NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2671

INVESTIGATION OF STRESS DISTRIBUTION IN RECTANGULAR PLATES WITH LONGITUDINAL STIFFENERS UNDER AXIAL COMPRESSION AFTER BUCKLING

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Washington
March 1952

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SUMMARY

An investigation was carried out to study the elastic behavior after buckling of a rectangular plate reinforced with longitudinal stiffeners and subjected to compressive loads in a direction parallel to the stiffeners. Two possible buckling modes were investigated; namely, the plate may buckle around the stiffeners as nodal lines and the plate may buckle with the intermediate stiffeners as a unit. The edge stiffeners were assumed to have finite torsional rigidity, infinite bending rigidity perpendicular to the plate, and either infinite or zero bending rigidity in the plane of the plate. For the first buckling mode the calculated results were compared with experimental results and the agreement was found to be good. The method of analysis used is a modified variational procedure. Instead of solving three nonlinear partial differential equations simultaneously, the transverse deflection w is assumed in the form of a function which satisfies the boundary conditions but contains undetermined parameters. In terms of this assumed expression for w, the other displacement components u and v may be solved from the differential equations. Then the unknown parameters in w are determined from the condition that the potential energy of the system must be stationary. It was found that this modified variational procedure will give much better results with the same amount of computational labor.

INTRODUCTION

Aircraft engineers are confronted with the difficult task of designing monocoque structures to fulfill all strength requirements and yet to remain within the limits prescribed by weight economy. To obtain an efficient monocoque structure, the engineer must have accurate information regarding the stress distribution in the structure. One of the most important component parts of the monocoque structure is the reinforced plate under the action of compressive loads.

A reinforced plate considered in this report is a sheet reinforced with stiffeners in the direction of the applied loads and equally spaced dividing the sheet into panels (fig. 1(a)). Under the action of the applied compressive load, the plate may deflect around the stiffeners as nodal lines as in case 1 or the system may displace as a complete unit, as shown in case 2, the stiffeners bowing with the plate (fig. 1(b)). In case 1, it is possible to consider the plate as one of the panels of the continuous plate attached to the stiffeners. In case 2, it is necessary to consider the stiffeners and the plate together.

As mentioned previously, to obtain the apportionment of the applied loads to the plate and the stiffeners it is necessary to know the stress distribution in the elastic structure. Up to and including buckling, the stress distribution in the panels may be satisfactorily predicted by the linear- or small-deflection theory. After buckling, however, the linear theory fails and the large-deflection theory of plates must be used. The large-deflection theory requires the solution of nonlinear differential equations which are generally intractable. In many cases, however, good approximate solutions can be obtained by the energy method or the variational method.

In this report an investigation was carried out to determine the stress distribution after buckling in rectangular plates reinforced with longitudinal stiffeners and subjected to compressive loads in a direction parallel to the stiffeners. Instead of following the energy or variational method as generally employed, a modified procedure is used. It may be recalled that the fundamental differential equations to be solved consist of three simultaneous nonlinear equations in terms of displacements u, v, and w. However, if w is assumed to be of some form, two of the differential equations become linear in u and v and can then be solved. Therefore an alternative way of carrying out the energy method may be as follows. First assume w in terms of a series which satisfies the boundary conditions but contains undetermined parameters. Substitution of w into two of the differential equations makes it possible to solve for u and v in terms of these undetermined parameters instead of by the usual way of assuming u and v in the form of series with additional parameters. The values of these parameters may now be ascertained so that the potential energy of the system is stationary. This condition implies that the third differential equation is approximately satisfied. This alternative procedure has two advantages. First, it is relatively easy to represent accurately the deflection surface w by a series of a few terms because of the existing experimental observations. Second, without any additional parameters in u and v only the parameters in w need to be determined in this case and much more accurate results can be obtained than could be obtained otherwise by the same amount of computational labor.

The behavior of a rectangular panel after buckling according to case 1 of the buckling mode has been treated by Dunn (reference 1). One may, however, take issue with Dunn's work on two counts. First, instead of satisfying the three equilibrium equations, Dunn's solution satisfies only two of the three equations and the third equation is replaced by an empirical relationship. Second, in determining the buckling load Dunn used one set of boundary conditions, but in studying the behavior after buckling another set of boundary conditions was used which was not at all compatible with the first set.

In this report, analysis has been carried out for both case 1 and case 2 of the buckling modes. Deflection pattern, stress distribution, and effective width are calculated to compare with Dunn's results and the experimental results obtained by Ramberg, McPherson, and Levy (reference 2).

The work was carried out at the Daniel Guggenheim School of Aeronautics, College of Engineering, New York University, under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. The authors are indebted to Professor F. K. Teichmann, Chairman of the Department of Aeronautics, for his understanding cooperation.

SYMBOLS

a	length of rectangular plate
A_{st}	stiffener area
b	width of rectangular plate
C	torsional stiffness of a sturdy stiffener when attached to rectangular plate
C_{BT}	torsion-bending constant of stiffener sectional area
d	distance between stiffeners
D	plate stiffness per unit width $\left(Et^3/12(1 - \nu^2)\right)$
E,E_{st}	Young's modulus for plate and stiffener, respectively
G	modulus of rigidity
I	moment of inertia

I_p polar moment of inertia of stiffener sectional area about axis of rotation

J torsion factor of stiffener

K buckling constant $\left(\sigma_{cr} b^2 t / \pi^2 D\right)$

m number of half waves into which plate buckles in x-direction

n, n_{st} number of bays and stiffeners, respectively

p lateral pressure

V strain energy

u, v x- and y-components of displacements, respectively

w lateral plate deflection

w_e effective width

x, y, z coordinate axes (fig. 2)

β aspect factor (λ/b)

γ shearing strain

$\delta = E_{st} A_{st} / E\,dt$

$\delta' = E_{st} A_{st} (1 - \mu^2) / E\,dt$

ϵ_x, ϵ_y axial strain in x- and y-direction, respectively

η constant of proportionality defining location of stiffener

λ half-wave length (a/m)

$\mu = bD/C$

$\mu' = dD/2C$

ν Poisson's ratio

$\rho = 2E_{st} I_{st} / dD$

σ compressive stress in stiffener

σ_x, σ_y	normal stress in x- and y-direction, respectively
$\sigma_{st}, \epsilon_{st}$	stiffener stress and strain, respectively
τ	shear stress
$\varphi = \pi m/a$	
cr	quantity at buckling, used as a subscript

GOVERNING DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

In terms of the displacement components u, v, and w and considering the case where w is large compared with u and v, the differential equations of equilibrium for a thin flat plate may be written as

$$u_{xx} + u_{yy} + \frac{1 + \nu}{1 - \nu}\left(u_{xx} + v_{xy}\right) + \frac{2}{1 - \nu}\left(w_x w_{xx} + w_y w_{xy}\right) + w_x w_{yy} - w_y w_{xy} = 0 \tag{1}$$

$$v_{xx} + v_{yy} + \frac{1 + \nu}{1 - \nu}\left(u_{xy} + v_{yy}\right) + \frac{2}{1 - \nu}\left(w_x w_{xy} + w_y w_{yy}\right) + w_{xx} w_y - w_x w_{xy} = 0 \tag{2}$$

$$\nabla^4 w - \frac{p}{D} - \frac{12}{t^2}\left[\left(u_x + \frac{1}{2} w_x^2\right)\left(w_{xx} + \nu w_{yy}\right) + \left(v_y + \frac{1}{2} w_y^2\right)\left(w_{yy} + \nu w_{xx}\right) + 2(1 - \nu)\left(u_y + v_x + w_x w_y\right) w_{xy}\right] = 0 \tag{3}$$

where $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$, p is the lateral loading on the plate, and the subscripts denote partial differentiation. The median-fiber strains are

$$\left.\begin{aligned} \epsilon_x' &= u_x + \frac{1}{2} w_x^2 \\ \epsilon_y' &= v_y + \frac{1}{2} w_y^2 \\ \gamma_{xy}' &= u_y + v_x + w_x w_y \end{aligned}\right\} \tag{4}$$

and the median-fiber stresses are

$$\left.\begin{aligned} \sigma_x' &= \frac{E}{1 - \nu^2}\left(\epsilon_x' + \nu\epsilon_y'\right) \\ \sigma_y' &= \frac{E}{1 - \nu^2}\left(\epsilon_y' + \nu\epsilon_x'\right) \\ \tau_{xy}' &= \frac{E}{2(1 + \nu)} \gamma_{xy}' \end{aligned}\right\} \tag{5}$$

The extreme-fiber bending and shearing stresses are

$$\left.\begin{aligned} \sigma_x'' &= -\frac{Et}{2\left(1 - \nu^2\right)}\left(w_{xx} + \nu w_{yy}\right) \\ \sigma_y'' &= -\frac{Et}{2\left(1 - \nu^2\right)}\left(w_{yy} + \nu w_{xx}\right) \\ \tau_{xy}'' &= -\frac{Et}{2(1 + \nu)} w_{xy} \end{aligned}\right\} \tag{6}$$

Equations (1) to (3) may be obtained by substituting relations (4) and (5) into the three equilibrium equations as given in reference 3, page 305.

To formulate the boundary conditions, it is necessary to survey briefly the general picture of the structure to be analyzed, the conditions which exist within it, and the loads that are imposed on it. The structure consists of a thin flat plate to which are continuously attached parallel and equidistant stiffeners. It is assumed that the length parallel to the stiffeners is large in comparison with the stiffener spacing. The structure is loaded with a compressive load parallel to the stiffeners in such a way that the axial loads are identical in each stiffener.

The stiffeners along the edges of the plate are assumed to have infinite bending rigidity in the z-direction; that is,

$$w = 0 \quad \text{at} \quad y = 0 \text{ and } b \tag{7}$$

In the y-direction, the sizes of the stiffeners are usually such that the bending rigidity is much smaller than that in the z-direction. In the theoretical treatment, two limiting cases may be taken. First, the stiffeners may be assumed to have zero bending rigidity. In such a case, the boundary condition is that along the edges the median-fiber stress σ_y' is zero; that is,

$$\sigma_y' = 0 \quad \text{at} \quad y = 0 \text{ and } b \tag{8a}$$

On the other hand, the stiffeners may be assumed to have infinite bending rigidity, and, in such a case, the boundary condition is

$$v = 0 \quad \text{at} \quad y = 0 \text{ and } b \tag{8b}$$

In the x-direction, it is assumed that the stiffeners are under a uniform compressive strain $-\epsilon_{st}$; that is,

$$u = -\epsilon_{st}x \quad \text{at} \quad y = 0 \text{ and } b \tag{9}$$

The slopes of the deflection surface along the edges depend on the torsional rigidity of the stiffeners. Along the edge of the plate adjacent to the stiffener at $y = 0$ the internal bending moment may be written in the form:

$$M_y = -D\left(w_{yy} + \nu w_{xx}\right) \tag{10}$$

where D is the flexural stiffness of the plate.

Assuming the plate cut along the stiffener, the moment given by equation (10) may be considered as an externally applied moment acting on the stiffener. This bending moment, which acts at any point x, produces twisting in the stiffener. The moment per unit length of the edge is numerically equal to the rate of change of the twisting moments in the stiffener at the point x. Expressed analytically,

$$M_y = -\left(\frac{dM_t}{dx}\right) \tag{11}$$

For a sturdy stiffener attached to a plate in compression, it was shown by Lundquist and Stowell (reference 4) that the moment required to rotate the plate through 1 radian is

$$C = \frac{\pi^2}{\lambda^2}\left(GJ - \sigma I_p + \frac{\pi^2}{\lambda^2} EC_{BT}\right) \tag{12}$$

where λ is the half wave of the buckled pattern of the plate in the direction of the applied load, GJ is the torsional rigidity of the stiffener, σ is the uniformly distributed compressive stress in the stiffener, I_p is the polar moment of inertia of stiffener sectional area about axis of rotation, and C_{BT} is the torsion-bending constant of the stiffener sectional area about the axis of rotation at or near the edge of the plate. Thus

$$M_t = C \frac{d\theta}{dx} \tag{13}$$

where θ is the angle of twist. Since

$$\theta \approx \frac{\partial w}{\partial y}$$

the boundary condition (11), after substitution, becomes

$$D\left(w_{yy} + \nu w_{xx}\right) = -Cw_{xxy} \tag{14}$$

Similarly, along the edge $y = b$ one obtains

$$D\left(w_{yy} + \nu w_{xx}\right) = Cw_{xxy} \tag{15}$$

Consider the limiting cases of zero torsional stiffness and infinite torsional stiffness. In the case of zero torsional stiffness, $C = 0$ and equations (14) and (15) reduce to $M_y = 0$ at $y = 0$ and b, which corresponds to the case usually referred to as simply supported edges. In the case of infinite torsional stiffness, $C = \infty$ and equations (14) and (15) reduce to $\theta = dw/dy = 0$ at $y = 0$ and b, which corresponds to the case usually referred to as clamped edges.

METHOD OF SOLUTION

Equations (1), (2), and (3) are three nonlinear partial differential equations to be solved simultaneously. The exact solution of these equations is very difficult, if not impossible, with the present boundary conditions. Approximate solutions, however, can be obtained by means of a variational method or the finite-differences method. In the present work, a modified variational method is to be used.

In the theory of elasticity, it is a well-known fact that the first variation of the potential energy of the elastic body with respect to the displacements u, v, and w leads to the differential equations of equilibrium which are, in the present case, equations (1), (2), and (3). The potential energy is defined as the difference between the strain energy and the virtual work which the surface stresses do over that portion of the boundary where these stresses are prescribed. For stable equilibrium, this stationary value is a minimum. The fact that the potential energy has a stationary value can be easily utilized in transforming the variational problem into an ordinary extreme-value problem by the Rayleigh-Ritz procedure as follows. Assume expressions for u, v, and w in terms of infinite series which satisfy the given boundary conditions but contain undetermined parameters. Substitute these expressions into the potential-energy expression and set the first derivatives of the resulting expression with respect to these parameters equal to zero. These parameters are then determined by solving the resultant simultaneous equations. With the values of the parameters so determined, the resulting expressions for u, v, and w satisfy both the boundary conditions and the differential equations. If infinite terms of the series are retained and the series consist of complete sets of functions, the solution may be proved to be exact. For the numerical determination of these parameters, however, only a finite number of terms can be retained. In such a case, the solution is then only approximate.

In the case of a thin plate, the deflection surface form can be easily observed from experimental tests, and consequently the deflection w can be represented fairly accurately by a series with only a few terms. It may also be observed that equations (1) and (2) reduce to linear differential equations if w is known. Therefore, for the same amount of computational labor, much more accurate results can be obtained by carrying out the variational procedure as follows. Assume an expression for w in terms of a series which can represent well the deflection from observed tests. The series satisfies the boundary conditions and contains undetermined parameters. Substitution of this assumed w into equations (1) and (2) results in two linear differential equations in u and v. Solving these equations simultaneously, "exact" solutions of u and v in terms of the parameters may be obtained. With the expression for w so assumed and expressions for u and v so obtained, these parameters can be determined by the usual Rayleigh-Ritz procedure. Following this procedure, only a few parameters will be used and therefore there are only a few resulting simultaneous equations to be solved. The amount of labor in the computation can thus be greatly reduced.

To satisfy the boundary conditions of w, the deflection surface may be assumed to be of the form:

$$\frac{w}{t} = \left[W_o\left(1 - \cos\frac{2\pi y}{b}\right) + W_1 \sin\frac{\pi y}{b}\right] \sin\frac{m\pi x}{a} \tag{16}$$

where W_o and W_1 are arbitrary deflection amplitudes to be determined and t is the thickness of the plate. The form of equation (16) is selected so that w satisfies the conditions of simply supported edges when $W_o = 0$ and the conditions of clamped edges when $W_1 = 0$. The ratio W_o/W_1 is therefore a measure of edge restraint.

This ratio W_o/W_1 is related to the elastic restraint of the sides through the boundary condition given by equation (14):

$$D\left(w_{yy} + \nu w_{xx}\right) = -Cw_{xxy} \tag{17}$$

Since $w = 0$ at the side edges and w is assumed to be of the form given by equation (16), the foregoing equation may be written in the form

$$Dw_{yy} = C\varphi^2 w_y \tag{18}$$

at $y = 0$, where $\varphi = \pi m/a$.

Substitution of equation (16) for w in equation (18) yields the following relationship

$$W_1 = \frac{4}{\pi} \mu\beta^2 W_o \tag{19}$$

where $\mu = bD/C$ is the ratio of the flexural rigidity of the plate between the edge stiffeners to the torsional rigidity of the stiffeners and $\beta = \lambda/b$.

The deflection-surface form may now be given in terms of the maximum amplitude W_o, with the replacing of W_1 in equation (16) by equation (19), and is as follows:

$$\frac{w}{t} = W_o(1 - \cos 2\beta\varphi y + A \sin \beta\varphi y) \sin \varphi x \tag{20}$$

where $A = \frac{4}{\pi} \mu\beta^2$.

It may be noted that at $y = b$ the condition given by equation (15) is also satisfied. For plates with simply supported edges μ becomes infinite, W_o becomes zero, and then AW_o becomes W_1 as defined before.

Expressions of u and v

The expressions for the displacement functions u and v may be obtained by solving the simultaneous differential equations (1) and (2) with the given w expression (20). These equations, being linear in u and v, may be solved as follows.

The complementary solution of equations (1) and (2) may be obtained by solving the following homogeneous equations:

$$\left.\begin{aligned} u_{xx} + u_{yy} + \frac{1 + \nu}{1 - \nu}\left(u_{xx} + v_{xy}\right) &= 0 \\ v_{xx} + v_{yy} + \frac{1 + \nu}{1 - \nu}\left(u_{xy} + v_{yy}\right) &= 0 \end{aligned}\right\} \tag{21}$$

To fulfill the boundary condition (7), the appropriate form of the complementary solution can be shown to be as follows:

$$u = \varphi t^2 W_o^{\,2}\left[\left(C_1 + C_2\right) \cosh 2\varphi y + 2C_2\varphi y \sinh 2\varphi y + \left(C_3 + C_4\right) \sinh 2\varphi y + 2C_4\varphi y \cosh 2\varphi y\right] \sin 2\varphi x - \varphi^2 t^2 \bar{\epsilon}_{st} x \tag{22}$$

$$v = \varphi t^2 W_o^{\,2}\left\{\left[\left(\frac{1 - \nu}{1 + \nu} 2C_2 - C_1\right) \sinh 2\varphi y - 2C_2\varphi y \cosh 2\varphi y + \left(\frac{1 - \nu}{1 + \nu} 2C_4 - C_3\right) \cosh 2\varphi y - 2C_4\varphi y \sinh 2\varphi y\right] \cos 2\varphi x + C_5\beta + C_6\beta^2\varphi y\right\} \tag{23}$$

where $\varphi^2 t^2 \bar{\epsilon}_{st} = \epsilon_{st}$, and C_1, C_2, C_3, C_4, C_5, and C_6 are integration constants. Physically, the complementary solution gives the displacements due to bending.

