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THEORETICAL AND EXPERIMENTAL INVESTIGATION OF RANDOM GUST LOADS. PART II - THEORETICAL FORMULATION OF ATMOSPHERIC GUST RESPONSE PROBLEM

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SUMMARY

Equations of motion are derived for the dynamic response of an aircraft to random atmospheric gust loads. These equations include the degrees of freedom of plunging, pitching, rolling, and an arbitrary number of elastic normal modes. Solutions of these equations are expressed in terms of a number of so-called primitive solutions obtainable by introducing the Dirac delta function. The solutions for center-of-gravity acceleration response and wing-root bending-moment response depend upon certain autocorrelations and cross correlations which enter the analysis. Results for simplified cases show that unsteady aerodynamic theory is not important for increasingly large values of the turbulence scale compared with values of the wing chord. However, the pitching degree of freedom exhibits an important effect as the turbulence scale increases. The results are also compared with the results of the usual sharp-edged-gust formula.

INTRODUCTION

The subject of aircraft gust loads has been studied by theoreticians and experimentalists for many years. It is one of the principal considerations in the design of both military and commercial aircraft; that is, airworthiness requirements for all types of aircraft include gust-load criteria. These criteria in all cases are based on the sharp-edged-gust formula developed by Rhode in 1931 (ref. 1) and in most cases take into account the effects of gust alleviation due to rigid-body motion.

The engineer's understanding of the gust-load problem has steadily increased over the years through basic research investigation and operational experience. Foremost among these investigations is the work carried out by the National Advisory Committee for Aeronautics. A great body of experimental data is now available to engineers and aircraft designers through these research efforts. Important statistical information relating to peak gust loads experienced by aircraft of various types subject to atmospheric gusts in various types of atmospheric and topographical environment has been collected by VGH recorders (ref. 2). Such information may be used to determine the probability density distribution of peak gust loads as a function of airspeed, altitude, and several important aircraft parameters.

More recently, important information has become available on the gust structure in the atmosphere through the application of generalized harmonic analysis (refs. 3 and 4). This type of analysis has been used to predict gust loads. In addition to these analyses, which are in the nature of applied research, more basic theoretical investigations which bear directly on the gust-loads problem may be cited. For example, recent papers by Liepmann (refs. 5 and 6) and Fung (ref. 7) are among the important contributions to the present state of knowledge.

A fundamental aspect of all of these investigations is the necessity for giving the gust structure a mathematical form convenient for inclusion in a rational analysis of gust loads. Unfortunately, this is still a weak area which can be strengthened only by continued experimental and theoretical investigation. In this connection effective use of turbulence theories may prove to be very fruitful. Certainly, the problem of atmospheric turbulence bears too close a resemblance to many of the turbulence problems in aeronautics already studied to be ignored. It is well known, of course, that many of these latter "classical" problems are far from being solved. Indeed, the subject of turbulence in general has stimulated some of the keenest intellects in historical times, yet many areas of the turbulence problem remain unconquered.

In the present report the concepts derived by Liepmann and Fung are extended to include the complications introduced by a multi-degree-of-freedom system, the airplane. In some respects the analysis parallels the work of Diederich (ref. 8). Related experimental investigations are reported in references 9 and 10.

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SYMBOLS

$A_r(x,y)$	mode shape of rth normal mode, positive up
b	wing semispan
b_c	wing semichord
$C(k)$	Theodorsen function

C_L	lift coefficient
$c_{l_\alpha}(y)$	section lift-curve slope of wing at station y
$c(y)$	chord of wing at station y
E_e	external work of applied forces
E_i	internal strain energy
$\left[e\right]$	generalized stiffness matrix
$\left[F_{jj}\right]$	forcing-function matrix
$f_{jj}(t)$	diagonal element of $\left[F_{jj}\right]$
$h_s(t)$	system function
J	pitch radius of gyration referred to wing semichord
k	reduced frequency, $\omega b_c/U$
L	integral scale of turbulence
$L(x,y,t)$	lift on the wing at point (x,y)
l	distance traveled in arbitrary time interval, Ut
l'	tail length measured from airplane center of gravity
l_o	distance between wing midchord and tail midchord in plane of symmetry
M	total number of modes
M_A	mass of airplane
M_B	bending moment
M_r	generalized mass, $\int_{S'} dm\ A_r(x,y)$
$M_\mu^*(\omega)$	transfer function

