frac{1}{U^2} \frac{d}{dx}\left(\int_0^\delta \overline{u_r'^2}\, \bar{u}_r^2 \frac{1}{\bar{u}^2}\, dy \right) \tag{10}$$

Let $\xi = \frac{y}{\delta}$ so that the term on the right becomes

$$\frac{1}{U^2} \frac{d}{dx}\left(\delta \int_0^1 \overline{u_r'^2}\, \bar{u}_r^2 \frac{1}{\bar{u}^2}\, d\xi \right)$$

or

$$\frac{1}{U^2} \frac{d}{dx}\left(\frac{\delta}{U^2} \int_0^1 \overline{u_r'^2}\, \bar{u}_r^2 \frac{U^2}{\bar{u}^2}\, d\xi \right)$$

If the assumption is made that $\dfrac{\overline{u}}{U} = f_1(H,\xi)$ and θ is introduced into both the numerator and denominator, equation (10) becomes

$$\frac{1}{U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy = \frac{\overline{u_r'^2}\ \overline{u}_r^2}{U^2}\frac{d}{dx}\left[\frac{\theta}{U^2}\frac{\delta}{\theta}\int_0^1 f_1(H,\xi)d\xi\right]$$

$$= \frac{\overline{u_r'^2}\ \overline{u}_r^2}{U^2}\frac{d}{dx}\left[\frac{\theta}{U^2}f(H)\right]$$

Upon differentation, this equation becomes

$$\frac{1}{U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy = \frac{\overline{u_r'^2}\ \overline{u}_r^2}{U^4}\left[g(H)\theta\frac{dH}{dx} + f(H)\frac{d\theta}{dx} - f(H)\frac{\theta}{q}\frac{dq}{dx}\right] \qquad (11)$$

where

$$g(H) = \frac{d\left[f(H)\right]}{dH}$$

It should be noted that the parameters of equation (11) which, although not complete, control the term $\dfrac{1}{U^2}\displaystyle\int_0^\delta \dfrac{\partial}{\partial x}(\overline{u'u'})dy$ also apply to the term $\dfrac{1}{U^2}\displaystyle\int_0^\delta f\,\dfrac{\partial}{\partial x}(\overline{u'u'})dy$ since

$$f = \frac{\overline{u}}{U} = f_1(H,\xi)$$

As was stated previously, the results are not based on a rigorous analysis and the list of parameters obtained may not be considered complete.

Determination of the Parameters for the

Dissipation Coefficient

The purpose of this section is to suggest parameters of the dissipation-coefficient term which appears in equation (7), which is

$$\frac{\tau_o}{2q} \int_0^\delta f \frac{\partial g}{\partial y}\, dy$$

or

$$\frac{\tau_o}{2q} \int_0^\delta g \frac{\partial f}{\partial y}\, dy$$

This integral is made dimensionless by dividing y by δ to obtain

$$\frac{\tau_o}{2q} \int_0^1 g \frac{\partial f}{\partial \xi}\, d\xi$$

If the shear distribution across the boundary layer were known, the above integral could be solved either numerically or graphically. The work of Fediaevsky (ref. 14), who evaluated the shear distribution in terms of a power series in ξ based on the parameter A_1 where

$$A_1 = \frac{\delta}{\tau_o} \frac{d\overline{p}}{dx}$$

$$= -\frac{1}{2q} \frac{dq}{dx} \frac{\delta}{\tau_o} \rho U^2 \tag{12}$$

suggests a possible approximate shear distribution. The Fediaevsky analysis is known to provide only an approximation to the shearing-stress distribution across the boundary layer. It thus follows that the list of parameters obtained from equation (12) is not complete.

A relationship was derived in an analysis made by C. duP. Donaldson at the Langley Laboratory in 1952 as follows:

$$\tau_o = A(H) q R_\delta^{-\frac{2}{r+1}}$$

Thus, equation (12) becomes

$$A_1 = -\frac{1}{2q}\frac{dq}{dx}\,\delta\,\frac{\rho U^2}{qA(H)}\,R_\delta^{\frac{2}{r+1}}$$

or

$$A_1 = -\frac{\theta}{q}\frac{dq}{dx}\frac{\delta}{\theta}\frac{1}{A(H)}\left(\frac{U\delta}{\nu}\right)^{\frac{2}{r+1}}$$

The factor $(\theta/\theta)^{2/(r+1)}$ is introduced into equation (12) to give

$$A_1 = -\frac{\theta}{q}\frac{dq}{dx}\left(\frac{\delta}{\theta}\right)^{\frac{r+3}{r+1}}\left(\frac{U\theta}{\nu}\right)^{\frac{2}{r+1}}\frac{1}{A(H)}$$

Since δ/θ is some function of H,

$$A_1 = A_1\left[\frac{\theta}{q}\frac{dq}{dx},\ R_\theta^{\frac{2}{r+1}},\ B(H)\right]$$

Thus

$$g = g\left[\frac{\theta}{q}\frac{dq}{dx},\ R_\theta,\ C(H),\ \xi\right]$$

and since $\dfrac{\partial f}{\partial \xi} = f(H,\xi)$,

$$\frac{\tau_0}{2q}\int_0^\delta f\,\frac{\partial g}{\partial y}\,dy = h\left[\frac{\theta}{q}\frac{dq}{dx},\ R_\theta,\ D(H)\right] \tag{13}$$

where the right-hand side of equation (13) contains the approximate list of controlling parameters for the dissipation coefficient.