The particular integrals of the differential equations (1) and (2) give the additional displacements u and v due to large deflection. The method for determining these particular integrals is given in appendix A where the following forms for u and v are obtained:

$$u = \varphi t^2 W_o^2 \left(E_0 + E_1 \sin \beta\varphi y + E_2 \cos 2\beta\varphi y + E_3 \sin 3\beta\varphi y + E_4 \cos 4\beta\varphi y\right) \sin 2\varphi x \tag{24}$$

$$v = \varphi t^2 W_o^2 \beta \left[\left(F_1 \cos \beta\varphi y + F_2 \sin 2\beta\varphi y + F_3 \cos 3\beta\varphi y + F_4 \sin 4\beta\varphi y\right) \cos 2\varphi x + \left(F_5 \cos \beta\varphi y + F_6 \sin 2\beta\varphi y + F_7 \cos 3\beta\varphi y + F_8 \sin 4\beta\varphi y\right)\right] \tag{25}$$

The expressions for the E's and F's are to be found in appendix A.

The general solutions for the displacement functions u and v are the sums of the complementary solutions and the particular integrals. The integration constants C_1, C_2, C_3, C_4, C_5, and C_6 may now be determined from the given boundary conditions as is carried out in appendix B. With the boundary conditions satisfied, the general solutions may now be written as:

$$u = \varphi t^2 W_o^2 \left[\left(C_1 + C_2 - 2C_2\varphi y\right)(\cosh 2\varphi y - \sinh 2\varphi y) + E_0 + E_1 \sin \beta\varphi y + E_2 \cos 2\beta\varphi y + E_3 \sin 3\beta\varphi y + E_4 \cos 4\beta\varphi y + E_5 \sin 5\beta\varphi y\right] \sin 2\varphi x - \varphi^2 t^2 \bar{\epsilon}_{st} x \tag{26}$$

$$v = \varphi t^2 w_o^2 \left\{\left[\left(C_1 - \frac{1 - \nu}{1 + \nu} 2C_2 - 2C_2\varphi y\right)(\cosh 2\varphi y - \sinh 2\varphi y) + \beta\left(F_1 \cos \beta\varphi y + F_2 \sin 2\beta\varphi y + F_3 \cos 3\beta\varphi y + F_4 \sin 4\beta\varphi y\right)\right] \cos 2\varphi x + \beta\left(C_5 + C_6\varphi\beta y + F_5 \cos \beta\varphi y + F_6 \sin 2\beta\varphi y + F_7 \cos 3\beta\varphi y + F_8 \sin 4\beta\varphi y\right)\right\} \tag{27}$$

In equations (26) and (27), C_1, C_2, C_5, and C_6 are now known functions of the deflection parameters. The expressions for these coefficients are to be found in appendix B.

Evaluation of Potential Energy

As mentioned previously, the potential energy is the difference between the strain energy and the virtual work. The total strain energy of the panel is the sum of the strain energies of the plate and stiffeners. The strain energy of the plate is composed of two parts, one due to the bending of the plate V_{bp} and the other due to the extension or stretching of the plate V_{ep}. Let V_p denote the total plate strain energy, then

$$V_p = V_{bp} + V_{ep} \tag{28}$$

where

$$V_{bp} = \frac{Et^3}{24\left(1 - \nu^2\right)} \int\int \left[\left(w_{xx} + w_{yy}\right)^2 - 2(1 - \nu)\left(w_{xx}w_{yy} - w_{xy}^2\right)\right] dx\, dy \tag{29}$$

and

$$V_{ep} = \frac{Et}{2\left(1 - \nu^2\right)} \int\int \left[\left(\epsilon_x'\right)^2 + \left(\epsilon_y'\right)^2 + 2\nu\epsilon_x'\epsilon_y' + \frac{1}{2}(1 - \nu)\left(\gamma_{xy}'\right)^2\right] dx\, dy \tag{30}$$

The above integrations are to be taken over the total plate area. For rectangular plates with edges held in the original plane it can be proved that

$$\iint\left(w_{xx}w_{yy} - w_{xy}^2\right) dx\ dy = 0 \tag{31}$$

With the assumed expression for w in equation (20), and the expressions for u and v given by equations (26) and (27), integration of equations (29) and (30) gives

$$V_{bp} = \frac{E\varphi^4t^5ab}{1 - \nu^2} M_1W_o^2 \tag{32}$$

and

$$V_{ep} = \frac{E\varphi^4t^5ab}{1 - \nu^2}\left(M_2W_o^4 + M_3\bar{\epsilon}_{st}W_o^2 + M_4\bar{\epsilon}_{st}^2\right) \tag{33}$$

The evaluation of these expressions is carried out in detail in appendix C. The expressions for M_1, M_2, M_3, and M_4 are given by equations (C3), (C6), and (C7) in appendix C.

The strain energy of a stiffener V_s is composed of three parts, namely, the extensional strain energy V_{es}, the bending strain energy V_{bs}, and the twisting strain energy V_{ts}. The total strain energy is therefore

$$V_s = V_{es} + V_{bs} + V_{ts} \tag{34}$$

and the expressions for V_{es}, V_{bs}, and V_{ts} are:

$$V_{es} = \frac{1}{2} E_{st}A_{st}\int_0^a \left(\epsilon_x\right)^2_{y=\eta b}\ dx \tag{35}$$

$$V_{bs} = \frac{1}{2} E_{st}I_{st}\int_0^a \left(w_{xx}\right)^2_{y=\eta b}\ dx \tag{36}$$

$$V_{ts} = \frac{1}{2} C \int_0^a \left(w_{xy}\right)^2_{y=\eta b} dx \tag{37}$$

where ηb gives the location of the stiffeners $\eta = j/n$, n is the number of bays, and j is an integer which is equal to or less than n. The evaluation of equations (35), (36), and (37) is also carried out in appendix C. After some simplification, it is found that

$$V_{es} = \frac{1}{2} \varphi^4 t^4 E_{st} A_{st} a \left[W_o{}^4\left(\frac{N_1{}^2}{2} + N_2{}^2\right) - 2W_o{}^2 N_2 \bar{\epsilon}_{st} + \bar{\epsilon}_{st}{}^2\right] \tag{38}$$

$$V_{bs} = \frac{1}{2} \varphi^4 t^2 E_{st} I_{st} a W_o{}^2 \frac{N_3}{2} \tag{39}$$

$$V_{ts} = \frac{1}{2} \varphi^4 t^2 C a W_o{}^2 \frac{N_4}{2} \tag{40}$$

The expressions for N_1, N_2, N_3, and N_4 are given in appendix C.

Introducing the following parameters

$$\delta = \frac{E_{st} A_{st}}{Edt}$$

$$\delta' = \frac{E_{st} A_{st}\left(1 - \nu^2\right)}{Edt}$$

$$\rho = \frac{E_{st} I_{st}}{dD}$$

$$\mu' = \frac{dD}{2C}$$

where $d = b/n$ is the distance between stiffeners, the strain energy of a stiffener then becomes

$$V_s = \frac{E\varphi^4 t^5 ab}{1 - \nu^2}\left[\frac{\delta'}{4n}\left(N_1^2 + 2N_2^2\right)W_o^4 + \left(\frac{\delta'}{n} N_2\bar{\epsilon}_{st} + \frac{\rho N_3}{48n} + \frac{N_4}{48n\mu'}\right)W_o^2 + \frac{\delta'}{2n}\bar{\epsilon}_{st}^2\right] \tag{41}$$

The virtual work W is the product of the external load and the displacement in the direction of the load over that portion of the boundary where the boundary load is prescribed. In the present case, W is

$$\begin{aligned} W &= \int\int\left(\sigma_x' u\right)_{x=a} dy\, dz \\ &= \epsilon_{st}^2 a\left(E_{st} n_{st} A_{st} + Ebt\right) \end{aligned} \tag{42}$$

where n_{st} is the number of stiffeners.

The potential energy U of the panel is the sum of the strain energy of the plate and the strain energy of the stiffeners minus the virtual work

$$\begin{aligned} U = \frac{E\varphi^4 t^5 ab}{1 - \nu^2}\Bigg\{&\left[M_2 + \frac{\delta'}{4n}\sum\left(N_1^2 + 2N_2^2\right)\right]W_o^4 + \left[\left(M_3 - \frac{\delta'}{n}\sum N_2\right)\bar{\epsilon}_{st} + \right. \\ &\left.\left(M_1 + \frac{\rho\sum N_3}{48n} + \frac{\sum N_4}{48n\mu'}\right)\right]W_o^2 + \left[M_4 + \frac{n_{st}\delta'}{2n} - \right. \\ &\left.\left(1 - \nu^2\right)\left(\frac{E_{st}}{E}\frac{A_{st}}{bt} n_{st} + 1\right)\right]\bar{\epsilon}_{st}^2\Bigg\} \end{aligned} \tag{43}$$

In writing equation (43) it is assumed that all the stiffeners have the same properties, and $\sum N_i$ indicates the sum of N_i with $\eta b = d, 2d, 3d,$ and so forth.

Determination of Unknown Parameters

In the previous paragraphs, the solution of the differential equations (1), (2), and (3) has been carried out in the following manner. Assume an expression for w which satisfies the boundary conditions but contains undetermined parameters. Solve equations (1) and (2) in terms of the assumed w expression. If this assumed w also satisfies equation (3), then the solution is exact. In the present analysis an attempt will be made to satisfy equation (3) approximately by using the condition that at equilibrium the potential energy has a stationary value. In reference 1 a similar problem was treated by Dunn; the difference between Dunn's method and the method used in this report is that Dunn determined the parameters in w by empirical relations instead of by equation (3).

The parameters to be determined in w are W_o and β. Inspection of equation (43) shows that the potential energy U is of the form

$$U = R_1 W_o^{\,4} - \left(R_2 \bar{\epsilon}_{st} - R_3\right) W_o^{\,2} + R_4 \bar{\epsilon}_{st}^{\,2} \tag{44}$$

where R_1, R_2, and R_3 are functions of β and R_4 is a constant.

It is observed from experiments that once a plate buckles, the number of buckled waves remains the same when the buckling strain is exceeded. The factor β, which defines the number of buckles in the direction of loading, is therefore to be taken as a constant after buckling and equal to the value at buckling.

The parameter W_o may now be determined from the condition that the potential energy U has a stationary value with respect to W_o. The condition $\frac{\partial U}{\partial W_o} = 0$ leads to

$$W_o \left[2R_1 W_o^{\,2} - \left(R_2 \bar{\epsilon}_{st} - R_3\right)\right] = 0 \tag{45}$$

Equation (45) gives three roots for W_o, namely

$$W_o = 0 \tag{46}$$

and

$$W_o = \pm\sqrt{\frac{R_2 \bar{\epsilon}_{st} - R_3}{2R_1}} \tag{47}$$

The root $W_o = 0$ represents the unbuckled state. The positive and negative signs in equation (47) indicate that the plate may buckle indifferently in or out. Since these two possibilities are irrelevant to the present analysis, the positive sign will be taken in the subsequent discussions. Equation (47) also shows that when

$$\epsilon_{st} < \frac{R_3}{R_2}$$

there is no real root and therefore $W_o = 0$ is the only possible solution for equation (45). Physically, it indicates that when ϵ_{st} is below a certain value, the panel will not buckle and the unbuckled state is the only possible equilibrium form. The limiting condition occurs when

$$\left(\bar{\epsilon}_{st}\right)_{cr} = \frac{R_3}{R_2} \tag{48}$$

which indicates the buckling of the panel.

Substituting equation (48) into equation (47), one obtains

$$W_o = \sqrt{\frac{R_3}{2R_1}\left(\frac{\epsilon_{st}}{\epsilon_{cr}} - 1\right)} \tag{49}$$

The buckling stress of a plate is usually written in the following form:

$$\sigma_{cr} = \frac{K\pi^2 E t^2}{12\left(1 - \nu^2\right)b^2} \tag{50}$$

where K is a nondimensional coefficient that depends on conditions of edge restraint and shape of the plate. The buckling strain ϵ_{cr} is of course related to σ_{cr} according to Hooke's law. Rewrite equation (5) as follows:

$$\left.\begin{aligned} \epsilon_x' &= \frac{1}{E}\left(\sigma_x' + \nu\sigma_y'\right) \\ \epsilon_y' &= \frac{1}{E}\left(\sigma_y' + \nu\sigma_x'\right) \end{aligned}\right\} \tag{51}$$

When the boundary conditions are that $\sigma_y' = 0$ along $y = 0$ and $y = b$, <u>at buckling stress</u> $\sigma_y' = 0$ everywhere. Therefore, one obtains

$$\epsilon_{cr} = \sigma_{cr}/E$$

and

$$\bar{\epsilon}_{cr} = \left(\bar{\epsilon}_{st}\right)_{cr}$$

$$= \frac{K\beta^2}{12}$$

$$= \frac{R_3}{R_2}$$

Hence

$$K = \frac{12}{\beta^2}\frac{R_3}{R_2} \tag{52}$$

On the other hand, when the boundary conditions are that $v = 0$ along $y = 0$ and $y = b$, <u>at buckling stress</u> it can be seen from equation (B4) that $\epsilon_y' = 0$ everywhere. From equation (51) $\sigma_y' = -\nu\sigma_x'$ and $\epsilon_x' = \left(1 - \nu^2\right)\sigma_x'/E$. Therefore, one obtains

$$\epsilon_{cr} = \left(1 - \nu^2\right)\sigma_{cr}/E$$

and

$$K = \frac{12\left(1 - \nu^2\right)}{\beta^2}\frac{R_3}{R_2} \tag{53}$$

Equations (52) and (53) give K as a function of $\beta = a/mb$. For a given value of m, K may be plotted against a/b for a given value of μ which specifies the edge-support conditions. In a given problem, when a/b and μ are known, the value of K may be obtained by choosing the minimum value from the curves. The corresponding value of m is the number of half waves which actually occur. In practical cases, the dimensions of the panels are usually such that the curves K against a/b become straight lines and the value of K approaches the minimum value. For these cases, the values of β may be taken to be those which correspond to the minimum values of K. With β determined, W_o may be found from equation (49) at any ϵ_{st}. The median-fiber strains and stresses may be computed from equations (4) and (5) and the extreme-fiber bending stresses, from equations (6).

RESULTS AND DISCUSSION

Case 1: Rectangular Plate with Edge Stiffeners

As mentioned in the introduction, when a stiffened rectangular panel is under the action of compressive load two buckling modes may occur. One case is that the plate may deflect around the stiffeners as nodal lines. In such a case, it is possible to consider only one of the panels between two stiffeners and to regard it as a rectangular plate elastically supported along the two unloaded edges by stiffeners having infinite bending rigidity in the z-direction but finite torsional rigidity.

Neglecting the extensional strain energy of the stiffeners and letting b in this case be the width of the panel, the potential energy of the panel becomes

$$U = \frac{E\varphi^4 t^5 ab}{1 - \nu^2}\left\{M_2 W_o^4 + \left(M_3\bar{\epsilon}_{st} + M_1 + \frac{N_4}{24\mu}\right)W_o^2 + \left[M_4 - (1 - \nu^2)\left(\frac{E_{st}}{E}\frac{A_{st}}{bt}n_{st} + 1\right)\right]\bar{\epsilon}_{st}^2\right\} \tag{54}$$

The condition $\partial U/\partial W_o = 0$ becomes

$$W_o\left\{2M_2 W_o^2 + \left[M_3\bar{\epsilon}_{st} + M_1 + \left(N_4/24\mu\right)\right]\right\} = 0 \tag{55}$$

At buckling strain $\left(\bar{\epsilon}_{st}\right)_{cr}$,

$$M_3\left(\bar{\epsilon}_{st}\right)_{cr} + M_1 + \left(N_4/24\mu\right) = 0 \tag{56}$$

In terms of K as defined by equations (52) and (53), if the edge stiffeners are assumed to have negligible bending stiffness in the y-direction, that is, $\sigma_y{}' = 0$ at $y = 0$ and $y = b$,

$$K = \left[\mu^2\beta^4\left(0.4053 + 0.8106\beta^2 + 0.4053\beta^4\right) + \mu\beta^2\left(1.0808 + 2.1615\beta^2 + 2.7019\beta^4\right) + \left(0.75 + 2\beta^2 + 4\beta^4\right)\right] \Big/ \beta^2\left(0.4053\beta^4\mu^2 + 1.0807\beta^2\mu + 0.75\right) \tag{57}$$

and if the edge stiffeners are assumed to have infinite bending stiffness in the y-direction, that is, $v = 0$ at $y = 0$ and $y = b$,

$$K = \left[\frac{1}{4} + \frac{1}{8}\left(1 + 4\beta^2\right)^2 + \frac{16}{3\pi^2}\beta^2\left(1 + \beta^2\right)^2\mu + \frac{2}{\pi^2}\beta^4\left(1 + \beta^2\right)^2\mu^2 + \frac{8}{\pi^2}\mu\beta^6\right] \Big/ \beta^2\left[\frac{1}{8}\left(3 + 4\nu\beta^2\right) + \frac{16}{3\pi^2}\beta^2\left(1 + \nu\beta^2\right)\mu + \frac{2}{\pi^2}\beta^4\left(1 + \nu\beta^2\right)\mu^2\right] \tag{58}$$

For given values of μ and β, K for these cases can be calculated from equations (57) or (58) and is plotted against β for various values of μ in figure 3. For a large a/b ratio, it is pointed out in a previous paragraph that the minimum values of K or K_{min} may be used. The values of K_{min} and the corresponding values of β are plotted in figures 4 and 5, respectively. The values of β corresponding to K_{min} may be calculated from the condition $dK/d\beta = 0$, and K_{min} may then be computed. The condition $dK/d\beta = 0$, however, results in an equation with twelfth power in β and is difficult to solve. But with the approximate values of β obtained from figure 3, one may use Newton's method to calculate more-accurate values.

