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ANALYSIS OF ELASTIC THERMAL STRESSES IN THIN PLATE WITH SPANWISE AND CHORDWISE VARIATIONS OF TEMPERATURE AND THICKNESS

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SUMMARY

An approximate method for computing thermal stresses in a thin plate of variable thickness is presented. The temperature and the thickness can both vary chordwise and spanwise, and the effect of the free end is automatically included. The method makes use of polynomial approximations for the stress function to reduce the partial differential equation to a set of ordinary differential equations. This results in satisfying the differential equation everywhere spanwise and at a finite number of stations chordwise. The boundary conditions are everywhere satisfied.

Several examples of the method are presented in detail, and curves showing the effects of chordwise and spanwise temperature distribution and chordwise and spanwise thickness variation are shown for several cases. It is indicated from these few examples that the effects of relatively large thickness variation and spanwise temperature variation are not of major importance for a plate with a free end.

INTRODUCTION

Jet engines, high-speed airplanes and missiles, and nuclear powerplants are examples of modern devices in which large temperature gradients exist. Such temperature gradients can produce large thermal stresses which, by themselves or in conjunction with stresses produced by various external loads, can cause serious component failures. Thus jet-engine turbine blades possess large temperature gradients during operation which cause internal forces that superimposed upon the centrifugal and aerodynamic loads can contribute to blade failure. During acceleration, temperature gradients occur in wings of high-speed aircraft because of aerodynamic heating. Serious thermal stresses may thus be produced which must be of concern to the designer. These and other new applications have caused a renewed and urgent interest in the relatively old subject of thermal stresses.

The geometries that exist in these various applications are usually very complicated and difficult to treat exactly. It is usual, therefore, to approximate the various configurations by simple bodies such as plates, cylinders, and spheres. Thus, the turbine blade and the supersonic wing are approximated in this investigation by the thin plate of variable thickness. 4098

The thermal stresses in a thin plate are treated in detail in reference 1. However, the plate is assumed to be of constant thickness, the temperature is assumed to vary chordwise only, and the end effects are neglected, use being made of Saint Venant's principle. All these assumptions are invalid in many practical cases. In a turbine blade, for example, there is a large variation both in temperature and in thickness in both chordwise and spanwise directions. In addition, the length-to-chord ratio is often less than 2, and the end effects may therefore be important over a large part of the blade. It is suggested in reference 1 (p. 400) that the end effects can be calculated using the strain energy method.

In order to avoid some of these assumptions, an approximate solution to the equations of elasticity is given in reference 2. A solution is obtained for the biharmonic equation for the stress function in terms of an infinite series. However, only two terms of the infinite series are used, and the boundary conditions are not exactly satisfied but only in an average fashion over the surface. The results at the free end of the plate are compared with those obtained by the energy method, and it is indicated that the method of reference 2, although more laborous, has inherently higher accuracy. The calculations made in reference 2 are for a uniform-thickness plate with chordwise temperature variation only. It is indicated that, although this method, as well as the energy method, could be adapted to a variable-thickness plate, the amount of labor involved would become prohibitive and should not be attempted without the aid of a high-speed computing machine.

The equations for calculating thermal stresses in a supersonic wing with chordwise variations in thickness and temperature are presented in reference 3. The method of reference 1 (p. 399) is used in deriving these equations, and again the results as stated therein are not valid at a distance less than a chord length from the tip.

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The object of the present paper is to present an approximate method for calculating accurately the thermal stresses in a thin plate with both chordwise and spanwise variations in temperature and chordwise but restricted spanwise variations in thickness. Numerical examples of the effects of the free end and of thickness variations will be presented. The method used is essentially a collocation procedure applied to a partial differential equation and is used successfully in reference 4 for calculating transient thermal stresses. In this procedure, the differential equation is satisfied everywhere in x but at only a finite number of values of y. Use is made of a polynomial approximation for the stress function to reduce the biharmonic equation to a set of ordinary linear differential equations which can be readily solved. The extension to other problems involving the biharmonic or Laplace equation can be made.

ANALYSIS

Stress Function

Consider a thin flat plate of variable thickness, as shown in figure 1. The origin of coordinates is taken at the middle of the free edge, and the midplane of the plate is assumed to lie in the xy-plane. The width of the plate runs from $y = -1$ to $y = 1$; and the length, in the x-direction, can be either finite or infinite. The variable thickness is a function of both x and y and is assumed to have the following form:

$$h = h_1(x)h_2(y)$$

The function $h_2(y)$ is any continuous function of y having a continuous first derivative, and the function $h_1(x)$ is a function of x having the form $h_1(x) = h_0e^{mx}$. (Symbols are defined in appendix A). For m equal to zero, the thickness is constant in the x-direction. At any instant of time, the plate is assumed to have a temperature distribution varying with x and y but independent of z. Thus

$$T = T(x,y)$$

It is assumed that the plate is sufficiently thin so that a state of plane stress exists, the only significant stresses being σ_x, σ_y, and τ_{xy}, which are independent of z. The assumptions that $\sigma_z = \tau_{yz} = \tau_{xz} = 0$ and that σ_x, σ_y, and τ_{xy} are independent of z lead to an inconsistency in that all the compatibility conditions are not necessarily satisfied. It can be shown (see, e.g., ref. 1, p. 241), however, that the error is proportional to z^2 and is therefore small for thin plates. It is further assumed that any curvature of the midplane of the plate can be neglected.

Under these conditions it is shown in appendix B that the elastic stresses σ_x, σ_y, and τ_{xy} are derivable from a stress function φ satisfying the following differential equation:

$$\nabla^2\left(\frac{1}{h}\nabla^2\varphi + E\alpha T\right) = (1+\nu)\left[\left(\frac{1}{h}\right)_{xx}\varphi_{yy} + \left(\frac{1}{h}\right)_{yy}\varphi_{xx} - 2\left(\frac{1}{h}\right)_{xy}\varphi_{xy}\right] \quad \text{(1) or (B6)}$$

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where the subscripts indicate differentiation with respect to that variable, and ∇^2 is the Laplacian operator

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The stresses are given in terms of the stress function as follows

$$\left.\begin{aligned} \sigma_x &= \frac{1}{h}\frac{\partial^2\varphi}{\partial y^2} \\ \sigma_y &= \frac{1}{h}\frac{\partial^2\varphi}{\partial x^2} \\ \tau_{xy} &= -\frac{1}{h}\frac{\partial^2\varphi}{\partial x\,\partial y} \end{aligned}\right\} \quad \text{(2) or (B5)}$$

The boundary conditions are ones of no normal or shear forces on the surface of the plate. These conditions lead to the following equations that must be satisfied at the edges of the plate:

at $x = 0$

$$\left.\begin{aligned} h\sigma_x &= \frac{\partial^2\varphi}{\partial y^2} = 0 \\ h\tau_{xy} &= -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0 \end{aligned}\right\} \quad (3)$$

at $y = \pm 1$

$$\left.\begin{aligned} h\sigma_y &= \frac{\partial^2\varphi}{\partial x^2} = 0 \\ h\tau_{xy} &= -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0 \end{aligned}\right\} \quad (4)$$

For an infinite plate the stresses at infinity must be finite. For a finite plate with both ends free, the same boundary conditions exist at both ends.

For a fixed edge the boundary conditions are given in terms of displacements. This necessitates setting up the differential equations in terms of displacements rather than stresses.

Polynomial Approximations

Equation (1) will be solved approximately by a collocation procedure whereby the differential equation is satisfied everywhere in x but at only a finite number of values of y. Thus, as indicated in figure 1, n stations are taken along y. The y-coordinates of these stations are $y_1, y_2, \ldots\ y_n$. The stress function φ is then assumed to have the following form:

$$\varphi = \sum_{j=1}^{n} P_j(y)\varphi_j(x) \tag{5}$$

where $P_j(y)$ is a polynomial in y associated with the j^{th} station and satisfying the following conditions:

$$\left.\begin{aligned} P_j(y_j) &= 1 \\ P_j(y_i) &= 0 \qquad j \neq i \end{aligned}\right\} \tag{6}$$

Since φ_j will, in general, be a nonlinear function of x, the following conditions must be satisfied in order to ensure that equations (4) hold for any φ_j:

$$\left.\begin{aligned} P_j(\pm 1) &= 0 \\ P_j'(\pm 1) &= 0 \end{aligned}\right\} \tag{7}$$

Polynomials having these properties can readily be obtained. Thus

$$P_j(y) = \frac{(y^2 - 1)^2}{(y_j^2 - 1)^2} \prod_{i \neq j} (y - y_i) \Big/ \prod_{i \neq j} (y_j - y_i) \qquad y_j \neq \pm 1 \tag{8}$$

where $\prod_{i \neq j}$ is the product for all values of i except $i = j$. Equation (8) satisfies equations (6) since $P_j = 0$ at $y = y_i$ and $P_j = 1$ at $y = y_j$. It also satisfies equations (7) since P_j and P_j' are both zero at $y = \pm 1$. Equations (8) and (5) are now substituted into equation (1), and the equation is then evaluated at each of the n stations. This results in a set of simultaneous fourth-order ordinary differential equations in $\varphi_j(x)$ of the following form: 4098

$$\sum_{j=1}^{n} \left(a_{ij}\varphi_j'''' + b_{ij}\varphi_j''' + c_{ij}\varphi_j'' + d_{ij}\varphi_j' + e_{ij}\varphi_j\right) = -h(x,y_i)\nabla^2\left[E\alpha T(x,y_i)\right]$$

$$i = 1, 2, \ldots, n \qquad (9)$$

where

$$\left.\begin{aligned}
a_{ij} &= P_j(y_i) \\
b_{ij} &= -\frac{2P_j}{h}\frac{\partial h}{\partial x}\bigg|_{y=y_i} = -\frac{2}{h_1}h_1'P_j(y_i) \\
c_{ij} &= \left\{2P_j'' - \frac{2P_j'}{h}\frac{\partial h}{\partial y} + P_j\left[\frac{2}{h^2}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{1}{h}\frac{\partial^2 h}{\partial x^2} - \frac{2\nu}{h^2}\left(\frac{\partial h}{\partial y}\right)^2 + \frac{\nu}{h}\frac{\partial^2 h}{\partial y^2}\right]\right\}_{y=y_i} \\
&= \left\{2P_j'' - 2P_j'\frac{h_2'}{h_2} + P_j\left[2\left(\frac{h_1'}{h_1}\right)^2 - \frac{h_1''}{h_1} - 2\nu\left(\frac{h_2'}{h_2}\right)^2 + \nu\frac{h_2''}{h_2}\right]\right\}_{y=y_i} \\
d_{ij} &= 2\left[-\frac{P_j''}{h}\frac{\partial h}{\partial x} + (1+\nu)P_j'\left(\frac{2}{h^2}\frac{\partial h}{\partial y}\frac{\partial h}{\partial x} - \frac{1}{h}\frac{\partial^2 h}{\partial x\,\partial y}\right)\right]_{y=y_i} \\
&= 2\left[-P_j''\frac{h_1'}{h_1} + (1+\nu)P_j'\frac{h_1'h_2'}{h_1h_2}\right]_{y=y_i} \\
e_{ij} &= \left\{P_j'''' - \frac{2P_j'''}{h}\frac{\partial h}{\partial y} + P_j''\left[\frac{2}{h^2}\left(\frac{\partial h}{\partial y}\right)^2 - \frac{1}{h}\frac{\partial^2 h}{\partial y^2} - \frac{2\nu}{h^2}\left(\frac{\partial h}{\partial x}\right)^2 + \frac{\nu}{h}\frac{\partial^2 h}{\partial x^2}\right]\right\}_{y=y_i} \\
&= \left\{P_j'''' - 2P_j'''\frac{h_2'}{h_2} + P_j''\left[2\left(\frac{h_2'}{h_2}\right)^2 - \frac{h_2''}{h_2} - 2\nu\left(\frac{h_1'}{h_1}\right)^2 + \nu\frac{h_1''}{h_1}\right]\right\}_{y=y_i}
\end{aligned}\right\} \qquad (10)$$

It is to be noted that because of equations (6) all terms in equations (10) which are multiplied by P_j are zero except when $j = i$, in which case $P_j(y_j) = 1$.

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Special Cases of Thickness Variation

The coefficients defined in equations (10) are, in general, functions of x if the thickness varies spanwise. The solution of equation (9) can become very laborious under these conditions. If, however, $h_1(x)$ is chosen to be an exponential h_0e^{mx}, then the coefficients are all constant. Equation (9) can then be solved subject to the boundary conditions (eqs. (3)) by any of the standard methods for solving ordinary linear differential equations with constant coefficients. The solution will be given in terms of exponentials. By the proper choice of m, $h_1(x)$ can be made to give a reasonable approximation to the variation of thickness along the span of a turbine blade, particularly in the vicinity of the free edge. If there is no variation in thickness, spanwise, then m is taken as zero.

