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MINIMUM-DRAG BODIES OF REVOLUTION IN A NONUNIFORM SUPERSONIC FLOW FIELD

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SUMMARY

A general expression for the cross-sectional-area distribution of the minimum-drag body of revolution of given volume and length in a nonuniform supersonic flow field is derived on the basis of linearized theory. This result is restricted to cases where the potential of the disturbance to the uniform stream can be expanded in a Taylor's series about the body axis. The theory is applied to the determination of the minimum-drag body of revolution of given volume and length located in the flow field of a parabolic body. Several representative calculations show that the interference pressures from a main body have a negligible effect on the shape for minimum wave drag of the satellite body (minimum-drag body).

INTRODUCTION

The aerodynamic characteristics of airplanes designed for supersonic flight speeds are influenced by the interference effects between the various components of the configuration. In the analysis of interference effects based on linearized theory, it proves convenient to introduce an interference velocity potential which is defined as the difference between the velocity potential for a complete configuration and the sum of the velocity potentials of the isolated components. This potential arises from properly satisfying the boundary conditions for the flow past the complete configuration. The velocity or pressure at any point of the flow field is the sum of the velocities or pressures derived from the potentials of each of the components plus the interference potential. For example, the pressure acting on a given component, such as a wing, is considered to consist of three terms: the pressure that would act on the wing if it were in a uniform flow field, the pressures from the other components of the configuration evaluated on the wing, and the pressure derived from the interference velocity potential. The last two terms are the interference pressures acting on the wing.

The calculation of the forces acting on a configuration is complicated by the interactions between the components. Many investigations have been made of these interference effects, most of which have considered the lift of wing-body combinations. Interference effects on the wave drag of a configuration are also of importance, but to date little work has been done in this direction. Friedman and Cohen (ref. 1) have calculated the wave drag of a system of three bodies of revolution to determine the importance of the orientation of the bodies. Their study considered the interference wave drag arising from the pressures from one body acting on another. They concluded that the wave drag of such a configuration can be significantly less than the sum of the wave drags of the various components provided the bodies are located for favorable interference drag.

The bodies considered by Friedman and Cohen in their interference calculations were designed for minimum wave drag in a uniform stream. However, additional wave-drag reductions should be possible by properly shaping the bodies to take advantage of the interference pressures. The pressures from one body acting on another body can be considered as arising from a disturbance to a uniform stream; that is, a body lying in the flow field of another body can be considered to be in a nonuniform stream.

In the present paper, the wave drag of a body of revolution in a nonuniform supersonic stream is derived on the basis of slender-body theory. The area distribution of the minimum-drag body of revolution in a nonuniform supersonic stream is then obtained for the auxiliary conditions of given volume and length. This result is used to determine the area distribution of the minimum-drag body of revolution lying in the flow field of another body of revolution. Calculations for several representative cases are included.

SYMBOLS

a,b,c,d constants

$A_j(x)$,$B_j(x)$ singularity strengths

$$C_{p_b} = \frac{p_b - p_\infty}{\frac{\rho_\infty}{2} U^2}$$

D wave drag

D_{SH} wave drag of Sears-Haack body

ΔD difference between wave drag of Sears-Haack body in flow field of main body and wave drag of minimum-drag satellite body

$$G(x_1) = \int_{-1}^{x_1} \frac{H(\eta_1)}{\sqrt{1 - \eta_1^2}} d\eta_1$$

$$H(x_1) = \int_{-1}^{1} \frac{K(\xi_1)\sqrt{1 - \xi_1^2}}{x_1 - \xi_1} d\xi_1$$

$$K(x_1) = \phi_1(x_1,0,0) + \int_{-1}^{x_1} \left[\phi_{1_{y_1}}{}^2(\xi_1,0,0) + \phi_{1_{z_1}}{}^2(\xi_1,0,0)\right] d\xi_1$$

l length of satellite body

$\bar{l}$ length of main body

M Mach number

m outwardly directed normal to base contour

p local pressure

p_∞ stream pressure

$q = \frac{\rho_\infty}{2} U^2$

$R(x)$ body radius at any station x, $\frac{-l}{2} \leqq x \leqq \frac{l}{2}$

$R\left(\frac{l}{2},\theta\right)$ base radius

R_{max} maximum radius of parabolic body

r_o radius of cylinder of integration

x,r,θ cylindrical coordinates

$S(x) = \pi R^2(x)$

$$s(x_1) = \frac{S(x)}{(l/2)^2}$$

U stream velocity

$V = \dfrac{\text{Volume}}{(l/2)^3}$

x,y,z rectangular coordinates

$x_1 = \dfrac{x}{l/2}$

$y_1 = \dfrac{y}{l/2}$

$z_1 = \dfrac{z}{l/2}$

$\bar{x},\bar{y}$ coordinate system for location of satellite body with respect to main body

$\beta = \sqrt{M^2 - 1}$

ρ local density

ρ_∞ stream density

σ surface of integration

$\varphi_1(x,y,z)$ velocity potential of a small disturbance to a uniform stream

$\varphi_2(x,y,z)$ velocity potential of a distribution of singularities along body axis required to represent flow past body in a uniform stream

$\varphi_3(x,y,z)$ interference velocity potential of a second distribution of singularities along body axis which is required to cancel flow through surface boundary due to nonuniformity of flow field

$\phi_1(x_1,y_1,z_1) = \dfrac{\varphi_1(x,y,z)}{l/2}$

$\phi_2(x_1,y_1,z_1) = \dfrac{\varphi_2(x,y,z)}{l/2}$

$$\phi_3(x_1,y_1,z_1) = \frac{\varphi_3(x,y,z)}{l/2}$$

$\varphi = \varphi_1 + \varphi_2 + \varphi_3$

Φ velocity potential

$\mu = \dfrac{\bar{x}}{l} - \beta \dfrac{\bar{y}}{l}$

ξ,η,ξ_1,η_1 dummy variables of integration

A prime denotes the derivative with respect to the argument of the function.

Subscripts denote partial derivatives with respect to the indicated variable.