Now for a given μ the values of β corresponding to K_{min} may be determined from figure 5, and W_o may be calculated from equation (49) as a function of ϵ_{st}. In figure 6 the values of $W_o\Big/\sqrt{\left(\epsilon_{st}/\epsilon_{cr}\right) - 1}$ are plotted against μ.

With W_o and β calculated, it is thus possible to compute the stress distribution and the effective width of the plate at a given value of ϵ_{st}.

Substituting in equation (5) the expressions of ϵ_x' and ϵ_y' given by equations (B3) and (B4) in appendix B, the median-fiber stress σ_x' may be written in the following form:

$$\sigma_x' = \frac{E}{1 - \nu^2} \varphi^2 t^2 W_o^2 \Bigg\{ \bigg[2(1 - \nu)\left(C_1 - 2\varphi C_2 y + \frac{1 + 3\nu}{1 + \nu} C_2\right)(\cosh 2\varphi y - \sinh 2\varphi y) + \left(G_0 + \nu H_0\right) + \left(G_1 + \nu H_1\right) \sin \beta\varphi y + \left(G_2 + \nu H_2\right) \cos 2\beta\varphi y + \left(G_3 + \nu H_3\right) \sin 3\beta\varphi y \bigg] \cos 2\varphi x + \left(\nu\beta^2 C_6 + G_4 + \nu H_4\right) + \left(G_5 + \nu H_5\right) \sin \beta\varphi y + \left(G_6 + \nu H_6\right) \cos 2\beta\varphi y + \left(G_7 + \nu H_7\right) \sin 3\beta\varphi y + \left(G_8 + \nu H_8\right) \cos 4\beta\varphi y \Bigg\} - \frac{E}{1 - \nu^2} \varphi^2 t^2 \bar{\epsilon}_{st} \tag{59}$$

The distributions of σ_x' over the loaded edges of the plate with different boundary conditions are plotted in figure 7 as compared with those calculated by Dunn.

The total load carried by the plate for a given stiffener stress is

$$P = t \int_0^b \sigma_x \, dy$$

The effective width of a plate in compression is by definition

$$P = -2w_e t\sigma_{st}$$

where the negative sign indicates that σ_{st} is a compressive stress. The ratio of the effective width to the plate width may now be written as

$$\frac{w_e}{b} = \frac{1}{2\sigma_{st} b} \int_0^b \sigma_x' \, dy \tag{60}$$

It is shown in reference 1 that the distribution of stress at the nodal lines gives the actual load carried by the plate and any variation of the stress distribution between the nodal lines is balanced by the shearing stresses at the stiffener. Therefore, the effective width of the plate may be found from equation (60) when φx is taken to be equal to π, 2π, and so forth; carrying out the integration and taking $\varphi x = \pi$, equation (60) becomes

$$\frac{w_e}{b} = \frac{1}{2\left(1 - \nu^2\right)} \frac{E}{E_{st}}\left\{1 - \frac{W_o^2}{\bar{\epsilon}_{st}}\left[\frac{1}{2\varphi b}\left(C_1 + \frac{2\nu}{1 + \nu} C_2\right) + \left(G_0 + \nu H_0 + G_4 + \nu H_4 + \nu\beta^2 C_6\right) + \frac{2}{\pi}\left(G_1 + \nu H_1 + G_5 + \nu H_5\right) + \frac{2}{3\pi}\left(G_3 + \nu H_3 + G_7 + \nu H_7\right)\right]\right\} \quad (61)$$

The ratio of total effective width to initial width or $2w_e/b$ is plotted in figure 8 for the two limiting conditions $\mu = 0$ and $\mu = \infty$.

To compare these results with test results given in reference 2 where the panels were stiffened by z-shaped stiffeners, a panel with the following dimensions is chosen as a typical specimen:

$$\mu = 0.198$$

$$a = 19 \text{ inches}$$

$$b = 4.0 \text{ inches}$$

$$t = 0.025 \text{ inch}$$

$$P = 6800 \text{ pounds}$$

$$A_{st} = 0.39 \text{ square inch}$$

With these dimensions and the boundary conditions $\sigma_y' = 0$ one obtains $\beta = 0.688$ from figure 5. The calculated length of buckle developed is $\lambda = 0.688 \times 4 = 2.752$ inches. The corresponding number of buckles in the direction of the length of panel is $m = 19/2.752 = 6.90$ or 7 buckles since m must be an integer. Observed data showed the existence of seven buckles with a corresponding measured length of buckle of $\lambda = 2.70$ inches. The critical buckling stress may be calculated utilizing the coefficient K obtained from figure 4, and is found to be $\sigma_{cr} = 2570$ psi, to compare with the observed value of $\sigma_{cr} = 2500$ psi.

The wave form of the plate between the stiffeners under the compression load of 6800 pounds may be obtained in the following manner.

For $\mu = 0.198$ and $\beta = 0.688$, the amplitude factor $\dfrac{W_o}{\sqrt{(\epsilon_{st}/\epsilon_{cr}) - 1}}$ is 0.538. The stiffener strain corresponding to the test load of 6800 pounds is $\epsilon_{st} = 11 \times 10^{-4}$. Using equation (20), the results of the calculation of the wave form through the crest of a buckle $x = \lambda/2$ are shown in figure 9. The wave forms calculated by the present method show good agreement with those observed from the test.

With the boundary conditions $v = 0$, the value of β from figure 5 is 0.835. The calculated length of buckle developed is $\lambda = 0.835 \times 4 = 3.340$ inches. The corresponding number of buckles in the direction of the length of panel is $m = 19/3.340 = 5.70$ or 6 buckles since m must be an integer. This is compared with the test observance of seven buckles having a measured length of $\lambda = 2.70$ inches. The critical buckling stress, utilizing figure 4 for the value of K, is calculated to be 2170 psi, as compared with the observed data of $\sigma_{cr} = 2500$ psi.

The wave form of the plate between the stiffeners under the compression load of 6800 pounds is calculated in a similar manner as for the boundary condition $\sigma_y' = 0$ utilizing figure 6 for the amplitude factor $\dfrac{W}{\sqrt{(\epsilon_{st}/\epsilon_{cr}) - 1}}$, and is shown graphically in figure 9.

The experimental values of $2W_e/b$ from these tests are also plotted in figure 8 for comparison. It is seen from the figure that with boundary conditions $\sigma_y' = 0$, the calculated values give very good comparison. It thus seems that, for the z-shaped stringers used, the boundary condition can be well approximated by assuming $\sigma_y' = 0$ along the stiffeners.

It may be pointed out here that in reference 5 Dunn implicitly used the condition $\sigma_y' = 0$ in the determination of the buckling stress and then he took the results obtained in this case together with the boundary condition $v = 0$ in the study of the behavior after buckling. It is easily seen that these two boundary conditions are not compatible at all with each other.

Case 2: Rectangular Panel with Intermediate Stiffeners

When the longitudinal stiffeners have small bending and torsional rigidities, the rectangular panel will buckle as a unit with intermediate stiffeners bowing out with the plate. The edge stiffeners are assumed to have infinite bending rigidity in the z-direction. Four cases will be considered, namely, the edge stiffeners having either zero or infinite bending rigidity in the y-direction, together with either zero or infinite torsional rigidity.

Consider first the case where the edge stiffeners have zero bending rigidity in the y-direction and zero torsional rigidity; $\mu = \infty$ in this case and AW_0 becomes W_1. In terms of W, the potential energy U of the panel in equation (43) becomes now

$$U = \frac{E\varphi^4 t^5 ab}{1 - \nu^2}\left(\left\{M_2' + \frac{\delta'}{4n}\sum\left[\left(N_1'\right)^2 + 2\left(N_2'\right)^2\right]\right\}W_1^4 + \left[\left(M_3' - \frac{\delta'}{n}\sum N_2'\right)\overline{\epsilon}_{st} + \left(M_1' + \frac{\rho\sum N_3'}{48n} + \frac{\sum N_4'}{48n\mu'}\right)\right]W_1^2 + \left[M_4 + \frac{n_{st}\delta'}{2n} - \left(1 - \nu^2\right)\left(\frac{E_{st}}{E}\frac{A_{st}}{bt}n_{st} + 1\right)\right]\overline{\epsilon}_{st}^2\right) \tag{62}$$

where

$$\left.\begin{aligned}
M_1' &= \left(1 + \beta^2\right)^2/96 \\
M_2' &= \left(1 - \nu^2\right)\left[12 + 4\beta^4 - \left(3\beta^5/\pi\right)\right]/1024 \\
M_3' &= -\left(1 - \nu^2\right)/8 \\
M_4 &= \left(1 - \nu^2\right)/2 \\
N_1' &= \beta^2\left[\nu - \left(\nu + \frac{1 + \nu}{\beta}\pi\eta\right)(\cosh 2\varphi\eta b - \sinh 2\varphi\eta b)\right]/8 \\
N_2' &= \sin^2\pi\eta/4 \\
N_3' &= \sin^2\pi\eta \\
N_4' &= \beta^2\cos^2\pi\eta
\end{aligned}\right\} \tag{63}$$

The condition $\partial U/\partial W_1 = 0$ leads to

$$W_1\left(2W_1^2\left\{M_2' + \frac{\delta'}{4n}\sum\left[\left(N_1'\right)^2 + 2\left(N_2'\right)^2\right]\right\} + \left[\left(M_3' - \frac{\delta'}{n}\sum N_2'\right)\bar{\epsilon}_{st} + \left(M_1' + \frac{\rho\sum N_3'}{48n} + \frac{\sum N_4'}{48n\mu'}\right)\right]\right) = 0 \tag{64}$$

The buckling strain is therefore

$$\left(\bar{\epsilon}_{st}\right)_{cr} = \frac{M_1' + \sum\left[\rho N_3' + \left(N_4'/\mu'\right)\right]\Big/48n}{-M_3' + \left(\sum \delta' N_2'/n\right)} \tag{65}$$

and

$$K = \frac{\left(1 - \nu^2\right)n^2}{4\beta^2}\,\frac{48nM_1' + \sum\left[\rho N_3' + \left(N_4'/\mu'\right)\right]}{-nM_3' + \sum \delta' N_2'} \tag{66}$$

From equation (64), one obtains

$$W_1 = \left\{\frac{48nM_1' + \sum\left[\rho N_3' + \left(N_4'/\mu'\right)\right]}{96nM_2' + 24\delta'\sum\left[\left(N_1'\right)^2 + 2\left(N_2'\right)^2\right]}\left(\frac{\epsilon_{st}}{\epsilon_{cr}} - 1\right)\right\}^{\frac{1}{2}} \tag{67}$$

With these equations, one may proceed with the computation as in case 1 when the number of intermediate stiffeners is given.

Consider next the case where the edge stiffeners have zero bending rigidity in the y-direction and infinite torsional rigidity. The potential energy U of the panel is given by equation (43). With $\mu = 0$, one obtains

$$\left.\begin{aligned}
M_1 &= \left[2 + \left(1 + 4\beta^2\right)^2\right]\Big/96 \\
M_2 &= \left[35\left(1 - \nu^2\right)\Big/256\right] + \left\{\left(1 - \nu^2\right)\beta^4\left[2 + \left(1 + \beta^2\right)^{-2}\right]\Big/32\right\} - \\
&\quad \left\{\left(1 - \nu^2\right)\beta^9\left(1 + \beta^2\right)^{-5}\left[7 + (17 + 8\nu)\beta^2 + 13\beta^4 + 3\beta^6\right]\Big/64\pi\right\} \\
M_3 &= -3\left(1 - \nu^2\right)\Big/8 \\
M_4 &= \left(1 - \nu^2\right)\Big/2 \\
N_1 &= \beta^4\left\{2\left[(1 + 2\nu) + \nu\beta^2 - \varphi\eta b\left(1 + \beta^2\right)(1 + \nu)\right](\cosh 2\varphi\eta b - \right. \\
&\quad \left.\sinh 2\varphi\eta b) + \frac{\nu\beta^2}{2} + \frac{\beta^2\left(\beta^2 - \nu\right)}{2\left(1 + \beta^2\right)^2}\cos 2\pi\eta\right\}\Big/4\left(1 + \beta^2\right)^2 \\
N_2 &= \frac{3}{8} - \frac{1}{2}\cos 2\pi\eta + \frac{1}{8}\cos 4\pi\eta \\
N_3 &= \sin^2\pi\eta/4 \\
N_4 &= 4\beta^2\sin^2 2\pi\eta
\end{aligned}\right\} \tag{68}$$

Then $\left(\bar{\epsilon}_{st}\right)_{cr}$, K, and W_o are given by the following expressions:

$$\left(\bar{\epsilon}_{st}\right)_{cr} = \frac{M_1 + \left\{\sum\left[\rho N_3 + \left(N_4/\mu\right)\right] 48n\right\}}{-M_3 + \left(\sum \delta N_2/n\right)} \tag{69}$$

$$K = \frac{\left(1 - \nu^2\right)n^2}{4\beta^2} \frac{48nM_1 + \sum\left[\rho N_3 + \left(N_4/\mu\right)\right]}{-nM_3 + \sum \delta N_2} \tag{70}$$

$$W_o = \left\{\frac{48nM_1 + \sum\left[\rho N_3 + (N_4/\mu)\right]}{96nM_2 + 24\delta' \sum\left(N_1^2 + 2N_2^2\right)}\left(\frac{\epsilon_{st}}{\epsilon_{cr}} - 1\right)\right\}^{\frac{1}{2}} \tag{71}$$

The third case to be considered is that of the edge stiffeners having infinite bending rigidity in the y-direction and zero torsional rigidity. The potential energy U of the panel is given by equation (62) with

$$\left.\begin{aligned} M_2' &= \left[\left(3 - \nu^2\right)\left(1 + \beta^4\right) + 4\nu\beta^2 + \frac{1 - \nu}{3 - \nu}\frac{\nu^2\beta^5}{\pi}\right]\Big/256 \\ M_3' &= -\left(1 + \nu\beta^2\right)/8 \\ N_1' &= \nu\beta^2\left[1 - \left(1 - \frac{1 + \nu}{3 - \nu}2\varphi\eta b\right)(\cosh 2\varphi\eta b - \sinh 2\varphi\eta b)\right]\Big/8 \end{aligned}\right\} \tag{72}$$

M_1', M_4, N_2', N_3', and N_4' being the same as given by equation (63). One can calculate $(\epsilon_{st})_{cr}$ and W_1 from equations (65) and (67) by the use of new M_2', M_3', and N_1' as given by equations (72). From the definition of K in this case

$$K = \frac{n^2}{4\beta^2}\frac{48nM_1' + \sum\left[\rho N_3' + (N_4'/\mu')\right]}{-nM_3' + \sum \delta' N_2'} \tag{73}$$

The fourth case is that the edge stiffeners have infinite bending rigidity in the y-direction and infinite torsional rigidity. The potential energy U of the panel is again given by equation (43) with

$$
\left.
\begin{aligned}
M_2 &= \frac{\left(35 - 17\nu^2\right)}{256} + \frac{3\nu\beta^2}{16} + \left\{\beta^4\left[2\left(3 - \nu^2\right) + \frac{\left(1 - \nu^2\right)}{\left(1 + \beta^2\right)^2}\right]\right\}\Big/32 + \\
&\qquad \frac{(1 - \nu)^9\left(1 + 2\nu + \nu\beta^2\right)}{16\pi(3 - \nu)\left(1 + \beta^2\right)^4} \\
M_3 &= -\left(3 + 4\nu\beta^2\right)\big/8 \\
N_1 &= \frac{\beta^4(1 + 2\nu + \nu\beta^2)(1 + \nu)}{2\left(1 + \beta^2\right)^2(3 - \nu)}\left[(2\varphi\eta b - 3)(\cosh 2\varphi\eta b - \sinh 2\varphi\eta b)\right] + \\
&\qquad \frac{\nu\beta^2}{2} + \frac{\beta^2\left(\beta^2 - \nu\right)}{2\left(1 + \beta^2\right)^2}\cos 2\pi\eta
\end{aligned}
\right\} \tag{74}
$$

M_1, M_4, N_1, N_2, N_3, and N_4 being the same as given by equations (68). Equations (69) and (71) give $(\bar{\epsilon}_{st})_{cr}$ and W_o, while K is

$$
K = \frac{n^2}{4\beta^2}\,\frac{48nM_1 + \sum\left[\rho N_3 + \left(N_4/\mu'\right)\right]}{-nM_3 + \sum \delta' N_2} \tag{75}
$$

Numerical calculations have been carried out in the case of one stiffener in the middle of the plate. This case is important because in the case of multistiffeners approximate analysis can be made by the substitution of the single-stiffener method. In this case, $n = 2$. The bending rigidity ratio ρ is plotted against K_{min} in figure 10 with δ or δ' equal to 0 and 0.4; the corresponding β is plotted in figure 11. In figure 12 are plotted $W_o\Big/\sqrt{(\epsilon_{st}/\epsilon_{cr}) - 1}$ and $W_1\Big/\sqrt{(\bar{\epsilon}_{st}/\bar{\epsilon}_{cr}) - 1}$. Finally, the distribution of σ_x' along the loaded edge of the plate is plotted in figure 13 at $\epsilon_{st}/\epsilon_{cr} = 10$ for boundary condition $\sigma_y' = 0$ and in figure 14 at $\epsilon_{st}/\epsilon_{cr} = 10$ for boundary condition $v = 0$.