If the thickness varies only in the chordwise direction, b_{ij} and d_{ij} and all terms in c_{ij} and e_{ij} containing h_1 vanish. Equation (9) then becomes

$$\sum_{j=1}^{n}\left(a_{ij}\varphi_j'''' + c_{ij}\varphi_j'' + e_{ij}\varphi_j\right) = -h(y_i)\nabla^2\left[E\alpha T(x,y_i)\right] \qquad (11)$$

$$i=1,2,\ldots,n$$

If the thickness is constant throughout the plate, equation (1) becomes

$$\nabla^4\varphi = -\nabla^2(E\alpha T) \qquad (12)$$

where

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}$$

is the biharmonic operator, and h is deleted from equations (2), which define the stress function φ. Equation (9) reduces to

$$\sum_{j=1}^{n}\left[P_j(y_i)\varphi_j'''' + 2P_j''(y_i)\varphi_j'' + P_j''''(y_i)\varphi_j\right] = -\nabla^2\left[E\alpha T(x,y_i)\right] \qquad (13)$$

Special Cases of Temperature Distribution

If the thickness and the temperature are even functions in y (i.e., symmetric about $y = 0$), then only that half of the plate between $y = 0$ and $y = 1$ need be considered. This will reduce the number of stations required for a given accuracy. The polynomials P_j can be taken as even polynomials; thus

$$P_j(y) = \frac{(y^2 - 1)^2}{(y_j^2 - 1)^2} \prod_{i \neq j} (y^2 - y_i^2) \Bigg/ \prod_{i \neq j} (y_j^2 - y_i^2) \qquad y_j \neq \pm 1 \qquad (14)$$

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Similarly, if the thickness is an even function of y and the temperature is an odd function of y (antisymmetric), the polynomials are taken as odd polynomials; thus

$$P_j(y) = \frac{(y^2 - 1)^2}{(y_j^2 - 1)^2} \prod_{i \neq j} y(y^2 - y_i^2) \Bigg/ \prod_{i \neq j} y_j(y_j^2 - y_i^2) \qquad y_j \neq \pm 1 \qquad (15)$$

Again only that half of the plate between $y = 0$ and $y = 1$ need be considered. Since any temperature distribution can be split up into the sum of an even and an odd function, it is possible to solve the problem of an arbitrary temperature distribution in two steps, namely, solving for the stresses for an even distribution and for an odd distribution, and adding the two. This will generally result in some reduction in labor, since the work involved in solving two problems each with $n/2$ stations is usually less then solving one problem with n stations. It is to be noted, however, that this procedure cannot be followed if h is not an even function of y. In that case the general polynomial (eq. (8)) must be used and stations taken ranging from $y = 1$ to $y = -1$.

EXAMPLES

Uniform Thickness, Parabolic Chordwise Temperature Distribution

As a first example, consider a semi-infinite thin plate of constant thickness with a temperature distribution independent of x and given by

$$T = \left(y^2 - \frac{1}{3}\right)T_0$$

Since this temperature distribution has zero mean and zero moment about the x-axis, the stress at a distance far from the free end is given by

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the method of reference 1 (p. 401) as

$$\sigma_x = -E\alpha T = E\alpha\left(\frac{1}{3} - y^2\right)T_0$$

$$\sigma_y = \tau_{xy} = 0$$

The only problem, therefore, is to obtain the stresses near the free end. Since the thickness is constant and the temperature distribution is symmetric, equations (13) and (14) will be used. Two stations are chosen at $y_1 = 1/4$ and $y_2 = 3/4$. The polynomials then become

$$P_1 = \frac{(y^2 - 1)^2\left(y^2 - \frac{9}{16}\right)}{\left(\frac{1}{16} - 1\right)^2\left(\frac{1}{16} - \frac{9}{16}\right)} = -\frac{512}{225}(y^2 - 1)^2\left(y^2 - \frac{9}{16}\right)$$

$$P_2 = \frac{(y^2 - 1)^2\left(y^2 - \frac{1}{16}\right)}{\left(\frac{9}{16} - 1\right)^2\left(\frac{9}{16} - \frac{1}{16}\right)} = \frac{512}{49}(y^2 - 1)^2\left(y^2 - \frac{1}{16}\right)$$

Evaluating P_1 and P_2 and their derivatives at $y_1 = 1/4$ and at $y_2 = 3/4$ gives

$$P_1(1/4) = 1 \qquad P_1(3/4) = 0$$

$$P_1''(1/4) = -5.564 \qquad P_1''(3/4) = 8.089$$

$$P_1''''(1/4) = 88.75 \qquad P_1''''(3/4) = -320.8$$

$$P_2(1/4) = 0 \qquad P_2(3/4) = 1$$

$$P_2''(1/4) = 8.571 \qquad P_2''(3/4) = -22.78$$

$$P_2''''(1/4) = -282.1 \qquad P_2''''(3/4) = 1599$$

Substituting now into equation (13) gives

$$\varphi_1'''' - 11.13\varphi_1'' + 88.75\varphi_1 + 17.14\varphi_2'' - 282.1\varphi_2 = -2E\alpha T_0$$

$$16.18\varphi_1'' - 320.8\varphi_1 + \varphi_2'''' - 45.55\varphi_2'' + 1599\varphi_2 = -2E\alpha T_0$$

The particular solutions are readily obtained as

$$\varphi_{1p} = -0.07324E\alpha T_0$$

$$\varphi_{2p} = -0.01595E\alpha T_0$$

The homogeneous solution is obtained by the usual exponential substitution. The complete solution is then

$$\varphi_1 = \sum_{k=1}^{8} A_k e^{\lambda_k x} - 0.07324E\alpha T_0$$

$$\varphi_2 = \sum_{k=1}^{8} B_k e^{\lambda_k x} - 0.01595E\alpha T_0$$

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where the terms λ_k are the roots of the determinantal equation and are equal to

$$\lambda_1 = -2.120 + 1.117i \qquad \lambda_3 = -5.682 + 2.681i$$

$$\lambda_2 = \bar{\lambda}_1 \qquad \lambda_4 = \bar{\lambda}_3$$

$$\lambda_5 = -\lambda_1 \qquad \lambda_7 = -\lambda_3$$

$$\lambda_6 = \bar{\lambda}_5 \qquad \lambda_8 = \bar{\lambda}_7$$

where the bar signifies the complex conjugate.

The term B_k is given in terms of A_k as follows:

$$B_k = -\frac{\lambda_k^4 - 11.13\lambda_k^2 + 88.75}{17.14\lambda_k^2 - 282.1} A_k$$

In order for the stresses to remain finite as x approaches infinity, A_5, A_6, A_7, and A_8 must vanish. This leaves for φ_1 and φ_2

$$\varphi_1 = \sum_{k=1}^{4} A_k e^{\lambda_k x} - 0.07324E\alpha T_0$$

$$\varphi_2 = -\sum_{k=1}^{4} \frac{\lambda_k^4 - 11.13\lambda_k^2 + 88.75}{17.14\lambda_k^2 - 282.1} A_k e^{\lambda_k x} - 0.01595E\alpha T_0$$

In order to determine the values of the A_k's, use is now made of the boundary conditions of equations (3). It is to be noted that all other boundary conditions are already satisfied in the choice of the P function. In order to satisfy equations (3), it is necessary that

$$\varphi_1(0) = \varphi_1'(0) = \varphi_2(0) = \varphi_2'(0) = 0$$

since $\frac{\partial^2\varphi}{\partial y^2}$ is not necessarily zero at $x = 0$. Thus, four equations result for the four A_k's. The solution of these four equations gives

$$A_1 = (0.03625 - 0.07114i)E\alpha T_0 \qquad A_2 = \bar{A}_1$$

$$A_3 = (0.0003671 + 0.0001926i)E\alpha T_0 \qquad A_4 = \bar{A}_3$$

After adding all the complex conjugates, the imaginary parts drop out and the final solution is

$$\begin{aligned}\varphi_1 = {} & 2e^{-2.120x}[0.03625\cos(1.117x) + 0.07114\sin(1.117x)]E\alpha T_0 + \\ & 2e^{-5.682x}[0.0003671\cos(2.681x) - 0.0001926\sin(2.681x)]E\alpha T_0 - \\ & 0.07324E\alpha T_0\end{aligned}$$

$$\begin{aligned}\varphi_2 = {} & 2e^{-2.120x}[0.008941\cos(1.117x) + 0.01249\sin(1.117x)]E\alpha T_0 - \\ & 2e^{-5.682x}[0.0009662\cos(2.681x) + 0.0001790\sin(2.681x)]E\alpha T_0 - \\ & 0.01595E\alpha T_0\end{aligned}$$

The stress function is given by

$$\varphi = P_1\varphi_1 + P_2\varphi_2$$

and the stresses are

$$\begin{aligned}\sigma_x &= P_1''\varphi_1 + P_2''\varphi_2 \\ &= -\frac{512}{225}\left(30y^4 - \frac{123}{4}y^2 + \frac{17}{4}\right)\varphi_1 + \frac{512}{49}\left(30y^4 - \frac{99}{4}y^2 + \frac{9}{4}\right)\varphi_2\end{aligned}$$

$$\begin{aligned}\sigma_y &= P_1\varphi_1'' + P_2\varphi_2'' \\ &= -\frac{512}{225}(y^2 - 1)^2\left(y^2 - \frac{9}{16}\right)\varphi_1'' + \frac{512}{49}(y^2 - 1)^2\left(y^2 - \frac{1}{16}\right)\varphi_2''\end{aligned}$$

$$\begin{aligned}\tau_{xy} &= -P_1'\varphi_1' - P_2'\varphi_2' \\ &= \frac{512}{225}\left(6y^5 - \frac{41}{4}y^3 + \frac{17}{4}y\right)\varphi_1' - \frac{512}{49}\left(6y^5 - \frac{33}{4}y^3 + \frac{9}{4}y\right)\varphi_2'\end{aligned}$$

It is to be noted that at $x = \infty$

$$\varphi_1 = -0.07324E\alpha T_0$$

$$\varphi_2 = -0.01595E\alpha T_0$$

$$\varphi_1' = \varphi_1'' = \varphi_2' = \varphi_2'' = 0$$

Therefore,

$$\sigma_y = 0$$

$$\tau_{xy} = 0$$

$$\sigma_x = (-y^2 + 0.333)E\alpha T_0$$

which are the exact stresses at $x = \infty$. The particular solution of the differential equation gives the stresses far from the end. The homogeneous solution gives the correction due to the free end and dies out rapidly away from this end.

The preceding stresses are plotted in figures 2 and 3. These results will be discussed in RESULTS AND DISCUSSION.

The values of the λ's, given previously, can be used for any constant-thickness two-station symmetric problem, since the λ's are independent of the temperature distribution. For any other temperature distribution new values for the particular solutions to the differential equation must be obtained, the homogeneous solution remaining the same. New values of the constants A_k and B_k can then be determined from the boundary conditions.

For a three-station solution of the same problem $\left(\text{stations at } y_1 = \frac{1}{6}, y_2 = \frac{3}{6}, \text{ and } y_3 = \frac{5}{6}\right)$, the following values can be used:

$$P_1(y) = 7.141(y^2 - 1)^2\left(y^2 - \frac{1}{4}\right)\left(y^2 - \frac{25}{36}\right)$$

$$P_2(y) = -18.00(y^2 - 1)^2\left(y^2 - \frac{1}{36}\right)\left(y^2 - \frac{25}{36}\right)$$

$$P_3(y) = 36.15(y^2 - 1)^2\left(y^2 - \frac{1}{36}\right)\left(y^2 - \frac{1}{4}\right)$$

$$\lambda_1 = -2.10568 - 1.12509i \qquad \lambda_2 = \bar{\lambda}_1$$

$$\lambda_3 = -5.42265 - 1.27826i \qquad \lambda_4 = \bar{\lambda}_3$$

$$\lambda_5 = -9.23538 - 3.37145i \qquad \lambda_6 = \bar{\lambda}_5$$

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The equations for the φ_j functions in this case are

$$\varphi_1 = \sum_{k=1}^{6} A_k e^{\lambda_k x} + \varphi_{1p}$$

$$\varphi_2 = \sum_{k=1}^{6} K_k A_k e^{\lambda_k x} + \varphi_{2p}$$

$$\varphi_3 = \sum_{k=1}^{6} N_k A_k e^{\lambda_k x} + \varphi_{3p}$$

where the φ_{jp}'s are the particular solutions for the given temperature distributions and the K_k and N_k values are as follows:

$$K_1 = 0.5637 - 0.03947i \qquad K_2 = \bar{K}_1$$

$$K_3 = -1.236 - 0.2541i \qquad K_4 = \bar{K}_3$$

$$K_5 = -2.968 - 1.061i \qquad K_6 = \bar{K}_5$$

$$N_1 = 0.08273 - 0.01802i \qquad N_2 = \bar{N}_1$$

$$N_3 = -0.4420 + 0.04044i \qquad N_4 = \bar{N}_3$$

$$N_5 = 2.019 + 4.497i \qquad N_6 = \bar{N}_5$$

the A_k's can be obtained from the boundary conditions; that is, $\varphi_j = \varphi_j' = 0$ at $x = 0$.