RESULTS

The consideration of the rate of increase of the Reynolds normal stress in the stream direction led to the derivation of the modified momentum equation (eq. 4):

$$\frac{d\theta}{dx} + \frac{\theta}{U}\frac{dU}{dx}(H + 2) = \frac{\tau_0}{\rho U^2} + \frac{1}{\rho U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy$$

and the modified kinetic-energy equation (eq. 7):

$$\theta\frac{dH}{dx} = \frac{\theta}{U}\frac{dU}{dx}\left[\frac{K(H-1)}{K'}\right] - \frac{\tau_0}{\rho U^2}\left(\frac{-K + 2\int_0^\delta f\frac{\partial g}{\partial y}dy}{K'}\right) -$$

$$\frac{1}{U^2}\left[\frac{K\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy - 2\int_0^\delta f\frac{\partial}{\partial x}(\overline{u'u'})dy}{K'}\right]$$

Equation (4) differs from the Von Kármán momentum equation by the term

$$\frac{1}{U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy$$

which is the additional stress due to the inclusion of the Reynolds normal stress. Values of this term computed from the data of reference 8 are plotted in figure 3. The magnitude of this term is shown to increase sharply as separation is approached. The data in figure 3 have also been presented in references 7 and 9.

Values of $d\theta/dx$ computed from equation (4) by use of the experimental values of θ and H from reference 8 and the skin-friction formula of reference 5 are plotted in figure 4. For comparison, values from the Von Kármán equation are shown. It is seen that equation (4) agrees better with the faired data of reference 8. The data of figure 4 were also previously presented in reference 9.

Equation (7) differs from the kinetic-energy equation of reference (11) by the expression

$$-\frac{1}{U^2}\left[\frac{K\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy - 2\int_0^\delta f\frac{\partial}{\partial x}(\overline{u'u'})dy}{K'}\right]$$

where K is the ratio of the kinetic-energy thickness to momentum thickness. The curves and data for figures 5 and 6 are the same as those in reference 9 (figs. 24(a) and 24(b)), which were obtained by using equation (7) with power profiles being assumed. The effect of the Reynolds normal stress is seen to be negligible.

On the assumption that the streamwise fluctuations vary according to equation (8), it was found that the term

$$\frac{1}{U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy$$

was controlled approximately by the mean-flow parameters $\theta\frac{dH}{dx}$, $\frac{\theta}{q}\frac{dq}{dx}$, and H (eq. (11)). It can be shown that $\frac{d\theta}{dx}$ depends upon these same parameters if the skin friction depends on Reynolds number and the shape factor H. In figure 7, which is taken from figure 8 of reference 9, the term

$$\frac{\tau_R}{2q} = \frac{1}{U^2}\int_0^\delta \frac{\partial}{\partial x}(\overline{u'u'})dy$$

is plotted against $\theta\frac{dH}{dx}$ and the scatter is rather large. This result may be due to failure to include the other parameters suggested herein. When an attempt was made to correlate the data of figure 7 by using the parameters H and $\frac{\theta}{q}\frac{dq}{dx}$ as well as $\theta\frac{dH}{dx}$, no correlation was obtained. It should be noted, however, that the values of $\theta\frac{dH}{dx}$ and $\frac{\theta}{q}\frac{dq}{dx}$ are obtained by taking slopes from curves faired through experimental points and are, therefore, subject to some uncertainty. Moreover, it should be

noted that $\dfrac{\tau_R}{2q}$ is obtained from

$$\frac{\tau_R}{2q} = \frac{d\theta}{dx} - \left[-\frac{\theta}{U}\frac{d}{dx}(H + 2) + \frac{\tau_o}{\rho U^2}\right]$$

where $\dfrac{d\theta}{dx}$ is positive, $\dfrac{\theta}{U}\dfrac{d}{dx}(H + 2)$ is negative, and $\dfrac{\tau_o}{\rho U^2}$ is positive.

Thus $\dfrac{\tau_R}{2q}$ is the difference between two quantities, each of which is much

larger than $\dfrac{\tau_R}{2q}$ and each of which involves slopes of experimental curves.
The lack of correlation can thus be caused by the difficulty in obtaining
accurate values of $\theta\dfrac{dH}{dx}$, of $\dfrac{\theta}{q}\dfrac{dq}{dx}$, and especially of $\dfrac{\tau_R}{2q}$. On the other
hand, the lack of correlation can also be caused by the incorrect choice
of parameters. More data are necessary to determine more definitely the
reason for the inability to correlate the data of figure 7 by the sug-
gested parameters.

The shear-work term of equation (7) or

$$\frac{\tau_o}{2q}\int_0^\delta f\,\frac{\partial g}{\partial y}\,dy = \frac{\tau_s}{2q}$$

was found to depend on the parameters $\dfrac{\theta}{q}\dfrac{dq}{dx}$, R_θ, and $D(H)$. This
dependency agrees with the results shown in figure 8, which is taken from
reference 9, figure 6.

The results as given herein indicate the need for more experimental
measurements of turbulent axial fluctuating velocities in turbulent
boundary layers. If sufficient data were available, it could be deter-
mined more definitely whether the Reynolds stress terms depend upon the
suggested parameters. In addition, the two additional terms of the
modified kinetic-energy equation could be evaluated so as to determine
whether their net effect is still negligible for a large range of
conditions.

CONCLUDING REMARKS

A general integral form of the boundary-layer equation which included the Reynolds normal-stress term was derived. From this general integral form, two special equations were obtained. They were the modified momentum equation and the modified kinetic-energy equation. Both expressions included the effect of the Reynolds normal stress in the stream direction.

The parameters for the dissipation of the mean-flow kinetic energy by the shearing stress were suggested by the use of the Fediaevsky analysis for the shearing stress across turbulent boundary layers and by the use of a friction formula that makes the surface shear depend on the shape of the velocity profile as well as on the Reynolds numbers. The parameters obtained in this way were found to be the same as those previously assumed by other investigators.

The parameters for the Reynolds normal stress were suggested by assuming that a relation between the local fluctuating axial velocity and the local mean velocity that is useful in wind tunnels can be used to obtain some of the parameters upon which the Reynolds normal stress may depend. A test of these parameters by a limited amount of data showed no better correlation than previously obtained by other investigators who used only one of the three parameters obtained in this analysis. This result may be caused by incorrect parameters or insufficient data.