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New York, N. Y., December 29, 1949

APPENDIX A

DETERMINATION OF PARTICULAR INTEGRALS

FOR EQUATIONS (1) AND (2)

The nonhomogeneous differential equations are:

$$u_{xx} + u_{yy} + \frac{1 + \nu}{1 - \nu}\left(u_{xx} + v_{xy}\right) = \zeta(w) \tag{A1}$$

$$v_{xx} + v_{yy} + \frac{1 + \nu}{1 - \nu}\left(u_{xy} + v_{yy}\right) = \xi(w) \tag{A2}$$

where

$$\left.\begin{aligned} \zeta(w) &= -\frac{2}{1 - \nu}\left(w_x w_{xx} + w_y w_{xy}\right) - \left(w_x w_{yy} - w_y w_{xy}\right) \\ \xi(w) &= -\frac{2}{1 - \nu}\left(w_x w_{xy} + w_y w_{yy}\right) - \left(w_{xx} w_y - w_x w_{xy}\right) \end{aligned}\right\} \tag{A3}$$

Substituting the expression for w, equation (20), into equations (A3), $\zeta(w)$ and $\xi(w)$ are obtained as follows:

$$\zeta(w) = \frac{\varphi^3 t^2 W_o^2}{1 - \nu}\left\{\frac{1}{2}\left[\left(3 - 4\nu\beta^2\right) + \left(1 - \nu\beta^2\right)A^2\right] + \right.$$

$$\frac{1}{4}\left[12 + (3 - 11\nu)\beta^2\right]A \sin \beta\varphi y - \frac{1}{4}\left[4 + (9 - \nu)\beta^2\right]A \sin 3\beta\varphi y -$$

$$\frac{1}{4}\left[8 + 8(1 - \nu)\beta^2 + \left(2 + 2\beta^2\right)A^2\right] \cos 2\beta\varphi y +$$

$$\left.\frac{1}{2}\left(1 + 4\beta^2\right) \cos 4\beta\varphi y\right\} \sin 2\varphi x$$

$$\xi(w) = \frac{\varphi^3 t^2 w_o^2 \beta}{1 - \nu} \left\{ \left[-\frac{1}{2}\left(3 - 2\beta^2\right) A \cos \beta\varphi y + \right. \right.$$

$$\frac{1}{2}\left(3 + 6\beta^2\right) A \cos 3\beta\varphi y - \left(2 + \frac{1 + \beta^2}{2} A^2\right) \sin 2\beta\varphi y +$$

$$\left. \left(1 + 4\beta^2\right) \sin 4\beta\varphi y \right] \cos 2\varphi x - \frac{1}{2}\left(3\nu + 2\beta^2\right) A \cos \beta\varphi y +$$

$$\frac{1}{2}\left(3\nu - 6\beta^2\right) A \cos 3\beta\varphi y - \left(2\nu + \frac{\nu - \beta^2}{2} A^2\right) \sin 2\beta\varphi y +$$

$$\left. \left(\nu - 4\beta^2\right) \sin 4\beta\varphi y \right\}$$

Substituting the expressions just obtained into equations (A1) and (A2) for $\zeta(w)$ and $\xi(w)$, it is possible now to solve the differential equations and to obtain the particular integrals of the displacement functions u and v in terms of the undetermined deflection parameter W_o. The method of solution used here is the so-called method of undetermined coefficients. Examining the expressions for $\zeta(w)$ and $\xi(w)$, the particular integrals for u and v may be assumed to be of the following form:

$$u = \varphi t^2 W_o^2 \left(E_0 + E_1 \sin \beta\varphi y + E_2 \cos 2\beta\varphi y + E_3 \sin 3\beta\varphi y + \right.$$

$$\left. E_4 \cos 4\beta\varphi y\right) \sin 2\varphi x \tag{A4}$$

$$v = \varphi t^2 W_o^2 \beta \left[\left(F_1 \cos \beta\varphi y + F_2 \sin 2\beta\varphi y + F_3 \cos 3\beta\varphi y + \right. \right.$$

$$\left. F_4 \sin 4\beta\varphi y\right) \cos 2\varphi x + \left(F_5 \cos \beta\varphi y + F_6 \sin 2\beta\varphi y + \right.$$

$$\left. \left. F_7 \cos 3\beta\varphi y + F_8 \sin 4\beta\varphi y\right)\right] \tag{A5}$$

Substituting equations (A4) and (A5) into equations (A1) and (A2) and equating the coefficients of like terms, the E and F coefficients may be determined to be as follows:

$$E_0 = -\frac{1}{16}\left[\left(3 - 4\nu\beta^2\right) + \left(1 - \nu\beta^2\right)A^2\right]$$

$$E_1 = -\frac{1}{4\left(4 + \beta^2\right)^2}\left[24 + 2(6 - 11\nu)\beta^2 + 7\beta^4\right]A$$

$$E_2 = \frac{1}{16\left(1 + \beta^2\right)^2}\left\{4\left[1 + (2 - \nu)\beta^2 + 2\beta^4\right] + \left(1 + \beta^2\right)^2 A^2\right\}$$

$$E_3 = \frac{1}{4\left(4 + 9\beta^2\right)^2}\left[8 + 2(18 - \nu)\beta^2 + 45\beta^4\right]A$$

$$E_4 = -\frac{1}{16}$$

$$F_1 = \frac{1}{4\left(4 + \beta^2\right)^2}\left[12 - (16 + 11\nu)\beta^2 - 2\beta^4\right]A$$

$$F_2 = \frac{1}{16\left(1 + \beta^2\right)^2}\left[4\left(1 - \nu\beta^2\right) + \left(1 + \beta^2\right)^2 A^2\right]$$

$$F_3 = -\frac{3}{4\left(4 + 9\beta^2\right)^2}\left[4 + (16 - \nu)\beta^2 + 18\beta^4\right]A$$

$$F_4 = -\frac{1}{8}$$

$$F_5 = \frac{1}{4\beta^2}\left(3\nu + 2\beta^2\right)A$$

$$F_6 = \frac{1}{16\beta^2}\left[4\nu + \left(\nu - \beta^2\right)A^2\right]$$

$$F_7 = -\frac{1}{12\beta^2}\left(\nu - 2\beta^2\right)A$$

$$F_8 = -\frac{1}{32\beta^2}\left(\nu - 4\beta^2\right)$$

APPENDIX B

DETERMINATION OF INTEGRATION CONSTANTS

OF EQUATIONS (1) AND (2)

The general solutions for the displacements u and v are the sums of the complementary solutions and the particular integrals:

$$u = \varphi t^2 w_o^2 \Big[\left(C_1 + C_2\right) \cosh 2\varphi y + 2C_2 \varphi y \sinh 2\varphi y + \left(C_3 + C_4\right) \sinh 2\varphi y + 2C_4 \varphi y \cosh 2\varphi y + E_0 + E_1 \sin \beta\varphi y + E_2 \cos 2\beta\varphi y + E_3 \sin 3\beta\varphi y + E_4 \cos 4\beta\varphi y \Big] \sin 2\varphi x - \varphi^2 t^2 \bar{\epsilon}_{st} x \tag{B1}$$

$$v = \varphi t^2 w_o^2 \Bigg\{ \Bigg[\left(\frac{1 - \nu}{1 + \nu} 2C_2 - C_1\right) \sinh 2\varphi y - 2C_2 \varphi y \cosh 2\varphi y + \left(\frac{1 - \nu}{1 + \nu} 2C_4 - C_3\right) \cosh 2\varphi y - 2C_4 \varphi y \sinh 2\varphi y + \beta\Big(F_1 \cos \beta\varphi y + F_2 \sin 2\beta\varphi y + F_3 \cos 3\beta\varphi y + F_4 \sin 4\beta\varphi y\Big) \Bigg] \cos 2\varphi x + \beta\Big(C_5 + C_6 \varphi\beta y + F_5 \cos \beta\varphi y + F_6 \sin 2\beta\varphi y + F_7 \cos 3\beta\varphi y + F_8 \sin 4\beta\varphi y\Big) \Bigg\} \tag{B2}$$

Consider first the case where the edge stiffeners have negligible bending stiffness in the y-direction; that is,

$\sigma_y' = \dfrac{E}{1 - \nu^2}\left(\epsilon_y' + \nu\epsilon_x'\right) = 0$ along the edges $y = 0$ and $y = b$.

The strains in the median surface of the plate are given by equations (4). Substitution of the expressions for u, v, and w given by equations (B1), (B2), and (20) into equations (4) gives the following strain expressions:

$$\epsilon_x' = \varphi^2 t^2 W_o^2 \Big\{ \Big[2\left(C_1 + C_2 + 2\varphi C_4 y\right) \cosh 2\varphi y + 2\left(C_3 + C_4 + 2\varphi C_2 y\right) \sinh 2\varphi y + G_0 + G_1 \sin \beta\varphi y + G_2 \cos 2\beta\varphi y + G_3 \sin 3\beta\varphi y \Big] \cos 2\varphi x + G_4 + G_5 \sin \beta\varphi y + G_6 \cos 2\beta\varphi y + G_7 \sin 3\beta\varphi y + G_8 \cos 4\beta\varphi y \Big\} - \varphi^2 t^2 \bar{\epsilon}_{st} \tag{B3}$$

$$\epsilon_y' = \varphi^2 t^2 W_o^2 \Big\{ \Big[2\left(\frac{1 - 3\nu}{1 + \nu} C_2 - C_1 - 2\varphi C_4 y\right) \cosh 2\varphi y + 2\left(\frac{1 - 3\nu}{1 + \nu} C_4 - C_3 - 2\varphi C_2 y\right) \sinh 2\varphi y + H_0 + H_1 \sin \beta\varphi y + H_2 \cos 2\beta\varphi y + H_3 \sin 3\beta\varphi y \Big] \cos 2\varphi x + \beta^2 C_6 + H_4 + H_5 \sin \beta\varphi y + H_6 \cos 2\beta\varphi y + H_7 \sin 3\beta\varphi y + H_8 \cos 4\beta\varphi y \Big\} \tag{B4}$$

$$\gamma_{xy}' = \varphi^2 t^2 W_o^2 \Big[4\left(C_1 + \frac{2\nu}{1 + \nu} C_2 + 2\varphi C_4 y\right) \sinh 2\varphi y + 4\left(C_3 + \frac{2\nu}{1 + \nu} C_4 + 2\varphi C_2 y\right) \cosh 2\varphi y + J_1 \cos \beta\varphi y + J_2 \sin 2\beta\varphi y + J_3 \cos 3\beta\varphi y \Big] \sin 2\varphi x \tag{B5}$$

where

$$G_0 = \frac{1}{8} \nu\beta^2\left(4 + A^2\right)$$

$$G_1 = \frac{11\beta^2 A}{4\left(4 + \beta^2\right)^2}\left(4\nu - \beta^2\right)$$

$$G_2 = -\frac{1}{2\left(1 + \beta^2\right)^2} \beta^2\left(\nu - \beta^2\right)$$

$$G_3 = -\frac{1}{4\left(4 + 9\beta^2\right)^2} \beta^2 A\left(4\nu - 9\beta^2\right)$$

$$G_4 = \frac{1}{8}\left(3 + A^2\right)$$

$$G_5 = \frac{3A}{4}$$

$$G_6 = -\frac{1}{8}\left(4 + A^2\right)$$

$$G_7 = -\frac{A}{4}$$

$$G_8 = \frac{1}{8}$$

$$H_0 = -\frac{1}{8} \beta^2\left(4 + A^2\right)$$

$$H_1 = -\frac{11}{4\left(4 + \beta^2\right)^2} \beta^2 A\left(4 - \nu\beta^2\right)$$

$$H_2 = \frac{1}{2\left(1 + \beta^2\right)^2} \beta^2\left(1 - \nu\beta^2\right)$$

$$H_3 = \frac{1}{4\left(4 + 9\beta^2\right)^2} \beta^2 A\left(4 - 9\nu\beta^2\right)$$

$$H_4 = \frac{1}{8} \beta^2\left(4 + A^2\right)$$

$$H_5 = -\frac{3\nu A}{4}$$

$$H_6 = \frac{1}{8} \nu\left(4 + A^2\right)$$

$$H_7 = \frac{1}{4} \nu A$$

$$H_8 = -\frac{1}{8} \nu$$

$$J_1 = \frac{1}{\left(4 + \beta^2\right)^2} 11\beta^3 A(1 + \nu)$$

$$J_2 = \frac{1}{\left(1 + \beta^2\right)^2} \beta^3(1 + \nu)$$

$$J_3 = -\frac{1}{\left(4 + 9\beta^2\right)^2} 3\beta^3 A(1 + \nu)$$

The boundary condition (8a) leads to the following three equations:

$$\left.\begin{aligned}
&\left(H_0 + \nu G_0\right) + \left(H_2 + \nu G_2\right) + 2(1 - \nu)\left(\frac{1 - \nu}{1 + \nu} C_2 - C_1\right) = 0 \\
&W_o{}^2\left[\beta^2 C_6 + \left(H_4 + \nu G_4\right) + \left(H_6 - \nu G_6\right) + \left(H_8 + \nu G_8\right)\right] - \nu\bar{\epsilon}_{st} = 0 \\
&\left(H_0 + \nu G_0\right) + \left(H_2 + \nu G_2\right) + 2(1 - \nu)\left(\frac{1 - \nu}{1 + \nu} C_2 - C_1 - 2\varphi C_4 b\right) \cosh 2\varphi b + \\
&2(1 - \nu)\left(\frac{1 - \nu}{1 + \nu} C_4 - C_3 - 2\varphi C_2 b\right) \sinh 2\varphi b = 0
\end{aligned}\right\} \text{(B6)}$$

These equations result from the stipulation that the boundary condition must be satisfied for all values of x; thus it is necessary that the coefficients of each trigonometric term in x vanish at the boundaries.

The boundary condition (9) that $u = -\epsilon_{st}x$ at $y = 0$ and $y = b$ gives the following relations:

$$\left.\begin{aligned}
&C_1 + C_2 + E_0 + E_2 + E_4 = 0 \\
&\left(C_1 + C_2 + 2C_4\varphi b\right) \cosh 2\varphi b + \left(C_3 + C_4 + 2C_2\varphi b\right) \sinh 2\varphi b + \\
&E_0 + E_2 + E_4 = 0
\end{aligned}\right\} \text{(B7)}$$

Equations (B6) and (B7) consist of five equations for the determination of the six integration constants. One more equation therefore is needed in order to determine these constants uniquely. This equation may be obtained by specifying that v is a constant at a certain point. However, since the actual value of v is immaterial to the problem, equations (B6) and (B7) will be used to determine C_1, C_2, C_3, C_4, and C_6, with C_5 undetermined. Solving these equations, one obtains:

$$
\left.
\begin{aligned}
C_1 &= \frac{1 + \nu}{4(1 - \nu)}\Bigg[\left(H_0 + \nu G_0\right) + \left(H_2 + \nu G_2\right) - \frac{(1 - \nu)^2}{1 + \nu}\left(G_0 + G_2 + \right. \\
&\qquad \left. G_4 + G_6 + G_8\right)\Bigg] \\
C_2 &= -\frac{1 + \nu}{4(1 - \nu)}\Bigg[\left(H_0 + \nu G_0\right) + \left(H_2 + \nu G_2\right) + (1 - \nu)\left(G_0 + G_2 + \right. \\
&\qquad \left. G_4 + G_6 + G_8\right)\Bigg] \\
C_3 &= -\frac{1 + \nu}{4(1 - \nu)}\Bigg[\left(H_0 + \nu G_0\right) + \left(H_2 + \nu G_2\right) - \frac{(1 - \nu)^2}{1 + \nu}\left(G_0 + G_2 + \right. \\
&\qquad \left. G_4 + G_6 + G_8\right)\Bigg] \frac{\cosh 2\varphi b - 1}{\sinh 2\varphi b} \\
C_4 &= \frac{1 + \nu}{4(1 - \nu)}\Bigg[\left(H_0 + \nu G_0\right) + \left(H_2 + \nu G_2\right) + (1 - \nu)\left(G_0 + G_2 + \right. \\
&\qquad \left. G_4 + G_6 + G_8\right)\Bigg] \frac{\cosh 2\varphi b - 1}{\sinh 2\varphi b} \\
C_6 &= \frac{1}{\beta^2}\left[\nu \frac{\overline{\epsilon}_{st}}{W_o^2} - \left(H_4 + \nu G_4\right)\right]
\end{aligned}
\right\} \qquad \text{(B8)}
$$

Since $2\varphi b = 2\pi/\beta$ and $\beta = a/mb$, for usual panel dimension $e^{2\varphi b}$ is large compared with $e^{-2\varphi b}$ and 1. It is therefore sufficiently accurate to take

$$
\begin{aligned}
\sinh 2\varphi b &\approx \cosh 2\varphi b \\
&\approx \frac{1}{2} e^{2\varphi b}
\end{aligned}
\qquad \text{(B9)}
$$

With these simplifications, it is obvious that

$$\left.\begin{aligned} C_1 &\approx -C_3 \\ C_2 &\approx -C_4 \end{aligned}\right\} \tag{B10}$$

Now consider the case where the edge stiffeners have infinite bending rigidity in the y-direction. The boundary condition that $v = 0$ at $y = 0$ and $y = b$ then yields the following relations:

$$\left.\begin{aligned} &\frac{1 - \nu}{1 + \nu} 2C_4 - C_3 + \left(F_1 + F_3\right)\beta = 0 \\ &C_5 + F_5 + F_7 = 0 \\ &\left(\frac{1 - \nu}{1 + \nu} 2C_2 - C_1 - 2C_4\varphi b\right) \sinh 2\varphi b + \left(\frac{1 - \nu}{1 + \nu} 2C_4 - C_3 - \right. \\ &\left. 2C_2\varphi b\right) \cosh 2\varphi b - \beta\left(F_1 + F_3\right) = 0 \\ &C_5 + \pi C_6 - F_5 - F_7 = 0 \end{aligned}\right\} \tag{B11}$$

Solving simultaneously equations (B11) and equations (B7) which result from the boundary condition $u = -\epsilon_{st}x$ at $y = 0$ and $y = b$, one obtains

$$
\left.
\begin{aligned}
C_1 &= -\left[\left(E_0 + E_2 + E_4\right)\left(\frac{1 - \nu}{1 + \nu} 2 \sinh 2\varphi b - 2\varphi b\right) + \beta\left(F_1 + F_3\right)\left(1 + \cosh 2\varphi b\right)\right] \Big/ \left(\frac{3 - \nu}{1 + \nu} \sinh 2\varphi b - 2\varphi b\right) \\
C_2 &= -\left[\left(E_0 + E_2 + E_4\right) \sinh 2\varphi b - \beta\left(F_1 + F_3\right)\left(1 + \cosh 2\varphi b\right)\right] \Big/ \left(\frac{3 - \nu}{1 + \nu} \sinh 2\varphi b - 2\varphi b\right) \\
C_3 &= \left[\frac{1 - \nu}{1 + \nu} 2\left(E_0 + E_2 + E_4\right)\left(\cosh 2\varphi b - 1\right) + \beta\left(F_1 + F_3\right)\left(\sinh 2\varphi b - 2\varphi b\right)\right] \Big/ \left(\frac{3 - \nu}{1 + \nu} \sinh 2\varphi b - 2\varphi b\right) \\
C_4 &= \left[\left(E_0 + E_2 + E_4\right)\left(\cosh 2\varphi b - 1\right) - \beta\left(F_1 + F_3\right) \sinh 2\varphi b\right] \Big/ \left(\frac{3 - \nu}{1 + \nu} \sinh 2\varphi b - 2\varphi b\right) \\
C_5 &= -\left(F_5 + F_7\right) \\
C_6 &= \frac{2}{\pi}\left(F_5 + F_7\right)
\end{aligned}
\right\} \quad \text{(B12)}
$$

With the simplification given by equation (B9), the values of C_1, C_2, C_3, and C_4 become

$$
\left.
\begin{aligned}
C_1 \approx -C_3 &\approx -\left[\frac{1 - \nu}{3 - \nu} 2\left(E_0 + E_2 + E_4\right) + \frac{1 + \nu}{3 - \nu} \beta\left(F_1 + F_3\right)\right] \\
C_2 \approx -C_4 &\approx -\frac{1 + \nu}{3 - \nu}\left[\left(E_0 + E_2 + E_4\right) - \beta\left(F_1 + F_3\right)\right]
\end{aligned}
\right\} \quad \text{(B13)}
$$

Since $C_1 \approx -C_3$ and $C_2 \approx -C_4$ in both cases, the general solutions for u and v may be written in the simpler form as given by equations (26) and (27).