Variable Chordwise Thickness, Parabolic Chordwise Temperature Distribution

As a second example, the plate is assumed to have a variable thickness in the chordwise direction given by

$$h = h_2 = (1 - 0.9y^2)h_0$$

The temperature distribution is taken the same as before:

$$T = \left(y^2 - \frac{1}{3}\right)T_0$$

Again two stations will be taken at $y_1 = 1/4$ and at $y_2 = 3/4$. Since conditions of symmetry exist, the same polynomials as used in the first example will be used here. The derivatives of these polynomials evaluated at the two stations have previously been listed. Since the thickness is no longer constant, equation (13) cannot be used. Instead, equation (11) is used in this case. The coefficients a_{ij}, c_{ij}, and b_{ij} are evaluated using equations (10). Equation (11) then becomes

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$$\varphi_1'''' - 13.81\varphi_1'' + 104.9\varphi_1 + 21.52\varphi_2'' - 366.5\varphi_2 = -1.888h_0E\alpha T_0$$

$$12.61\varphi_1'' - 226.4\varphi_1 + \varphi_2'''' - 72.22\varphi_2'' + 1947\varphi_2 = -0.9875h_0E\alpha T_0$$

These equations can be solved in the same way as for the uniform-thickness plate. The final results for φ_1 and φ_2 are

$$\frac{\varphi_1}{h_0E\alpha T_0} = 2e^{-2.569x}\left[0.01644 \cos(1.159x) + 0.03761 \sin(1.159x)\right] + 2e^{-6.386x}\left[0.0002130 \cos(1.741x) + 0.000002496 \sin(1.741x)\right] - 0.03330$$

$$\frac{\varphi_2}{h_0E\alpha T_0} = 2e^{-2.569x}\left[0.002629 \cos(1.159x) + 0.003692 \sin(1.159x)\right] - 2e^{-6.386x}\left[0.0004389 \cos(1.741x) + 0.0001872 \sin(1.741x)\right] - 0.004381$$

The stresses are then given by

$$\sigma_x = \frac{1}{h} P_1''\varphi_1 + \frac{1}{h} P_2''\varphi_2$$

$$= -\frac{512}{225} \frac{\left(30y^4 - \frac{123}{4}y^2 + \frac{17}{4}\right)}{\left(1 - 0.9y^2\right)}\varphi_1 + \frac{512}{49} \frac{\left(30y^4 - \frac{99}{4}y^2 + \frac{9}{4}\right)}{\left(1 - 0.9y^2\right)}\varphi_2$$

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$$\sigma_y = \frac{1}{h} P_1\varphi_1'' + \frac{1}{h} P_2\varphi_2''$$

$$= -\frac{512}{225}\frac{(y^2 - 1)^2\left(y^2 - \frac{9}{16}\right)}{(1 - 0.9y^2)}\varphi_1'' + \frac{512}{49}\frac{(y^2 - 1)^2\left(y^2 - \frac{1}{16}\right)}{(1 - 0.9y^2)}\varphi_2''$$

$$\tau_{xy} = -\frac{1}{h} P_1'\varphi_1' - \frac{1}{h} P_2'\varphi_2'$$

$$= \frac{512}{225}\frac{\left(6y^5 - \frac{41}{4}y^2 + \frac{17}{4}y\right)}{(1 - 0.9y^2)}\varphi_1' - \frac{512}{49}\frac{\left(6y^5 - \frac{33}{4}y^2 + \frac{9}{4}y\right)}{(1 - 0.9y^2)}\varphi_2'$$

At $x = \infty$,

$$\sigma_y = \tau_{xy} = 0$$

$$\frac{\sigma_x}{E\alpha T_0} = 0.07578\,\frac{(30y^4 - 30.75y^2 + 4.25)}{(1 - 0.9y^2)} - 0.04578\,\frac{(30y^4 - 24.75y^2 + 2.25)}{(1 - 0.9y^2)}$$

$$= 0.219 - y^2$$

The exact solution for this case at $x = \infty$ is given by (ref. 3)

$$\frac{\sigma_x}{E\alpha T_0} = -\frac{T}{T_0} + \frac{\int_{-1}^{1}\frac{T}{T_0}\,h\,dy}{\int_{-1}^{1} h\,dy}$$

Substituting the expressions for T and h gives

$$\frac{\sigma_x}{E\alpha T_0} = -y^2 + \frac{1}{3} + \frac{\int_{-1}^{1}\left(y^2 - \frac{1}{3}\right)(1 - 0.9y^2)dy}{\int_{-1}^{1}(1 - 0.9y^2)dy}$$

$$= 0.219 - y^2$$

Thus the two-station solution again gives the same answer as the exact solution at x equal infinity.

The stresses σ_y, τ_{xy}, and σ_x for this variable-thickness case are plotted in figure 4. These results will be discussed in RESULTS AND DISCUSSION.

Uniform Thickness, Nonsymmetric Chordwise

Temperature Distribution

As a final example, consider the case of a uniform-thickness plate with a nonsymmetric chordwise temperature distribution given by

$$T = \left(y^3 + y^2 - \frac{1}{3}\right)T_0$$

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Since this temperature distribution is neither even nor odd, the general polynomial of equation (8) will be used with three stations taken at $y_1 = -2/3$, $y_2 = 0$, and $y_3 = 2/3$. The polynomials then are

$$P_1 = \frac{(y^2 - 1)^2}{\frac{25}{81}} \frac{y\left(y - \frac{2}{3}\right)}{\left(-\frac{2}{3}\right)\left(-\frac{4}{3}\right)} = \frac{729}{200} y\left(y - \frac{2}{3}\right)(y^2 - 1)^2$$

$$P_2 = \frac{(y^2 - 1)^2}{1} \frac{\left(y + \frac{2}{3}\right)\left(y - \frac{2}{3}\right)}{\left(\frac{2}{3}\right)\left(-\frac{2}{3}\right)} = -\frac{9}{4}\left(y + \frac{2}{3}\right)\left(y - \frac{2}{3}\right)(y^2 - 1)^2$$

$$P_3 = \frac{(y^2 - 1)^2}{\frac{25}{81}} \frac{\left(y + \frac{2}{3}\right)y}{\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)} = \frac{729}{200} y\left(y + \frac{2}{3}\right)(y^2 - 1)^2$$

When these polynomials and their required derivatives are evaluated at the three stations and the results substituted into equation (13), the following three equations are obtained:

$$\varphi_1'''' - 30.06\varphi_1'' + 797.0\varphi_1 - 228.0\varphi_2 - 9.900\varphi_3'' + 213.8\varphi_3 = 2E\alpha T_0$$

$$14.58\varphi_1'' - 175.0\varphi_1 + \varphi_2'''' - 17.00\varphi_2'' + 132.0\varphi_2 + 14.58\varphi_3'' - 175.0\varphi_3 = -2E\alpha T_0$$

$$-9.900\varphi_1'' + 408.2\varphi_1 + 15.00\varphi_2'' - 228.0\varphi_2 + \varphi_3'''' - 30.06\varphi_3'' + 602.6\varphi_3 = -6E\alpha T_0$$

These equations are solved giving φ_1, φ_2, and φ_3 as functions of x. The stresses are now computed as before from equations (2) with h deleted.

The previous problem could also have been solved by separating the temperature distribution into even and odd functions. Thus the stresses for the distribution $T = \left(y^2 - \frac{1}{3}\right)T_0$ have already been computed in the first example. It remains only to obtain the stresses due to the distribution $T = y^3T_0$. This can be done accurately by using only two stations at $y_1 = 1/4$ and $y_2 = 3/4$ with the antisymmetric polynomials of equation (15). The two stress distributions can then be added to give the complete stress distribution. This has been done for this problem and the results are compared with the previous solution in figure 5. Figure 5 also includes a four-station solution of this same problem using the nonsymmetrical temperature distribution.

RESULTS AND DISCUSSION

In order to determine the relative accuracy of the method presented herein, a comparison is made in figure 2 between this method using one, two, and three stations and the energy and elasticity methods presented in reference 2. A uniform-thickness plate with the parabolic temperature distribution of the first example is used for comparison. The spanwise stress σ_x, the shear stress τ_{xy} at the tip $(x = 0)$, and the chordwise stress σ_y at 1/4 chord from the tip $(x = 1/2)$ are plotted for the three methods. It is seen that for all practical purposes the method presented herein has converged using just two stations, giving a far superior answer than the energy method and a slightly better answer than the elasticity method of reference 2. Even the one-station solution gives an approximate answer almost as good as the energy method and may be useful for many practical engineering problems.

The effect of the free end for this problem is briefly shown in figure 3(a), where σ_x and σ_y at the midchord $(y = 0)$ are plotted against x. The chordwise stress σ_y is large at the tip but has dropped to practically zero after 1 chord length from the tip. The spanwise stress σ_x starts at zero at the free end and rapidly rises to a constant value after 1 chord length from the tip. The shearing stress, although not shown here, follows the same pattern as σ_y. It is to be noted that an approximate simple check can be obtained for σ_y from the fact that the integral under the curve must vanish. It is further seen from this curve that there is practically no difference between the two- and three-station solutions for this problem. The complete stress distributions for this case at various distances from the free end are plotted in figures 3(b) to (d).

The effect of variation in chordwise thickness is shown in figure 4. Again the temperature distribution is parabolic and the thickness variation is given by $h = h_0(1 - 0.9y^2)$, where h_0 is the thickness at the midchord. The edges therefore are one-tenth as thick as the middle. A comparison is made between a two-station and a three-station calculation, and it is seen that the two-station calculation gives good results except for the spanwise stress σ_x near the thin edge, where there is some difference between the two- and three-station solutions.

A comparison between the uniform-thickness plate and the variable-thickness plate in figure 4(d) indicates, as would be expected, that the effect of thinning out the edges is to raise the edge stress and lower the stress at the midchord. However, in spite of the large variation in thickness, the edge stress is raised by only about 15 percent for this case.

The case of a nonsymmetric chordwise temperature distribution $T = T_0\left(y^3 + y^2 - \frac{1}{3}\right)$ (the third example) is shown in figure 5. Three methods were used for this example: a three-station solution; a four-station solution; and a solution obtained by splitting the temperature into even and odd functions, using a two-station solution for each temperature distribution, and adding the results. It is seen from the figure that the sum of the two-station solutions gives the same results as that for the four-station solution. It is generally less work to perform two two-station calculations than one four-station calculation. Therefore, if the chordwise temperature distribution is not symmetric, it is best to divide it into even and odd parts and perform two separate calculations. This should be done if extreme accuracy is desired. However, it is seen from figure 5 that even three stations give sufficient accuracy for most engineering purposes; therefore, using more stations or dividing the problem into two parts would generally seem unnecessary.

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The effect of variations in both chordwise and spanwise temperatures is shown in figure 6. The thickness is assumed constant and the temperature distribution is given by $T = T_0\left(y^2 - \frac{1}{3}\right)(1 + 0.3e^{-x})$. This would correspond to the temperature decreasing exponentially by about 30 percent from the tip to the base of a turbine blade. This relatively large variation in spanwise temperature affects only the chordwise stress σ_y, leaving the spanwise stress σ_x unaltered. Since the maximum stress is generally the spanwise stress σ_x, the effect of the spanwise variation in temperature doesn't seem to be of great importance and can be neglected as a first approximation.

As a final calculation, the effect of spanwise variation of thickness was briefly investigated. The temperature distribution was taken to be parabolic and the thickness was assumed to be given by $h = h_1 = h_0e^{0.3x}$. This is a reasonable type of spanwise thickness variation for turbine blades. The results are shown in figure 7. A comparison of figure 7 with figure 3 shows that this variation in spanwise thickness produces no large changes in the stress distribution.

Although the effects of thickness variation are shown to be not very great in these examples, this conclusion may not be valid for extreme thickness variation.

It is to be noted that, although the cases considered herein are all for a plate with a free end, any other edge conditions can be treated in the same way, provided the boundary conditions are adequately specified.

CONCLUSIONS

An approximate method was presented for calculating the elastic thermal stresses in a thin plate of variable thickness. From the examples shown, it would seem that a two- or three-station solution should suffice for most practical engineering problems.

