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TRANSIENT TEMPERATURE DISTRIBUTIONS IN SIMPLE CONDUCTING BODIES STEADILY HEATED THROUGH A LAMINAR BOUNDARY LAYER

By Hermon M. Parker

Langley Aeronautical Laboratory
Langley Field, Va.

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SUMMARY

An analysis is made of the transient heat-conduction effects in three simple semi-infinite bodies: the flat insulated plate, the conical shell, and the slender solid cone. The bodies are assumed to have constant initial temperatures and, at zero time, to begin to move at a constant speed and zero angle of attack through a homogeneous atmosphere. The heat input is taken as that through a laminar boundary layer. Radiation heat transfers and transverse temperature gradients are assumed to be zero.

The appropriate heat-conduction equations are solved by an iteration method, the zeroeth-order terms describing the situation in the limit of small time. The method is presented and the solutions are calculated to three orders which are sufficient to give reasonably accurate results when the forward edge has attained one-half the total temperature rise (nose half-rise time). Flight Mach number and air properties occur as parameters in the result. Approximate expressions for the extent of the conduction region and nose half-rise times as functions of the parameters of the problem are presented.

INTRODUCTION

One of the major problems arising in supersonic and hypersonic flight of aircraft is that of aerodynamic heating. It is apparent that the maximum feasible speed for a given aircraft at a given altitude is limited by this effect. This problem has been studied from many varying points of view. Usually (refs. 1 to 3) these investigations were concerned with determining the temperature attained at the surface of a given body in steady flight or during a given flight history of variable speed and perhaps variable altitude. Little attention has been devoted to the effect of heat conduction within the body on the transient temperature distribution of its surface. Kaye (ref. 2) considered the aerodynamic heating of an infinite-aspect-ratio, 8-percent-thick, solid-steel, symmetrical double-wedge wing of 5.5-foot chord in flight in a homogeneous atmosphere and accelerating at a constant rate from Mach number 1.4 to Mach number 6.0. Using a "coarse network" of integration points, he numerically integrated the appropriate two-dimensional heat-conduction equations, using the best available experimental values of the heat-transfer coefficient (ref. 4) and neglecting radiation transfers. He concluded that, except near the leading and trailing edges, the chordwise conduction is relatively unimportant but that the transverse temperature gradients, especially at midchord, are of considerable importance in estimating the transient surface temperature distribution.

For actual aircraft in high-speed flight, the maximum rate of rise of temperature occurs at the leading edge or nose. Under the assumptions of zero internal conduction, of no heat transfer except that through the boundary layer, and of steady flight beginning at zero time in a homogeneous atmosphere, the surface temperature approaches the local adiabatic wall temperature at a rate which is theoretically infinite at zero Reynolds number and decreases with increasing Reynolds number. With internal conduction the surface approaches an equilibrium temperature distribution at a rate which is finite everywhere and is appreciably different from the zero-conduction rate at the smaller Reynolds numbers. Nonweiler (ref. 3) has calculated the equilibrium temperatures near the leading edge of the flat plate in steady flight by taking into account conduction in the plate, heat input through a laminar boundary layer, and loss of heat by radiation with the assumption that the difference between the local equilibrium temperature and the adiabatic wall temperature is large and constant.

The temperature lag due to heat capacity and heat conduction permits aircraft to exceed for short times the speed for which the corresponding equilibrium temperature is the maximum permissible temperature of the body. Further, transient thermal stresses are important problems in the structural design of high-speed aircraft. Thus it is important to determine quantitatively the effect of heat conduction on the transient temperatures near the leading edge or nose of as general a class of bodies and flight conditions as possible.

The problem considered herein is that of determining the transient effect of heat conduction parallel to the airstream and within certain simple infinitely long bodies in steady flight at zero angle of attack in a uniform atmosphere. The bodies considered are the insulated flat plate, the conical shell, and the slender solid cone. Heat transfers due to radiation are neglected. Because the effect of heat conduction is important only near the leading edge or nose, the usual expressions for heat transfer through an incompressible laminar boundary layer are used. Corrections to the incompressible heat transfer which do not alter the temperature and Reynolds number dependence may be included in the analysis. Because the laminar boundary-layer equations do not apply exactly at the leading edge or nose, the results of the analysis are applicable for Reynolds number (based on distances from the leading edge) greater than the order of 10. Temperatures within the body perpendicular to the stream direction are assumed uniform.

SYMBOLS

a	conduction parameter, $\left(\frac{k_b}{\rho_b c_b}\right)^{1/2} \frac{U}{\nu}$, $sec^{-1/2}$
b	heat-transfer constant for flat plate, $\frac{0.33 k_g U}{\rho_b c_b \nu} \frac{Pr^{1/3}}{h}$, sec^{-1}
b_c	heat-transfer constant for slender solid cone, $\sqrt{3}\, \frac{0.33 k_g U^2 Pr^{1/3}}{\rho_b c_b \nu^2}\, g$, sec^{-1}
c	speed of sound, ft/sec
c_b	specific heat of material of body
g	geometry factor for slender solid cone, $\frac{\sin \theta}{1 - \cos \theta}$
h	thickness of flat plate or conical shell, ft
k_g	thermal conductivity of air, Btu/(sec)(ft)($^{\circ}$R)
k_b	thermal conductivity of material of body, Btu/(sec)(ft)($^{\circ}$R)
M	Mach number just outside boundary layer, U/c
n	summation variable
Pr	Prandtl number
q	rate of heat transfer through laminar boundary layer on flat plate, Btu/(sec)(ft^2)

q_c rate of heat transfer through laminar boundary layer on cone, $q\sqrt{3}$, Btu/(sec)(ft^2)

R Reynolds number, $\frac{Ux}{\nu}$ or $\frac{Us}{\nu}$

R_o Reynolds number at rear of conduction region at nose half-rise time

s distance variable on conical surface, measured from vertex, ft

T temperature of body, ^{o}R

T_{aw} adiabatic wall temperature, ^{o}R

T_i initial temperature of body, ^{o}R

t time, sec

t_o nose half-rise time, sec

U stream speed just outside boundary layer, fps

u stream speed in boundary layer (Blasius flat-plate solution), fps

$$K = \frac{T - T_i}{T_{aw} - T_i}$$

x distance variable on flat plate, measured from leading edge, ft

y distance perpendicular to wall (Blasius flat-plate solution), ft

α dimensionless time parameter

β dimensionless distance parameter

η dimensionless transient-heat-conduction parameter, $\frac{R}{at^{1/2}}$

η_o value of η corresponding to rear boundary of heat-conduction region

ρ_b density of material of body

θ	half vertex angle of slender solid cone
ν	kinematic viscosity

Primes with a symbol denote differentiation with respect to η.

Subscripts:

x	at distance x
s	at distance s
b	for body material
g	for gas (air)

METHOD OF ANALYSIS

Rate of Heat Transfer

For the simple semi-infinite bodies to be considered in this analysis, the effect of internal heat conduction parallel to the stream direction is significant only near the leading edge or nose. Therefore, it is natural to assume for the heat transfer those expressions appropriate to a laminar boundary layer. The present analysis is applicable to cases in which the expression for the heat transfer to the body is of the form

$$q = (\text{Constant}) \frac{T_{aw} - T}{\sqrt{x}}$$

or

$$q = (\text{Constant}) \frac{T_{aw} - T}{\sqrt{s}}$$

where x is the distance from the leading edge of the flat plate and s is the distance along the conical surface from the vertex of the conical shell or the slender solid cone. Thus, the real assumption regarding the heat input is the temperature and distance dependence. Compressibility and nonuniform-surface-temperature effects which do not alter the temperature and distance dependence may be included in the present analysis by suitable changes in the constant of proportionality. For simplicity the usual expression for the heat transfer in the incompressible isothermal case is assumed.

The rate of heat transfer through a unit area of an incompressible laminar boundary layer on the flat plate is

$$q = 0.33 k_g \frac{U}{\nu} Pr^{1/3} \frac{T_{aw} - T}{R_x^{1/2}} \tag{1}$$

where R_x is based on the distance x from the leading edge. Equation (1) may be derived from the first equation on page 626 of reference 5. Hantzsche and Wendt (ref. 6) have shown that the laminar boundary-layer heat transfer on a cone is $\sqrt{3}$ times that on the flat plate with the same free-stream conditions just outside the boundary layer. Thus it follows that the rate of heat transfer through unit area of a laminar boundary layer on the cone is

$$q_c = 0.33\sqrt{3}\ k_g \frac{U}{\nu} Pr^{1/3} \frac{T_{aw} - T}{R_s^{1/2}} = \sqrt{3}\ q \tag{2}$$

where R_s is based on the distance s from the vertex along the surface of the cone. Since no satisfactory representative length exists for the semi-infinite bodies to be considered in the analysis, the Reynolds number is chosen as a suitable dimensionless distance parameter.

The expressions (1) and (2) are based on the assumption that the Blasius flat-plate solution for the laminar boundary layer is correct. Inasmuch as it is known that the Blasius solution is not correct at very low Reynolds numbers, an estimate of the Reynolds number at which the Blasius solution fails was made by the following calculation:

The Blasius solution was used to calculate the term $\nu \frac{\partial^2 u}{\partial x^2}$, which was neglected in the Navier-Stokes equations, on the basis that it is small in comparison with $\nu \frac{\partial^2 u}{\partial y^2}$. With the curve $y(R_x)$ defined as that curve under which $\nu \frac{\partial^2 u}{\partial x^2}$ is equal to or larger than $\nu \frac{\partial^2 u}{\partial y^2}$, it was found that $y(R_x)$ equals the displacement thickness at $R_x \approx 5$ and, at large R_x, $y(R_x)$ is approximately proportional inversely to the square root of R_x. Thus a reasonable conclusion appears to be that the Blasius solution and therefore equations (1) and (2) should not be in serious error for Reynolds numbers larger than the order of 10.