APPENDIX C

EVALUATION OF STRAIN ENERGY

The bending strain energy of the plate is given by equation (29) and is

$$V_{bp} = \frac{Et^3}{24\left(1 - \nu^2\right)} \iint \left(w_{xx} + w_{yy}\right)^2 \, dx \, dy \tag{C1}$$

Since

$$w_{xx} = \varphi^2 t W_o(-1 + \cos 2\beta\varphi y - A \sin \beta\varphi y) \sin \varphi x$$

$$w_{yy} = \varphi^2 t W_o\left(4\beta^2 \cos 2\beta\varphi y - A\beta^2 \sin \beta\varphi y\right) \sin \varphi x$$

by substituting in expression (C1) and integrating over the area of the plate, the bending strain energy is found to be

$$V_{bp} = \frac{E\varphi^4 t^5 ab}{1 - \nu^2} M_1 {W_o}^2 \tag{C2}$$

where

$$M_1 = \frac{1}{288}\left[6 + 3\left(1 + 4\beta^2\right)^2 + 3\left(1 + \beta^2\right)^2 A^2 + \frac{32}{\pi}\left(1 + \beta^2\right)^2 A\right] \tag{C3}$$

The extensional strain energy of the plate is given by equation (30) and is

$$V_{ep} = \frac{Et}{2\left(1 - \nu^2\right)} \iint \left[\left({\epsilon_x}'\right)^2 + \left({\epsilon_y}'\right)^2 + 2\nu {\epsilon_x}' {\epsilon_y}' + \frac{1}{2}(1 - \nu)\left({\gamma_{xy}}'\right)^2\right] dx \, dy \tag{C4}$$

The strains in the median surface of the plate are given by equations (B3), (B4), and (B5). Recalling that $C_1 \approx -C_3$ and $C_2 \approx -C_4$, inserting these expressions for strain components in equation (C4), and integrating over the area of the plate, the extensional strain energy becomes:

$$V_{ep} = \frac{Et^5\varphi^4 ab}{1 - \nu^2}\left(M_2 {W_o}^4 + M_3 \bar{\epsilon}_{st} {W_o}^2 + M_4 {\bar{\epsilon}_{st}}^2\right) \tag{C5}$$

where with the boundary condition $\sigma_y' = 0$ at $y = 0$ and $y = b$

$$
\left.
\begin{aligned}
M_2 = {} & \frac{1}{8}\Big[2G_0^2 + G_1^2 + G_2^2 + G_3^2 + 4G_4^2\left(1 - \nu^2\right) + 2G_5^2 + 2G_6^2 + 2G_8^2 + 2H_0^2 + H_1^2 + H_2^2 + H_3^2 + 2H_5^2 + 2H_6^2 + 2H_7^2 + H_8^2\Big] + \\
& \frac{\nu}{4}\Big[2G_0H_0 + G_1H_1 + G_2H_2 + G_3H_3 + 2G_5H_5 + 2G_6H_6 + 2G_7H_7 + 2G_8H_8\Big] + \frac{1}{\pi}\Big[G_0G_1 + 2G_4G_5 + H_0H_1 + 2H_4H_5 + \\
& \nu\left(G_0H_1 + G_1H_0 + 2G_4H_5 + 2G_5H_4\right)\Big] + \frac{1}{3\pi}\Big[G_0G_3 - G_1G_2 - 2G_5G_6 + 2G_4G_7 + H_0H_3 - H_1H_2 - 2H_5H_6 + 2H_4H_7 + \\
& \nu\left(G_0H_3 + G_3H_0 - G_1H_2 - G_2H_1 + 2G_4H_7 + 2G_7H_4 - 2G_5H_6 - 2G_6H_5\right)\Big] + \frac{3}{5\pi}\Big[G_2G_3 + 2G_6G_7 + H_2H_3 + 2H_6H_7 + \\
& \nu\left(G_2H_3 + G_3H_2 + 2G_6H_7 + 2G_7H_6\right)\Big] - \frac{6}{7\pi}\Big[G_7G_8 + H_7H_8 + \nu\left(G_7H_8 + G_8H_7\right)\Big] - \frac{2}{15\pi}\Big[G_5G_8 + H_5H_8 + \nu\left(G_5H_8 + G_8H_5\right)\Big] + \\
& \frac{1 - \nu}{16}\Big[J_1^2 + J_2^2 + J_3^2 + \frac{16}{15\pi} J_2\left(5J_1 - 3J_3\right)\Big] + \frac{\beta(1 - \nu)}{\pi(1 + \nu)^2}\Big[C_1^2(1 + \nu)^2 - C_1C_2\left(1 - 2\nu - 3\nu^2\right) + C_2^2\left(1 - \nu + 2\nu^2\right)\Big] + \\
& \frac{\beta}{2\pi}\Big[C_1(1 - \nu) + \frac{2\nu(1 - \nu)}{1 + \nu} C_2\Big]G_0 - \frac{\beta}{2\pi}\Big[C_1(1 - \nu) - \frac{2(1 - \nu)}{1 + \nu} C_2\Big]H_0 + \frac{\beta^2(1 - \nu)}{(4 + \beta)^2}\Big[\left(C_1 + \frac{1 + 3\nu}{1 + \nu} C_2\right)\left(4 + \beta^2\right) - 8C_2\Big]G_1 - \\
& \frac{\beta^2(1 - \nu)}{\pi\left(4 + \beta^2\right)^2}\Big[\left(C_1 - \frac{1 - \nu}{1 + \nu} C_2\right)\left(4 + \beta^2\right) - 8C_2\Big]H_1 + \frac{\beta(1 - \nu)}{2\pi\left(1 + \beta^2\right)^2}\Big[\left(C_1 + \frac{1 + 3\nu}{1 + \nu} C_2\right)\left(1 + \beta^2\right) - C_2\left(1 - \beta^2\right)\Big]G_2 - \\
& \frac{\beta(1 - \nu)}{2\pi\left(1 + \beta^2\right)^2}\Big[\left(C_1 - \frac{1 - \nu}{1 + \nu} C_2\right)\left(1 + \beta^2\right) - C_2\left(1 - \beta^2\right)\Big]H_2 + \frac{3\beta^2(1 - \nu)}{\pi\left(4 + 9\beta^2\right)^2}\Big[\left(C_1 + \frac{1 + 3\nu}{1 + \nu} C_2\right)\left(4 + 9\beta^2\right) - 8C_2\Big]G_3 - \\
& \frac{3\beta^2(1 - \nu)}{\pi\left(4 + 9\beta^2\right)^2}\Big[\left(C_1 - \frac{1 - \nu}{1 + \nu} C_2\right)\left(4 + 9\beta^2\right) - 8C_2\Big]H_3 - \frac{2\beta(1 - \nu)}{\pi\left(4 + \beta^2\right)}\Big[C_1 + \left(\frac{2\nu}{1 + \nu} - \frac{4 - \beta^2}{4 + \beta^2}\right)C_2\Big]J_1 - \\
& \frac{\beta^2(1 - \nu)}{2\pi\left(1 + \beta^2\right)}\Big[C_1 + \left(\frac{2\nu}{1 + \nu} - \frac{2\nu}{1 + \beta^2}\right)C_2\Big]J_2 - \frac{2\beta(1 - \nu)}{\pi\left(4 + 9\beta^2\right)}\Big[C_1 + \left(\frac{2\nu}{1 + \nu} - \frac{4 - 9\beta^2}{4 + 9\beta^2}\right)C_2\Big]J_3 \\
M_3 = {} & -\left(1 - \nu^2\right)\Big[\frac{1}{8}\left(3 + A^2\right) + \frac{4A}{3\pi}\Big] \\
M_4 = {} & \frac{1}{2}\left(1 - \nu^2\right)
\end{aligned}
\right\} \quad \text{(C6)}
$$

and with the boundary condition $v = 0$ at $y = 0$ and $y = b$,

$$
\left.
\begin{aligned}
M_2 = {} & \frac{1}{8}\left[2G_0^2 + G_1^2 + G_2^2 + G_3^2 + 4G_4^2 + 2G_5^2 + 2G_6^2 + 2G_7^2 + 2G_8^2\right] + \frac{\nu}{4}\left[2G_0H_0 + G_1H_1 + G_2H_2 + G_3H_3 + 4G_4H_4 + 2G_5H_5 + 2G_6H_6 + 2G_7H_7 + 2G_8H_8\right] + \\
& \frac{1}{\pi}\left[G_0G_1 + 2G_4G_5 + H_0H_1 + 2H_4H_5 + \nu\left(G_0H_1 + G_1H_0 + 2G_4H_5 + 2G_5H_4\right)\right] + \frac{1}{3\pi}\left[G_0G_3 - G_1G_2 + 2G_4G_7 - 2G_5G_6 + H_0H_3 - H_1H_2 + 2H_4H_7 - 2H_5H_6 + \right. \\
& \left.\nu\left(G_0H_3 + G_3H_0 - G_1H_2 - G_2H_1 + 2G_4H_7 + 2G_7H_4 - 2G_5H_6 - 2G_6H_5\right)\right] + \frac{3}{5\pi}\left[G_2G_3 + 2G_6G_7 + H_2H_3 + H_6H_7 + \nu\left(G_2H_3 + G_3H_2 + 2G_6H_7 + 2G_7H_6\right)\right] - \\
& \frac{6}{7\pi}\left[G_7G_8 + H_7H_8 + \nu\left(G_7H_8 + G_8H_7\right)\right] - \frac{2}{15\pi}\left[G_5G_8 + H_5H_8 + \nu\left(G_5H_8 + G_8H_5\right)\right] + \frac{1 - \nu}{16}\left[J_1^2 + J_2^2 + J_3^2 + \frac{16}{15\pi}J_2\left(5J_1 - 3J_3\right)\right] + \\
& \frac{\beta}{\pi}\frac{1 - \nu}{(1 + \nu)^2}\left[C_1^2(1 + \nu)^2 - C_1C_2\left(1 - 2\nu - 3\nu^2\right) + C_2^2\left(1 - \nu + 2\nu^2\right)\right] + \frac{1}{2}\beta^4C_6^2 + \beta^2C_6\left(\nu G_4 + H_4\right) + \frac{\beta}{2\pi}\left[C_1(1 - \nu) + \frac{2\nu(1 - \nu)}{1 + \nu}C_2\right]G_0 - \\
& \frac{\beta}{2\pi}\left[C_1(1 - \nu) - \frac{2(1 - \nu)}{1 + \nu}C_2\right]H_0 + \frac{\beta^2(1 - \nu)}{\pi(4 + \beta^2)^2}\left[\left(C_1 + \frac{1 + 3\nu}{1 + \nu}C_2\right)\left(4 + \beta^2\right) - 8C_2\right]G_1 - \frac{\beta^2(1 - \nu)}{\pi(4 + \beta^2)^2}\left[\left(C_1 - \frac{1 - \nu}{1 + \nu}C_2\right)\left(4 + \beta^2\right) - 8C_2\right]H_1 + \\
& \frac{\beta(1 - \nu)}{2\pi\left(1 + \beta^2\right)^2}\left[\left(C_1 + \frac{1 + 3\nu}{1 + \nu}C_2\right)\left(1 + \beta^2\right) - C_2\left(1 - \beta^2\right)\right]G_2 - \frac{\beta(1 - \nu)}{2\pi\left(1 + \beta^2\right)^2}\left[\left(C_1 - \frac{1 - \nu}{1 + \nu}C_2\right)\left(1 + \beta^2\right) - C_2\left(1 - \beta^2\right)\right]H_2 + \\
& \frac{3\beta^2(1 - \nu)}{\pi\left(4 + 9\beta^2\right)^2}\left[\left(C_1 + \frac{1 + 3\nu}{1 + \nu}C_2\right)\left(4 + 9\beta^2\right) - 8C_2\right]G_3 - \frac{3\beta^2(1 - \nu)}{\pi\left(4 + 9\beta^2\right)^2}\left[\left(C_1 - \frac{1 - \nu}{1 + \nu}C_2\right)\left(4 + 9\beta^2\right) - 8C_2\right]H_3 - \frac{2\beta(1 - \nu)}{\pi\left(4 + \beta^2\right)}\left[C_1 + \left(\frac{2\nu}{1 + \nu} - \frac{4 - \beta^2}{4 + \beta^2}\right)C_2\right]J_1 - \\
& \frac{\beta^2(1 - \nu)}{2\pi\left(1 + \beta^2\right)}\left[C_1 + \left(\frac{2\nu}{1 + \nu} - \frac{2}{1 + \beta^2}\right)C_2\right]J_2 - \frac{2\beta(1 - \nu)}{\pi\left(4 + 9\beta^2\right)}\left[C_1 + \left(\frac{2\nu}{1 + \nu} - \frac{4 - 9\beta^2}{4 + 9\beta^2}\right)C_2\right]J_3 \\
M_3 = {} & -\left\{\frac{1}{8}\left[3 + 4\nu\beta^2 + A^2\left(1 + \beta^2\nu\right)\right] - \frac{4A}{3\pi}\left(1 + \nu\beta^2\right)\right\} \\
M_4 = {} & \frac{1}{2}
\end{aligned}
\right\} \quad (C7)
$$

The extensional strain energy of a stiffener is

$$V_{es} = \frac{1}{2} E_{st}A_{st}\int_0^a \left(\epsilon_x'\right)^2_{y=\eta b} dx \tag{C8}$$

at $y = \eta b$; write

$$\epsilon_x' = \varphi^2t^2\left[W_o^2\left(N_1 \cos 2\varphi x + N_2\right) - \bar{\epsilon}_{st}\right] \tag{C9}$$

where

$$N_1 = 2\left(C_1 + C_2\right)(\cosh 2\varphi\eta b - \sinh 2\varphi\eta b) - 4C_2\varphi\eta b(\cosh 2\varphi\eta b - \sinh 2\varphi\eta b) + G_0 + G_1 \sin \beta\varphi\eta b + G_2 \cos 2\beta\varphi\eta b + G_3 \sin 3\beta\varphi\eta b \tag{C10}$$

and

$$N_2 = G_4 + G_5 \sin \beta\varphi\eta b + G_6 \cos 2\beta\varphi\eta b + G_7 \sin 3\beta\varphi\eta b + G_8 \cos 4\beta\varphi\eta b \tag{C11}$$

Integration of equation (C8) therefore gives

$$V_{es} = \frac{1}{2} \varphi^4t^4E_{st}A_{st}a\left[W_o^4\left(\frac{N_1^2}{2} + N_2^2\right) - 2W_o^2N_2\bar{\epsilon}_{st} + \bar{\epsilon}_{st}^2\right] \tag{C12}$$

The bending strain energy of a stiffener is

$$V_{bs} = \frac{1}{2} E_{st}I_{st}\int_0^a \left(w_{xx}\right)^2_{y=\eta b} dx \tag{C13}$$

since

$$\left(w_{xx}\right)_{y=\eta b} = -\varphi^2tW_o(1 - \cos 2\beta\varphi\eta b + A \sin \beta\varphi\eta b) \sin \varphi x$$

integration of equation (C13) gives

$$V_{bs} = \frac{1}{2} \varphi^4t^2E_{st}I_{st}aW_o^2 \frac{N_3}{2} \tag{C14}$$

where

$$N_3 = (1 - \cos 2\beta\varphi\eta b + A \sin \beta\varphi\eta b)^2 \tag{C15}$$

The twisting strain energy of a stiffener is

$$V_{ts} = \frac{1}{2} C \int_0^a \left(w_{xy}\right)^2_{y=\eta b} dx \tag{C16}$$

Since

$$\left(w_{xy}\right)_{y=\eta b} = \varphi^2 t W_o \beta \left[2 \sin 2\beta\varphi\eta b + A \cos \beta\varphi\eta b\right] \cos \varphi x$$

integration gives

$$V_{ts} = \frac{1}{2} \varphi^4 t^2 C a W_o^2 \frac{N_4}{2} \tag{C17}$$

where

$$N_4 = \beta^2 (2 \sin 2\beta\varphi\eta b + A \cos \beta\varphi\eta b)^2 \tag{C18}$$