Calculations on the effects of spanwise variations in temperature and spanwise and chordwise variations in thickness indicated that for the examples chosen these variations did not have any large effects on the stresses.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, July 16, 1956

APPENDIX A

SYMBOLS

A_k, B_k	constants of integration
a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}	coefficients in differential equation for stress function (eq. (9))
E	Young's modulus of elasticity
h	thickness of plate, $h = h_1(x)h_2(y)$
h_0	reference thickness
$h_1(x)$	function expressing spanwise thickness variation
$h_2(y)$	function expressing chordwise thickness variation
K_k	constant
m	coefficient in equation $h_1(x) = h_0 e^{mx}$
N_k	constant
P_j	polynomial associated with j^{th} station, function of y
T	temperature, function of x and y
T_0	reference temperature
x	spanwise coordinate measured from free edge, half chord lengths
y	chordwise coordinate measured from centroid, y-axis coincident with axis of minimum moment of inertia
z	thickness coordinate measured from center, z-axis coincident with axis of maximum moment of inertia

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α	coefficient of linear thermal expansion
γ_{xy}	shearing strain
ε_x	strain in x-direction
ε_y	strain in y-direction
ν	Poisson ratio
$\prod_{i \neq j}$	product for all values of i except $i = j$
σ_x	normal stress in x-direction
σ_y	normal stress in y-direction
τ_{xy}	shear stress
φ	stress function, function of x and y
φ_j	stress function associated with j^{th} station, function of x
φ_{jp}	particular solution to differential equation
∇^2	Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇^4	biharmonic operator, $\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

Subscripts:

i, j	summation or multiplication dummy indices or refer to the i^{th} or j^{th} station
x,y	partial differentiation with respect to that subscript, except where otherwise defined

Superscripts:

'	indicates ordinary derivatives
$\overline{\ }$	indicates complex conjugates

APPENDIX B

DIFFERENTIAL EQUATION FOR STRESS FUNCTION

The stress-strain relations for the plane stress problem are

$$\left.\begin{aligned} E\varepsilon_x &= \sigma_x - \nu\sigma_y + E\alpha T \\ E\varepsilon_y &= \sigma_y - \nu\sigma_x + E\alpha T \\ \gamma_{xy} &= \frac{2(1 + \nu)}{E}\tau_{xy} \end{aligned}\right\} \tag{B1}$$

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The compatibility equation is

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} \tag{B2}$$

Substituting equations (B1) into equation (B2) gives

$$\nabla^2(\sigma_x + \sigma_y) = -\nabla^2(E\alpha T) + (1 + \nu)\left(\frac{\partial^2\sigma_x}{\partial x^2} + \frac{\partial^2\sigma_y}{\partial y^2} + 2\frac{\partial^2\tau_{xy}}{\partial x\,\partial y}\right) \tag{B3}$$

The equations for the equilibrium of forces in the plane of the plate are (ref. 5, p. 297)

$$\left.\begin{aligned} \frac{\partial(h\sigma_x)}{\partial x} + \frac{\partial(h\tau_{xy})}{\partial y} &= 0 \\ \frac{\partial(h\sigma_y)}{\partial y} + \frac{\partial(h\tau_{xy})}{\partial x} &= 0 \end{aligned}\right\} \tag{B4}$$

If a stress function φ is defined as follows:

$$\begin{aligned} \sigma_x &= \frac{1}{h}\frac{\partial^2\varphi}{\partial y^2} \\ \sigma_y &= \frac{1}{h}\frac{\partial^2\varphi}{\partial x^2} \\ \tau_{xy} &= -\frac{1}{h}\frac{\partial^2\varphi}{\partial x\,\partial y} \end{aligned} \tag{B5}$$

then equations (B4) are automatically satisfied. Substituting equations (B5) into equation (B3) gives the equation to be solved for the stress function:

$$\nabla^2\left(\frac{1}{h}\nabla^2\varphi\right) = -\nabla^2(E\alpha T) + (1 + \nu)\left[\left(\frac{1}{h}\varphi_{yy}\right)_{xx} + \left(\frac{1}{h}\varphi_{xx}\right)_{yy} - 2\left(\frac{1}{h}\varphi_{xy}\right)_{xy}\right]$$

or

$$\nabla^2\left[\frac{1}{h}\nabla^2\varphi\right] - (1 + \nu)\left[\left(\frac{1}{h}\right)_{xx}\varphi_{yy} + \left(\frac{1}{h}\right)_{yy}\varphi_{xx} - 2\left(\frac{1}{h}\right)_{xy}\varphi_{xy}\right] = -\nabla^2(E\alpha T) \quad (B6)$$

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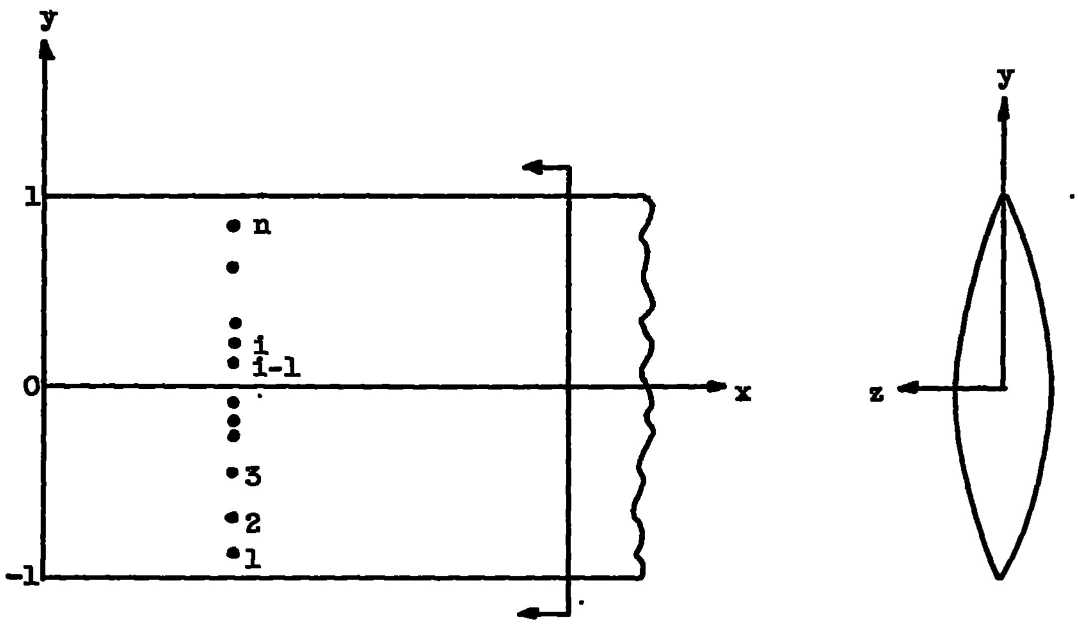

Figure 1. - Thin plate of variable thickness showing stations at which differential equation is satisfied.

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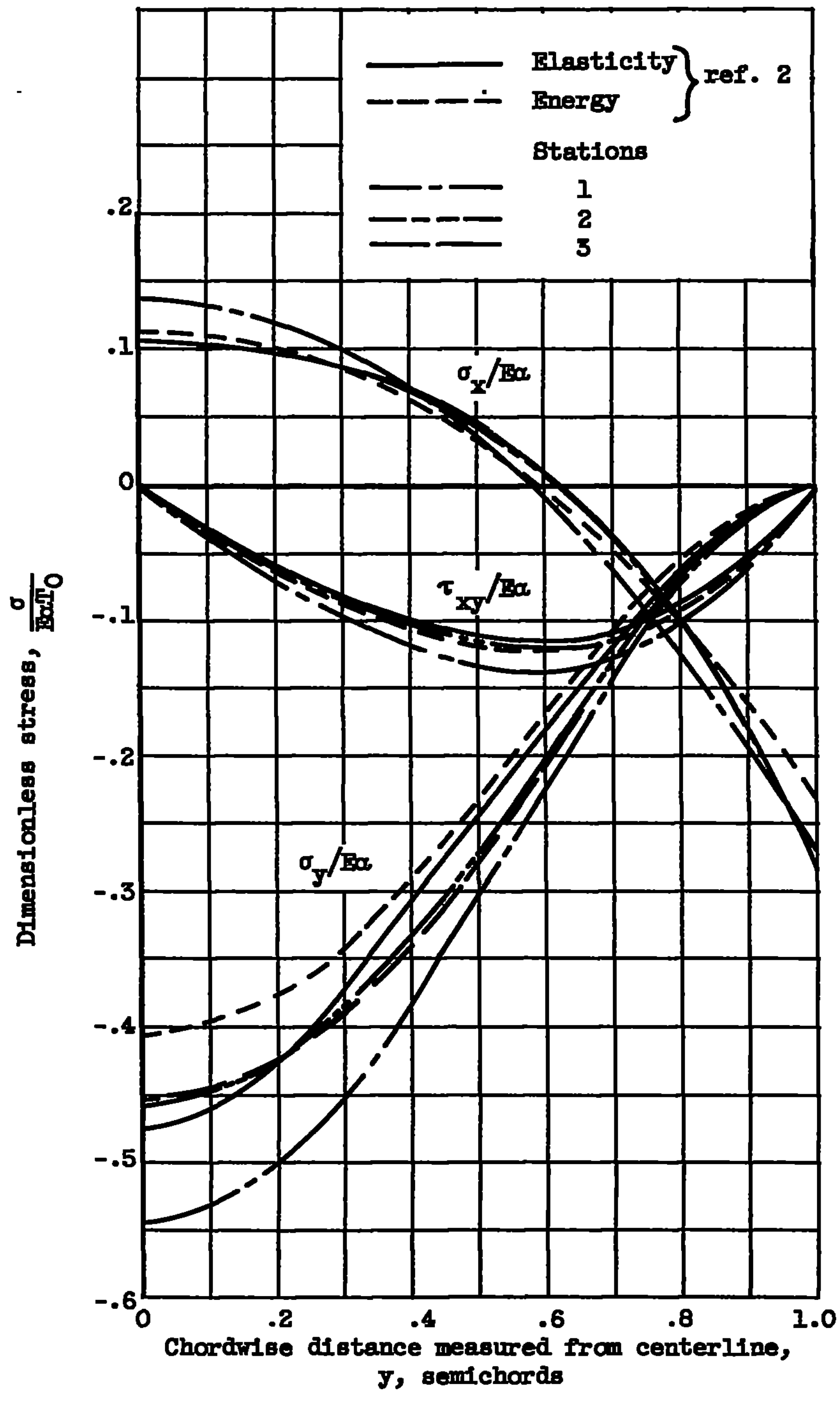
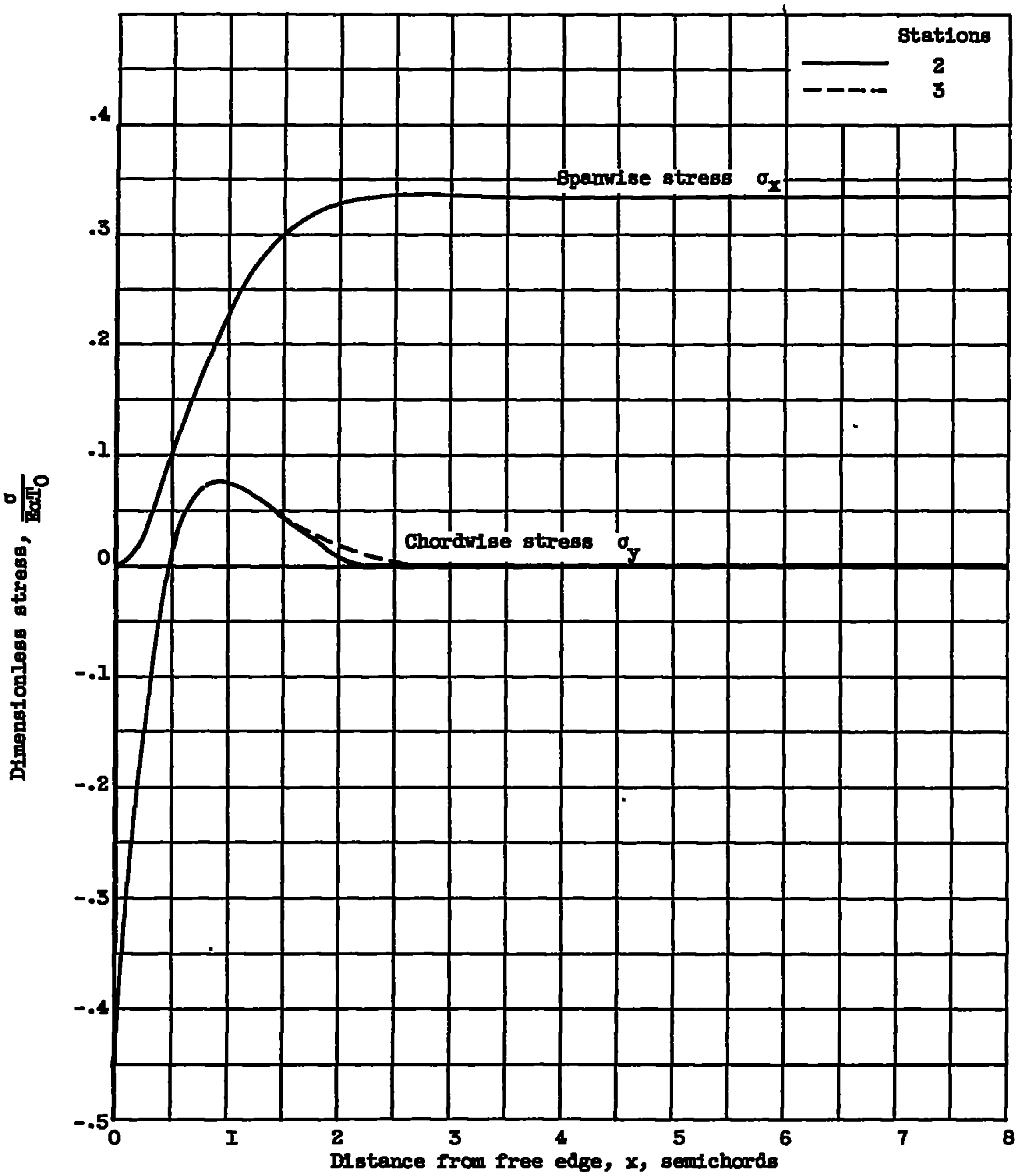
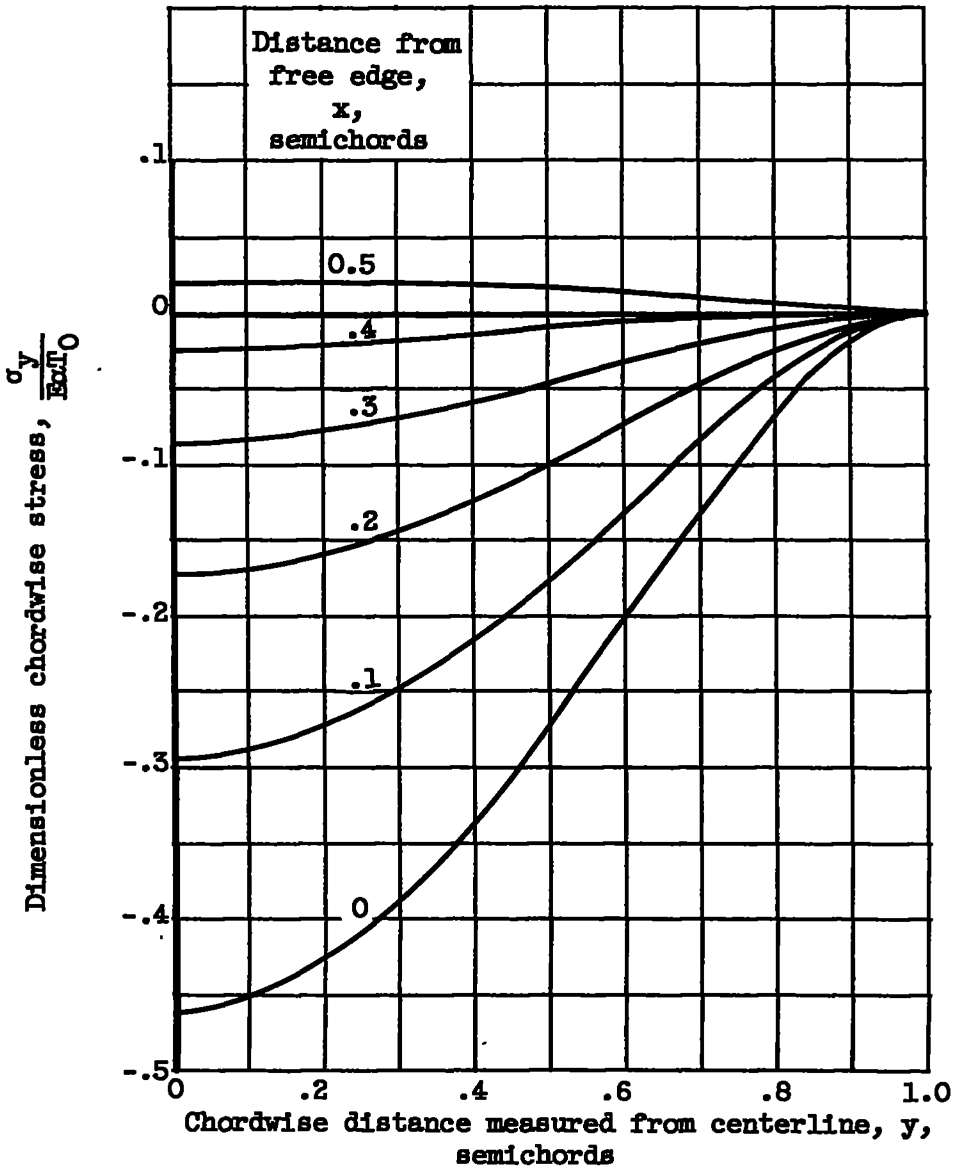