Equations (1) and (2) indicate that, at the leading edge or nose, the rate of heat transfer becomes infinite, which, of course, is physically impossible. Actually, at small Reynolds numbers the heat transfer must be limited by the finiteness of the random molecular motion, as happens in free-molecule flows. Comparison of the heat transfer in the free-molecule regime with equation (1) leads to the conclusion that the maximum q occurs at a Reynolds number of the order of 1/10 of the square of the Mach number. On the other hand, the failure of the Blasius solution at small Reynolds numbers is surely related to other approximations, such as the neglect of the term $\nu \frac{\partial^2 u}{\partial x^2}$. In any event, the effect of internal conduction is important over a range of Reynolds numbers very large compared with 10, so that the correctness of equations (1) and (2) for Reynolds numbers less than 10 is not of great importance. The integrability of the heat input assures no mathematical difficulties and no serious errors even at reasonably small Reynolds numbers.

The Flat Plate

The general three-dimensional heat-conduction equation is (ref. 7):

$$\rho_b c_b \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_b \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_b \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_b \frac{\partial T}{\partial z}\right) + Q(x,y,z)$$

where k_b is the thermal conductivity and $Q(x,y,z)$ is a heat-generation function specifying the rate at which heat is created per unit volume at the point (x,y,z). The problem of the flat plate heated through a boundary layer reduces to the one-dimensional time-dependent heat-conduction problem when temperature gradients normal to the stream direction are assumed to be zero. The heat-generation function becomes the rate of heat transfer per unit area q divided by the plate thickness h which is assumed to be constant. Thus, the temperature of the flat plate heated from one side is given by

$$\rho_b c_b \frac{\partial T}{\partial t} = k_b \frac{\partial^2 T}{\partial x^2} + 0.33 k_g \frac{U}{\nu} \frac{Pr^{1/3}}{h} \frac{T_{aw} - T}{R_x^{1/2}}$$

or

$$\frac{\partial K}{\partial t} = a^2 \frac{\partial^2 K}{\partial R_x^2} + \frac{b}{R_x^{1/2}}(1 - K) \tag{3}$$

where

$$a^2 = \frac{k_b}{\rho_b c_b} \frac{U^2}{\nu^2}$$

$$b = 0.33 \frac{k_g}{\rho_b c_b} \frac{U}{\nu} \frac{Pr^{1/3}}{h}$$

$$K = \frac{T - T_i}{T_{aw} - T_i}$$

Boundary and initial conditions are chosen as follows. At zero time the plate has a uniform temperature T_i so that

$$K_{t=0} = 0 \tag{4}$$

and at large values of time the surface temperature must be T_{aw}, that is,

$$K_{t \longrightarrow \infty} = 1 \tag{5}$$

A boundary condition on the temperature gradient at the leading edge may be established as follows. Assume that the plate is of unit width. Then the frontal area is $1 \times h$, a finite constant. The condition of no heat transfer through the frontal area then requires that the temperature gradient there be zero, that is,

$$\left(\frac{\partial K}{\partial R_x}\right)_{R_x=0} = 0 \tag{6}$$

The process of finding a solution of equation (3) proceeds as follows. At small values of time a good approximation (zero order) to equation (3) is obtained by putting K equal to zero in the second term of the right member; thus,

$$\frac{\partial K}{\partial t} = a^2 \frac{\partial^2 K}{\partial {R_x}^2} + \frac{b}{{R_x}^{1/2}} \tag{7}$$

It can be verified that

$$K = \frac{b}{\sqrt{a}} t^{3/4} F(\eta) \tag{8}$$

where

$$\eta = \frac{R_x}{at^{1/2}}$$

satisfies equation (7) provided F satisfies

$$\frac{d^2F}{d\eta^2} + \frac{\eta}{2} \frac{dF}{d\eta} - \frac{3}{4} F = -\frac{1}{\sqrt{\eta}} \tag{9}$$

In order to satisfy the boundary and initial conditions, F must remain finite as $\eta \longrightarrow \infty$ and $F'(0)$ must equal zero. Either a power-series method or numerical integration of equation (9) will give such a solution. An iteration process may be used for obtaining higher-order approximations to the solution of equation (3); for example, the first-order approximation is obtained by substituting the zero-order approximation (8) for K in the second term of the right member of equation (3). An equivalent process, suggested by the form of the zero-order approximation, is to write the solution of equation (3) in the form

$$K = \sum_{n=1}^{\infty} \left(\frac{b}{\sqrt{a}}\right)^n t^{3n/4} F_n(\eta) \tag{10}$$

where

$$\eta = \frac{R_x}{at^{1/2}}$$

substitute equation (10) into equation (3), collect terms, and equate the coefficients of the powers of $b/\sqrt{a}$ to zero. This solution is presented in appendix A, where the result is shown to be a set of ordinary differential equations for the functions F. The convergence of equation (10) for large values of η is easily demonstrable. The actual results for the first three terms indicate that equation (10) is convergent for small values of η. Values of F which permit equation (10) to satisfy conditions (4) and (6) may be calculated in a straightforward way (see appendix A). The functions F_n have the limiting values

$$F_n(\eta \longrightarrow \infty) = \frac{(-1)^{n+1}}{n!\ \eta^{n/2}}$$

and

$$F_n(\eta \longrightarrow 0) = C_n$$

where the coefficients C_n are constants positive for odd n and negative for even n.

In the case of zero conduction (a = 0) the solution of equation (3) which satisfies the boundary and initial conditions is

$$K_\infty = 1 - e^{-bt/R_x^{1/2}} \tag{11}$$

Equation (11) is also the asymptotic solution of equation (3) with conduction ($a \neq 0$) in the limit of large η and, therefore, correctly gives the temperature at finite values of R_x and sufficiently small values of time. This result is plausible physically, since at sufficiently small values of time and finite rate of heat transfer (R_x finite), conduction effects should have affected the temperatures by a negligible amount. The limit of large η also corresponds to large R_x and finite time. Therefore, equation (11) indicates that the effect of conduction is negligible at sufficiently large values of R_x. The functions F_1, F_2, and F_3 as derived in appendix A are plotted in figure 1 against η. For comparison, the asymptotic functions (zero conduction) are also shown. These results indicate that for values of η larger than about 5 the F_n functions are negligibly different from their asymptotic values and hence equation (10) is negligibly different from equation (11). Therefore, for $R_x > 5a\sqrt{t}$, the effect of conduction on the temperatures is negligible.

The solution for small values of η is perhaps of more interest. As η goes to zero, the values of F_n become the constants C_n. Therefore, the solution (eq. 10) at the leading edge ($R_x = 0$ and $\eta = 0$) becomes

$$K_{nose} = \sum_{n=1}^{\infty} \left(\frac{b}{\sqrt{a}}\right)^n t^{3n/4} C_n \tag{12}$$

where

$$C_1 = 1.9285; \quad C_2 = -2.05660; \quad C_3 = 1.59250$$

At small values of time, the temperature of the leading edge varies as the 3/4 power of the time. A curve of the form

$$f\left(\frac{b}{\sqrt{a}} t^{3/4}\right) = 1 - \left(1 + 0.2041 \frac{b}{\sqrt{a}} t^{3/4}\right)^{-9.447} \tag{13}$$

fits the leading-edge solution (12) in such a way that expanding equation (13) allows the first two terms of equation (12) to be given correctly and $C_3 = 1.605$ (0.79 percent difference).

The functions F_1, F_2, and F_3 were calculated and are plotted in figure 1. In order to present instantaneous temperature distributions over the flat plate, it is convenient to define a dimensionless time parameter

$$\alpha = \frac{bt^{3/4}}{\sqrt{a}}$$

and a dimensionless distance parameter

$$\beta = \frac{b^{2/3}R_x}{a^{4/3}}$$

These parameters are related to η in the following manner:

$$\eta = \frac{\beta}{\alpha^{2/3}}$$

Figure 2 shows the temperature distribution on the plate at a few chosen values of α obtained by using the first three terms of the summation in equation (10). The vertical marks on these curves indicate the contribution from the third term in the summation and, to some extent, the accuracy. The fourth term in the summation would give a negative contribution.

The Conical Shell

The heat conduction equation for the conical shell is derived in the following manner. The principle of conservation of heat is applied to a ring section of the conical shell of thickness ds parallel to the conical surface. Temperature gradients normal to the conical surface are assumed to be zero. When the shell is of constant thickness h and the axially symmetrical rate of heat input per unit area through the outer conical surface is the value of q_c given in equation (2), the temperature of the shell is

$$\rho_b c_b h \frac{\partial T}{\partial t} = k_b h \left(\frac{\partial^2 T}{\partial s^2} + \frac{1}{s} \frac{\partial T}{\partial s}\right) + 0.33\sqrt{3}\, k_g \frac{U}{\nu} Pr^{1/3} \frac{(T_{aw} - T)}{R_s^{1/2}}$$

or

$$\frac{\partial K}{\partial t} = a^2 \left(\frac{\partial^2 K}{\partial R_s^2} + \frac{1}{R_s} \frac{\partial K}{\partial R_s}\right) + \frac{\sqrt{3}b}{R_s^{1/2}}(1 - K) \tag{14}$$

where s is the distance along the conical surface from the vertex and

$$a^2 = \frac{k_b}{\rho_b c_b} \frac{U^2}{\nu^2}$$

$$b = 0.33 \frac{k_g}{\rho_b c_b} \frac{U}{\nu} \frac{Pr^{1/3}}{h}$$

$$K = \frac{T - T_i}{T_{aw} - T_i}$$

The fact that the initial temperature is T_i and the final temperature is T_{aw} gives the boundary conditions

$$K_{t=0} = 0 \tag{15}$$

and

$$K_{t \longrightarrow \infty} = 1 \tag{16}$$

Since the frontal area of the conical shell is zero, the condition of zero heat transfer through the frontal surface does not lead directly to a boundary condition on the temperature gradient at the nose. However, the condition that

$$K \text{ remains finite for all values of } t \text{ and } R_s \tag{17}$$

(which, incidentally, is implied in equations (15) and (16)) is adequate to select a unique solution. It is interesting to note that the solution selected by condition (17) has a zero temperature gradient at the nose. The assumption that the conical-shell thickness h is constant all the way to the vertex is an obvious mathematical fiction for values of s less than $h/\tan\theta$ where θ is the cone half angle. This error in the analysis of the conical shell is in such a direction as to give a rate of temperature rise that is less than the actual value, that is, too large a correction due to the effect of heat conduction. (This fact, partly at least, prompted the analysis of the next section.)