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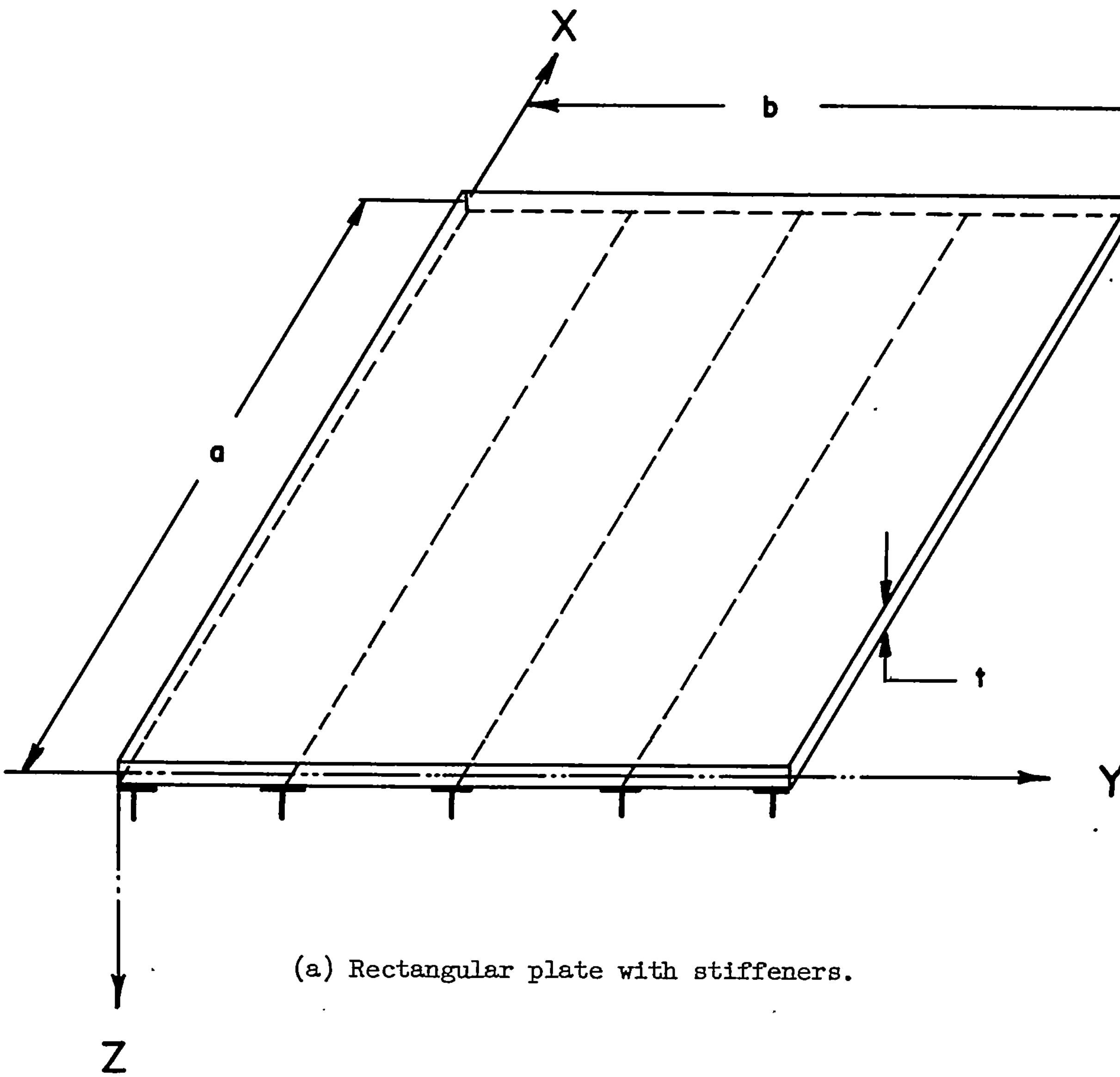
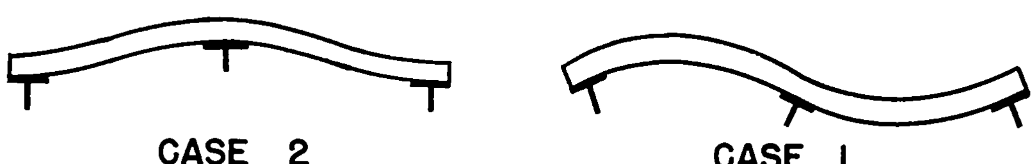
4. Lundquist, Eugene E., and Stowell, Elbridge Z.: Restraint Provided a Flat Rectangular Plate by a Sturdy Stiffener along an Edge of the Plate. NACA Rep. 735, 1942.

5. Dunn, Louis G.: An Investigation of Sheet-Stiffener Panels Subjected to Compression Loads with Particular Reference to Torsionally Weak Stiffeners. NACA TN 752, 1940.



(a) Rectangular plate with stiffeners.



(b) Case 1, plate deflecting around stiffeners; case 2, system displacing as complete unit.

Figure 1.- Sketch of reinforced plate.

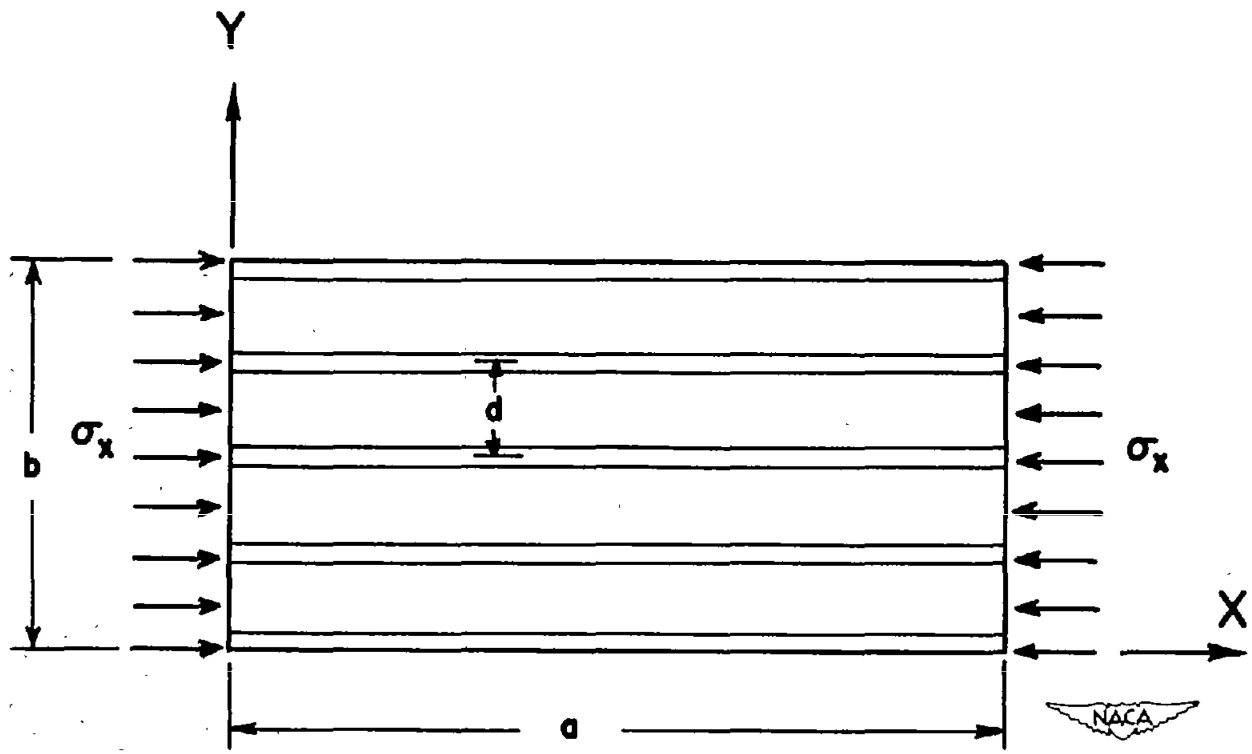


Figure 2.- Coordinate system for rectangular plate with longitudinal stiffeners.

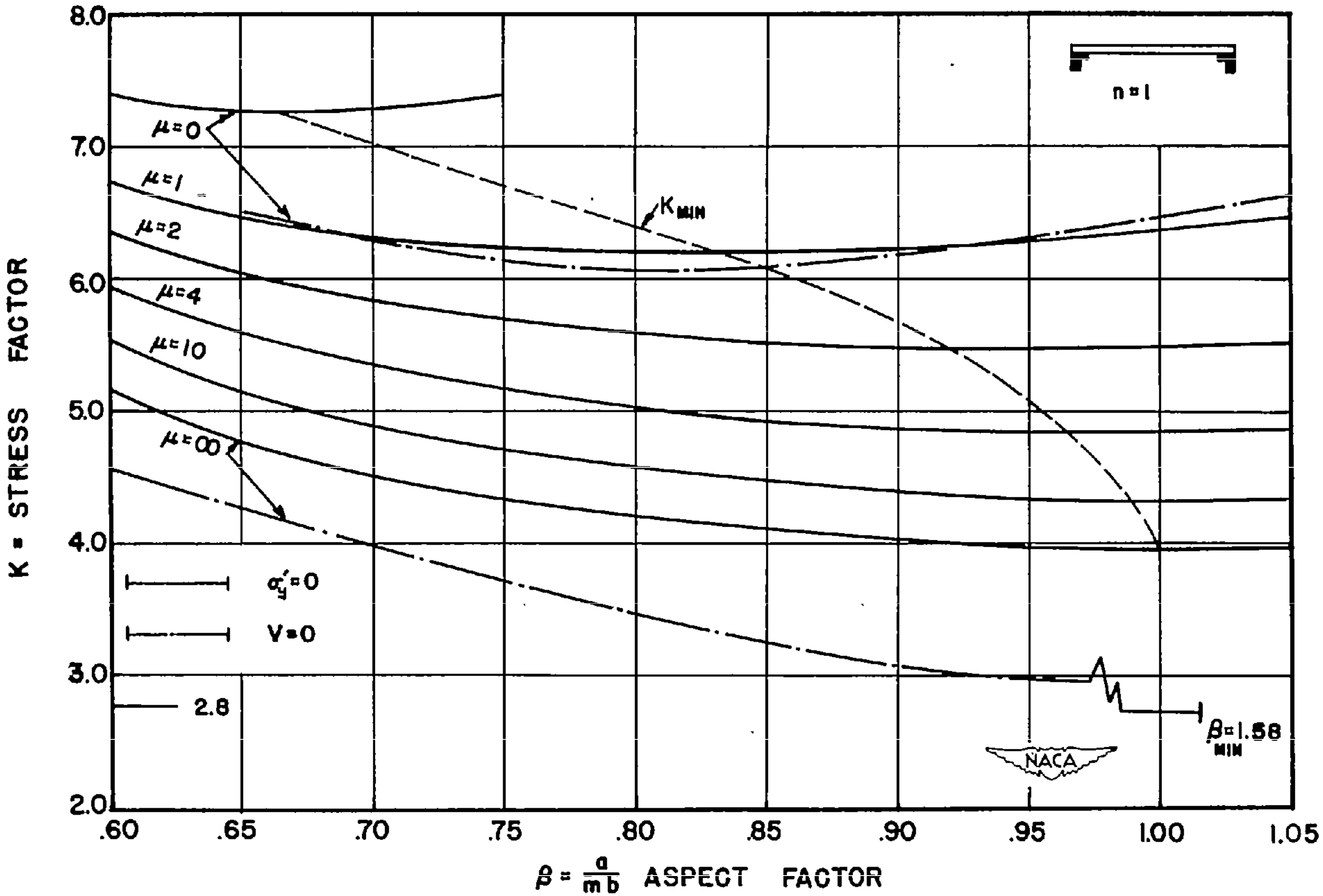


Figure 3.- Variation of K with aspect factor β.

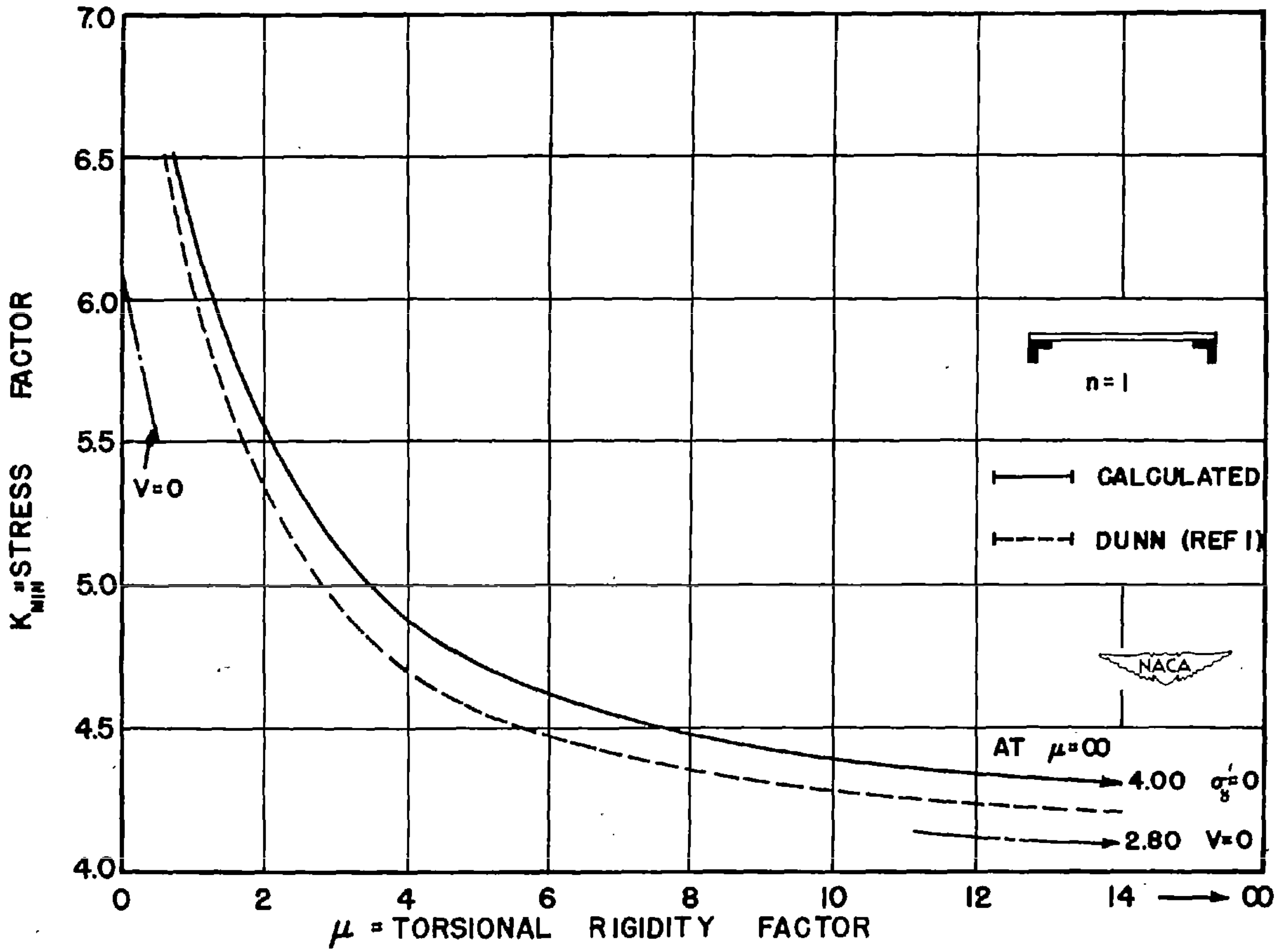


Figure 4.- Variation of K_{min} with μ.

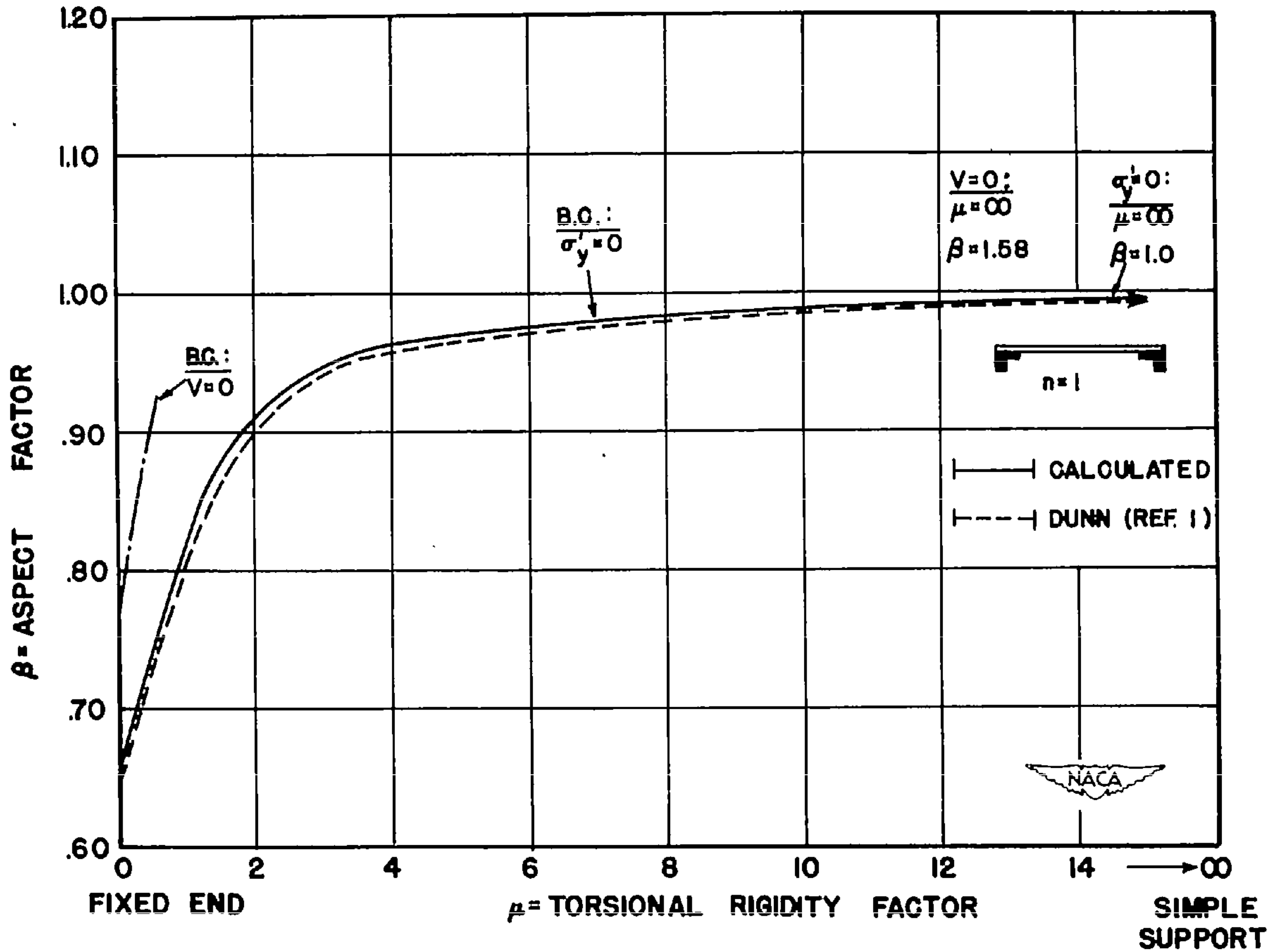


Figure 5.- Variation of β corresponding to K_{min} with μ.
B.C. denotes "boundary condition."