Figure 2. - Comparison of dimensionless stress for several methods of solution. Chordwise stress σ_y plotted at free end, $x = 0$; spanwise stress σ_x and shear stress τ_{xy} plotted at 1/4 chord from free end, $x = 1/2$; plate thickness, constant; $T = T_0(y^2 - \frac{1}{3})$.



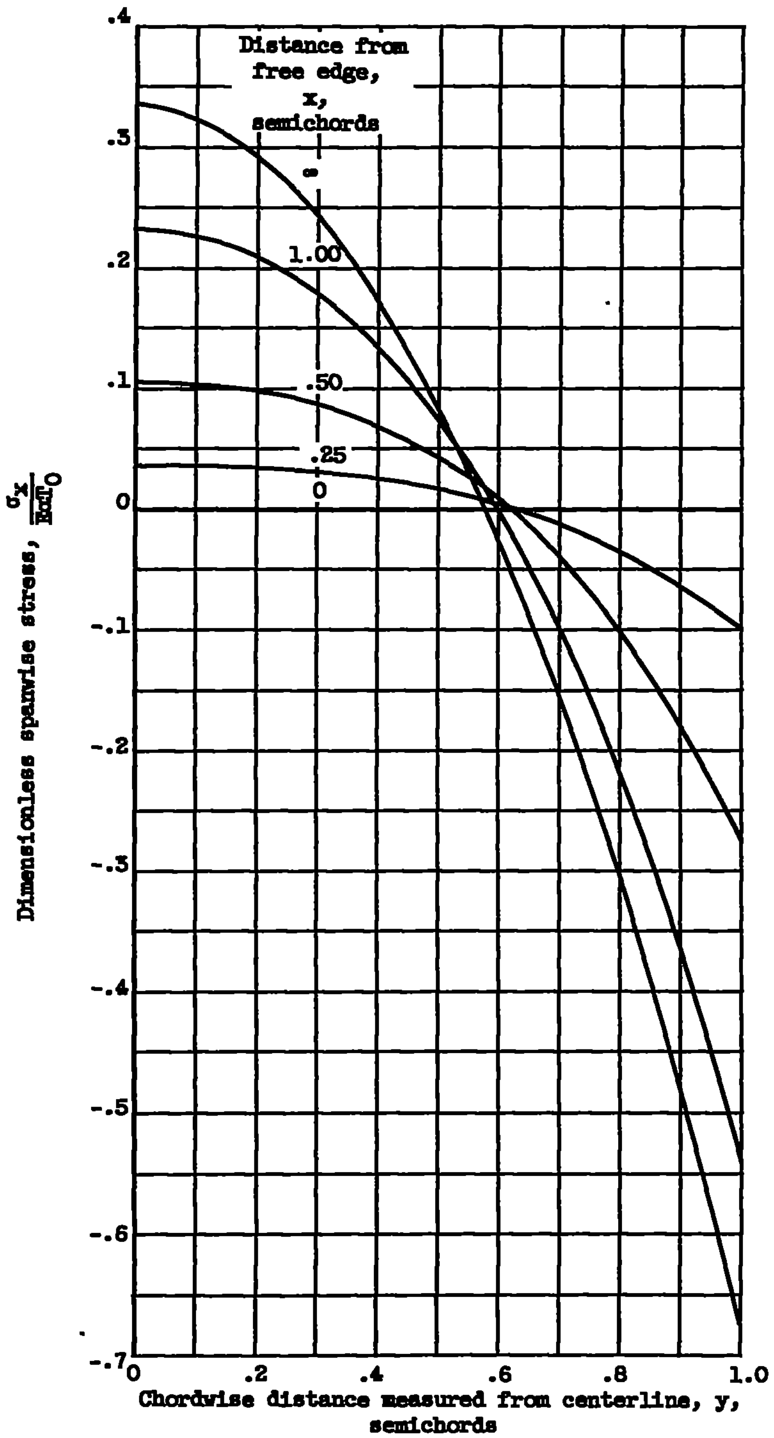
(a) Comparison of stresses at midchord for two- and three-station solutions.

Figure 3. - Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})$.



(b) Chordwise stress; two-station solution.

Figure 3. - Continued. Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})$.



(c) Spanwise stress; two-station solution.

Figure 3. - Continued. Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})$.

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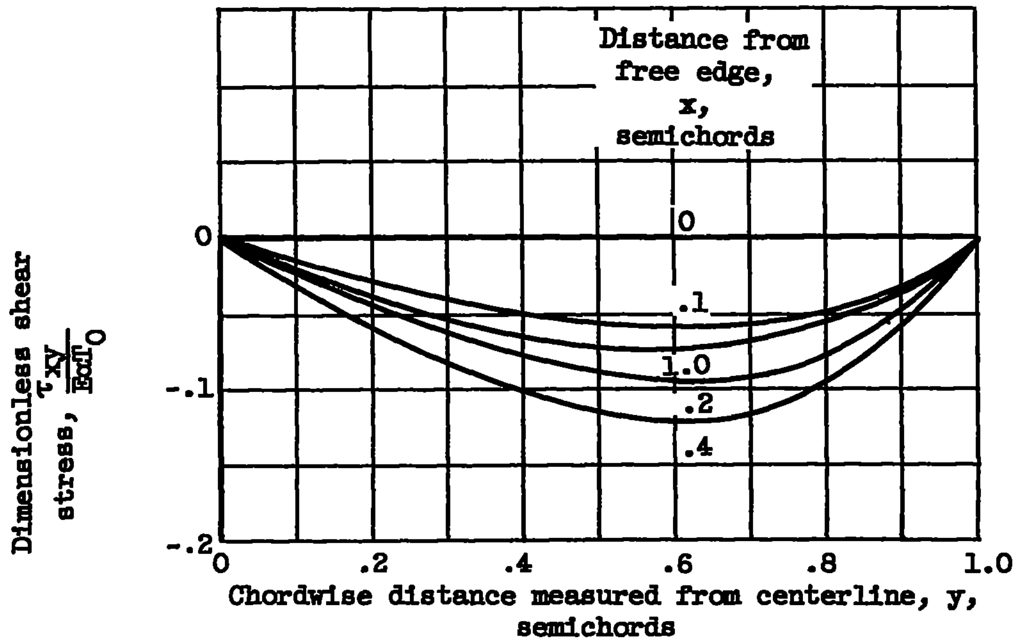

(d) Shear stress; two-station solution.

Figure 3. - Concluded. Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})$.

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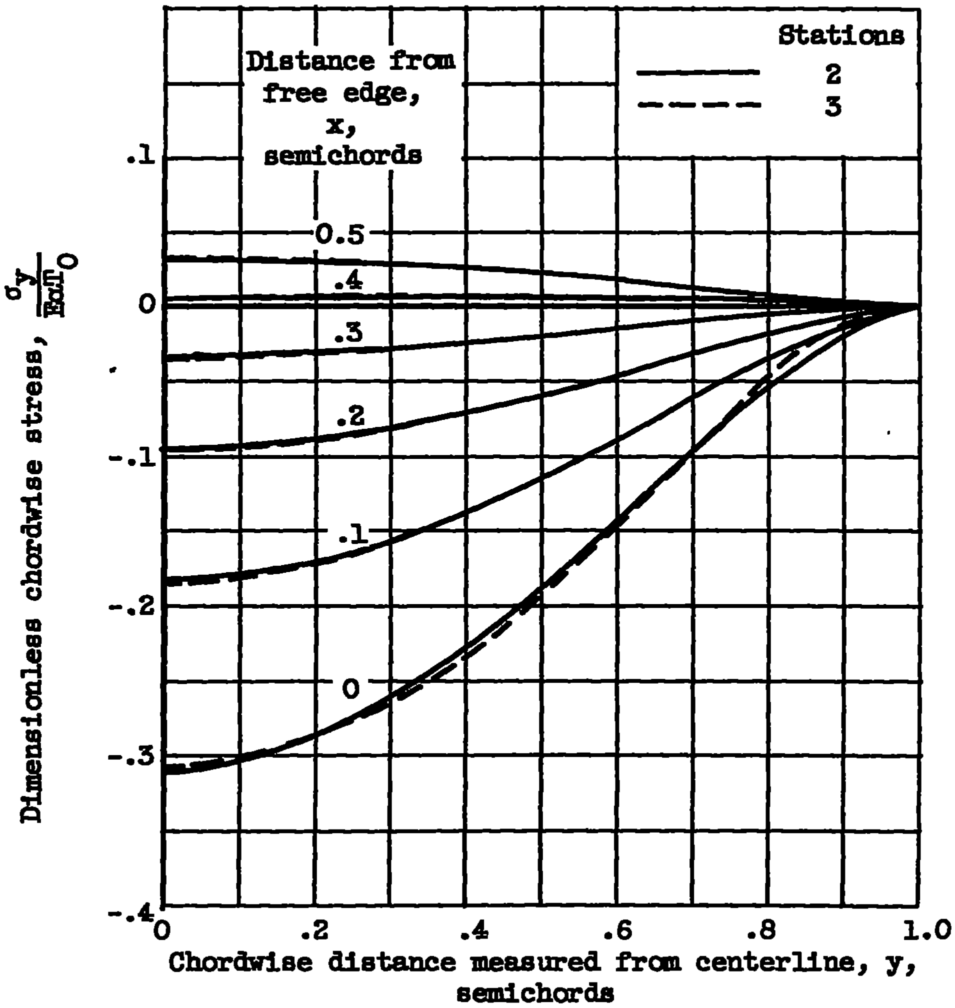

(a) Chordwise stress.