Langley Aeronautical Laboratory,
        National Advisory Committee for Aeronautics,
                Langley Field, Va., August 25, 1953.

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TABLE I

CALCULATED TURBULENCE INTENSITIES FOR REFERENCE STATION x = 17.5 FEET

y, in.	$(u'/U)_r$ (faired)	y, in.	u'/U		y, in.	u'/U	
			Computed	Experimental (faired)		Computed	Experimental (faired)
x = 17.5 ft		x = 22.5 ft			x = 25.4 ft		
0	-------	0	-----	-----	0	-----	-----
.10	0.0926	.180	0.119	0.109	.950	0.234	0.108
.20	.0875	.375	.118	.115	1.100	.215	.105
.30	.0827	.550	.110	.117	1.525	.188	.115
.40	.0783	.725	.103	.116	1.850	.162	.120
.50	.0745	.875	.099	.115	2.125	.146	.124
.75	.0670	1.275	.089	.109	2.775	.118	.129
1.00	.0596	1.650	.072	.100	3.370	.090	.126
1.50	.0457	2.340	.052	.080	4.310	.060	.115
2.00	.0280	3.020	.031	.058	5.200	.033	.094
2.50	.0088	3.675	.009	.034	6.025	.010	.069
2.70	.0062	3.925	.006	.025	6.325	.006	.060
3.25	------	5.000	-----	-----	8.150	-----	-----