ANALYSIS

Drag Equation

In the analysis of the flow past a slender body in a nonuniform stream, based on linearized theory, the nonuniform field is considered to arise from a small disturbance to a uniform stream of velocity U. The potential Φ of the flow about the body can then be expressed as

$$\Phi = \frac{l}{2} U\left(x_1 + \phi_1 + \phi_2 + \phi_3\right) \tag{1}$$

where

ϕ_1 nondimensional velocity potential of a small disturbance to the uniform stream

ϕ_2 nondimensional velocity potential of a distribution of singularities along the body axis required to represent the flow past the body in the uniform stream

ϕ_3 nondimensional interference velocity potential of a second distribution of singularities along the body axis which is required to cancel the flow through the surface boundary due to the nonuniformity of the flow field

The wave drag is calculated from the momentum transfer through a surface enclosing the body. The velocities and pressures occurring in the expression for the momentum transfer are related to the disturbance velocity potentials through equation (1) and a linearized pressure relationship. In order to facilitate the integration of the resulting equations, the potential ϕ_1 is expanded in a Taylor's series about the body axis $\left(x_1\text{-axis}\right)$ as

$$\phi_1(x_1,y_1,z_1) = \phi_1(x_1,0,0) + z_1\phi_{1_{z_1}}(x_1,0,0) + y_1\phi_{1_{y_1}}(x_1,0,0) + \ldots \quad (2)$$

where a consistent approximation is obtained by retaining the first three terms. The details of this analysis are presented in the appendix, where it is shown that the wave drag of a slender, closed body of revolution can be written as

$$\frac{D}{q(l/2)^2} = -\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} s''(x_1)s''(\xi_1)\log_e\left|x_1 - \xi_1\right|d\xi_1\,dx_1 -$$

$$2\int_{-1}^{1} s'(x_1)K'(x_1)dx_1 \quad (3)$$

where

$$K'(x_1) = \phi_{1_{x_1}}(x_1,0,0) + \phi_{1_{y_1}}{}^2(x_1,0,0) + \phi_{1_{z_1}}{}^2(x_1,0,0)$$

and $s(x_1)$ is the nondimensional body cross-sectional-area distribution. The double-integral term of equation (3) is the wave drag of a slender, closed body of revolution in a uniform supersonic flow field as derived by Von Kármán (ref. 2) and the single-integral term is the interference wave drag, which depends primarily on the pressures from the disturbance to the uniform stream integrated over the body surface.

Minimum-Drag Body

The body cross-sectional-area distribution $s(x_1)$, which minimizes equation (3), will be determined for the auxiliary conditions of given volume and length of the body. This is an isoperimetric problem of the calculus of variations, and, when the method outlined in reference 3 is employed, an integral equation for $s(x_1)$ is obtained as

$$\int_{-1}^{1} \frac{s'(\xi_1)}{x_1 - \xi_1} d\xi_1 = a + bx_1 + cx_1^2 + 2\pi K(x_1) \tag{4}$$

where a, b, and c are arbitrary constants. The solution of equation (4) for $s'(x_1)$ is (ref. 4)

$$s'(x_1) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x_1^2}}\left[d + \frac{b}{2} + \left(\frac{c}{2} - a\right)x_1 - bx_1^2 - cx_1^3 - 2H(x_1)\right] \tag{5}$$

where

$$H(x_1) = \int_{-1}^{1} \frac{K(\xi_1)\sqrt{1 - \xi_1^2}}{x_1 - \xi_1} d\xi_1$$

and d is an arbitrary constant. Integration of equation (5) gives for $s(x_1)$

$$s(x_1) = \frac{1}{\pi}\left[d\left(\sin^{-1}x_1 + \frac{\pi}{2}\right) + \left(a + \frac{c}{2}\right)\sqrt{1 - x_1^2} + \frac{b}{2}x_1\sqrt{1 - x_1^2} - \right.$$

$$\left. \frac{c}{3}\left(1 - x_1^2\right)^{3/2} - 2G(x_1)\right] \tag{6}$$

where

$$G(x_1) = \int_{-1}^{x_1} \frac{H(\eta_1)}{\sqrt{1 - \eta_1^2}} d\eta_1$$

and use has been made of the condition $s(-1) = 0$.

The arbitrary constants a, b, c, and d occurring in equation (6) are determined by use of the end-point conditions

$$s'(-1) = s'(1) = s(1) = 0$$

and the auxiliary condition

$$\int_{-1}^{1} s(x_1)dx_1 = V = \text{Constant}$$

where V is the volume of the body divided by $\left(\frac{l}{2}\right)^3$. Upon the determination of these constants, the area distribution of the optimum body of given volume and length in a nonuniform supersonic stream is expressed as

$$s(x_1) = \frac{1}{\pi}\left(\frac{8}{3} V\left(1 - x_1^2\right)^{3/2} + \frac{2G(1)}{\pi}\left(\sin^{-1}x_1 + \frac{\pi}{2}\right) + \left[H(-1) - H(1)\right]\sqrt{1 - x_1^2} + \left[\frac{2}{\pi} G(1) - H(1) - H(-1)\right]x_1\sqrt{1 - x_1^2} - \left(1 - x_1^2\right)^{3/2}\left\{\frac{16}{3\pi} G(1) + \frac{4}{3}\left[H(-1) - H(1)\right] - \frac{16}{3\pi}\int_{-1}^{1} G(x_1)dx_1\right\} - 2G(x_1)\right) \tag{7}$$

where

$$G(x_1) = \int_{-1}^{x_1} \frac{H(\eta_1)}{\sqrt{1 - \eta_1^2}} d\eta_1$$

$$H(\eta_1) = \int_{-1}^{1} \frac{K(\xi_1)\sqrt{1 - \xi_1^2}}{\eta_1 - \xi_1} d\xi_1$$

For a uniform supersonic stream, $\phi_1(x_1,y_1,z_1) \equiv 0$ and hence $K(\xi_1) = 0$. Then

$$s(x_1) = \frac{8}{3\pi} V\left(1 - x_1^2\right)^{3/2}$$

which is the Sears-Haack minimum-drag body of revolution for a given volume and length (refs. 5 and 6). Equation (7) is in the form of the Sears-Haack body plus a correction term which is a function of the parameters describing the nonuniformity of the flow field considered.