Figure 4. - Stresses in plate of variable chordwise thickness. $h = h_0(1 - 0.9y^2)$; $T = T_0(y^2 - \frac{1}{3})$.

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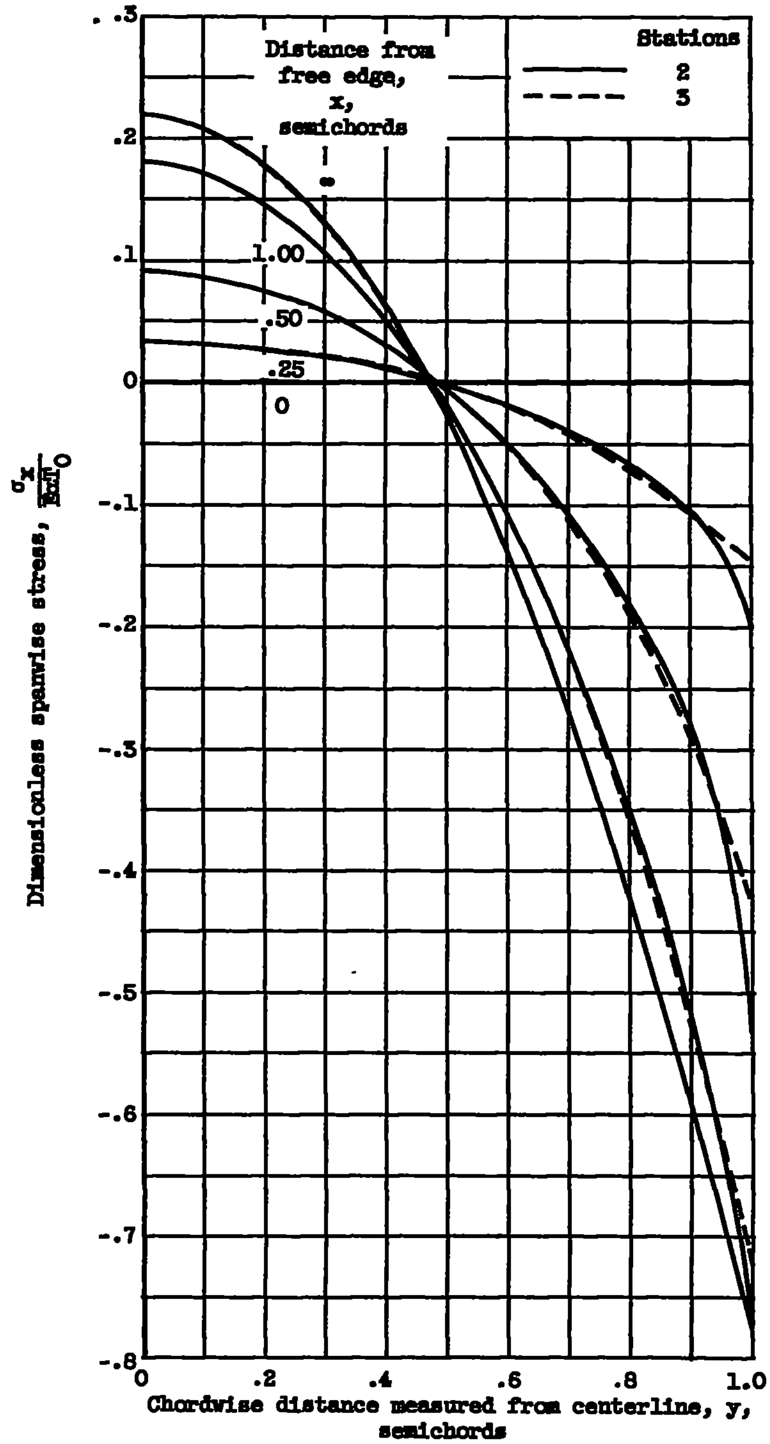

(b) Spanwise stress.

Figure 4. - Continued. Stresses in plate of variable chordwise thickness. $h = h_0(1 - 0.9y^2)$; $T = T_0(y^2 - \frac{1}{3})$.

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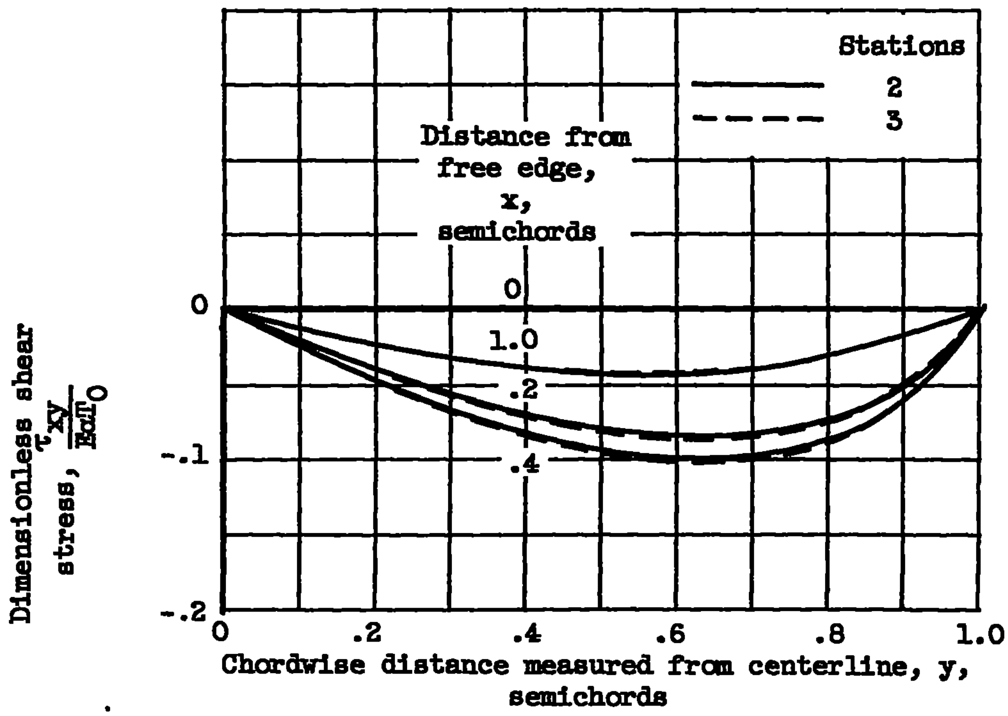
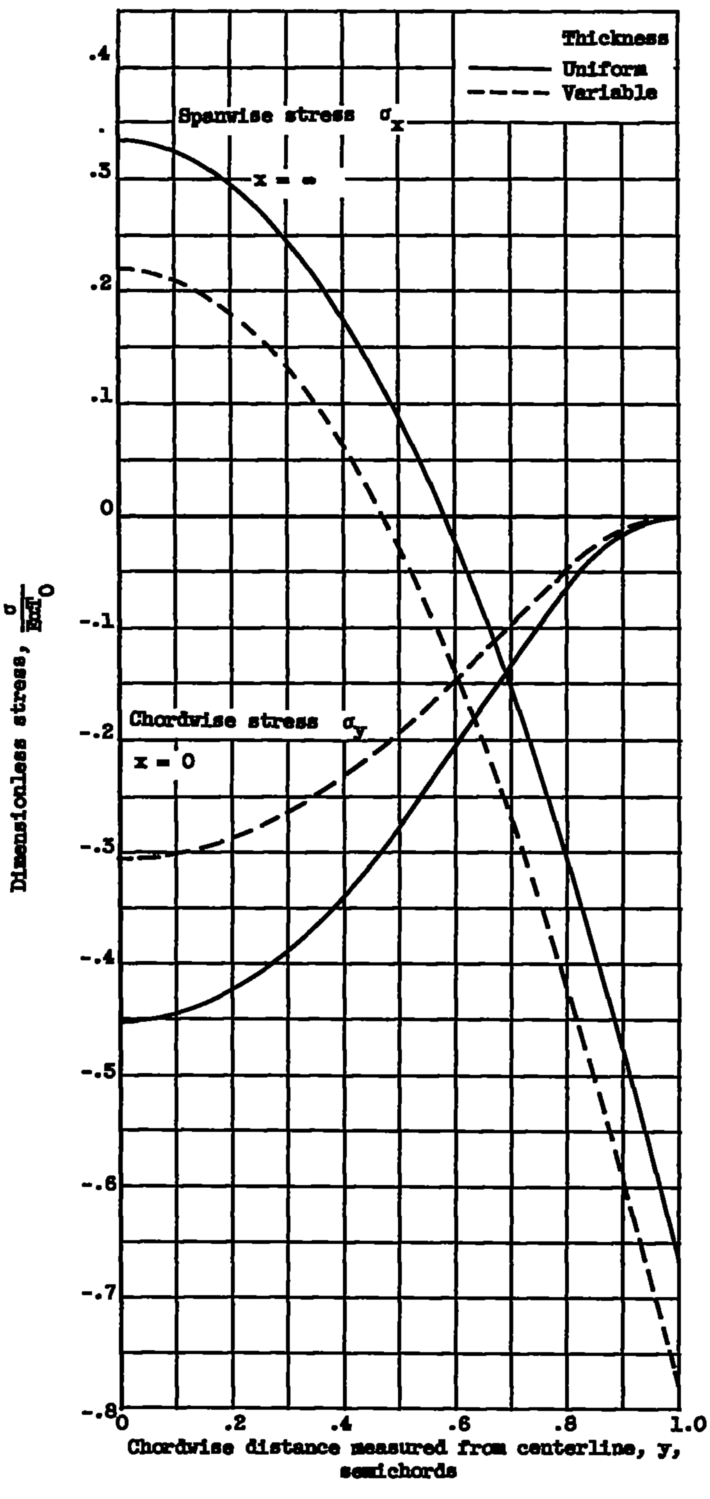


(c) Shear stress.

Figure 4. - Continued. Stresses in plate of variable chordwise thickness. $h = h_0(1 - 0.9y^2)$; $T = T_0(y^2 - \frac{1}{3})$.



(d) Comparison of variable-thickness plate with uniform-thickness plate.

Figure 4. - Concluded. Stresses in plate of variable chordwise thickness. $h = h_0(1 - 0.9y^2)$; $T = T_0(y^2 - \frac{1}{3})$.