TABLE II

CALCULATED TURBULENCE INTENSITIES FROM REFERENCE STATIONS x = 17.5 FEET AND 20.0 FEET

| y, in.	u'/U Experimental (faired)	y, in.	u'/U		y, in.	u'/U	
			Experimental (faired)	Computed		Experimental (faired)	Computed
x = 17.5 ft		x = 20.0 ft			x = 22.5 ft		
0	------	0	------	------	0	------	------
.10	0.0926	.120	0.0952	0.1030	.180	0.1090	0.1210
.20	.0875	.258	.0961	.0980	.375	.1150	.1220
.30	.0827	.370	.0929	.0921	.550	.1170	.1180
.40	.0783	.492	.0895	.0878	.725	.1160	.1140
.50	.0745	.700	.0842	.0804	.875	.1150	.1100
.75	.0670	.900	.0791	.0703	1.275	.1090	.0978
1.00	.0596	1.182	.0713	.0669	1.650	.1000	.0857
1.50	.0457	1.740	.0541	.0498	2.340	.0804	.0607
2.00	.0280	2.285	.0343	.0287	3.020	.0583	.0368
2.50	.0088	2.820	.0127	.0088	3.675	.0344	.0130
2.70	.0062	3.038	.0074	.0062	3.925	.0252	.0074
3.25	------	3.630	------	------	5.000	------	------
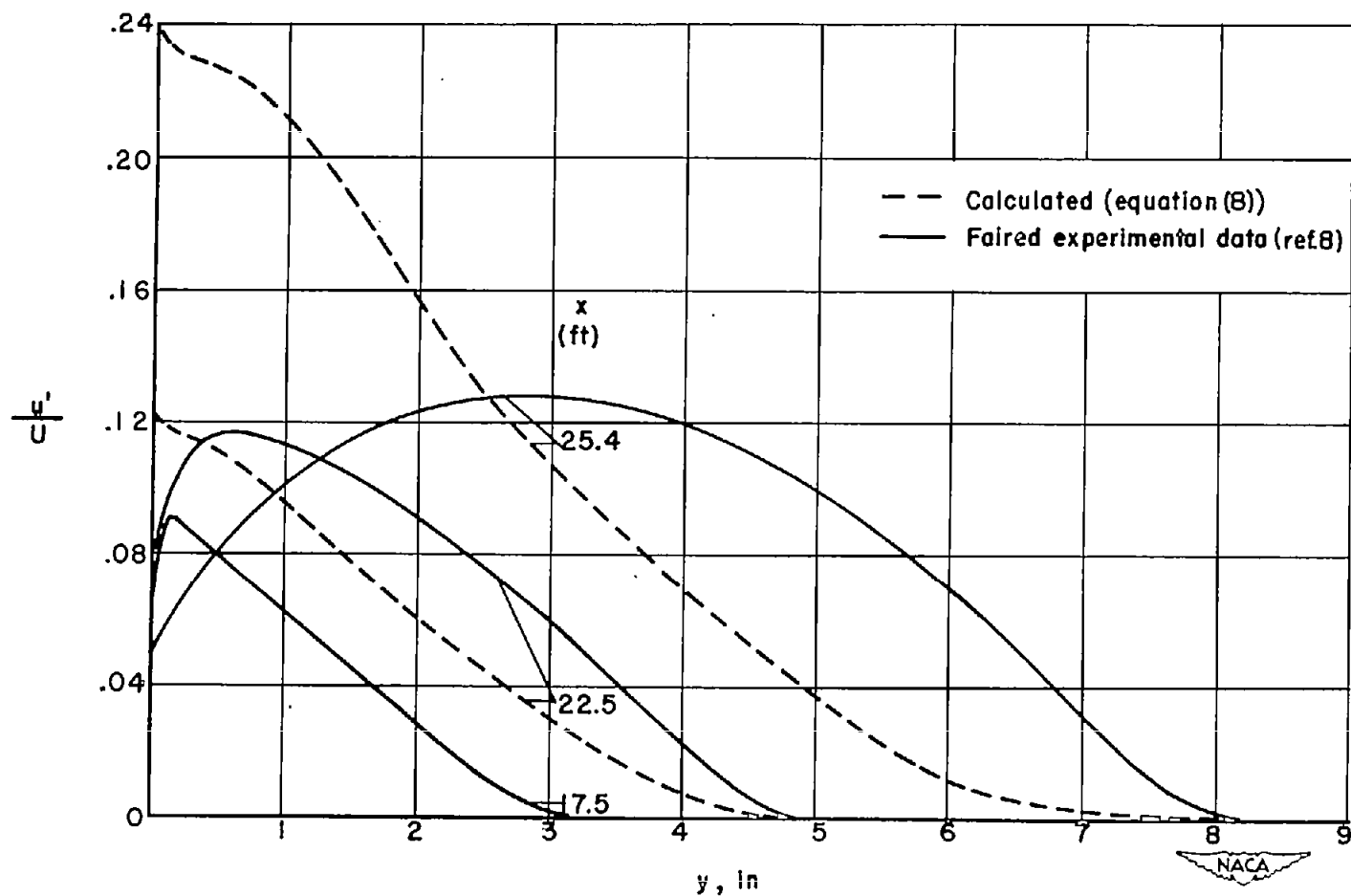
Figure 1.- Turbulence intensities from data of table I.
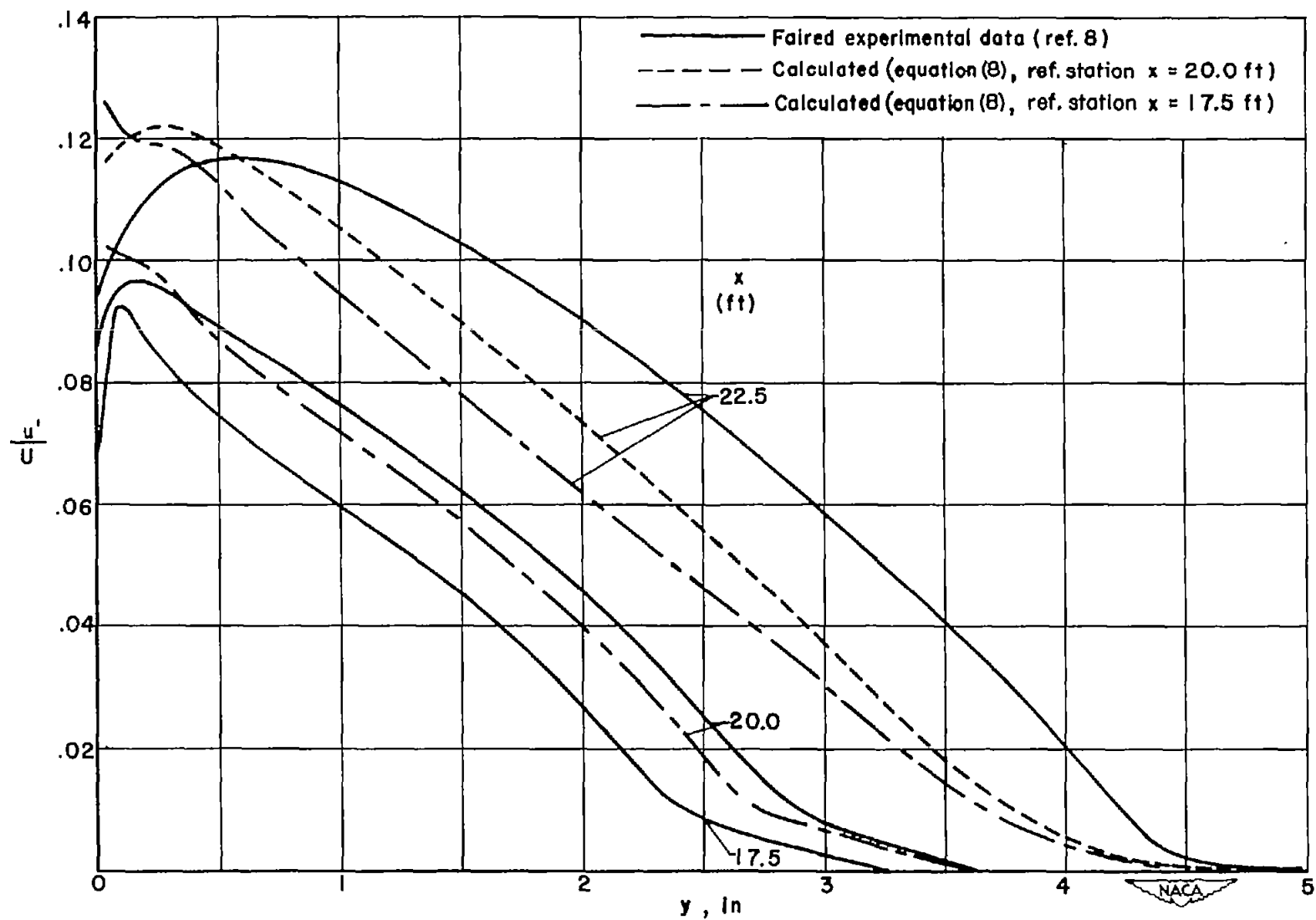
Figure 2.- Turbulence intensities from data of table II.

$$\frac{1}{U^2} \int_0^\delta \frac{\partial}{\partial x} \left(\overline{u'^2} \right) dy$$