APPLICATION

The area distribution for the minimum-drag body of revolution of given volume and length in an arbitrary nonuniform supersonic stream is presented in equation (7). In order to apply this equation to any given problem, an expression for $K(x_1)$ is required. In many cases, an analytical expression for $K(x_1)$ is not easily obtained, or, if it is, the integrations of this expression that must be performed to obtain the body area distribution appear to make any application of equation (7) almost prohibitive. However, it has been found that in some applications it is possible to approximate $K(x_1)$ by a polynomial of the fourth order or less, and that a polynomial expression for $K(x_1)$ can easily be integrated to obtain the required area distribution of the minimum-drag body of revolution. For the polynomial

$$K(x_1) = a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3 + a_4x_1^4$$

the area distribution of the minimum-drag body is, from equation (7),

$$s(x_1) = \frac{8V}{3\pi}\left(1 - x_1^2\right)^{3/2} + \left(1 - x_1^2\right)^{3/2}\left(\frac{a_4}{15} - \frac{a_3}{2}x_1 - \frac{2}{5}a_4x_1^2\right) \tag{8}$$

where $\frac{8V}{3\pi}\left(1 - x_1^2\right)^{3/2}$ is the area distribution of a Sears-Haack body of revolution. (In order that the area distribution $s(x_1)$ may always be positive, the volume-to-length-cubed ratio V of the body cannot become smaller than a limiting value dependent on a_3 and a_4. See eq. (8).) The wave drag of the minimum-drag body of revolution in this nonuniform stream is, from equations (3) and (8),

$$\frac{D}{q(l/2)^2} = \frac{8V^2}{\pi} + 4V\left(a_2 + a_4\right) - \frac{\pi}{16}\left(3a_3^2 + a_4^2\right) \tag{9}$$

Equation (8) for the area distribution is in the form of the area distribution of the Sears-Haack body of revolution plus a correction term independent of the volume of the body. This correction term is also independent of the constant, linear, and quadratic terms in $K(x_1)$. These terms do not affect the minimum problem since the constant and linear terms do not contribute to the wave drag and the quadratic term merely adds a constant amount of wave drag for a fixed volume, which is independent of the body cross-sectional-area distribution. (See eq. (3).) For cases where $K(x_1) \approx \phi_1(x_1,0,0)$, that is, when ${\phi_{1_{z_1}}}^2(x_1,0,0)$ and ${\phi_{1_{y_1}}}^2(x_1,0,0)$ are small compared with $\phi_{1_{x_1}}(x_1,0,0)$, which will be discussed subsequently, $K'(x_1)$ is -1/2 times the interference pressure caused by the nonuniform stream. Then the previous discussion concerning the terms in a polynomial $K(x_1)$, which affect the shape for minimum drag, can be restated in terms of the interference pressure coefficient. The shape for minimum drag is independent of the level of the pressure and the pressure gradient; only the curvature and higher order terms in the interference pressure distribution influence the minimum shape.

Equation (9) for the drag of the minimum-drag body is in the form of the drag of the Sears-Haack body in this nonuniform stream (first two terms of eq. (9)) plus a correction term which is independent of the volume of the body. This correction term is the difference between the wave drag of the Sears-Haack body in the nonuniform stream and the wave drag of the minimum-drag body and hence is the wave-drag reduction due to shaping the body for favorable interference drag. This drag reduction is constant for a given nonuniform stream, independent of the volume-to-length-cubed ratio of the body. This implies that the percent drag reduction will increase with decreasing volume for bodies of equal length.

Nonuniform Flow Field Produced by a Body of Revolution

One case where an expression for $K(x_1)$ can be obtained is when the nonuniform flow field is produced by a body in a uniform flow field; that is, the theory developed in the preceding sections can be used to determine the shape of the minimum-drag satellite body located in the flow field of a main body. In this case $\phi_1(x_1,0,0)$, which must be known to determine $K(x_1)$, is the disturbance potential of the main body evaluated along the satellite-body axis, provided disturbances originating at the satellite body and subsequently reflected from the main body do not influence the satellite body. This condition imposes a limiting minimum distance between the two bodies if any part of the satellite body lies between the nose Mach cone and the forecone from the tail of the main body. This limiting distance depends on the length of the satellite body and the Mach number. If the satellite body is closer to the main body than this limiting distance, $K(x_1)$ is difficult to obtain because there will be a contribution from the area distribution of the satellite body which is not known at the outset of the problem. Some type of iteration procedure would probably have to be employed in such cases.

If the nonuniform flow field under consideration arises from the disturbance field of a smooth body of revolution, some simplification in the form of $K(x_1)$ is possible. The derivative of $K(x_1)$ with respect to x_1, that is, $K'(x_1)$, is composed of three terms: $\phi_{1_{x_1}}(x_1,0,0)$, ${\phi_{1_{y_1}}}^2(x_1,0,0)$, and ${\phi_{1_{z_1}}}^2(x_1,0,0)$. In general if $\phi_1(x_1,0,0)$ arises from a smooth body of revolution, ${\phi_{1_{y_1}}}^2(x_1,0,0)$ and ${\phi_{1_{z_1}}}^2(x_1,0,0)$ will be small compared with $\phi_{1_{x_1}}(x_1,0,0)$ if the radial distance from the body is sufficiently large. For example, calculations performed with the linearized disturbance potential from a parabolic body of revolution of fineness ratio 10 showed that $\phi_{1_{x_1}}(x_1,0,0)$ was of the order of 10 times larger than ${\phi_{1_{y_1}}}^2(x_1,0,0)$ and ${\phi_{1_{z_1}}}^2(x_1,0,0)$ at stations that were 2.5 maximum body radii from the body. Consequently, only a small error is incurred in such cases by neglecting the contributions of ${\phi_{1_{y_1}}}^2(x_1,0,0)$ and ${\phi_{1_{z_1}}}^2(x_1,0,0)$ to $K'(x_1)$. Then $K'(x_1) \approx \phi_{1_{x_1}}(x_1,0,0)$ or $K(x_1) \approx \phi_1(x_1,0,0)$; that is, $K(x_1)$ is the linearized disturbance potential of the main body evaluated along the satellite-body axis, and a polynomial approximation for $K(x_1)$ is easily obtained.