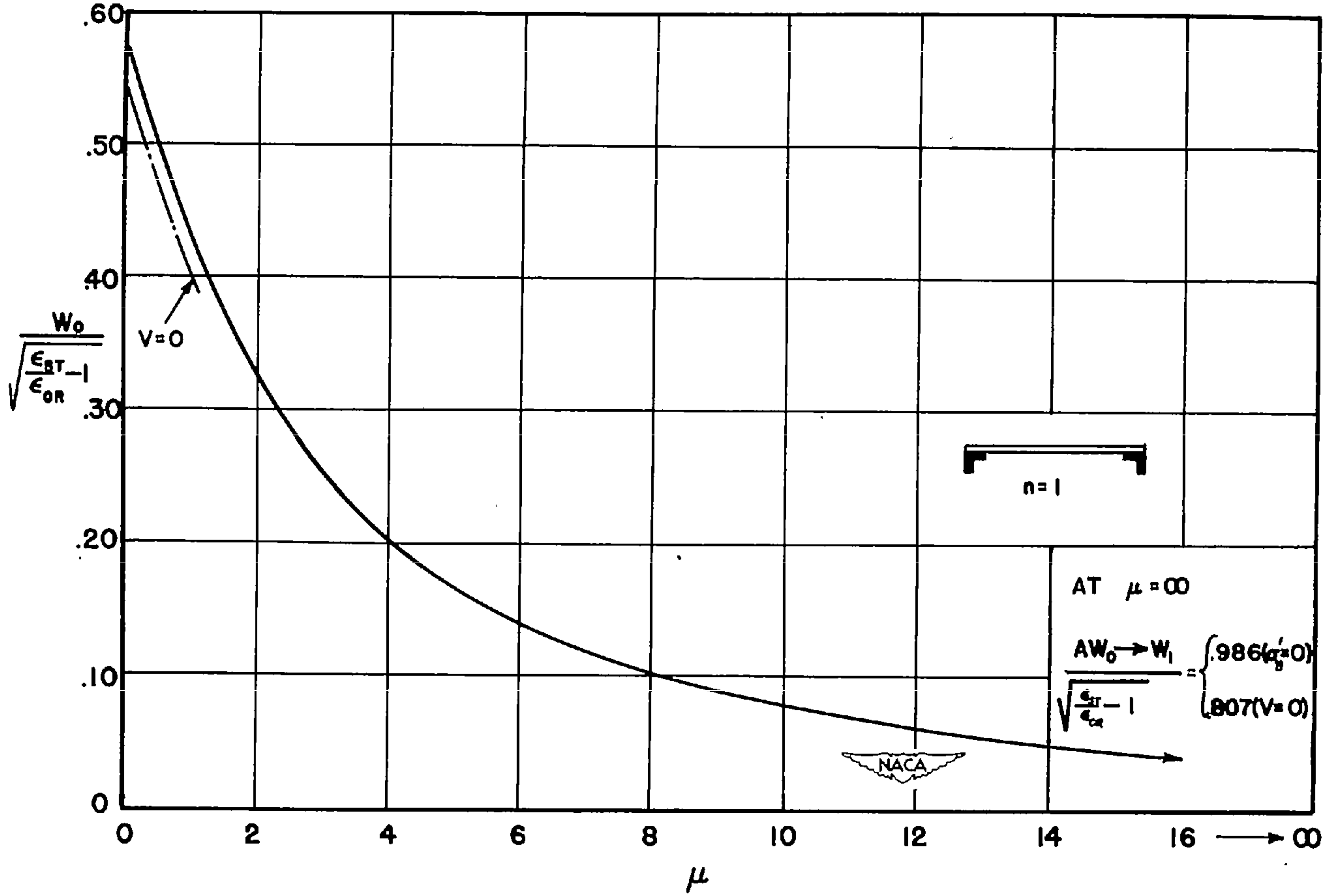


Figure 6.- Variation of amplitude factor W_0 with elastic restraints of edges.

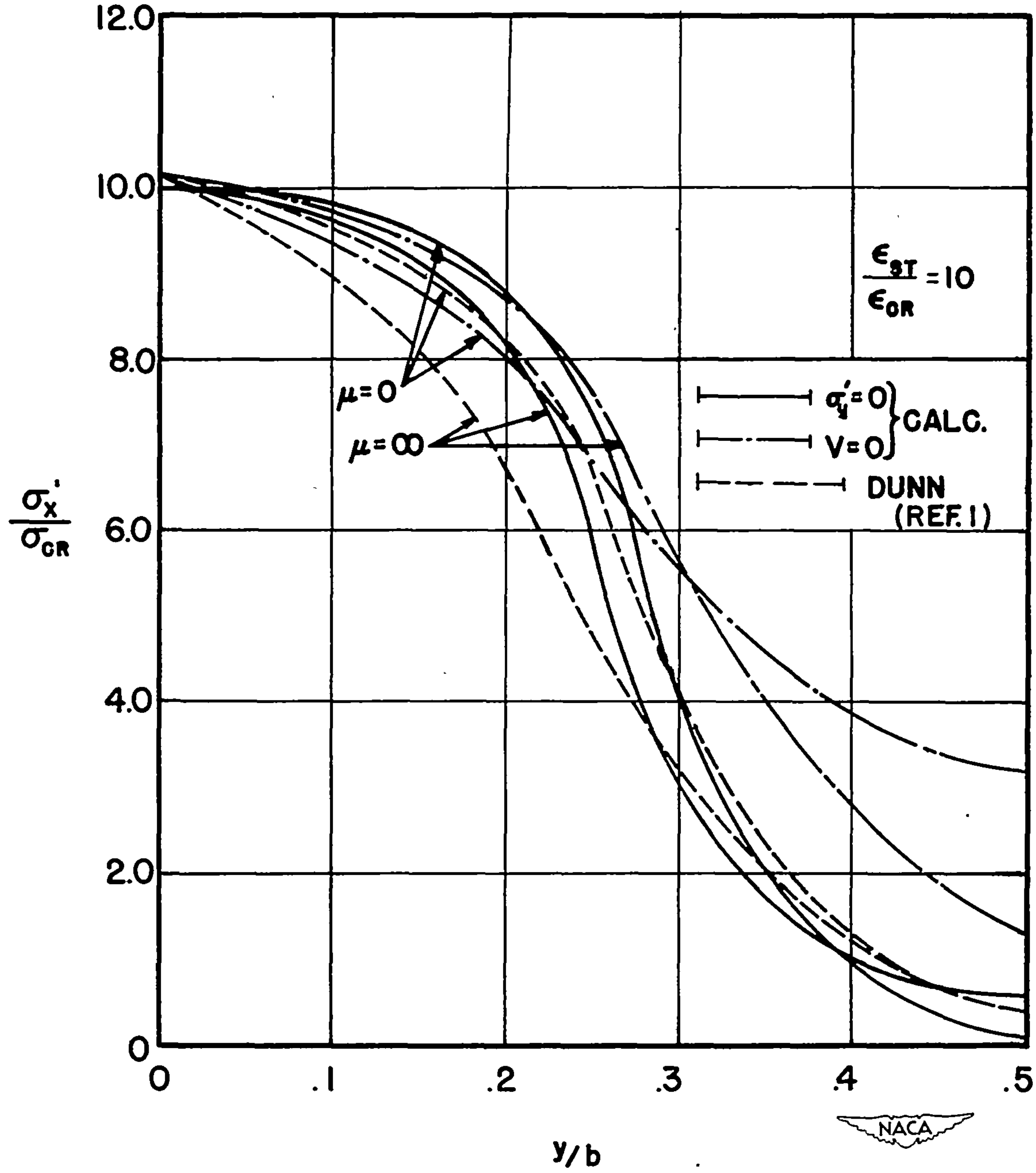


Figure 7.- Axial stress distribution. $x = \lambda$.

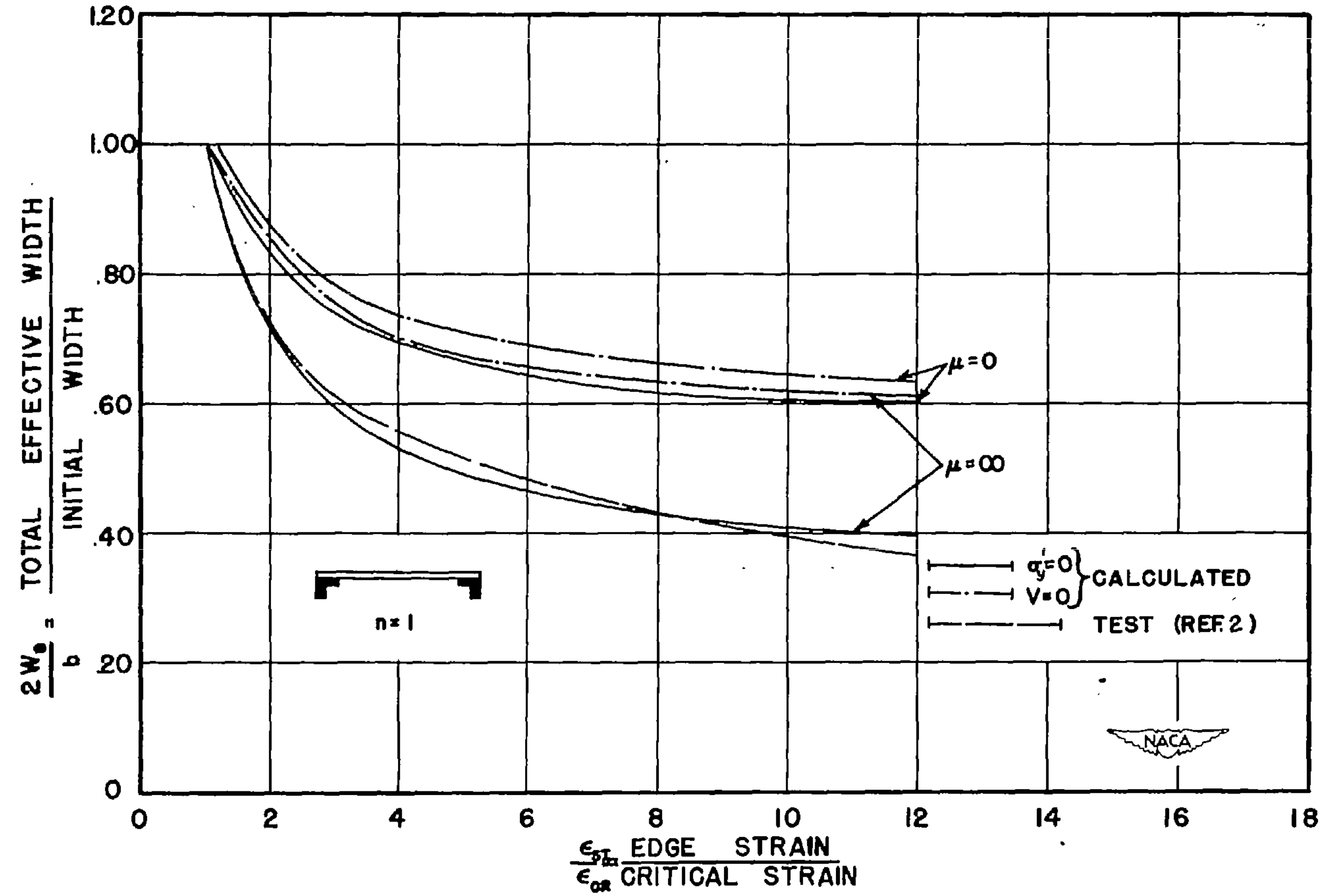


Figure 8.- Width ratio as a function of strain ratio.

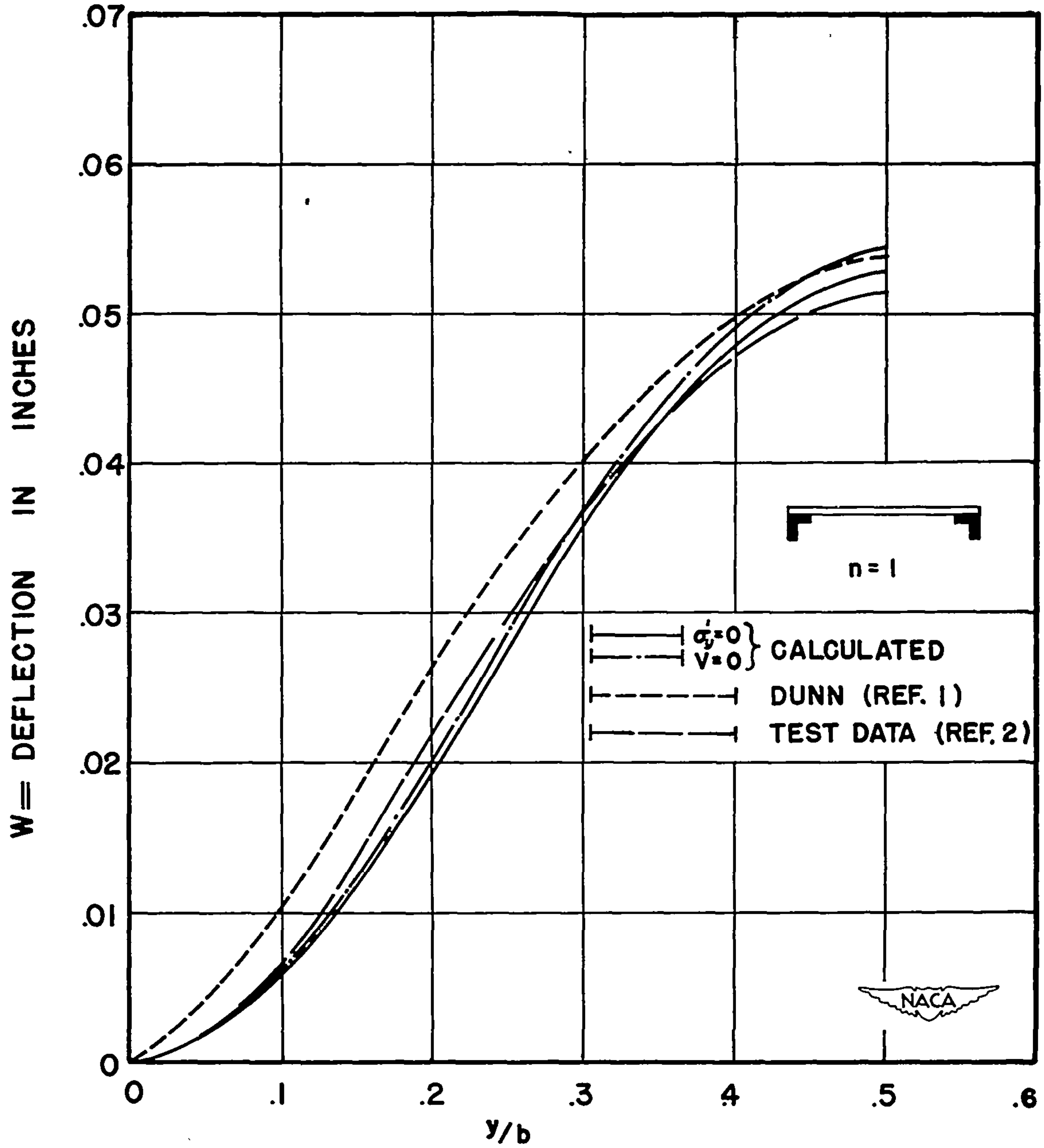


Figure 9.- Comparison of transverse wave form. $x = \lambda/2$.