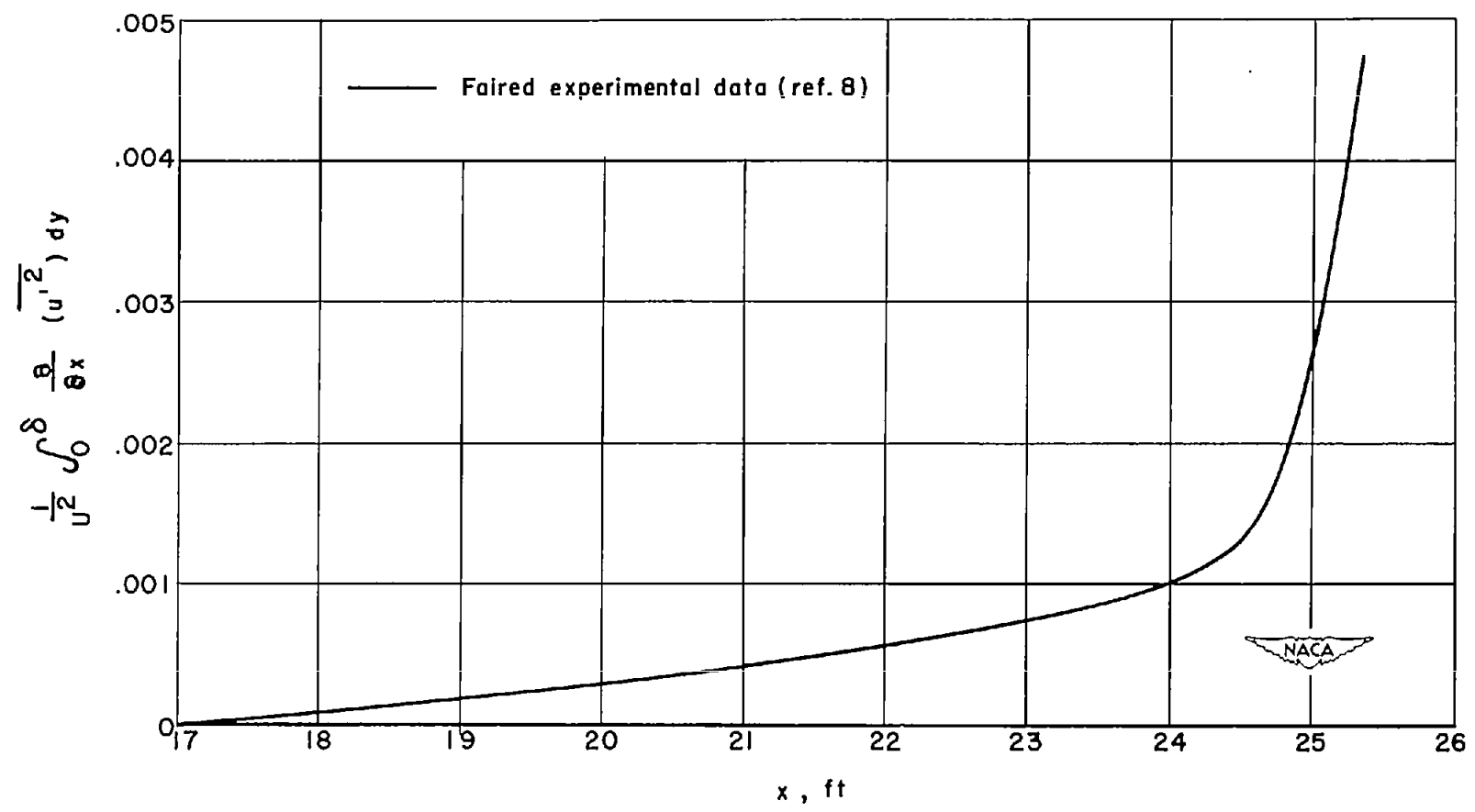—— Faired experimental data (ref. 8)

x , ft
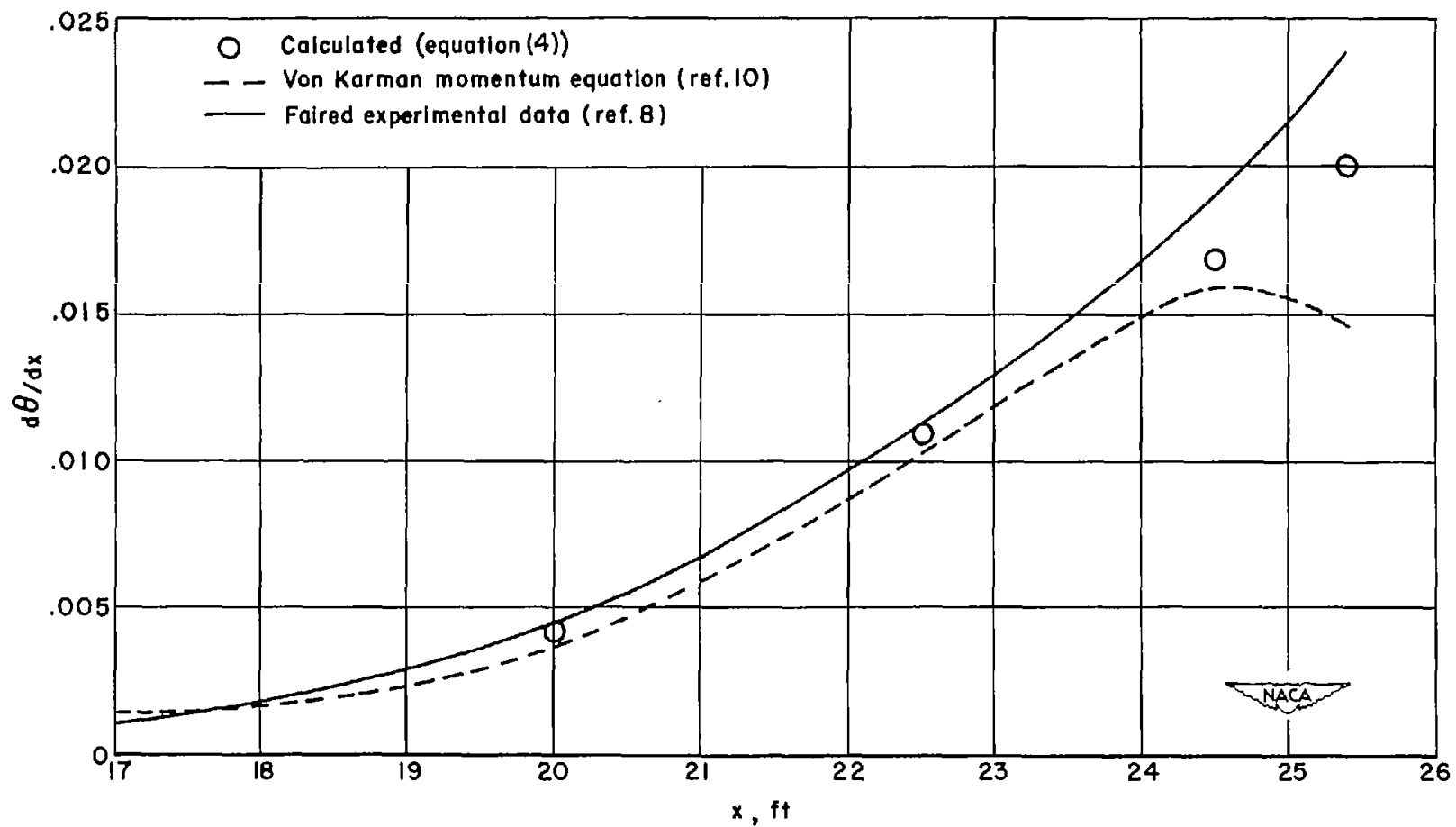
Figure 3.- Streamwise variation of turbulent fluctuation.