The drag of the minimum-drag satellite body has been calculated for two ratios of volume to $(l/2)^3$, $V = 0.037010$ and $V = 0.009253$, which correspond approximately to satellite-body fineness ratios of 10 and 20, respectively. A sketch of the configuration is shown in figure 1. The location of the satellite body is designated by the coordinates $\mu,\bar{y}$, where $\mu = \frac{\bar{x}}{\bar{l}} - \beta\frac{\bar{y}}{\bar{l}}$, $\bar{x}$ and $\bar{y}$ are the coordinates of the nose of the satellite body, $\bar{x} = \bar{y} = 0$ are the coordinates of the nose of the main body, and $\bar{l}$ is the length of the main body. The lines μ = Constant are Mach lines from the main body in the plane passing through the axes of both bodies. In both cases the main body is a parabolic body of revolution of fineness ratio 10 and the satellite body is one-fourth the length of the main body. The Mach number is $\sqrt{2}$ and $\bar{y} = 2.5R_{max}$, where R_{max} is the maximum radius of the main body. For the cases considered, this distance between the bodies corresponds to the limiting minimum distance. A polynomial expression for $K(x_1)$ was obtained for each location of the satellite body by fitting a polynomial to the linearized potential of the parabolic body evaluated along the satellite-body axis and the drag was calculated from equation (9).

The results of these calculations are presented in figure 2 as the difference between the wave drag of a Sears-Haack body in the flow field of the main body and the wave drag of the minimum-drag satellite body ΔD divided by the wave drag of the Sears-Haack body D_{SH}. This quantity $\Delta D/D_{SH}$ is plotted against μ, the location of the satellite body, and represents the fractional drag reduction obtained by shaping the satellite body for favorable interference drag. For both cases of figure 2, the minimum-drag satellite body has less drag than the Sears-Haack body in the same stream, as would be expected, but the percent drag reduction is less than 2 percent for most locations of the satellite body. Consequently, since these are fairly representative cases, little or no advantage can be expected from shaping satellite bodies for favorable interference drag, when the interference pressures arise from a smooth body of revolution sufficiently far removed that the satellite body is not influenced by reflected disturbances. The important parameter appears to be the location of the satellite body as shown by Friedman and Cohen in reference 1.

In order to illustrate the type of area distribution that gives minimum wave drag in the nonuniform flow field arising from a parabolic main body, the minimum-drag area distribution for a satellite body of fineness ratio 10 ($V = 0.037010$) located at $\mu = 0$ is presented in figure 3. The area distribution of the Sears-Haack body of the same volume and length is also included for comparison purposes. As would be expected, there is little difference between the area distributions of the two bodies.

Nonuniform Field Produced by Wings

The theory that has been presented is also applicable when the nonuniform flow field is produced by a wing, provided the body is off the wing. This restriction arises from the derivation of equation (1) where it was assumed that the potential of the disturbance to the uniform stream can be expanded in a Taylor's series about the body axis. However, for a wing, $\phi_{1_{z_1}}$ is discontinuous in the plane of the wing and a Taylor's series expansion in that region is not possible.

The calculations made for the two-body problem indicated that, for a nonuniform flow field produced by a smooth body, the correction to the Sears-Haack shape was determined primarily by the curvature of the pressure distribution in the nonuniform stream. In general, the curvature of the pressure distribution arising from a smooth wing will not be greatly different from that for a smooth body. Consequently, only small drag reductions are expected for satellite bodies located in the flow field of a wing.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
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APPENDIX

WAVE-DRAG EQUATION FOR A SLENDER BODY IN A NONUNIFORM STREAM

The wave drag of a slender body can be determined by the momentum transfer through a surface enclosing the body as

$$D = \int_\sigma \bar{n} \cdot \bar{i} p \, d\sigma + \int_\sigma \rho (\bar{n} \cdot \bar{q})(\bar{i} \cdot \bar{q}) \, d\sigma \tag{A1}$$

where the continuity equation is

$$\int_\sigma \rho \bar{q} \cdot \bar{n} \, d\sigma = 0 \tag{A2}$$

and where

σ	enclosing surface
$\bar{n}$	unit vector, directed inward, normal to σ
$\bar{i}$	unit vector in the x-direction
$\bar{q}$	velocity vector
p	local pressure
ρ	local density
x,r,θ	cylindrical coordinates

(See fig. 4.) It proves convenient to take the enclosing surface as two planes normal to the x-axis at $x = -\frac{l}{2}$ and $x = \frac{l}{2}$ and a cylinder $r = r_o$ = Constant. When a disturbance velocity potential φ defined by

$$\Phi = U(x + \varphi) \tag{A3}$$

is introduced into equations (A1) and (A2), the wave drag may be expressed as

$$D = \int_0^{r_o}\int_0^{2\pi}\left(p + \rho_\infty U^2\varphi_x\right)_{x=-\frac{l}{2}} r\ dr\ d\theta - \int_0^{2\pi}\int_{R\left(\frac{l}{2},\theta\right)}^{r_o}\left(p + \rho_\infty U^2\varphi_x\right)_{x=\frac{l}{2}} r\ dr\ d\theta -$$

$$\int_{-l/2}^{l/2}\int_0^{2\pi} \rho_\infty U^2\left(\varphi_r\varphi_x\right)_{r=r_o} r_o\ d\theta\ dx - p_b S_b \tag{A4}$$

where

p_b base pressure

S_b base area

$R\left(\frac{l}{2},\theta\right)$ base radius

and $\varphi_x{}^2$ has been neglected inasmuch as it is small in comparison with φ_x.

In order to relate the wave drag to the body geometry and to the disturbance to the uniform stream, the form of φ must be known. If the nonuniform field is considered to arise from a small disturbance to a uniform stream, φ can be expressed as

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 \tag{A5}$$

where

φ_1 velocity potential of a small disturbance to the uniform stream

φ_2 velocity potential of a distribution of singularities along the body axis required to represent the flow past the body in the uniform stream

φ_3 interference velocity potential of a second distribution of singularities along the body axis which is required to cancel the flow through the surface boundary due to the nonuniformity of the flow field

In order to facilitate the integration of the momentum equation, the potential φ_1 is expanded in a Taylor's series about the body axis (x-axis) as

$$\varphi_1(x,y,z) = \varphi_1(x,0,0) + z\varphi_{1_z}(x,0,0) + y\varphi_{1_y}(x,0,0) + \ . \ . \ . \qquad \text{(A6)}$$

The slender-body approximation for the potential $\varphi_2 + \varphi_3$ of the singularities distributed along the body axis is

$$\varphi_2 + \varphi_3 \approx \frac{A_0(x)}{2\pi} \log_e \frac{\beta r}{2} - \frac{1}{2\pi} \int_{-l/2}^{x} A_0'(\xi) \log_e |x - \xi| d\xi +$$

$$\sum_{j=1}^{\infty} \frac{1}{r^j} \left[A_j(x) \cos j\theta + B_j(x) \sin j\theta \right] \qquad \text{(A7)}$$

where $A_j(x)$ and $B_j(x)$ are the singularity strengths to be evaluated from the boundary conditions on the body. (See ref. 7.)