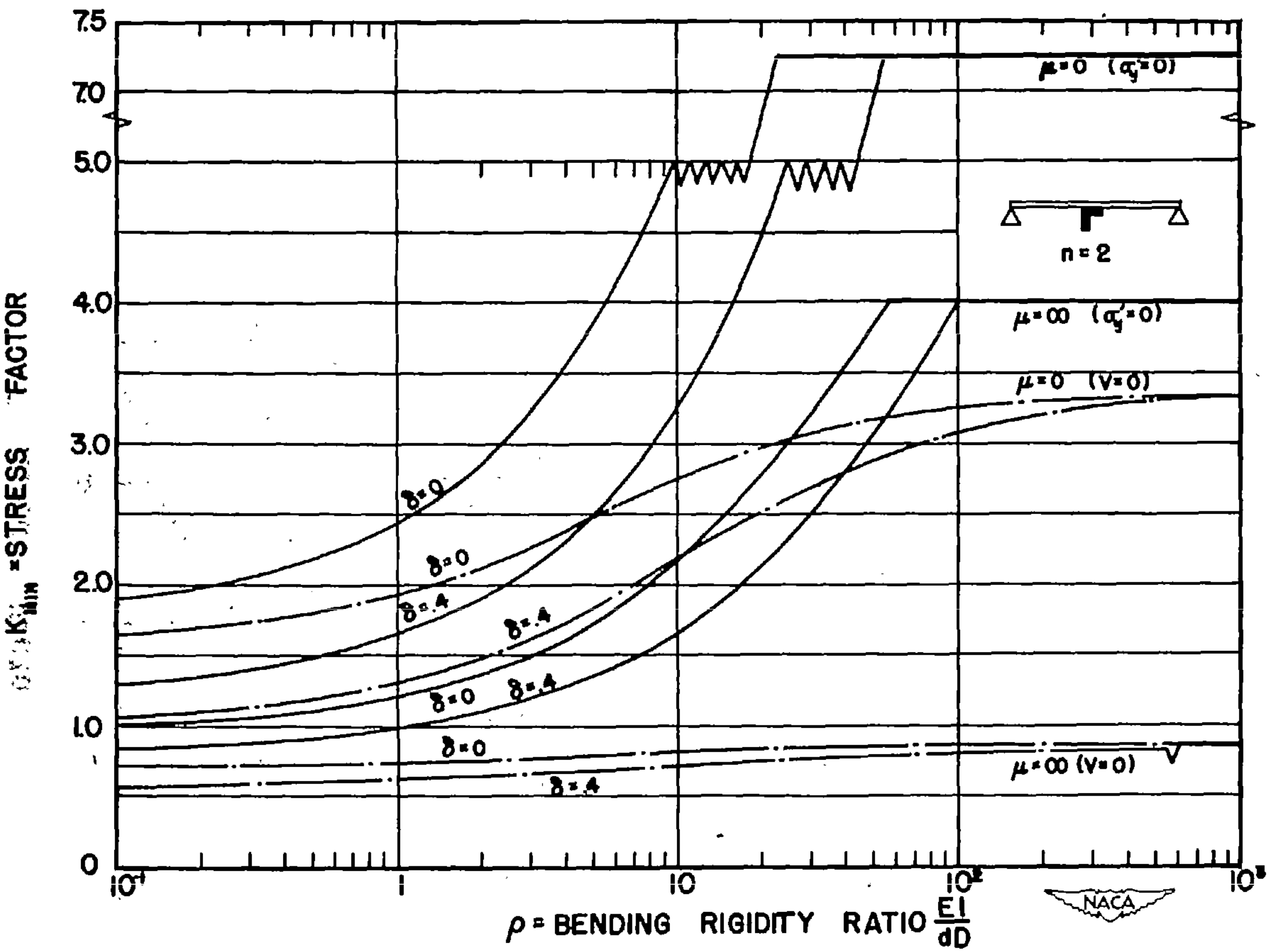


Figure 10.- Bending rigidity ratio ρ against K_{min} for stiffened plate.

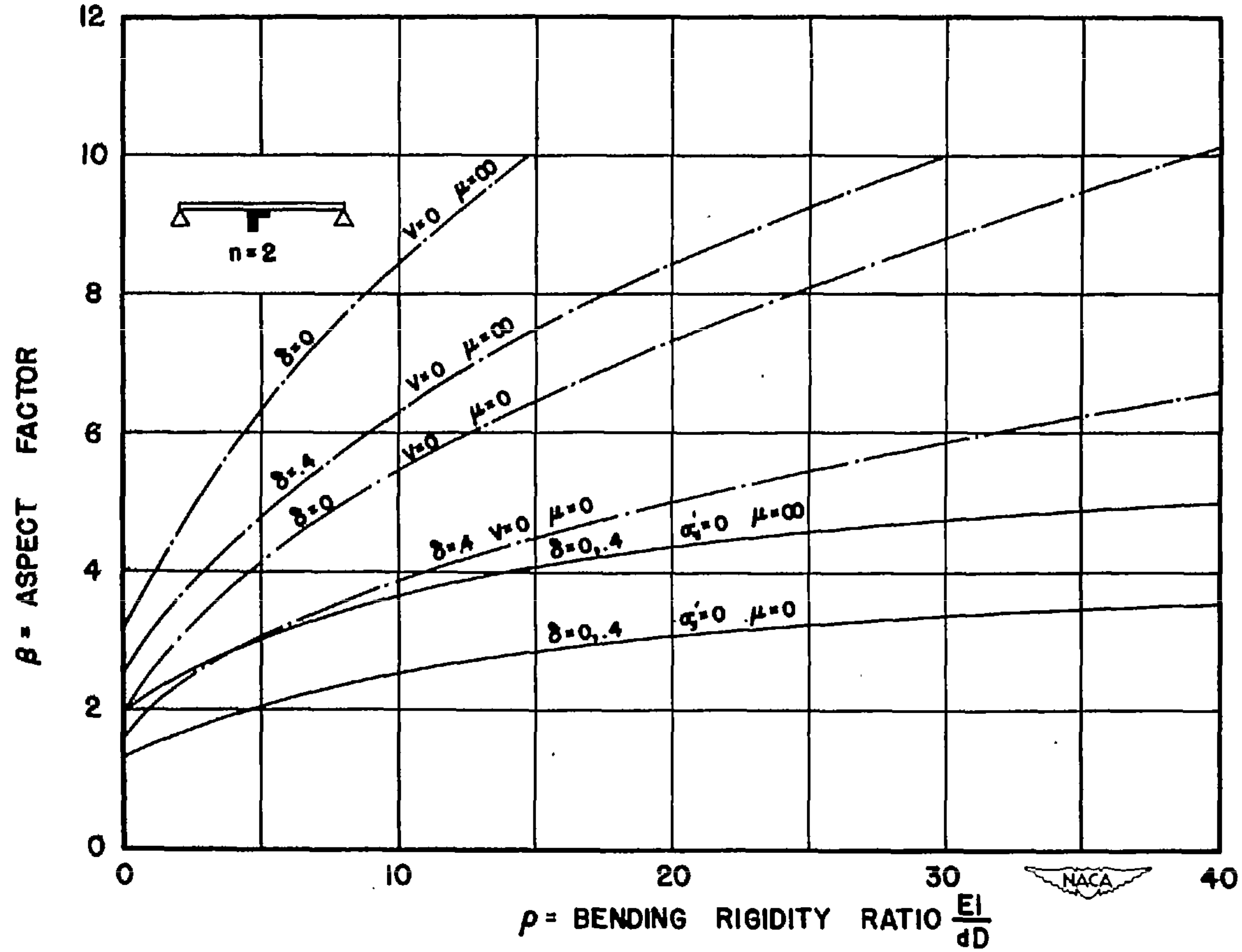


Figure 11.- Variation of β corresponding to K_{min} with ρ.

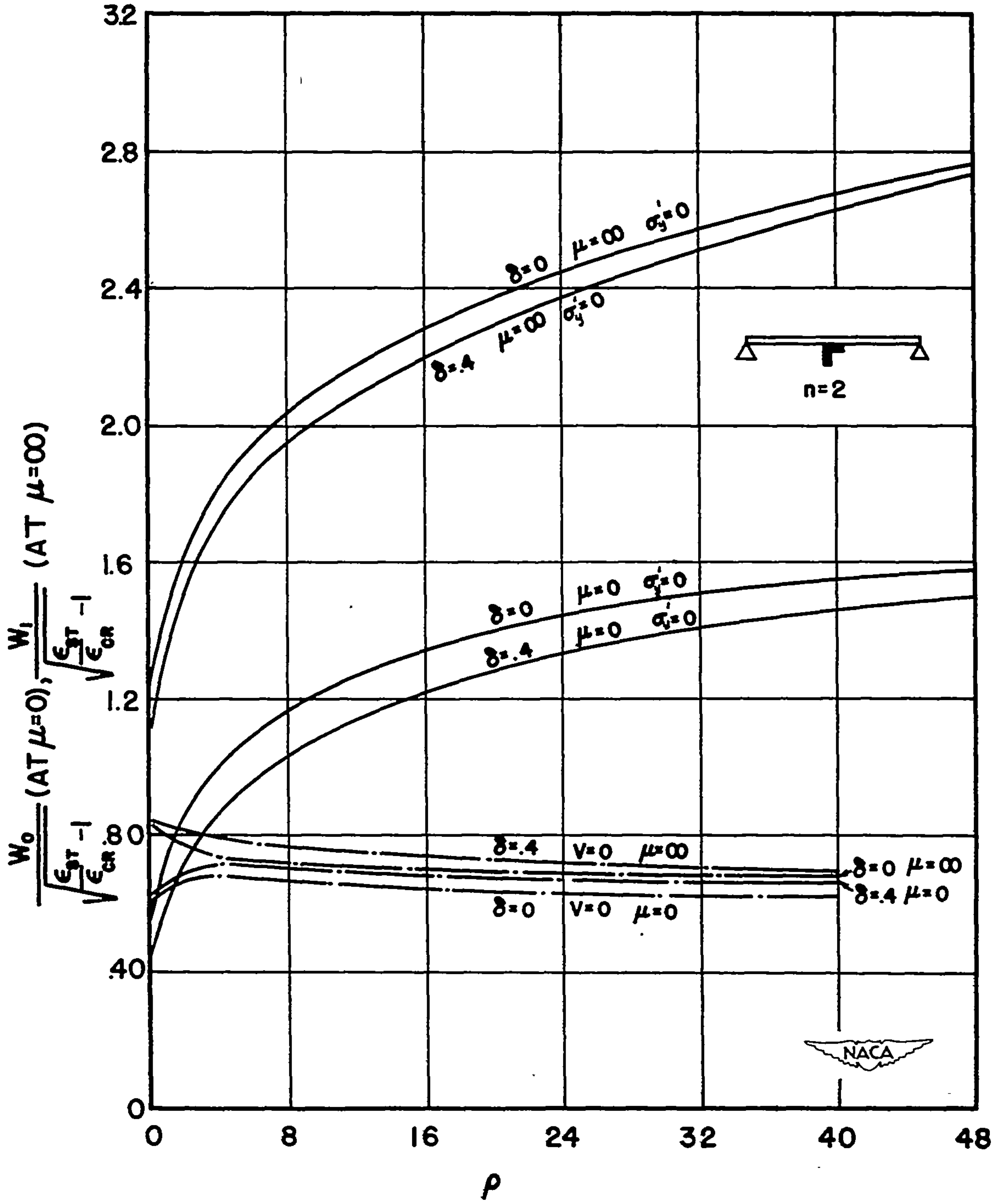


Figure 12.- Variation of amplitude factor with bending rigidity ratio.

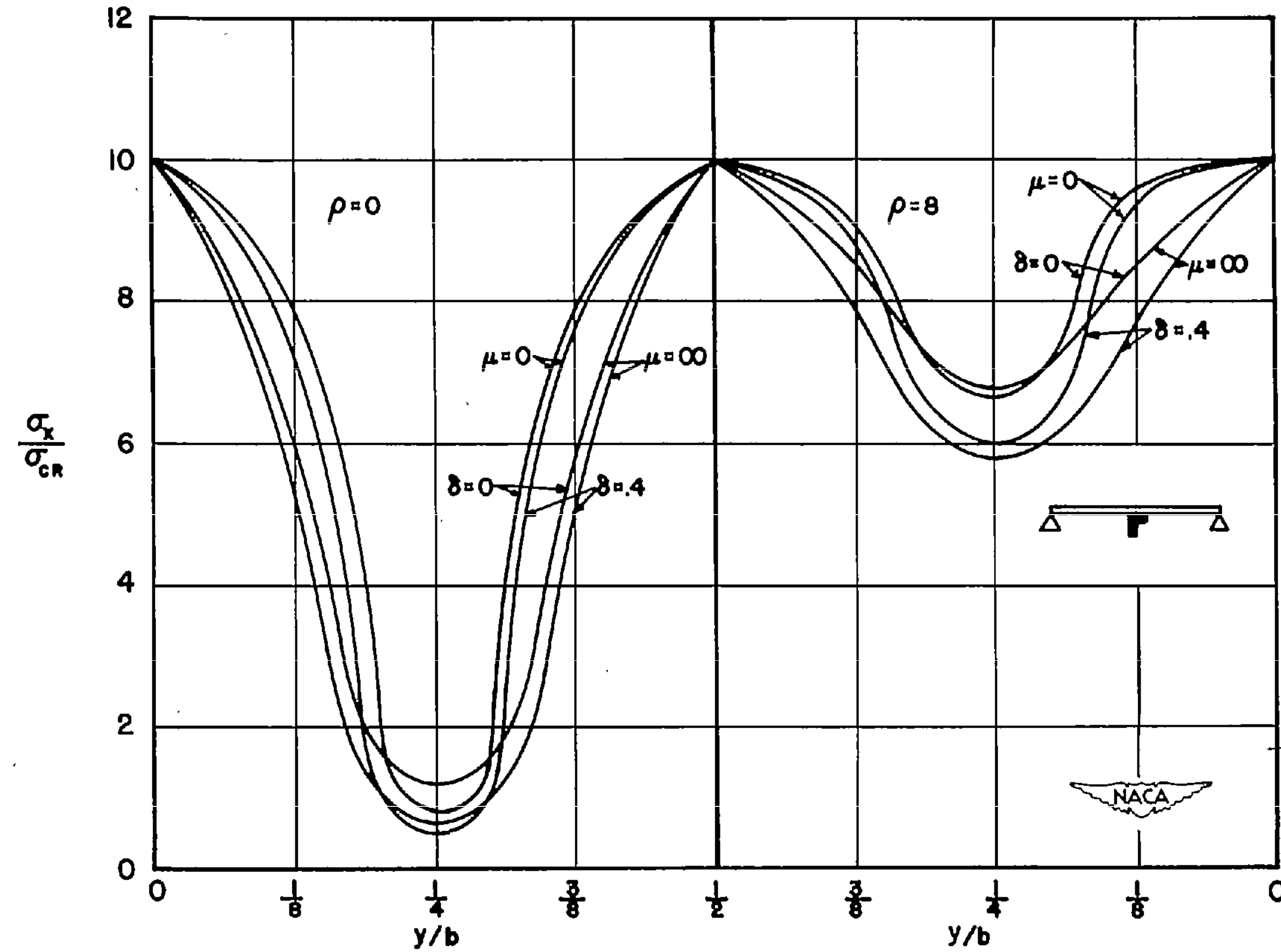


Figure 13.- Axial stress distribution for $\epsilon_{st}/\epsilon_{cr} = 10$. Boundary condition, $\sigma_y{}' = 0$.

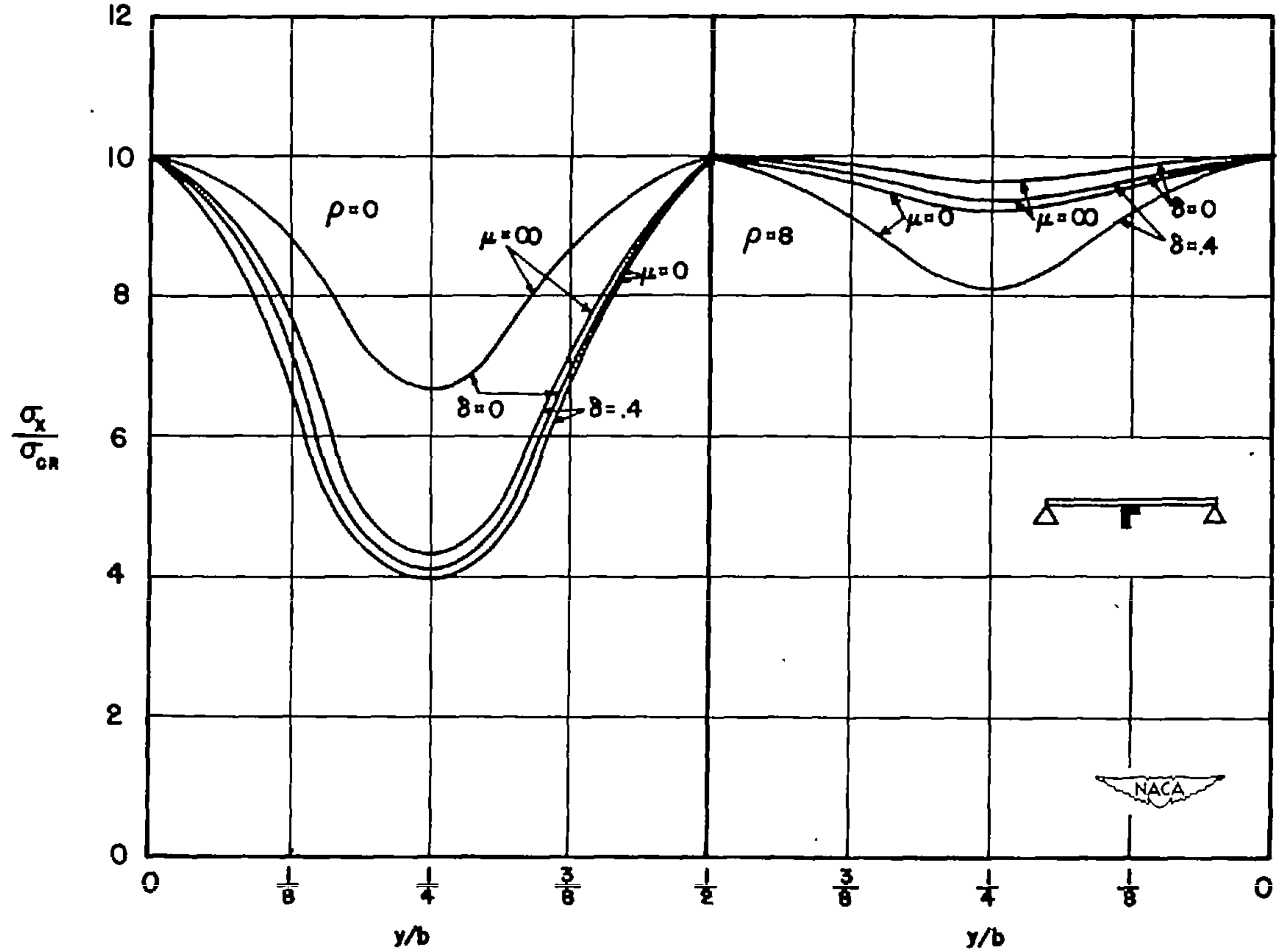


Figure 14.- Axial stress distribution for $\epsilon_{st}/\epsilon_{cr} = 10$. Boundary condition, $v = 0$.