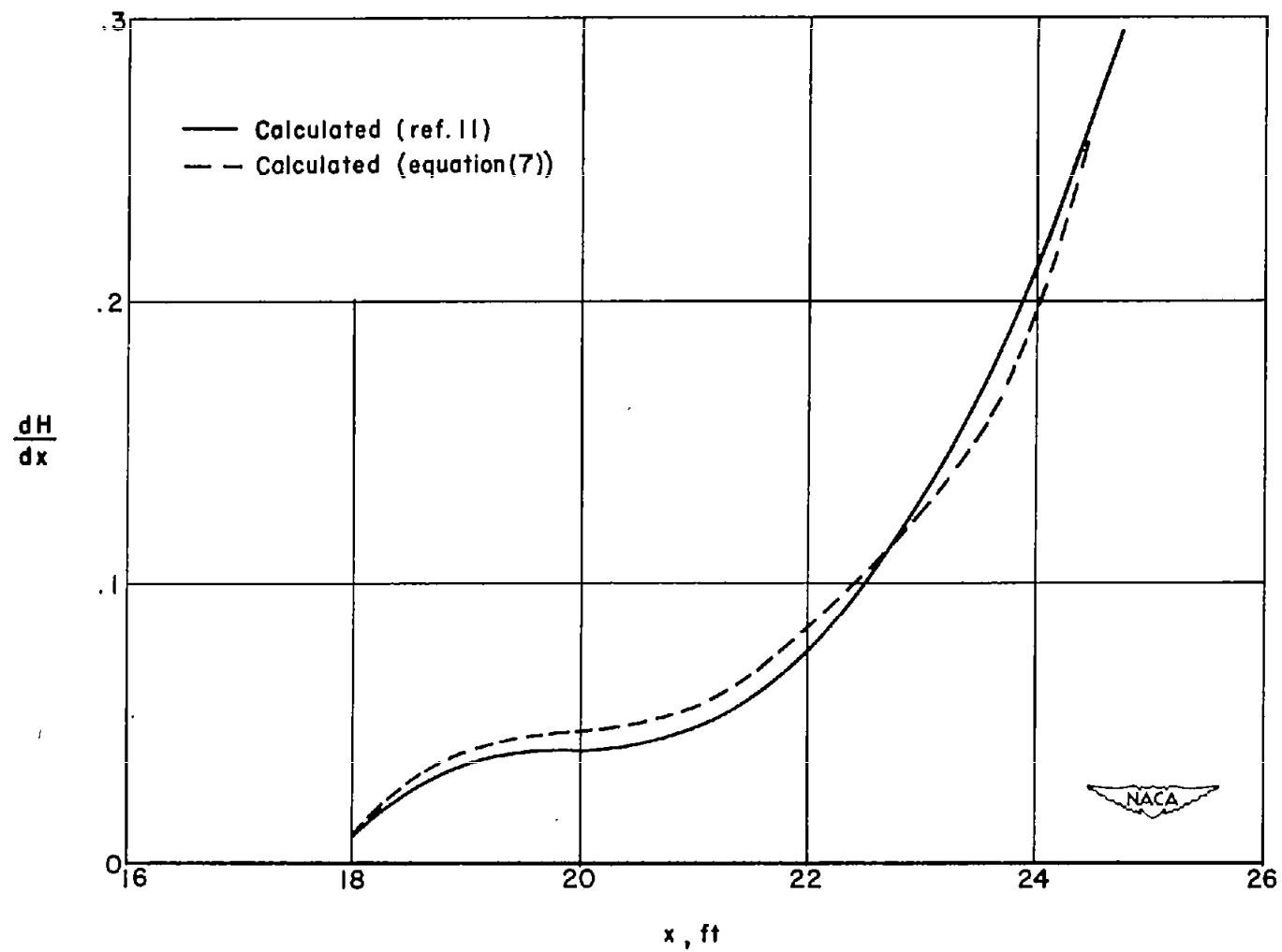Figure 4.- Streamwise variation of terms in the momentum and modified momentum equations.