The local pressure p is related to the stream pressure p_∞ and the disturbance potential φ by

$$p = p_\infty - \frac{\rho_\infty}{2} U^2 \left[2\varphi_x + \varphi_r^{\,2} + \left(\frac{\varphi_\theta}{r} \right)^2 \right] \qquad \text{(A8)}$$

where $\varphi_x^{\,2}$ has been neglected inasmuch as it is small in comparison with φ_x, $\varphi_r^{\,2}$, and $\left(\frac{\varphi_\theta}{r}\right)^2$. In many cases the contribution to $\varphi_r^{\,2}$ and $\left(\frac{\varphi_\theta}{r}\right)^2$ from the nonuniformity of the flow field $\left(\varphi_{1_r}^{\,2} \text{ and } \left(\frac{\varphi_{1_\theta}}{r}\right)^2\right)$ will be small compared with φ_{1_x}. However, these terms are retained in order to treat cases where they are important. The first three terms of equation (A6) are then retained as consistent with this approximation.

Combining equations (A4) and (A8) yields

$$\frac{D}{q} = \int_0^{2\pi} \int_{R\left(\frac{l}{2},\theta\right)}^{r_0} \left[\varphi_r^{\,2} + \left(\frac{\varphi_\theta}{r} \right)^2 \right]_{x=\frac{l}{2}} r \, dr \, d\theta - \int_0^{r_0} \int_0^{2\pi} \left[\varphi_r^{\,2} + \left(\frac{\varphi_\theta}{r} \right)^2 \right]_{x=-\frac{l}{2}} r \, dr \, d\theta -$$

$$2 \int_{-l/2}^{l/2} \int_0^{2\pi} \left(\varphi_r \varphi_x \right)_{r=r_0} r_0 \, d\theta \, dx - C_{p_b} S_b \qquad \text{(A9)}$$

where

$$C_{p_b} = \frac{p_b - p_\infty}{\frac{\rho_\infty}{2} U^2}$$

and

$$q = \frac{\rho_\infty}{2} U^2$$

The form of the potential $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ is such that it satisfies Laplace's equation in the cross-flow plane. Then

$${\varphi_r}^2 + \left(\frac{\varphi_\theta}{r}\right)^2 = \nabla\varphi\cdot\nabla\varphi$$

$$\nabla^2\varphi = 0$$

and equation (A9) becomes

$$\frac{D}{q} = \int_0^{2\pi} \left(\varphi\varphi_r\right)_{\substack{x=\frac{l}{2}\\ r=r_o}} r_o\, d\theta - \int_0^{2\pi} \left(\varphi\varphi_r\right)_{\substack{x=-\frac{l}{2}\\ r=r_o}} r_o\, d\theta - \oint_{base} \left(\varphi \frac{d\varphi}{dm}\right)_{x=\frac{l}{2}} dR\left(\frac{l}{2},\theta\right) -$$

$$2\int_{-l/2}^{l/2} \int_0^{2\pi} \left(\varphi_r\varphi_x\right)_{r=r_o} r_o\, d\theta\, dx - C_{p_b} S_b \tag{A10}$$

where $\oint_{base} \left(\varphi \frac{d\varphi}{dm}\right)_{x=\frac{l}{2}} dR\left(\frac{l}{2},\theta\right)$ is a contour integral about the base of the body and m is the outwardly directed normal to the base contour. When equations (A5), (A6), (A7), and (A10) are combined and the integrations over θ are performed, the following equation is obtained for the drag of a slender body in a nonuniform stream:

$$\frac{D}{q} = A_0\left(\frac{l}{2}\right)\left[\varphi_1\left(\frac{l}{2},0,0\right) - \frac{1}{2\pi}\int_{-l/2}^{l/2} A_0'(\xi)\log_e\left|\frac{l}{2} - \xi\right|d\xi\right] - 2\int_{-l/2}^{l/2} A_0(x)\left[\varphi_{1_x}(x,0,0) - \frac{1}{2\pi}\frac{d}{dx}\int_{-l/2}^{x} A_0'(\xi)\log_e|x - \xi|\,d\xi\right]dx -$$

$$2\pi\int_{-l/2}^{l/2}\left[\varphi_{1_{z_x}}(x,0,0)A_1(x) - \varphi_{1_z}(x,0,0)A_1'(x) - \varphi_{1_{y_x}}(x,0,0)B_1(x) + \varphi_{1_y}(x,0,0)B_1'(x)\right]dx - \oint_{\text{base}}\left(\varphi\,\frac{d\varphi}{dm}\right)_{x=\frac{l}{2}} dR\left(\frac{l}{2},\theta\right) - C_{p_b}S_b \qquad \text{(A11)}$$

where

$$A_j\left(-\frac{l}{2}\right) = B_j\left(-\frac{l}{2}\right) = 0$$

that is, all the singularity strengths are zero at the nose of the body.

For a body of revolution the contour integral of equation (A11) can be partially evaluated. Then

$$\oint_{\text{base}}\left(\varphi\,\frac{d\varphi}{dm}\right)_{x=\frac{l}{2}} dR\left(\frac{l}{2},\theta\right) = A_0\left(\frac{l}{2}\right)\left[\frac{A_0(l/2)}{2\pi}\log_e\frac{\beta R_b}{2} - \frac{1}{2\pi}\int_{-l/2}^{l/2} A_0'(\xi)\log_e\left|\frac{l}{2} - \xi\right|d\xi + \varphi_1\left(\frac{l}{2},0,0\right)\right] +$$

$$\pi R_b^{\,2}\left[\varphi_{1_z}^{\;2}\left(\frac{l}{2},0,0\right) + \varphi_{1_y}^{\;2}\left(\frac{l}{2},0,0\right)\right] - \pi\sum_{j=1}^{\infty}\frac{j}{R_b^{\,2j}}\left[A_j^{\,2}\left(\frac{l}{2}\right) + B_j^{\,2}\left(\frac{l}{2}\right)\right] \qquad \text{(A12)}$$

where R_b is the base radius for body of revolution.