The analysis of the conical shell (appendix B) is very similar to that for the flat plate. The solution is written as

$$K = \sum_{n=1}^{\infty} \left(\frac{\sqrt{3}b}{\sqrt{a}}\right)^n t^{3n/4} G_n(\eta) \tag{18}$$

where

$$\eta = \frac{R_s}{at^{1/2}}$$

The zero-conduction (or large η) solution is

$$K_{\infty} = 1 - e^{-\sqrt{3}\frac{bt}{R_s^{1/2}}} \tag{19}$$

The nose solution ($\eta = 0$) is

$$K_{nose} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{3}b}{\sqrt{a}}\right)^n t^{3n/4} D_n \tag{20}$$

where the coefficients in the first three terms of the summation are

$$D_1 = 1.15622$$

$$D_2 = -0.69618$$

$$D_3 = 0.29080$$

The expression

$$f\left(\frac{\sqrt{3}b}{\sqrt{a}}\ t^{3/4}\right) = 1 - \left(1 + 0.04801\ \frac{\sqrt{3}b}{\sqrt{a}}\ t^{3/4}\right)^{-24.08} \tag{21}$$

fits the nose solution (eq. (20)) almost perfectly for the first three terms. The functions G_1, G_2, and G_3 were calculated and plotted in figure 3.

In order to present instantaneous temperature distributions over the conical shell, it is convenient to define a dimensionless time parameter

$$\alpha = \frac{\sqrt{3}\ bt^{3/4}}{\sqrt{a}}$$

and a dimensionless distance parameter

$$\beta = \frac{R_s b^{2/3} 3^{1/3}}{a^{4/3}}$$

related to η by

$$\eta = \frac{\beta}{\alpha^{2/3}}$$

Figure 4 shows the temperature distributions obtained over the conical shell for a few chosen values of α by using the first three terms of the summation in equation (18). The vertical marks on these curves indicate the contributions from the third term.

The Slender Solid Cone

The transient one-dimensional heat-conduction problem applies also to the solid cone in the limit of small cone angles. The heat-conduction equation is derived by applying the principle of conservation of heat to an element of the solid cone, which is a spherical shell with center at the cone vertex of radius s and thickness ds. For the axially symmetrical case, q_c from equation (2) is the rate of heat input per unit area through the conical surface. Temperature gradients tangential to the spherical shell element are assumed to be zero. The resulting equation for the temperature of the slender solid cone is

$$\rho_b c_b \frac{\partial T}{\partial t} = k_b \left(\frac{\partial^2 T}{\partial s^2} + \frac{2}{s} \frac{\partial T}{\partial s} \right) + \frac{\sin \theta}{1 - \cos \theta} \frac{q_c}{s}$$

or

$$\frac{\partial K}{\partial t} = a^2 \left(\frac{\partial^2 K}{\partial {R_s}^2} + \frac{2}{R_s} \frac{\partial K}{\partial R_s} \right) + \frac{b_c}{{R_s}^{3/2}}(1 - K) \tag{22}$$

where

$$a^2 = \frac{k_b}{\rho_b c_b} \cdot \frac{U^2}{\nu^2}$$

$$b_c = 0.33\sqrt{3}\, \frac{k_g}{\rho_b c_b} \frac{U^2}{\nu^2} \mathrm{Pr}^{1/3} g$$

$$g = \frac{\sin \theta}{1 - \cos \theta}$$

$$K = \frac{T - T_i}{T_{aw} - T_i}$$

and θ is the half vertex angle of the cone.

Since the initial temperature is T_i and the final temperature is T_{aw}, the solutions of equation (22) must satisfy the conditions

$$K_{t=0} = 0 \tag{23}$$

and

$$K_{t \to \infty} = 1 \tag{24}$$

and the condition that:

$$K \text{ remains finite for all values of } t \text{ and } R_s \tag{25}$$

The condition (25) specifies a unique solution which has a gradient at the nose varying inversely as the square root of R_s.

The solution is straightforward (appendix C), and may be written as

$$K = \sum_{n=1}^{\infty} \left(\frac{b_c}{a^{3/2}}\right)^n t^{n/4} H_n(\eta) \tag{26}$$

where

$$\eta = \frac{R_s}{at^{1/2}}$$

The zero conduction (large η) solution is

$$K_\infty = 1 - e^{-b_c t / R_s^{3/2}} \tag{27}$$

The nose solution is

$$K_{nose} = \sum_{n=1}^{\infty} \left(\frac{b_c}{a^{3/2}}\right)^n t^{n/4} E_n \tag{28}$$

where

$$E_1 = 1.95940$$

$$E_2 = -2.20696$$

$$E_3 = 1.91106$$

The expression

$$f\left(\frac{b_c}{a^{3/2}} t^{1/4}\right) = 1 - \left(1 + 0.2933 \frac{b_c}{a^{3/2}} t^{1/4}\right)^{-6.681} \qquad (29)$$

fits the first two terms of the summation of equation (28) and gives the third term within 2.0 percent. The functions H_1, H_2, and H_3 were calculated and plotted in figure 5.

In order to present instantaneous temperature distributions over the slender solid cone, it is convenient to define a dimensionless time parameter by

$$\alpha = \frac{b_c t^{1/4}}{a^{3/2}}$$

and a dimensionless distance parameter

$$\beta = \frac{R_s {b_c}^2}{a^4}$$

related to η by

$$\eta = \frac{\beta}{\alpha^2}$$

Figure 6 shows the temperature distribution over the slender solid cone for a few chosen values of α obtained by using the first three terms of the summation in equation (26). The vertical marks on the curves indicate the contributions from the third term of this summation.

DISCUSSION OF RESULTS

Figures 2, 4, and 6 present the temperature distributions over the flat plate, conical shell, and slender solid cone, respectively, as a function of the parameters α and β. On the basis of expressions (1) and (2) for the heat transfer through the laminar boundary layer, α and β are known functions of the flow properties of the air and the variables of the problem in each of the three cases. However, since in each of the cases analyzed the flow is steady just outside the boundary layer, the only real assumption is that the temperature and Reynolds number dependence of the heat-transfer expressions is correct.

Generally it is evident that the conduction effects are important near the nose or leading edge. In the most forward region the temperature is always lower than the zero-conduction temperature. Immediately behind this region is a second region, for which the rearward boundary is not well defined and where the temperature is higher than the zero-conduction temperature. The boundary between these two regions moves rearward as time increases. Because of the R^2 and R^3 factors in the mass elements, the temperatures for the conical shell and for the slender solid cone are negligibly different in the second region from the zero-conduction temperature.

Nonweiler (ref. 3) calculated the equilibrium temperature of the flat plate by taking into account internal conduction on the assumption that the heat loss by radiation is of such importance that the equilibrium temperature is greatly different from the adiabatic wall temperature. He obtained an equilibrium temperature distribution which is quite similar to the instantaneous distributions of figure 2. However, Nonweiler's result derives from a different cause. A conduction effect is present in his time-independent case only because the adiabatic wall temperature is never reached because of radiation heat transfer, an effect omitted in the present analysis. Nevertheless, qualitatively, the effect of heat conduction near the leading edge is similar in the two cases.

For the flat plate and the conical shell the temperature at the leading edge or nose varies initially as the 3/4 power of the time, with a coefficient which is nearly the same for the two. Specifically,

$$\frac{\text{First Term of Flat-Plate Solution}}{\text{First Term of Conical-Shell Solution}} = \frac{C_1}{\sqrt{3}\,D_1} = \frac{1}{1.04}$$

If expressions (13) and (21) are assumed to be correct for the nose temperatures, the ratio of the time required for the flat-plate leading edge to attain 1/2 of its total temperature rise (nose half-rise time) to the time required for the conical-shell nose to attain 1/2 of its total rise is 1.08. It must be noted, however, that the assumption of a constant thickness all the way to the vertex of the shell results in a calculated rate of nose temperature rise less than the actual value.

The nose temperature of the slender solid cone varies initially as the 1/4 power of the time. Since the solid-cone solution depends on the flow and the body properties in a way different from that in the case of the flat plate and conical shell, it is not possible to compare them except for specific cases. The obvious and striking difference is that the temperature gradient is infinite at the nose and that the temperature falls off much more rapidly for the solid cone than for the plate and shell.