Figure 5.- Streamwise variation of shape-factor gradient.
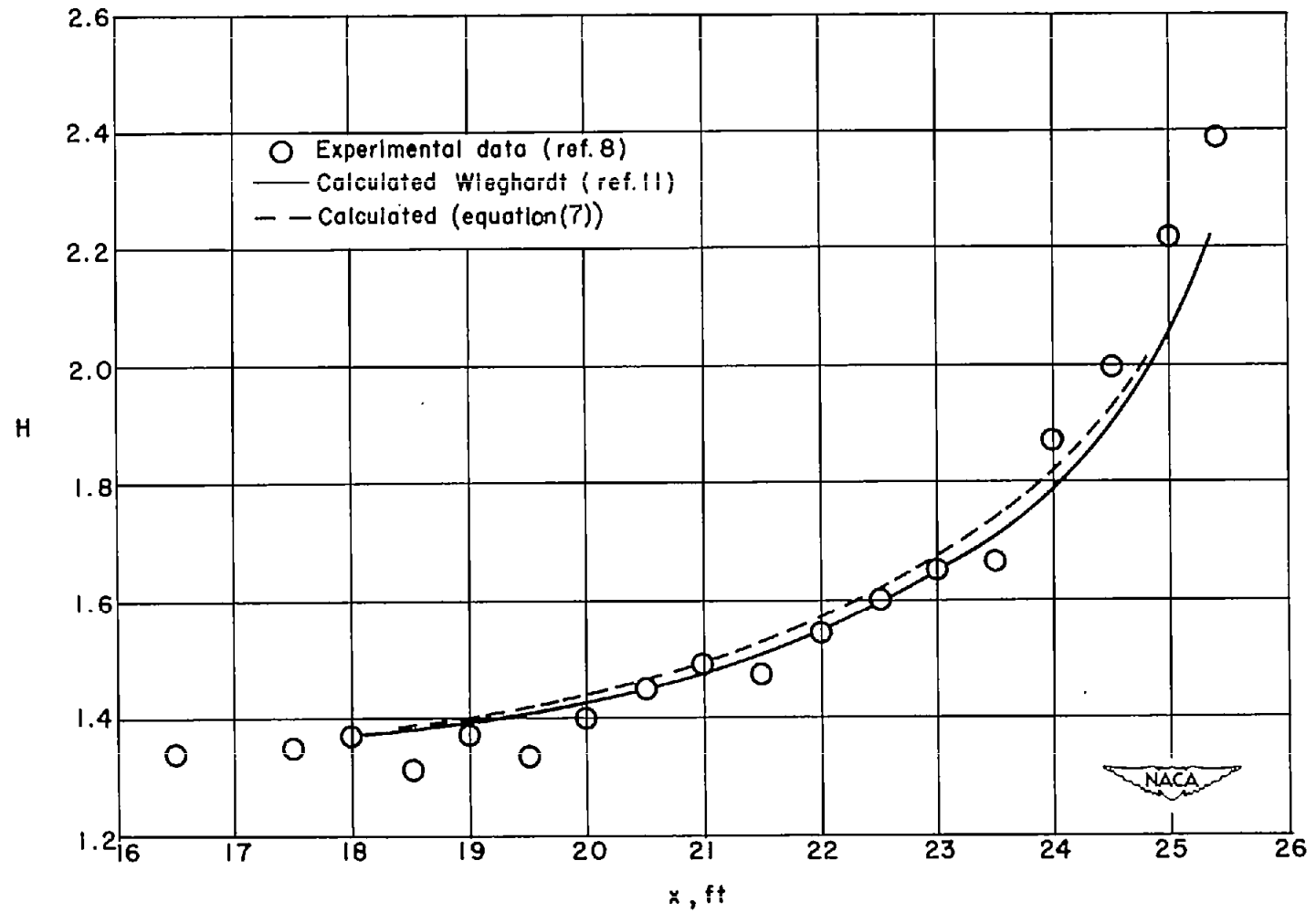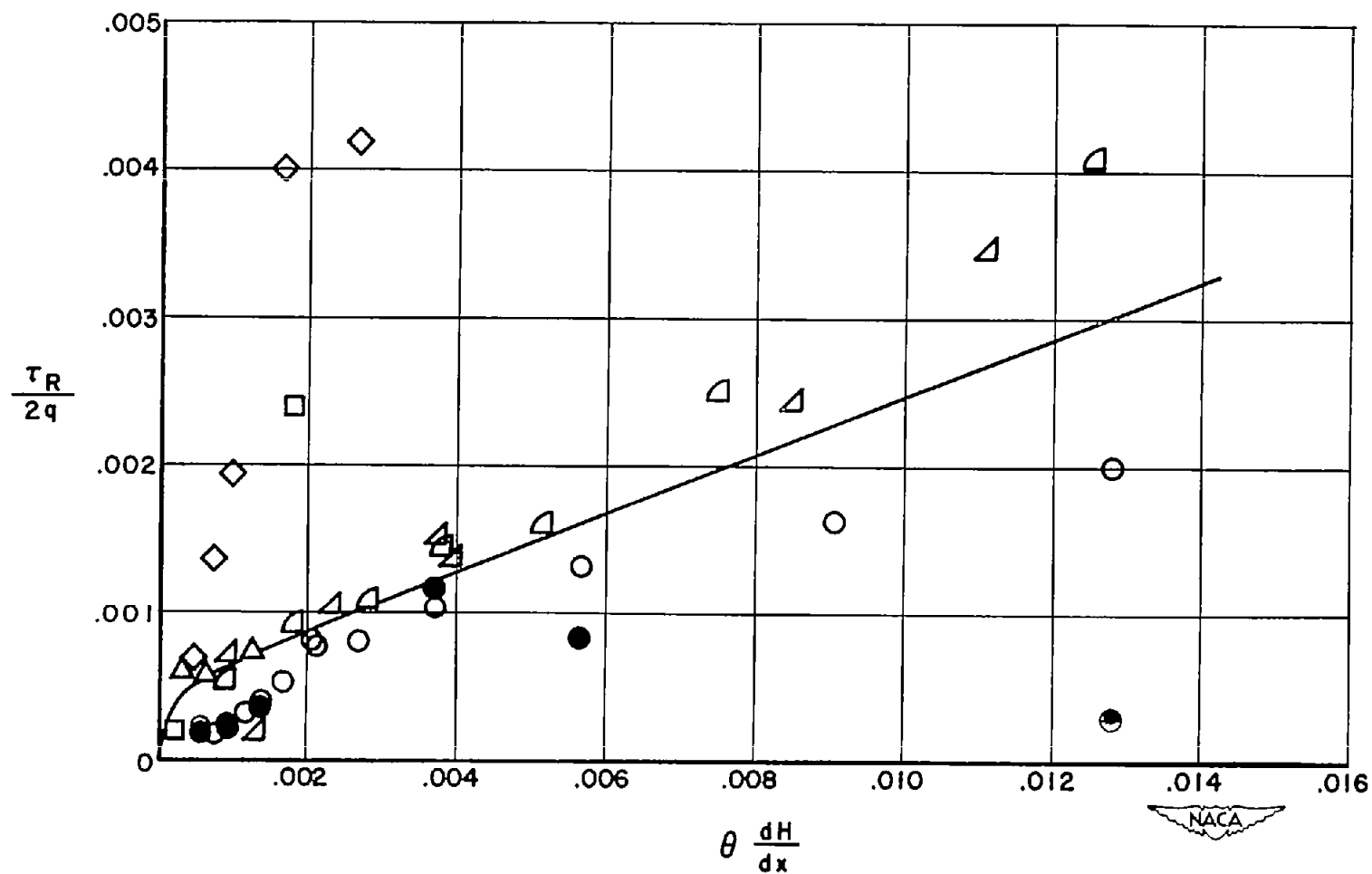
Figure 6.- Streamwise variation of shape factor.