In order to calculate the drag of a prescribed body, the strength of the singularities must be related to the geometry of the body. For a body of revolution, the following relation is obtained from the boundary condition:

$$\Phi_r = R_x(x)\,\Phi_x \qquad \text{(A13)}$$

where $R(x)$ is the radius of the body at any station x. From equations (A3), (A5), and (A13)

$$U\left(\varphi_{1_r} + \varphi_{2_r} + \varphi_{3_r}\right) = R_x(x)U\left(1 + \varphi_{1_x} + \varphi_{2_x} + \varphi_{3_x}\right)$$

or approximately

$$\varphi_{1_r} + \varphi_{2_r} + \varphi_{3_r} \approx R_x(x) \tag{A14}$$

and combining equations (A6), (A7), and (A14) gives

$$\frac{A_0(x)}{2\pi R(x)} - \sum_{j=1}^{\infty} \frac{j}{R(x)^{j+1}}\left[A_j(x)\cos j\theta + B_j(x)\sin j\theta\right] - \varphi_{1_z}(x,0,0)\cos\theta +$$

$$\varphi_{1_y}(x,0,0)\sin\theta = R_x(x) \tag{A15}$$

From equation (A15) the singularity strengths are related to the body geometry by setting

$$\left.\begin{array}{ll} A_0(x) = 2\pi R(x)R_x(x) = S'(x) & \dfrac{A_1(x)}{R^2(x)} = -\varphi_{1_z}(x,0,0) \\ \\ \dfrac{B_1(x)}{R^2(x)} = \varphi_{1_y}(x,0,0) & A_j(x) = B_j(x) = 0 \quad j > 1 \end{array}\right\} \tag{A16}$$

Combining equations (A11), (A12), and (A14) gives the expression

$$\frac{D}{q} = \frac{S'(l/2)}{\pi}\int_{-l/2}^{l/2} S''(x)\log_e\left|\frac{l}{2} - x\right| dx - \frac{1}{2\pi}\int_{-l/2}^{l/2}\int_{-l/2}^{l/2} S''(x)S''(\xi)\log_e\left|x - \xi\right| d\xi\, dx -$$

$$\frac{\left[S'(l/2)\right]^2}{2\pi}\log_e\frac{\beta R_b}{2} - C_{p_b}S_b - 2\int_{-l/2}^{l/2} S'(x)\left[\varphi_{1_x}(x,0,0) + \varphi_{1_y}{}^2(x,0,0) + \varphi_{1_z}{}^2(x,0,0)\right]dx \tag{A17}$$

for the wave drag of a body of revolution in a nonuniform stream. For a closed body of revolution $S'\left(\frac{l}{2}\right) = S\left(\frac{l}{2}\right) = 0$ and equation (A17) becomes

$$\frac{D}{q} = -\frac{1}{2\pi}\int_{-l/2}^{l/2}\int_{-l/2}^{l/2} S''(x)S''(\xi)\ \log_e\left|x - \xi\right|\ d\xi\ dx\ -$$

$$2\int_{-l/2}^{l/2} S'(x)\left[\varphi_{1_x}(x,0,0) + \varphi_{1_y}{}^2(x,0,0) + \varphi_{1_z}{}^2(x,0,0)\right] dx \tag{A18}$$

If $l/2$ is used as the unit of length, equation (A18) can be expressed in nondimensional form as

$$\frac{D}{q(l/2)^2} = -\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} s''\left(x_1\right)s''\left(\xi_1\right)\ \log_e\left|x_1 - \xi_1\right|\ d\xi_1\ dx_1\ -$$

$$2\int_{-1}^{1} s'\left(x_1\right)K'\left(x_1\right)\ dx_1$$

where

$$K'\left(x_1\right) = \phi_{1_{x_1}}\left(x_1,0,0\right) + \phi_{1_{y_1}}{}^2\left(x_1,0,0\right) + \phi_{1_{z_1}}{}^2\left(x_1,0,0\right)$$

and

$$x_1 = \frac{2x}{l}$$

$$\xi_1 = \frac{2\xi}{l}$$

$$s\left(x_1\right) = \frac{S(x)}{(l/2)^2}$$

and

$$\phi_1\left(x_1,0,0\right) = \frac{2\varphi_1(x,0,0)}{l}$$

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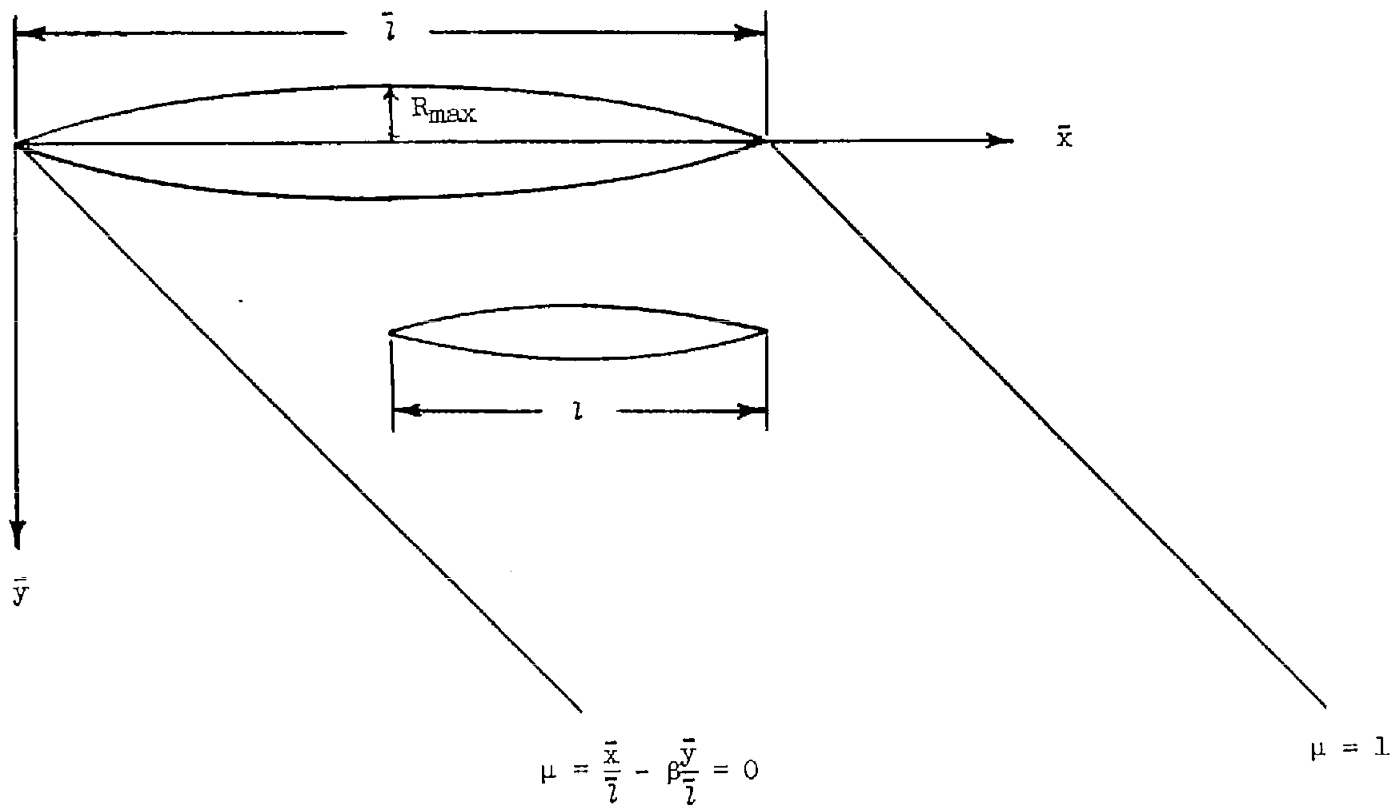