It is interesting to estimate the extent of the forward region over which heat conduction effects are significant. For the flat plate the second region (temperature higher than zero-conduction temperature) is significant. Approximately, $\eta = 3.5$ defines the extent of the conduction region; thus:

For the flat plate:

$$\eta_o = \frac{\beta}{\alpha^{2/3}} = \frac{R_x}{at^{1/2}} = 3.5$$

For the conical shell and the slender solid cone, the second region is not significant. The position at which the actual temperature equals the zero-conduction temperature is taken as defining the significant region. Approximately, these positions are:

For the conical shell:

$$\eta_o = \frac{\beta}{\alpha^{2/3}} = \frac{R_s}{at^{1/2}} = 1.75$$

For the slender solid cone:

$$\eta_o = \frac{\beta}{\alpha^2} = \frac{R_s}{at^{1/2}} = 2.0$$

Solving these defining criteria for Reynolds number gives:

For the flat plate:

$$R_x = 3.5\left(\frac{k_s}{\rho_b c_b}\right)^{1/2} \frac{c}{\nu} Mt^{1/2} = 9.3 \times 10^5 \; Mt^{1/2}$$

For the conical shell:

$$R_s = 1.75\left(\frac{k_s}{\rho_b c_b}\right)^{1/2} \frac{c}{\nu} Mt^{1/2} = 4.7 \times 10^5 \; Mt^{1/2}$$

For the solid slender cone:

$$R_s = 2.0\left(\frac{k_s}{\rho_b c_b}\right)^{1/2} \frac{c}{\nu} Mt^{1/2} = 5.3 \times 10^5 \; Mt^{1/2}$$

where the final expressions (here and subsequently) are calculated for standard air and copper.

If the approximate expressions (13), (21), and (29) for the forward-edge temperatures are used, values of α corresponding to the nose half-rise times are:

Flat plate:

$$\alpha = 0.37 = \frac{b}{\sqrt{a}} t^{3/4} = 0.000357 \frac{\sqrt{M}}{h} t^{3/4}$$

Conical shell:

$$\alpha = 0.61 = \frac{\sqrt{3}b}{\sqrt{a}} t^{3/4} = 0.000613 \frac{\sqrt{M}}{h} t^{3/4}$$

Slender solid cone:

$$\alpha = 0.37 = \frac{b_c}{a^{3/2}} t^{1/4} = 0.0200g\sqrt{M} \; t^{1/4}$$

Combining these results gives for the Reynolds number extent of the significant region at nose half-rise time:

Flat plate:

$$R_{x_o} = \frac{a^{4/3}}{b^{2/3}}(0.37)^{2/3} = Pr^{-2/9}\left(\frac{0.37}{0.33} \frac{k_b}{k_g} \frac{c}{\nu} Mh\right)^{2/3} = 9.66 \times 10^7 M^{2/3}h^{2/3}$$

Conical shell:

$$R_{s_o} = \frac{a^{4/3}}{b^{2/3}}(0.61)^{2/3} = Pr^{-2/9}\left(\frac{0.61}{0.33\sqrt{3}} \frac{k_b}{k_g} \frac{c}{\nu} Mh\right)^{2/3} = 4.69 \times 10^7 M^{2/3}h^{2/3}$$

Slender solid cone:

$$R_{s_o} = \frac{a^4}{{b_c}^2}(0.37)^2 = \left(\frac{0.37}{0.33\sqrt{3}} \frac{k_b}{k_g}\right)^2 \frac{1}{Pr^{2/3}g^2} = \frac{2.42 \times 10^8}{g^2}$$

The nose half-rise times are:

Flat plate:

$$t_o = (0.37)^{4/3} \frac{a^{2/3}}{b^{4/3}} = 1.09 \times 10^4 \frac{h^{4/3}}{M^{2/3}}$$

Conical shell:

$$t_o = \frac{(0.61)^{4/3}}{3^{2/3}} \frac{a^{2/3}}{b^{4/3}} = 1.01 \times 10^4 \frac{h^{4/3}}{M^{2/3}}$$

Slender solid cone:

$$t_o = (0.37)^4 \frac{a^6}{{b_c}^4} = \frac{2.07 \times 10^5}{g^4M^2}$$

It is evident that the results depend upon the properties of the material of which the bodies are made primarily through the conduction parameter a. The heat-transfer parameters b and b_c depend on the product $\rho_b c_b$, which varies only slightly from one metal to another. However, the parameter a is proportional to $\left(\frac{k_b}{\rho_b c_b}\right)^{1/2}$. For the flat plate and the conical shell the nose half-rise time varies as the 1/3 power of k_b and the extent of the conduction region varies as the 2/3 power of k_b. On the other hand, for the slender solid cone the nose half-rise time varies as the cube of k_b and the extent of the conduction region at nose half-rise time varies as the square of k_b.

A serious limitation of the present analysis results from the assumption of uniform temperatures in the bodies in a direction perpendicular to the stream direction, especially in the case of the slender solid cone. Undoubtedly the solid-cone results are valid only in cases of rather small half-vertex angles. The geometry factor g varies, for sufficiently small vertex angles, as $2/\theta$. Because of the strong dependence on the factor g, with cone angles for which the analysis is reasonably correct, equilibrium conditions near the vertex are attained in extremely short time intervals. The error is not so serious for the flat plate and the conical shell, although as the thickness h increases, the error increases.

For the flat plate and the conical shell the extent of the significant region and the nose half-rise time vary only slightly with Mach number and thickness. The dependence of the results upon Mach number and kinematic viscosity always occurs in the combination cM/ν, so that at constant temperature similar situations result if the Mach number is proportional to the kinematic viscosity.

No suitable experimental data are available to check the results of this analysis.

CONCLUSIONS

The analysis of the transient heat-conduction effects for the semi-infinite bodies the insulated flat plate, the conical shell, and the slender solid cone, subjected to steady flight conditions beginning at zero time and with the only heat transfer that through a laminar boundary layer, shows that the effect is important only over the most forward portions of the body. For all three bodies, the extent of the conduction region increases with the square root of time and linearly with the Mach number. All results depend upon Mach number and free-stream kinematic viscosity as the quotient of the two, so that similar results are obtained when the Mach number varies directly as the kinematic viscosity.

For the flat plate and conical shell in typical flight conditions, the time required for the nose to attain one half the total temperature rise (nose half-rise time) is of the order of a fraction of a minute and varies inversely as the 2/3 power of the Mach number. The extent of the conduction region at nose half-rise time is of the order of inches and varies linearly with the Mach number.

For the slender solid cone the nose half-rise time and extent of conduction region at nose half-rise time depend strongly on the cone half angle. With cones sufficiently slender that the analysis is reasonably correct and in the same flight conditions, the nose half-rise time and extent of the conduction region are one (or more) order of magnitude less than for the plate and shell.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., August 13, 1953.

APPENDIX A

SOLUTION OF FLAT-PLATE PROBLEM

The solution of equation (3)

$$\frac{\partial K}{\partial t} = a^2 \frac{\partial^2 K}{\partial R_x^{\ 2}} + \frac{b}{R_x^{\ 1/2}}(1 - K) \tag{A1}$$

is written in equation (10) as

$$K = \sum_{n=1}^{\infty} \left(\frac{b}{\sqrt{a}}\right)^n t^{3n/4}\ F_n(\eta) \tag{A2}$$

where

$$\eta = \frac{R_x}{at^{1/2}}$$

Substituting equation (A2) into (A1) and equating coefficients of the powers of $b/\sqrt{a}$ yields

$$\left.\begin{aligned}
F_1'' + \frac{\eta}{2} F_1' - \frac{3}{4} F_1 &= -\frac{1}{\sqrt{\eta}} \\
F_2'' + \frac{\eta}{2} F_2' - \frac{3}{2} F_2 &= \frac{F_1}{\sqrt{\eta}} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
F_n'' + \frac{\eta}{2} F_n' - \frac{3n}{4} F_n &= \frac{F_{n-1}}{\sqrt{\eta}}
\end{aligned}\right\} \tag{A3}$$

The asymptotic (large η) solutions for the F functions are

$$F_n(\eta \longrightarrow \infty) = \frac{(-1)^{n+1}}{n!\,\eta^{n/2}}$$

which when substituted into equation (A2) give the asymptotic solution (zero-conduction solution):

$$K_\infty = 1 - e^{-bt/R_x^{1/2}} \tag{A4}$$

The ordinary differential equations for the F functions may be solved in sequence; F_1 may be written as the sum of a particular solution and a power-series homogeneous solution

$$F_1 = -\frac{4}{3}\,\eta^{3/2} + C_1\left[1 + \frac{3}{8}\,\eta^2 - \frac{1}{128}\,\eta^4 + \ldots + (-1)^{r+1}d(1,1;r)\eta^{2r} + \ldots\right] \tag{A5}$$

where the recursion formula for the coefficients is

$$d(1,1;r+1) = \frac{|4r - 3|}{4(2r + 1)(2r + 2)}\,d(1,1;r)$$

The quantity $F_1'(0) = 0$ which is required by the boundary condition (6). Proceeding with similar solutions for F_2 and F_3 yields

$$F_2 = -\frac{2}{9}\,\eta^3 + C_1\left[\frac{4}{3}\,\eta^{3/2} + d(2,1;1)\eta^{7/2} - \ldots (-1)^{r+1}d(2,1;r)\eta^{\frac{4r+3}{2}} + \ldots\right] + C_2\left[1 + d(2,2;1)\eta^2 + d(2,2;2)\eta^4 - d(2,2;3)\eta^6 + \ldots (-1)^r d(2,2;r)\eta^{2r} + \ldots\right] \tag{A6}$$

where

$$d(2,1;r+1) = \frac{4d(1,1;r+1) + |4r - 3|d(2,1;r)}{(4r + 7)(4r + 5)}$$

$$d(2,2;r+1) = \frac{|2r - 3|}{2(2r + 1)(2r + 2)} d(2,2;r)$$

and

$$F_3 = -\frac{8}{567} \eta^{9/2} + C_1 \left[\frac{2}{9} \eta^3 + d(3,1;1)\eta^5 - \right.$$
$$\left. d(3,1;2)\eta^7 + . . . (-1)^{r+1}d(3,1;r)\eta^{2r+3} + . . . \right] +$$
$$C_2 \left[\frac{4}{3} \eta^{3/2} + d(3,2;1)\eta^{7/2} + d(3,2;2)\eta^{11/2} - \right.$$
$$\left. d(3,2;3)\eta^{15/2} + . . . (-1)^r d(3,2;r)\eta^{\frac{4r+3}{2}} + . . . \right] +$$
$$C_3 \left[1 + d(3,3;1)\eta^2 + d(3,3;2)\eta^4 + d(3,3;3)\eta^6 - \right.$$
$$\left. d(3,3;4)\eta^8 + . . . (-1)^{r+1}d(3,3;r)\eta^{2r} + . . . \right] \tag{A7}$$