Figure 7.- Correlation of turbulent-normal-stress coefficient as a
function of the nondimensional shape-factor gradient. (Test
points are identified in fig. 8 of ref. 9.)

$$\dfrac{\tau_{\theta}}{2q} R_{\theta}^{0.268} \times 10^{0.678H} \quad \dfrac{(3H-1)^2}{H(3H-0.9)-0.1}$$

1.4

1.2

1.0

.8

.6

.4

.2

0

H

2.0

1.8

1.6

1.5

1.4

1.286

.001 .002 .003 .004 .005 .006 .007 .008 .009 .010
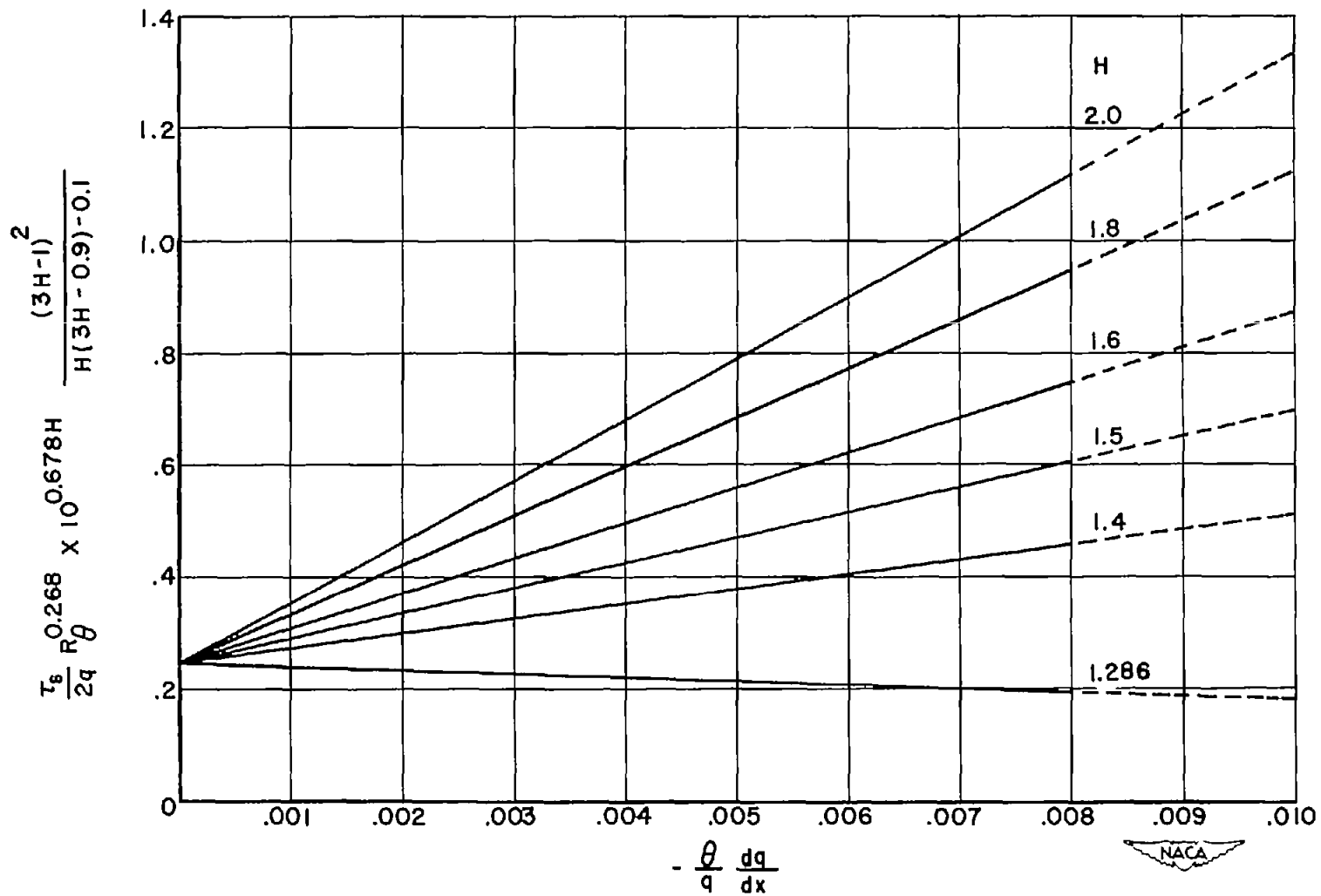
$$-\dfrac{\theta}{q}\dfrac{dq}{dx}$$

Figure 8.- Correlation of dissipation coefficient as a function of the nondimensional pressure gradient.