Figure 1.- Geometry of the two-body configuration.

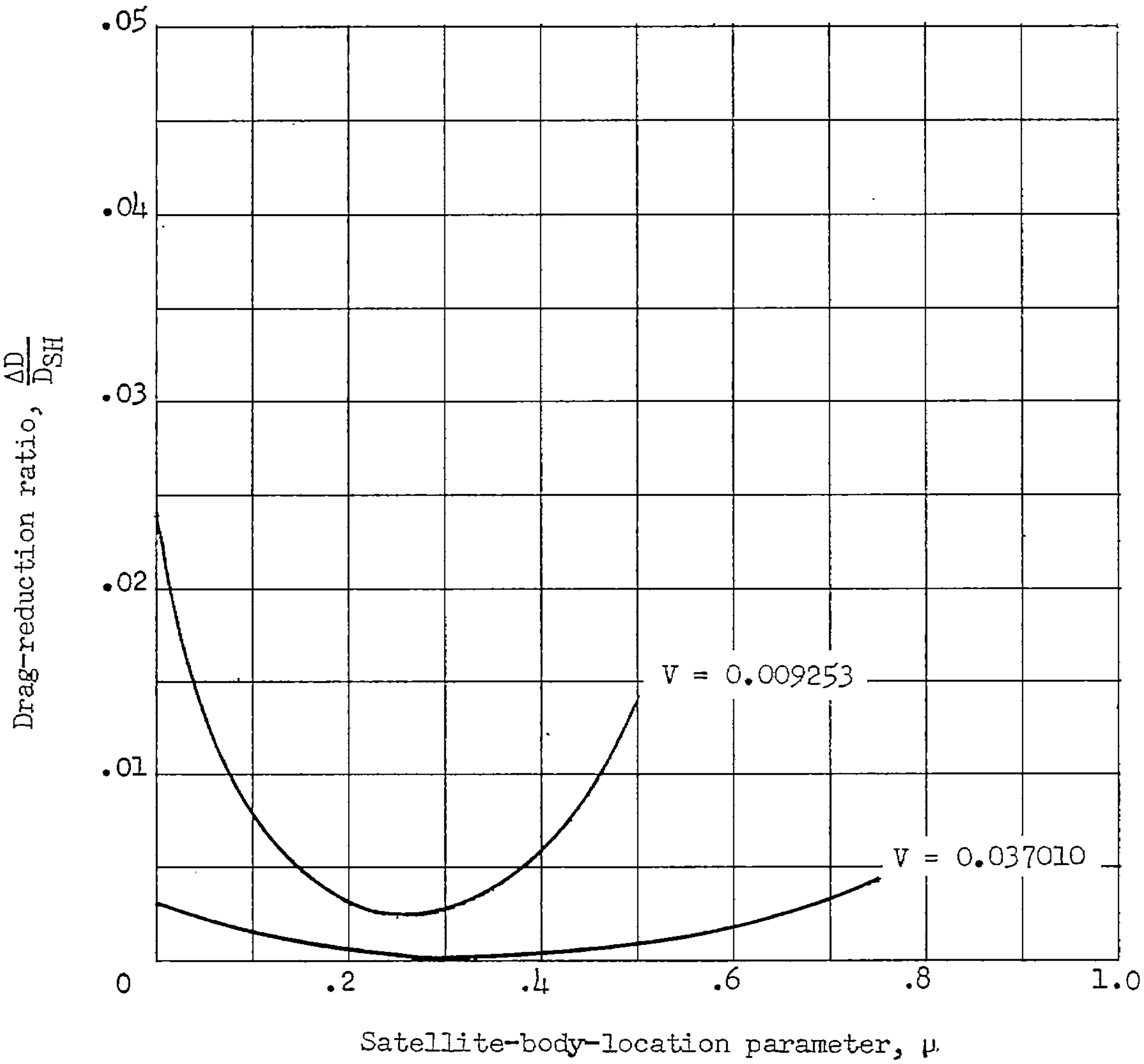

Figure 2.- Drag-reduction ratio for minimum-drag body of given volume and length as a function of satellite-body-location parameter. $M = \sqrt{2}$; $l = \frac{1}{4}\bar{l}$; $\bar{y} = 2.5R_{max}$.