where

$$d(3,1;r+1) = \frac{4d(2,1;r+1) + |4r - 3|d(3,1;r)}{4(2r + 5)(2r + 4)}$$

$$d(3,2;r+1) = \frac{4d(2,2;r+1) + |4r - 6|d(3,2;r)}{(4r + 7)(4r + 5)}$$

$$d(3,3;r+1) = \frac{|4r - 9|}{4(2r + 1)(2r + 2)} d(3,3;r)$$

The series expansions for the F's (that is, equations (A5), (A6), and (A7)) may be used to evaluate the F's, or the differential equations (A3) may be integrated numerically. The series expansions were used for the calculation presented herein. For $n > 3$ the series method probably involves more labor but can yield greater accuracy. In either case, the boundary condition (6) requires that

$$F_n'(0) = 0$$

and the constants C_n

$$F_n(0) = C_n$$

must be determined so that each F has its proper asymptotic (large η) value,

$$F_n(\eta \longrightarrow \infty) = \frac{(-1)^{n+1}}{n!\eta^{n/2}}$$

Each new F involves one new C. Utilizing the fact that at sufficiently small values of time the solution (eq. (A2)) reduces to one term permits C_1 to be calculated analytically.

The method consists in constructing a solution of the one-dimensional heat-conduction problem for boundary conditions consistent with the present problem at small values of time and then evaluating this solution at the leading edge of the plate. A well-known solution of

$$\frac{\partial T}{\partial t} = a^2 \frac{\partial^2 T}{\partial x^2} \tag{A8}$$

is

$$T_*(x,t) = \frac{1}{2a\sqrt{\pi t}} e^{-(x-\xi)^2/4a^2 t} \tag{A9}$$

This solution corresponds to an initial temperature distribution which is zero everywhere except at $x = \xi$ and is infinite at $x = \xi$ in such a way that

$$\int_{-\infty}^{+\infty} T_*(x,0)dx = 1$$

In other words, equation (A9) corresponds to an initial temperature distribution given by the Dirac δ function, $\delta(x - \xi)$. The essence of the method is to interpret equation (A9) as the solution corresponding to injecting a unit quantity of heat at $x = \xi$ at $t = 0$.

The construction of a solution satisfying the conditions of the present problem (at small time) proceeds as follows. In order to insure $\frac{\partial T}{\partial x} = 0$ at $x = 0$, the input is made symmetrical about $x = 0$:

$$T(x,\xi,t) = \frac{1}{2a\sqrt{\pi t}}\left[e^{-(x-\xi)^2/4a^2t} + e^{-(x+\xi)^2/4a^2t}\right] \tag{A10}$$

Equation (A10) corresponds to unit heat input at $t = 0$ at $x = \xi$ and $x = -\xi$. Then

$$T(x,t) = \frac{1}{2a\sqrt{\pi}}\int_0^\infty \frac{b}{\sqrt{\xi}\sqrt{t}}\left[e^{-(x-\xi)^2/4a^2t} + e^{-(x+\xi)^2/4a^2t}\right]d\xi \tag{A11}$$

corresponds to putting $\frac{b}{\sqrt{\xi}}\,d\xi$ units of heat per unit time at $t = 0$ into the interval $d\xi$ at ξ. Then, if the input rate is assumed constant in time for sufficiently small times, and is integrated over a small time t

$$F(x,t) = \frac{b}{2a\sqrt{\pi}}\int_0^t\int_0^\infty \frac{1}{\sqrt{\xi}\sqrt{t}}\left[e^{-(x-\xi)^2/4a^2t} + e^{-(x+\xi)^2/4a^2t}\right]d\xi\,dt \tag{A12}$$

the expression for the temperature at x and at a small time t is obtained. At the leading edge $(x = 0)$, equation (A12) reduces to an expression which can be evaluated

$$F(0,t) = \frac{b}{2a\sqrt{\pi}}\int_0^t\int_0^\infty \frac{2}{\sqrt{\xi}\sqrt{t}}\,e^{-\xi^2/4a^2t}\,d\xi\,dt \tag{A13}$$

Integrating first with respect to ξ gives

$$F(0,t) = \frac{b}{a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t}}\left[\sqrt{2at^{1/2}}\;2\Gamma\left(\frac{5}{4}\right)\right]dt \tag{A14}$$

and, finally, integrating with respect to t yields

$$F(0,t) = \frac{b}{\sqrt{a}} t^{3/4} \frac{2\sqrt{2}\,\Gamma\left(\frac{1}{4}\right)}{3\sqrt{\pi}} \tag{A15}$$

Comparing equation (A15) with the first term of equation (A2) at $x = 0$ $(\eta = 0)$ yields

$$C_1 = \frac{2\sqrt{2}\,\Gamma\left(\frac{1}{4}\right)}{3\sqrt{\pi}} = 1.9285$$

The coefficient C_1 could also be determined by equating the heat inputs in the cases of conduction and zero conduction (at small times) over a sufficiently large forward portion of the plate. This procedure would result in equating the integral of equation (A5) at, say, $\eta = 6.0$ to the integral of the asymptotic (zero-conduction) value of F_1 at the same η. The value of C_1 is checked by F_1 approaching its proper asymptotic value. The C_n's for $n > 1$ are determined by choosing them so that the corresponding F_n's have their proper asymptotic values.

APPENDIX B

SOLUTION OF THE CONICAL-SHELL PROBLEM

Substituting equation (18)

$$K = \sum_{n=1}^{\infty} \left(\frac{\sqrt{3}b}{\sqrt{a}}\right)^n t^{3n/4} G_n(\eta) \tag{B1}$$

where

$$\eta = \frac{R_s}{at^{1/2}}$$

into equation (14)

$$\frac{\partial K}{\partial t} = a^2\left(\frac{\partial^2 K}{\partial R_s^{\,2}} + \frac{1}{R_s}\frac{\partial K}{\partial R_s}\right) + \frac{\sqrt{3}b}{R_s^{\,1/2}}(1 - K) \tag{B2}$$

and equating the coefficients of the powers of $b/\sqrt{a}$ yields

$$\left.\begin{aligned}
G_1'' + \left(\frac{1}{\eta} + \frac{\eta}{2}\right)G_1' - \frac{3}{4} G_1 &= -\frac{1}{\sqrt{\eta}} \\
G_2'' + \left(\frac{1}{\eta} + \frac{\eta}{2}\right)G_2' - \frac{3}{2} G_2 &= \frac{G_1}{\sqrt{\eta}} \\
\cdots\cdots\cdots&\cdots\cdots\cdots \\
G_n'' + \left(\frac{1}{\eta} + \frac{\eta}{2}\right)G_n' - \frac{3n}{4} G_n &= \frac{G_{n-1}}{\sqrt{\eta}}
\end{aligned}\right\} \tag{B3}$$

The asymptotic (large η) solutions for the G functions are

$$G_n(\eta \longrightarrow \infty) = \frac{(-1)^{n+1}}{n!\,\eta^{n/2}} \tag{B4}$$

(the same, incidentally, as for the F functions). When the asymptotic values of the G functions (eq. (B4)) are substituted into the solution (B1), the asymptotic (zero-conduction) solution (eq. (19)) results.

The boundary condition (17) (K remains finite for all values of t and R_s) requires that

$$G_n \text{ remains finite for all values of } \eta \text{ and } n \tag{B5}$$

Series solutions for the G functions satisfying condition (B5) are

$$G_1 = -\frac{4}{9}\eta^{3/2} + D_1\left[1 + \frac{3}{16}\eta^2 - \frac{3}{1024}\eta^4 + \ldots (-1)^{r+1}d(1,1;r)\eta^{2r} + \ldots\right] \tag{B6}$$

where

$$d(1,1;r+1) = \frac{|4r - 3|}{4(2r + 2)^2}\, d(1,1;r)$$

Similarly,

$$G_2 = -\frac{4}{81}\eta^3 + D_1\left[\frac{4}{9}\eta^{3/2} + d(2,1;1)\eta^{7/2} - d(2,1;2)\eta^{11/2} + \ldots (-1)^{r+1}d(2,1;r)\eta^{\frac{4r+3}{2}} + \ldots\right] + D_2\left[1 + d(2,2;1)\eta^2 + d(2,2;2)\eta^4 - d(2,2;3)\eta^6 + \ldots (-1)^r d(2,2;r)\eta^{2r} + \ldots\right]$$

where

$$d(2,1;r) = \frac{4d(1,1;r) + |4r - 7|\, d(2,1;r-1)}{(4r + 3)^2}$$

$$d(2,2;r) = \frac{|2r - 5|}{8r^2}\, d(2,2;r-1)$$

and

$$G_3 = -\frac{16}{6561}\eta^{9/2} + D_1\left[\frac{4}{81}\eta^3 + d(3,1;1)\eta^5 - d(3,1;2)\eta^7 + \ldots (-1)^{r+1}d(3,1;r)\eta^{2r+3} + \ldots\right] + D_2\left[\frac{4}{9}\eta^{3/2} + d(3,2;1)\eta^{7/2} + d(3,2;2)\eta^{11/2} - d(3,2;3)\eta^{15/2} + \ldots (-1)^r d(3,2;r)\eta^{\frac{4r+3}{2}} + \ldots\right] + D_3\left[1 + d(3,3;1)\eta^2 + d(3,3;2)\eta^4 + d(3,3;3)\eta^6 - d(3,3;4)\eta^8 + \ldots (-1)^{r+1}d(3,3;r)\eta^{2r} + \ldots\right]$$

where

$$d(3,1;r+1) = \frac{4d(2,1;r+1) + |4r - 3|d(3,1;r)}{4(2r + 5)^2}$$

$$d(3,2;r+1) = \frac{4d(2,2;r+1) + |4r - 6|d(3,2;r)}{(4r + 7)^2}$$

$$d(3,3;r+1) = \frac{|4r - 9|}{4(2r + 2)^2} d(3,3;r)$$

No straightforward analytic method for determining the constant D_1 was devised. The D coefficients were determined by choosing them so that the series expressions for the values of G_n were close to their asymptotic value at $\eta = 4.5$. These constants appear in the nose solution $(x = 0)$ since $G_n(0) = D_n$.