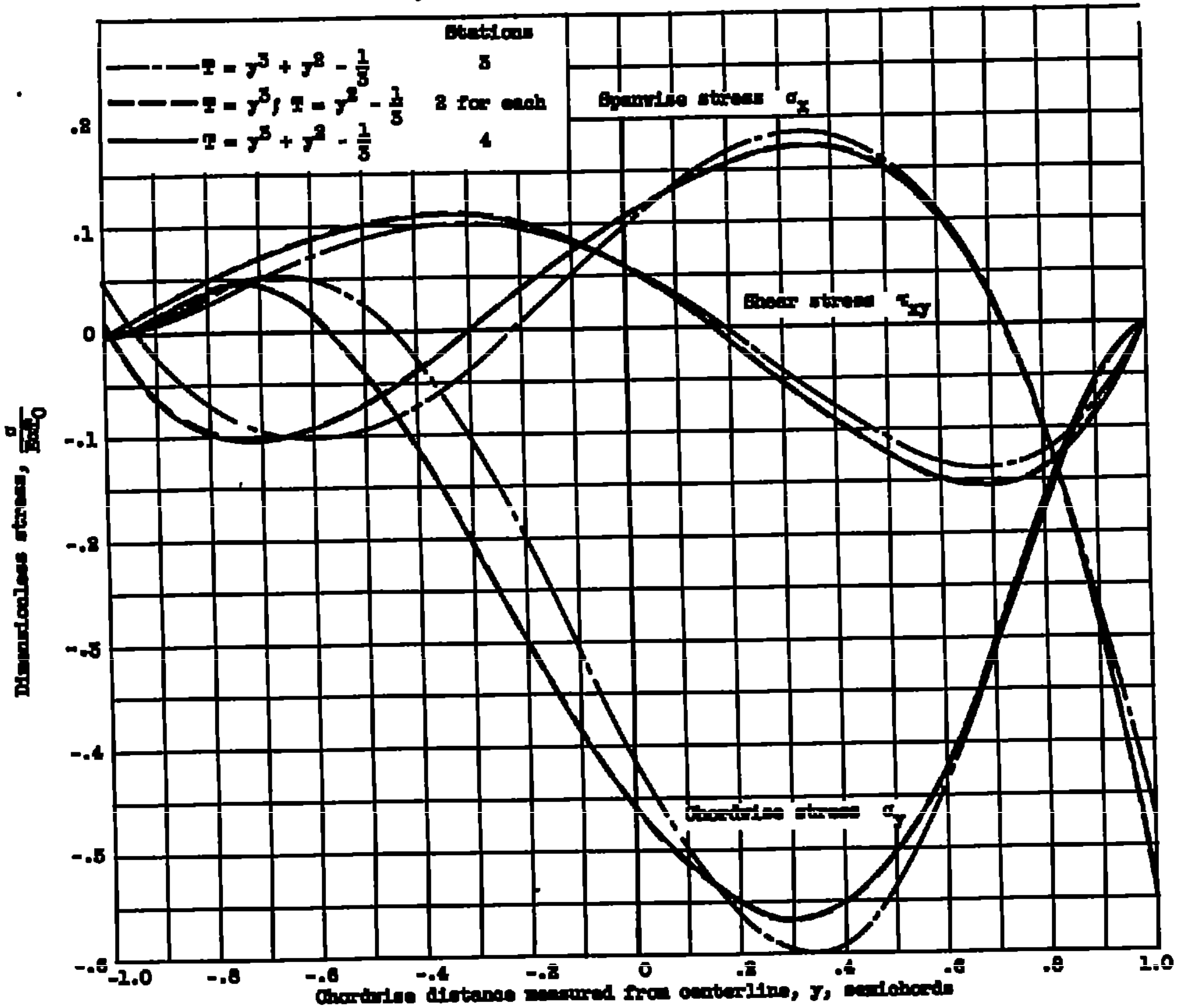
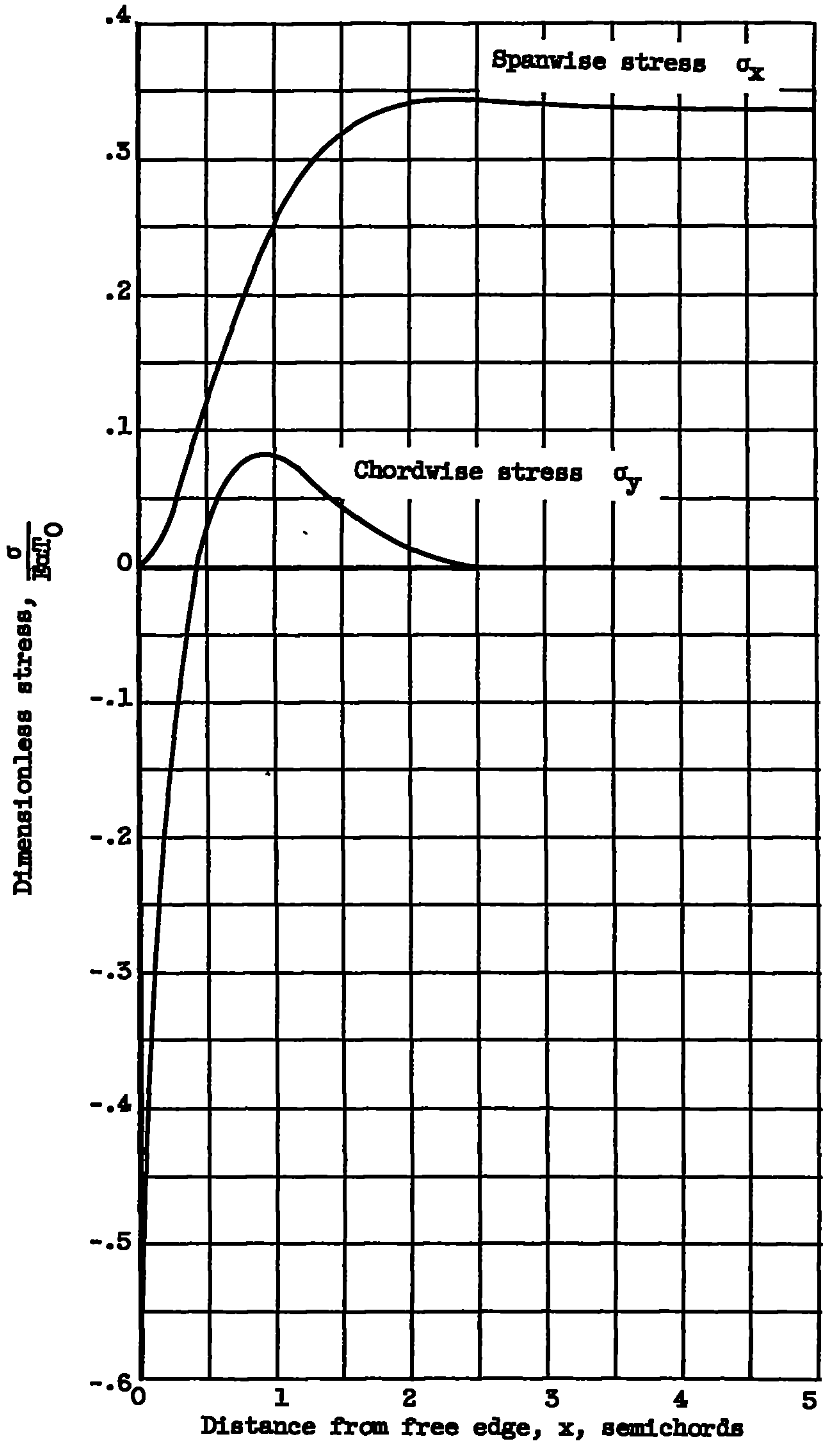
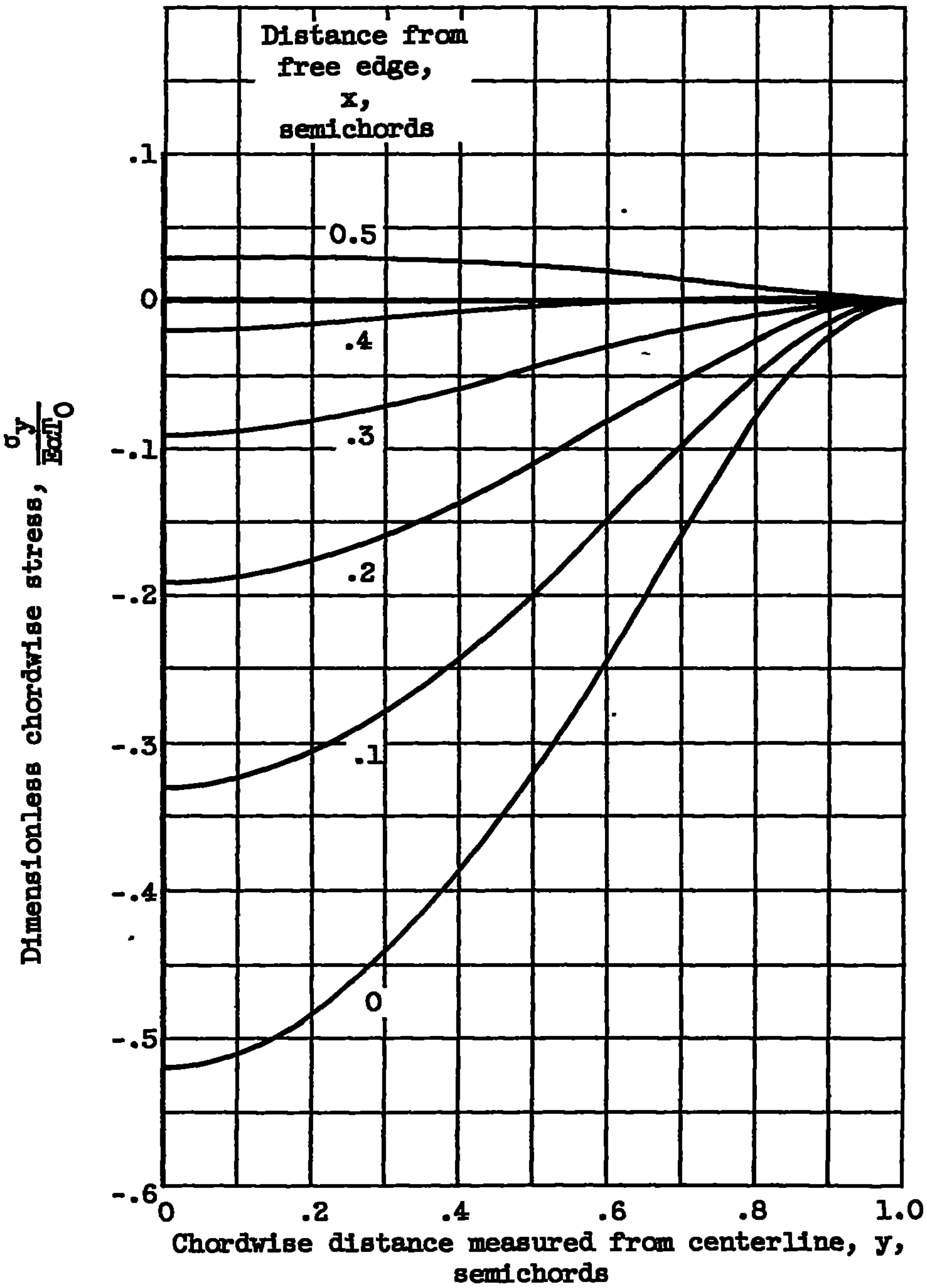


Figure 5. - Stresses in plate of uniform thickness with nonsymmetric chordwise temperature distribution. $T = T_0(y^3 + y^2 - \frac{1}{3})$. Chordwise stress σ_y plotted at free edge, $x = 0$; spanwise stress σ_x and shear stress τ_{xy} plotted at 1/4 chord from tip, $x = 1/2$.



(a) Midchord stress.

Figure 6. - Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})(1 + 0.3e^{-x})$.



(b) Chordwise stress.

Figure 6. - Continued. Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})(1 + 0.3e^{-x})$.

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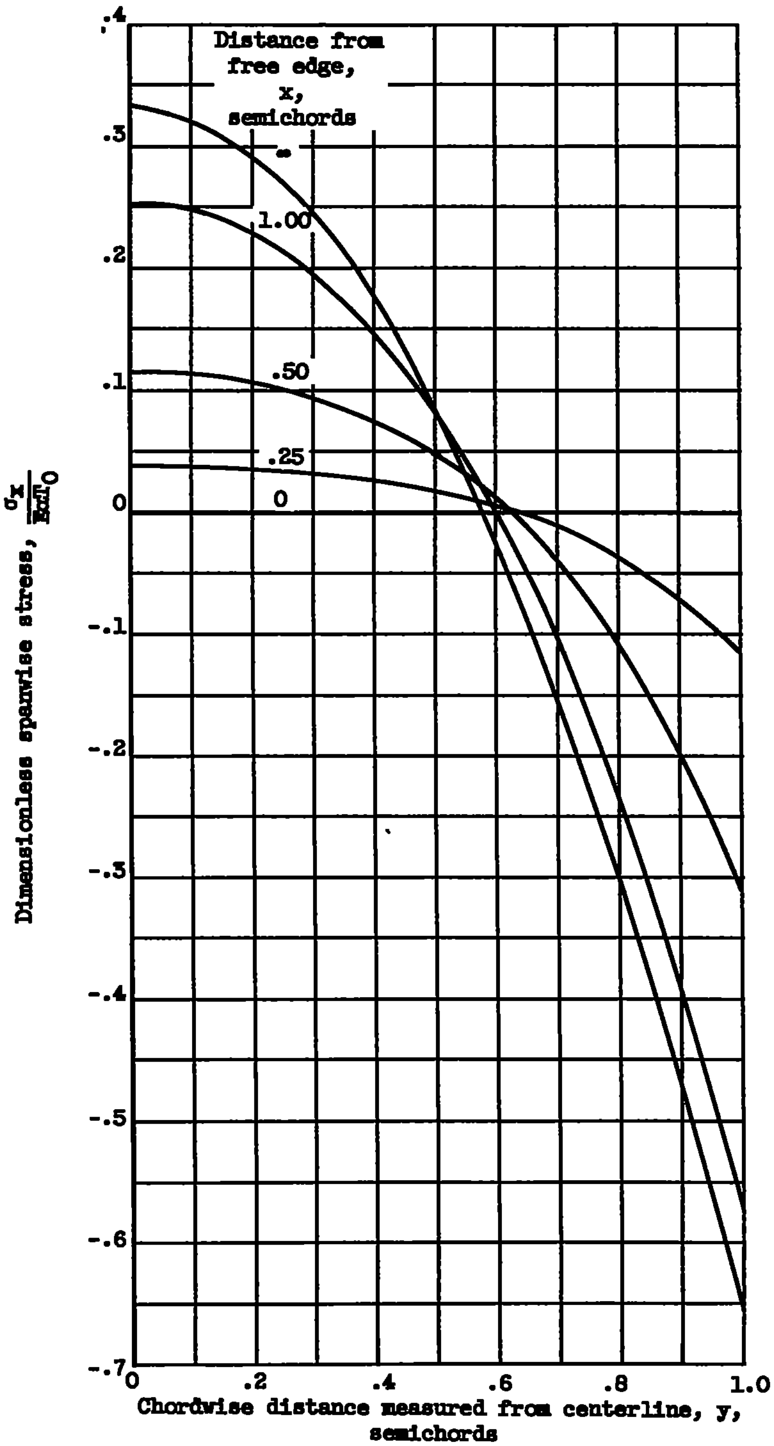

(c) Spanwise stress.