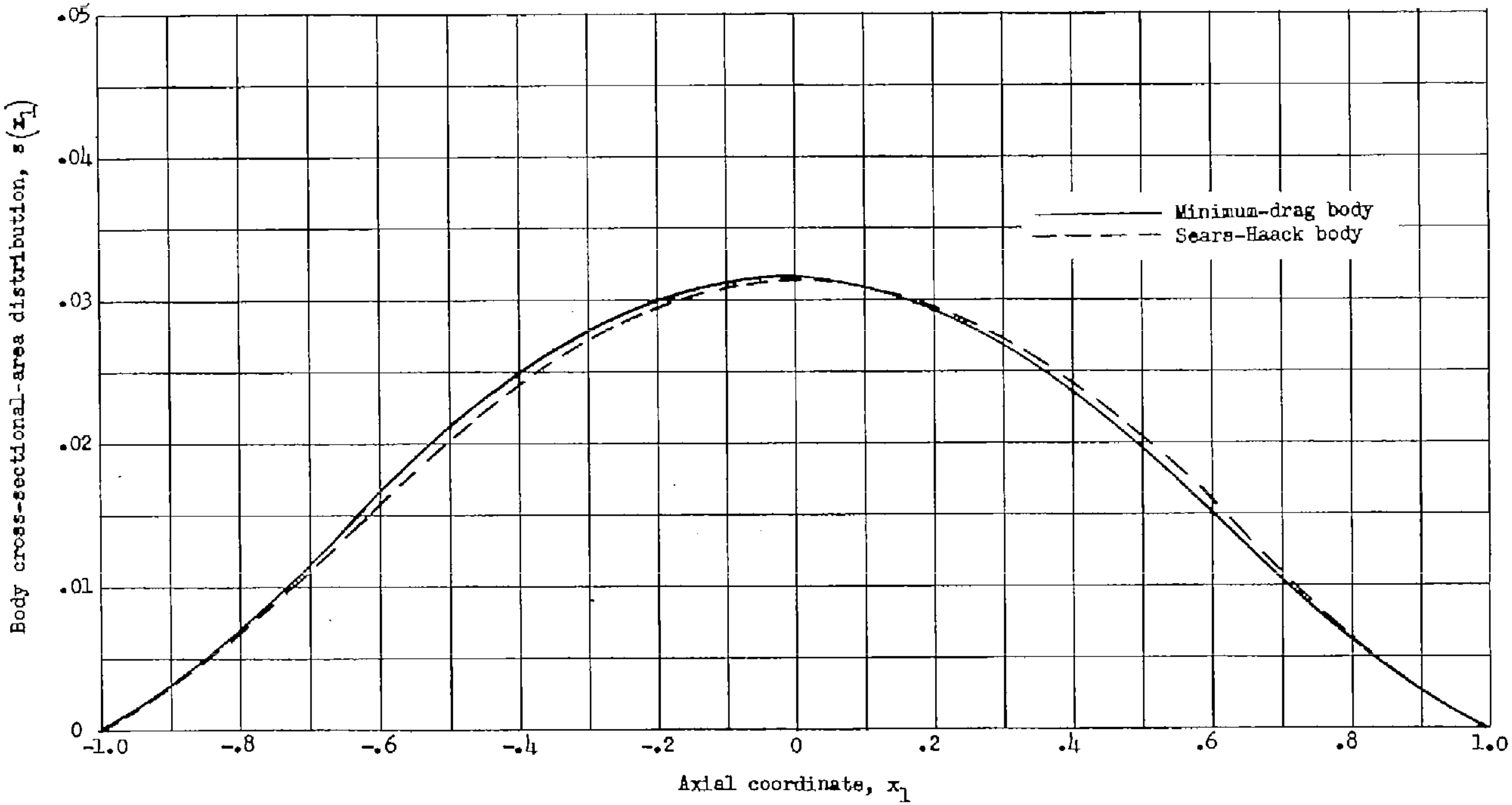

Figure 3.- Comparison between cross-sectional-area distributions of minimum body and Sears-Haack body. $M = \sqrt{2}$; $l = \frac{1}{4}\bar{l}$; $V = 0.037010$; $\mu = 0$; $\bar{y} = 2.5R_{max}$.

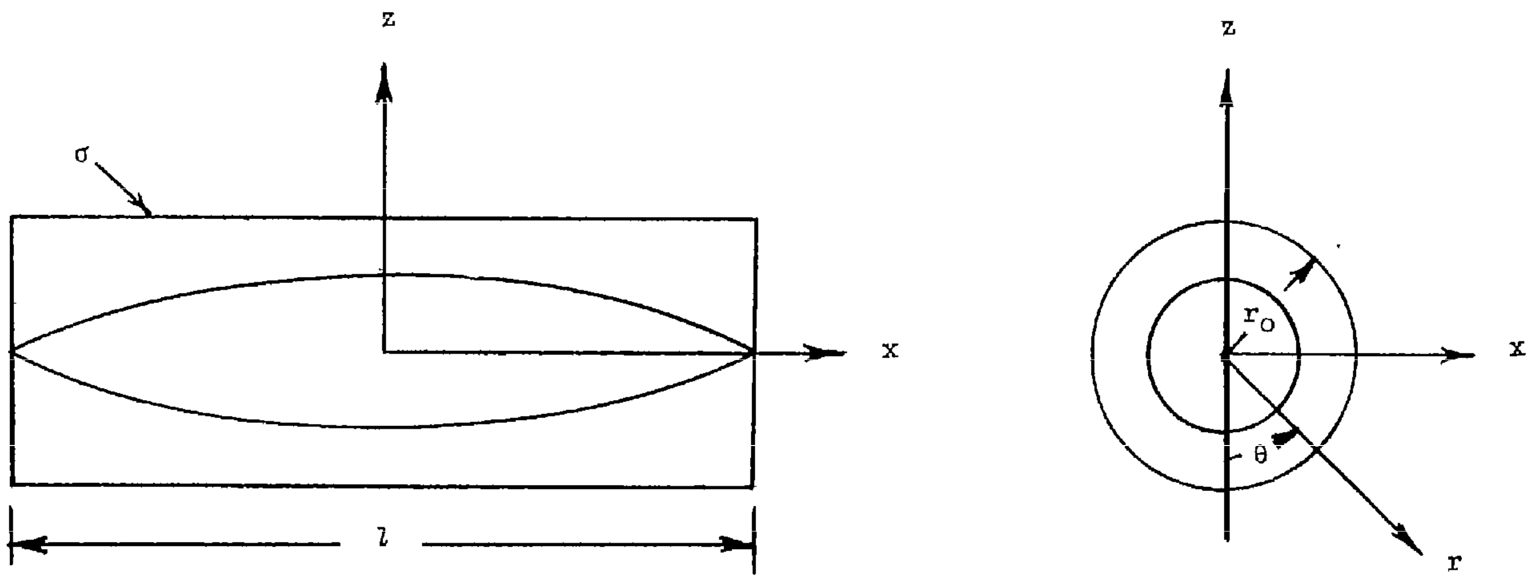

Figure 4.- Coordinate systems and integration surface for drag calculation.

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