APPENDIX C

SOLUTION OF THE SLENDER-SOLID-CONE PROBLEM

Substituting equation (26)

$$K = \sum_{n=1}^{\infty} \left(\frac{b_c}{a^{3/2}}\right)^n t^{n/4} H_n(\eta) \tag{C1}$$

where

$$\eta = \frac{R_s}{at^{1/2}}$$

into equation (22)

$$\frac{\partial K}{\partial t} = a^2\left(\frac{\partial^2 K}{\partial R_s^{\,2}} + \frac{2}{R_s}\frac{\partial K}{\partial R_s}\right) + \frac{b_c}{R_s^{\,3/2}}(1 - K) \tag{C2}$$

and equating the coefficients of the powers of $b_c/a^{3/2}$ yields

$$\left.\begin{aligned}
H_1'' + \left(\frac{2}{\eta} + \frac{\eta}{2}\right)H_1' - \frac{1}{4} H_1 &= -\frac{1}{\eta^{3/2}} \\
H_2'' + \left(\frac{2}{\eta} + \frac{\eta}{2}\right)H_2' - \frac{1}{2} H_2 &= \frac{H_1}{\eta^{3/2}} \\
\cdots\cdots\cdots&\cdots\cdots \\
H_n'' + \left(\frac{2}{\eta} + \frac{\eta}{2}\right)H_n' - \frac{n}{4} H_n &= \frac{H_{n-1}}{\eta^{3/2}}
\end{aligned}\right\} \tag{C3}$$

The asymptotic solutions for the H functions are

$$H_n(\eta \longrightarrow \infty) = \frac{(-1)^{n+1}}{n!\,\eta^{3n/2}} \tag{C4}$$

When the asymptotic values of the H's (eq. (C4)) are substituted into the solution (C1), the asymptotic (zero-conduction) solution (27) results.

The boundary condition (25) (K remains finite for all values of t and R_s) requires that none of the H functions diverge as $\eta \longrightarrow 0$. The series solution for H_1 satisfying this boundary condition is

$$H_1 = -\frac{4}{3}\eta^{1/2} + E_1\Big[1 + d(1,1;1)\eta^2 - d(1,1;2)\eta^4 + \ldots (-1)^{r+1}d(1,1;r)\eta^{2r} + \ldots\Big]$$

where

$$d(1,1;r+1) = \frac{|4r - 1|}{4(2r + 2)(2r + 3)}\,d(1,1;r)$$

The series solution for H_2 is

$$H_2 = -\frac{2}{3}\eta + E_1\Big[\frac{4}{3}\eta^{1/2} + d(2,1;1)\eta^{5/2} - d(2,1;2)\eta^{9/2} + \ldots (-1)^{r+1}d(2,1;r)\eta^{\frac{4r+1}{2}} + \ldots\Big] + E_2\Big[1 + d(2,2;1)\eta^2 - d(2,2;2)\eta^4 + \ldots (-1)^{r+1}d(2,2;r)\eta^{2r} + \ldots\Big]$$

where

$$d(2,1;r+1) = \frac{4d(1,1;r+1) + |4r - 1|\,d(2,1;r)}{(4r + 5)(4r + 7)}$$

$$d(2,2;r+1) = \frac{|2r - 1|}{2(2r + 2)(2r + 3)}\,d(2,2;r)$$

The result for H_3 is

$$H_3 = -\frac{8}{45}\eta^{3/2} + E_1\left[\frac{2}{3}\eta + d(3,1;1)\eta^3 - d(3,1;2)\eta^5 + \ldots(-1)^{r+1}d(3,1;r)\eta^{2r+1} + \ldots\right] +$$

$$E_2\left[\frac{4}{3}\eta^{1/2} + d(3,2;1)\eta^{5/2} - d(3,2;2)\eta^{9/2} + \ldots(-1)^{r+1}d(3,2;r)\eta^{\frac{4r+1}{2}} + \ldots\right] +$$

$$E_3\left[1 + d(3,3;1)\eta^2 - d(3,3;2)\eta^4 + \ldots(-1)^{r+1}d(3,3;r)\eta^{2r} + \ldots\right]$$

where

$$d(3,1;r+1) = \frac{4d(2,1;r+1) + |4r - 1|\,d(3,1;r)}{4(2r + 3)(2r + 4)}$$

$$d(3,2;r+1) = \frac{4d(2,2;r+1) + |4r - 2|\,d(3,2;r)}{(4r + 5)(4r + 7)}$$

$$d(3,3;r+1) = \frac{|4r - 3|}{4(2r + 2)(2r + 3)}\,d(3,3;r)$$

As $\eta \longrightarrow 0$, the H functions take the values $H_n = E_n$ which, when substituted into the solution (26), gives the nose solution (28). The constants E_n (up to E_3) were determined by choosing them so that the series expressions for the H functions were close to their asymptotic values at $\eta = 4.5$.

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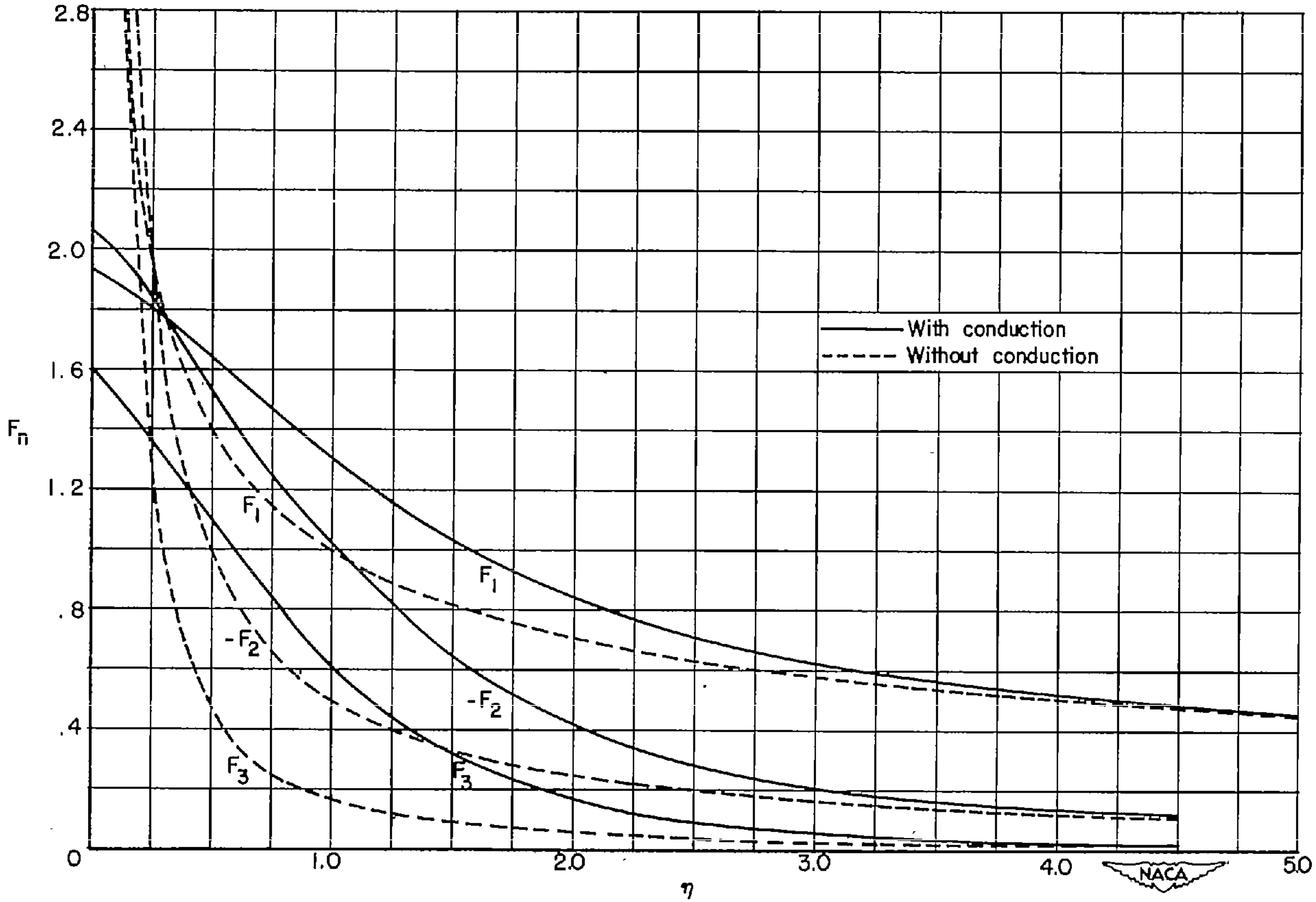

Figure 1.- The first three functions in the flat-plate solution. The asymptotic (zero-conduction) functions are shown for comparison.