Figure 6. - Continued. Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})(1 + 0.3e^{-x})$.

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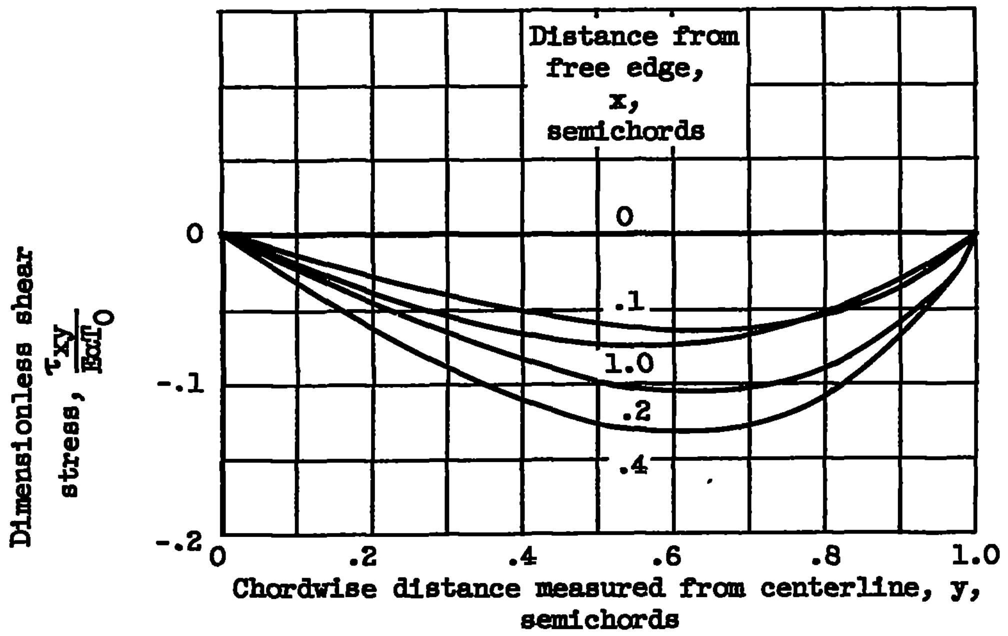

(d) Shear stress.

Figure 6. - Concluded. Stresses in plate of uniform thickness for parabolic temperature distribution. $T = T_0(y^2 - \frac{1}{3})(1 + 0.3e^{-x})$.

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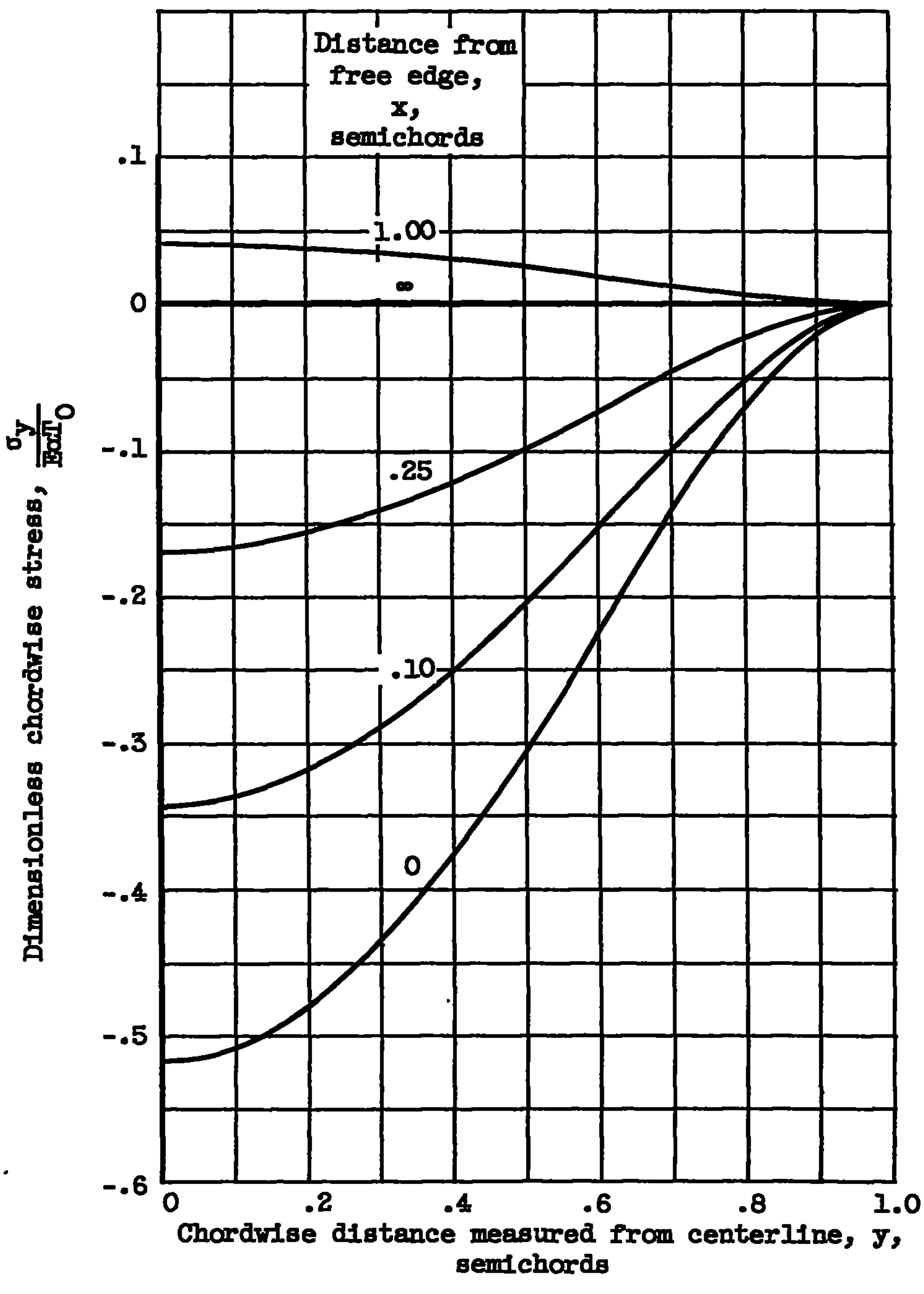
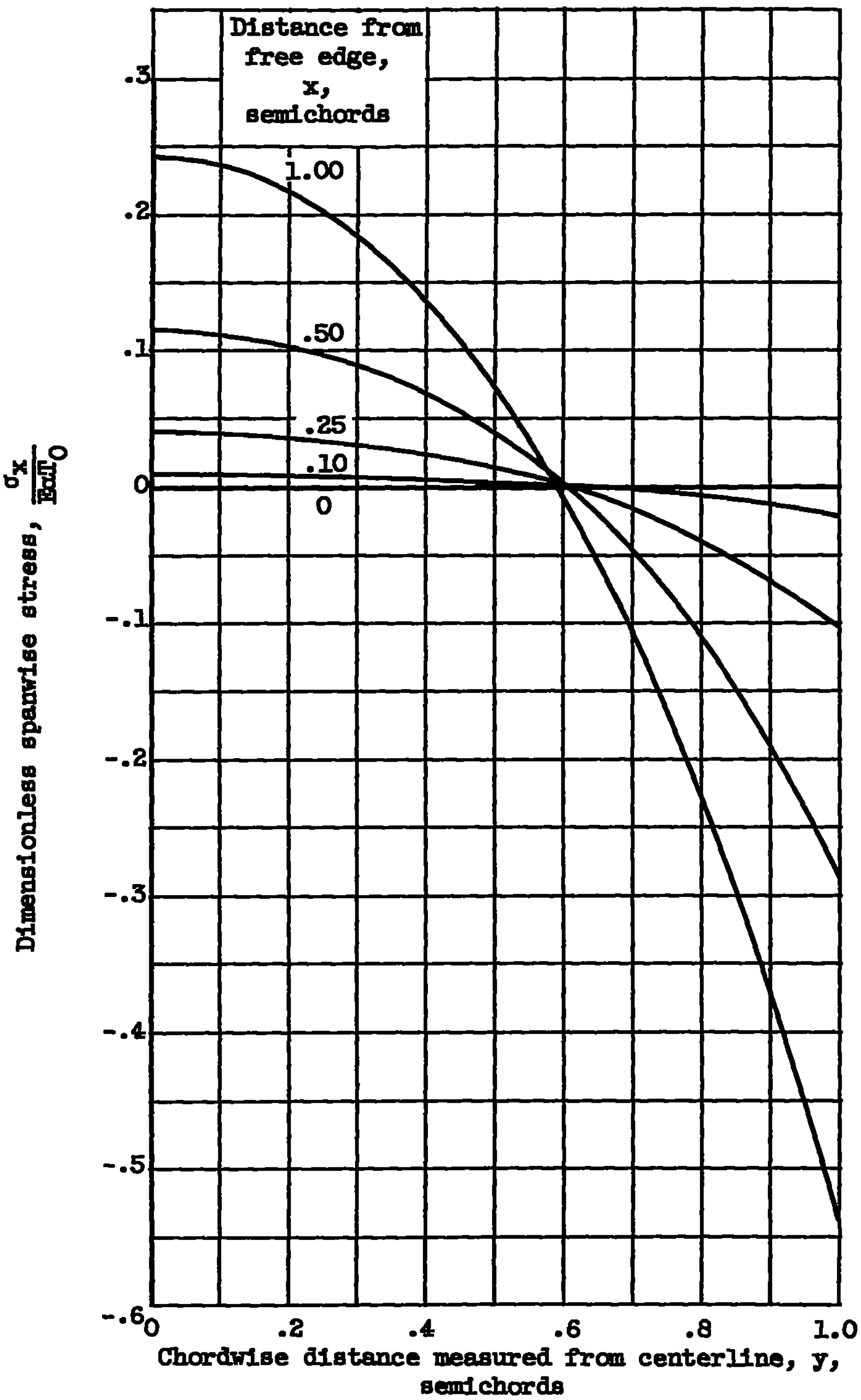


(a) Chordwise stress.

Figure 7. - Stresses in plate of variable thickness. $h = h_0 e^{0.3x}$; $T = T_0(y^2 - \frac{1}{3})$.



(b) Spanwise stress.

Figure 7. - Continued. Stresses in plate of variable thickness. $h = h_0 e^{0.3x}$; $T = T_0(y^2 - \frac{1}{3})$.

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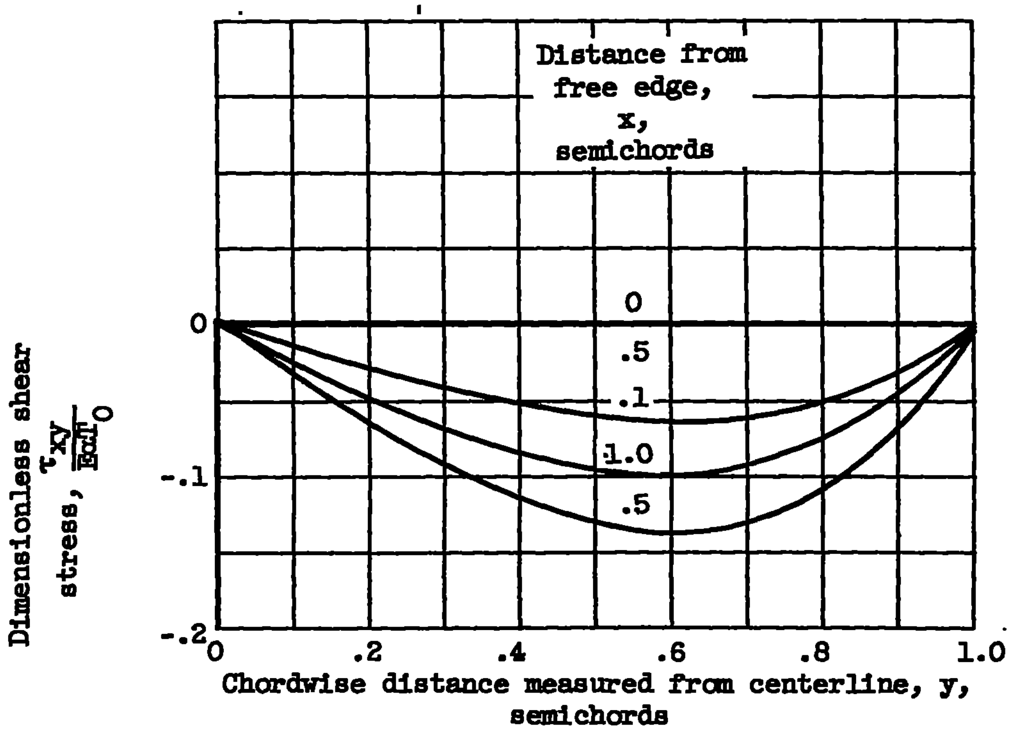

(c) Shear stress.

Figure 7. - Concluded. Stresses in plate of variable thickness. $h = h_0 e^{0.3x}$; $T = T_0(y^2 - \frac{1}{3})$.