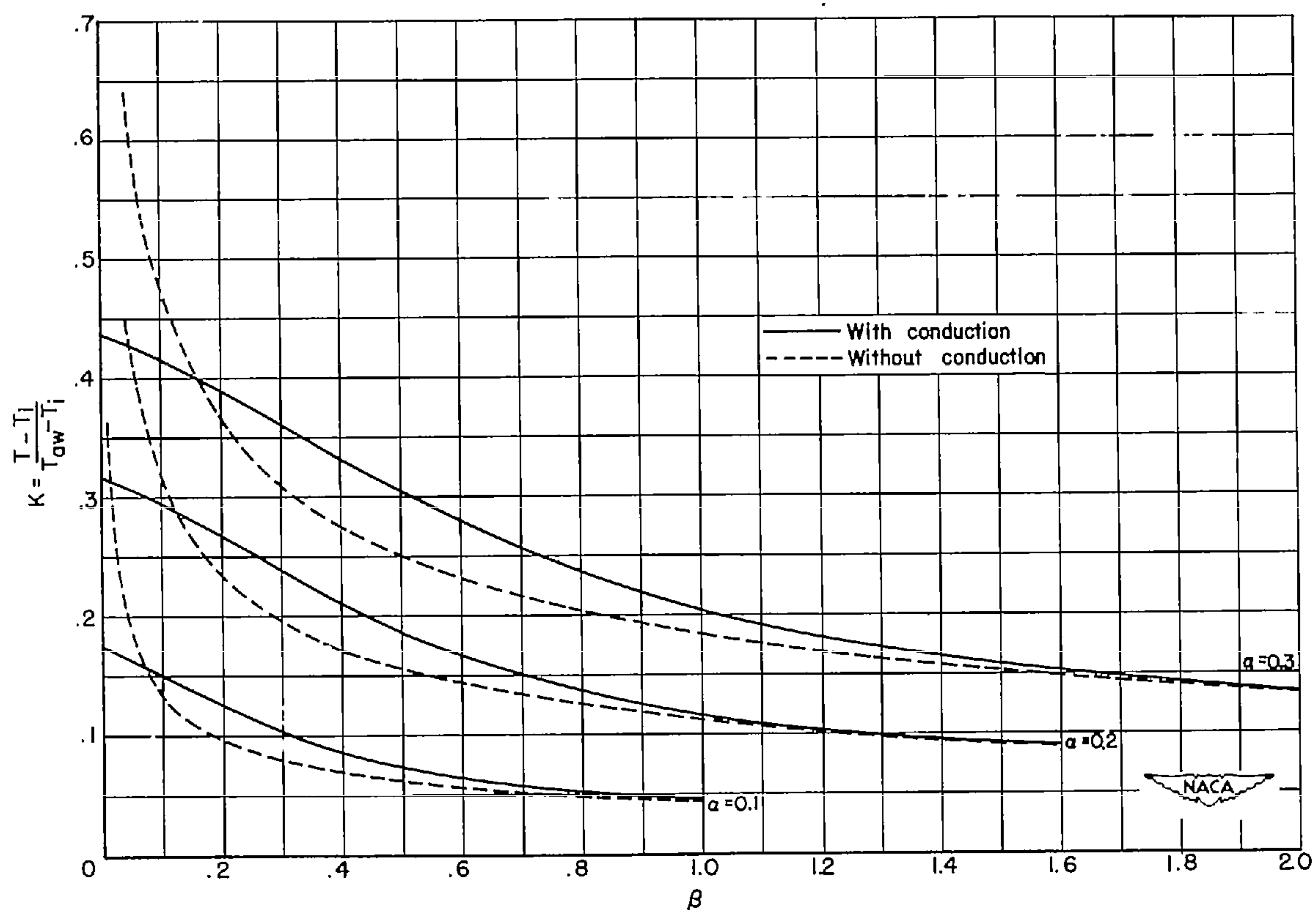

Figure 2.- Temperature distributions over the flat plate at three values of the time parameter α. $\alpha = \frac{bt^{3/4}}{\sqrt{a}}$; $\beta = \frac{b^{2/3}R_x}{a^{4/3}}$.

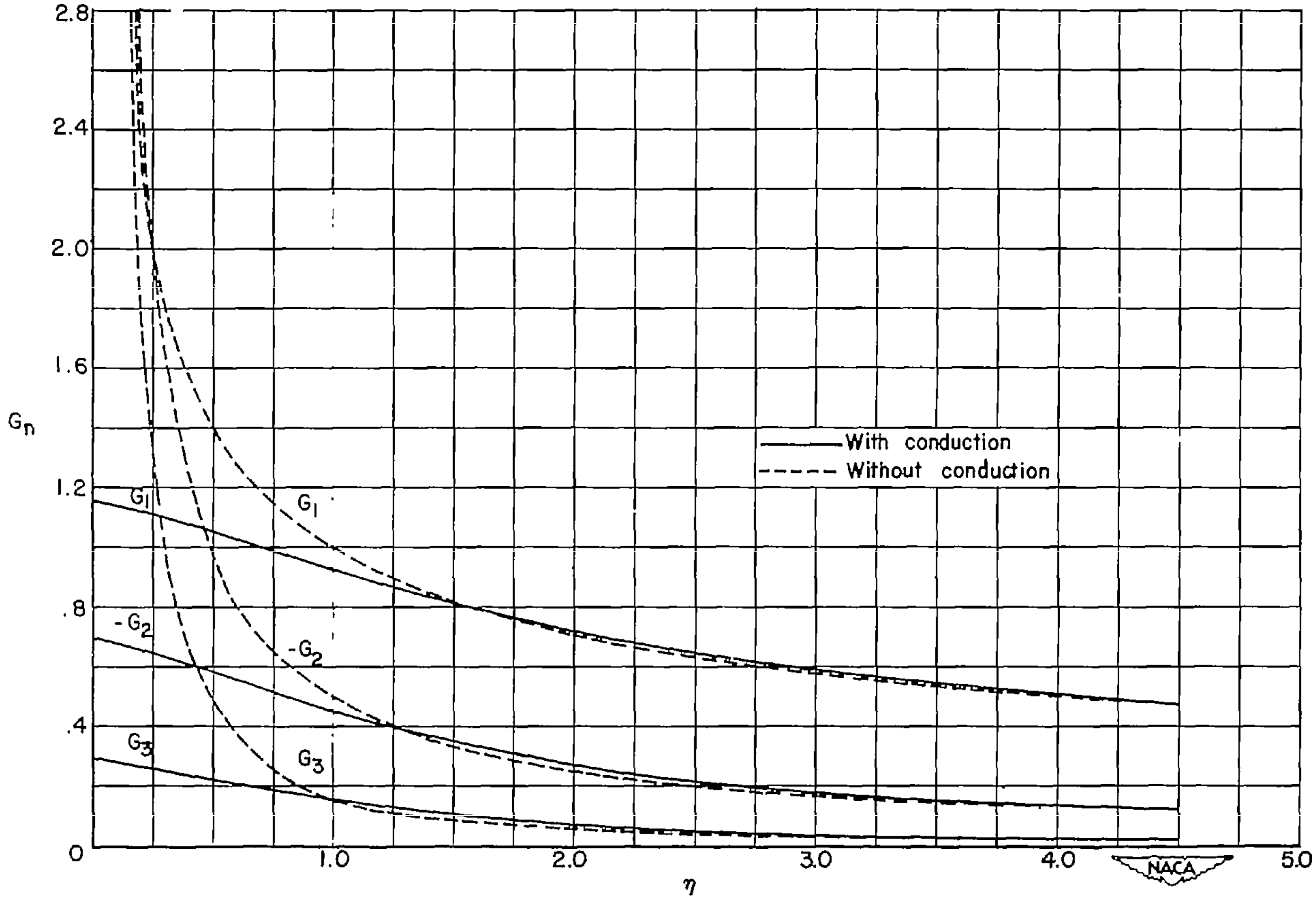

Figure 3.- The first three functions in the conical-shell solution. The asymptotic (zero-conduction) functions are shown for comparison.

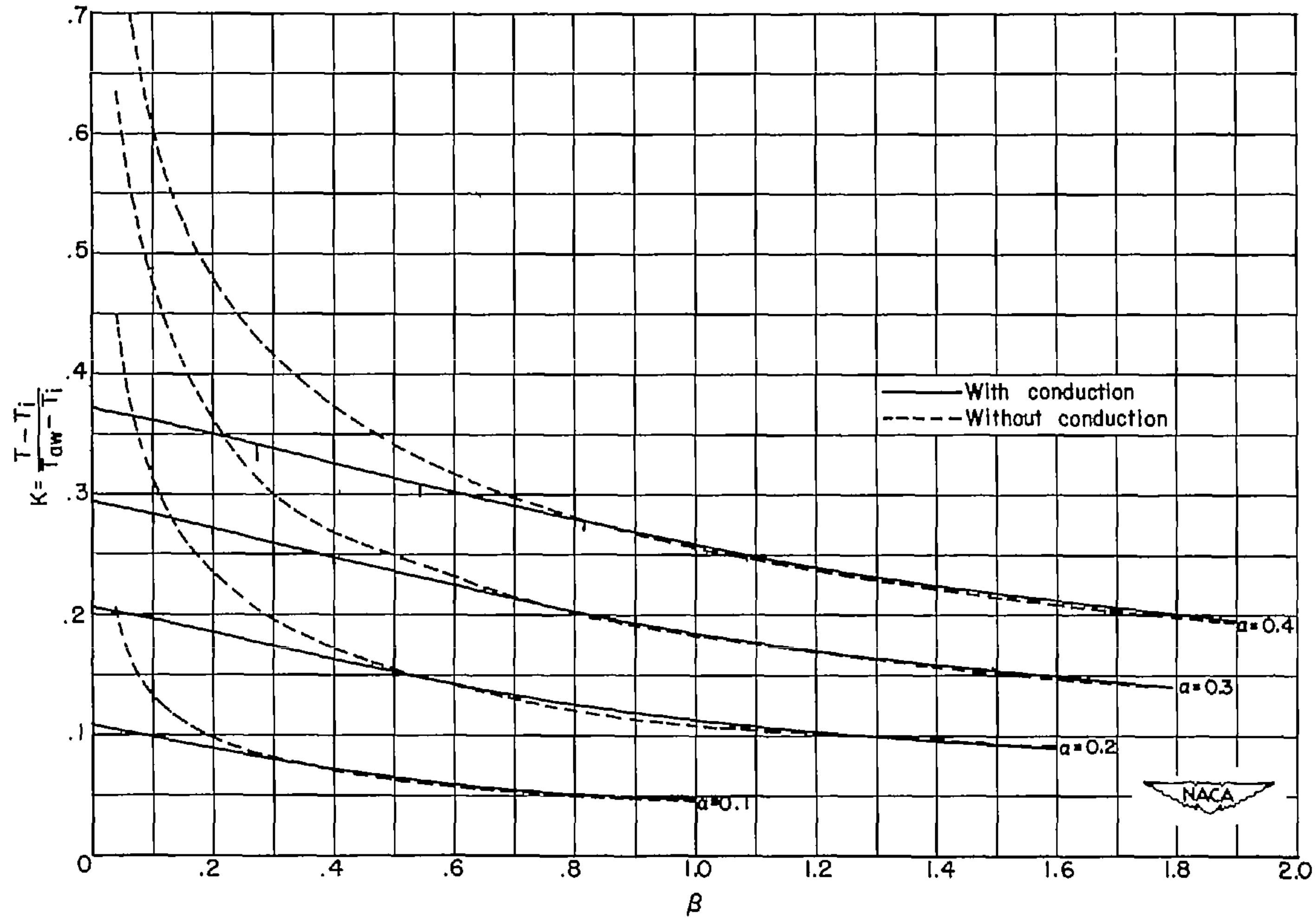

Figure 4.- Temperature distributions over the conical shell at four values of the time parameter α. $\alpha = \frac{3bt^{3/4}}{\sqrt{a}}$; $\beta = \frac{R_s b^{2/3} 3^{1/3}}{a^{4/3}}$.

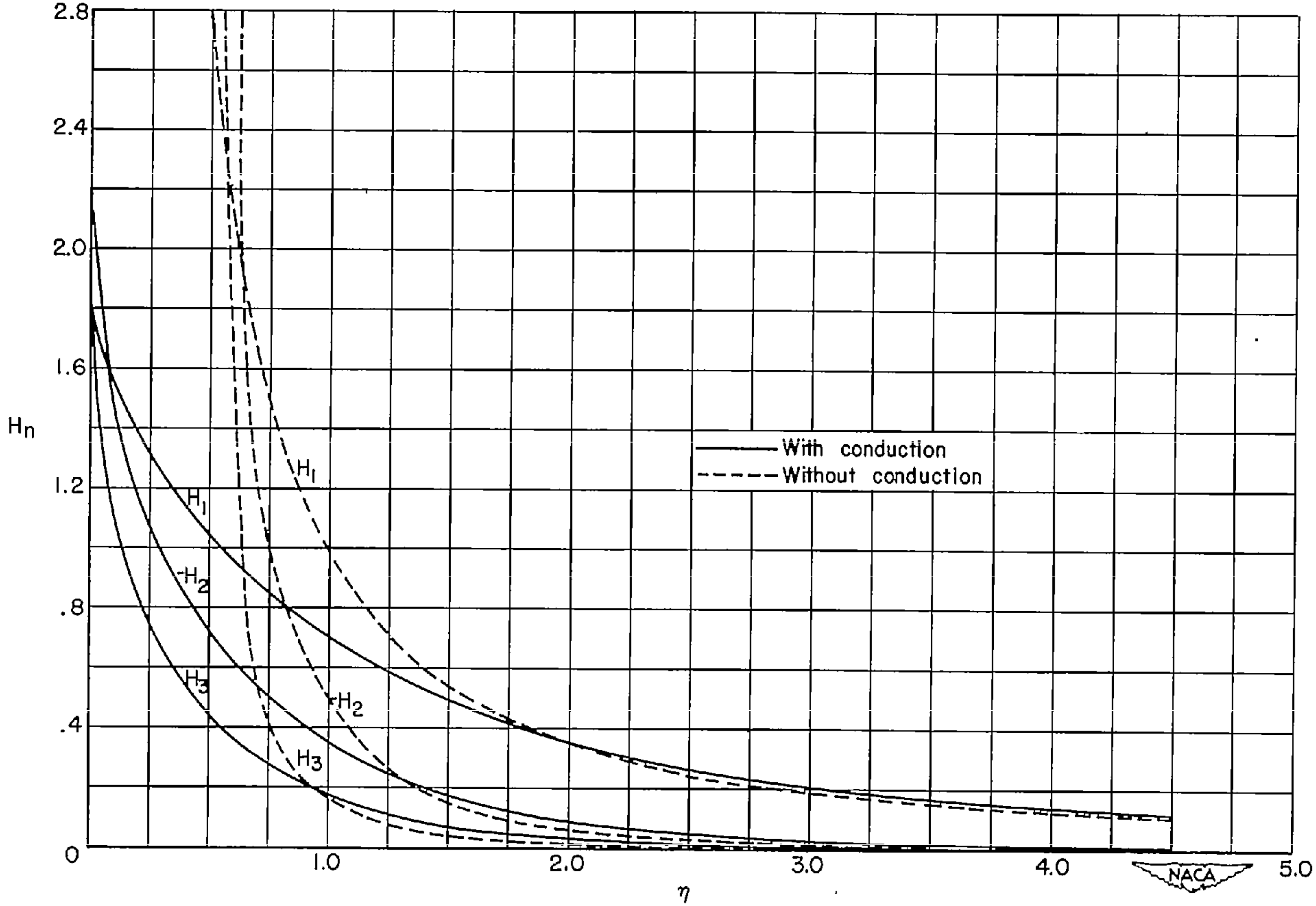

Figure 5.- The first three functions in the slender-solid-cone solution. The asymptotic (zero-conduction) functions are shown for comparison.

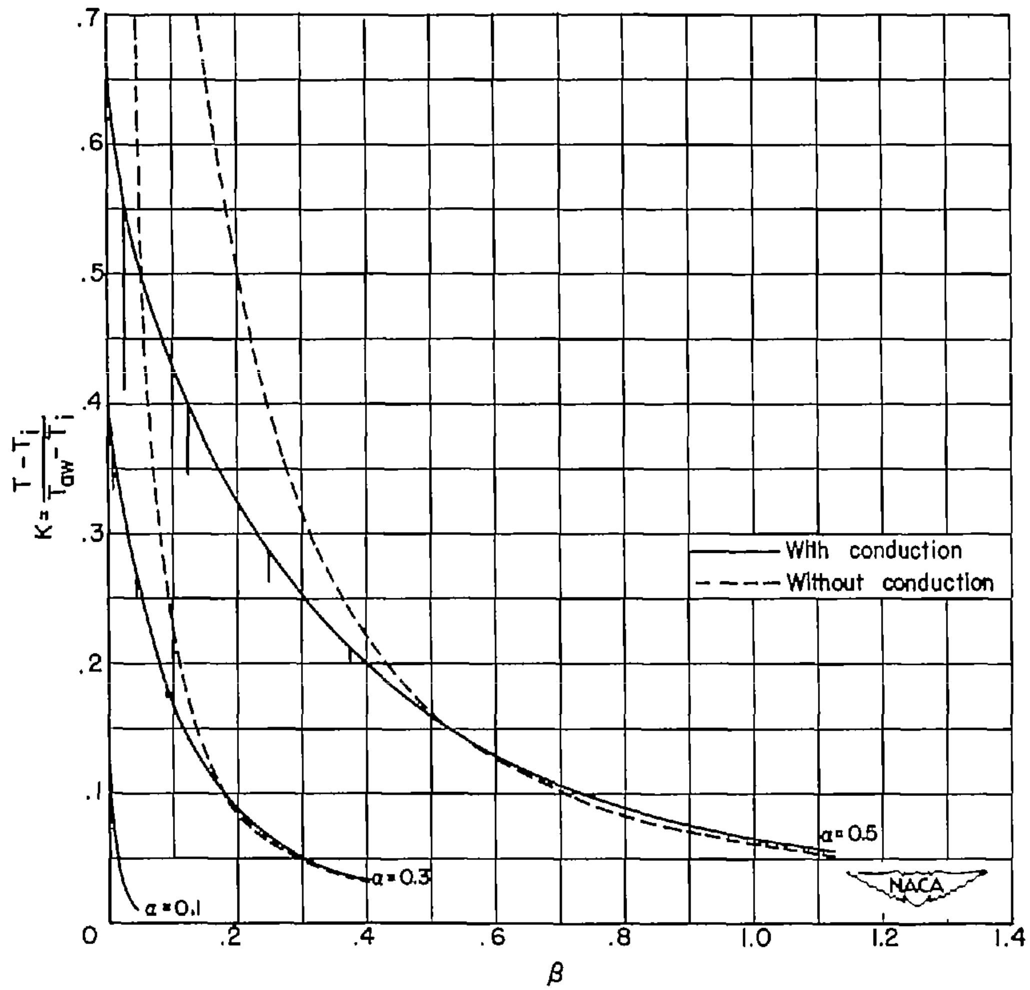

Figure 6.- Temperature distributions over the slender solid cone at three values of the time parameter α. $\alpha = \frac{b_c t^{1/4}}{a^{3/2}}$; $\beta = \frac{R_s b_c^2}{a^4}$.

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