$[m]$	mass matrix
N	number of generalized coordinates
P	velocity potential
P_{tt}	$\partial^2 P/\partial t^2$
p	pressure
$\Delta p(x,y,t)$	pressure difference at point (x,y), positive when it produces lift
q	generalized coordinate
$\bar{q}$	dynamic pressure
R	ratio of horizontal tail area to wing area
r,s	normal modes (subscript or superscript)
S	total wing area
S'	projection of airplane plan form in xy-plane
T	kinetic energy of airplane
t	time
U	mean forward speed of airplane

$$V = \dot{z} + x\dot{\theta} + y\dot{\phi} + \sum_r A_r(x,y)\dot{\xi}_r$$

v	crossflow
W	stochastic function with zero mean value
w	gust velocity
w'	root-mean-square intensity of turbulence
x,y	coordinate system, x positive forward, y positive along right wing

x_o,y	coordinates of midchord
$\bar{x},y$	coordinates of three-quarter chord
$\tilde{x},y$	coordinates of quarter chord
z	vertical displacement of center of gravity of rigid airplane relative to average position, positive up
α	angle of attack
α_i	equivalent gust angle of attack
$[\alpha],[\mathring{\alpha}]$	aerodynamic matrices
Γ	sweep angle
$[\Delta_{ii}]$	unspecified matrix
δ	unit impulse
δ_e	control angle
ζ	integration variable (streamwise)
η	integration variable (spanwise)
$\Theta_r(x,y)$	mode shape for twisted surface which is rigid chordwise
θ	angular displacement about center of gravity of rigid airplane, positive nose up
κ	airplane relative density
$\Lambda_j(y)$	spanwise weighting function
λ_r,λ_s	bending-moment coefficients for r,s normal modes
μ	generalized dependent variable
ν	integration variable (streamwise)
ξ_r,ξ_s	displacement of r,s normal modes, positive up
ρ	air mass density
σ	standard deviation

τ	integration variable (time)
Φ	power spectral density
ϕ	bank angle of rigid airplane, positive right wing up
φ	correlation function
χ	correlation interval (distance)
$\psi(t)$	Küssner function
ω	angular frequency
ω_r	natural (angular) frequency of rth normal mode
$\dot{()}, \ddot{()}$	derivatives with respect to time
$\overset{\circ}{()}$	derivatives with respect to distance (measured along x-axis)
$\overline{()}$	average

DERIVATION OF DYNAMICAL EQUATIONS OF MOTION

The problem of aircraft response to atmospheric turbulence (gust response) will be considered in a rather general manner in order to bring out some of the important features of the problem. The analysis is set up on the basis of familiar dynamical principles and is carried through as far as possible without introducing statistical concepts. However, a certain point is reached in the analysis, when solutions to the equations of motion are desired, where the introduction of statistical concepts becomes mandatory if the analysis is to continue. These concepts are then brought into the analysis and the final solution is obtained. The analysis is carried out this way in order to show that a solution to the problem depends upon the proper use of familiar dynamical analysis principles as well as upon statistical analytical tools.

The problem may be formulated by requiring that the "response" of an aircraft flying with velocity U through turbulent air is to be determined. The particular response of interest may be the acceleration, velocity, displacement, bending moment, shear, stress, and so forth at any arbitrary point of the aircraft structure.

The analysis will include three rigid-body degrees of freedom, heaving, pitching, and rolling (z, θ, and ϕ, respectively), and an unspecified number of elastic degrees of freedom (ξ_r where $r = 1, 2, 3, \ldots$) defined by the principal vibration modes of the aircraft; that is, the analysis is constructed for normal modes of the entire aircraft defined by $A_r(x,y)$ (fig. 1).

Use is made of Lagrange's equation to establish the equations of motion for the system, for small perturbations from equilibrium, namely (ref. 11):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial E_i}{\partial q} = \frac{\partial E_e}{\partial q}$$

The kinetic energy is

$$T = \frac{1}{2} \int_{S'} V^2 \, dm$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = \frac{d}{dt} \int_{S'} V \frac{\partial V}{\partial \dot{q}} \, dm$$

where

$$V = \dot{z} + x\dot{\theta} + y\dot{\phi} + \sum_r A_r(x,y)\dot{\xi}_r$$

The internal strain energy is (ref. 12)

$$E_i = \frac{1}{2}\sum_r M_r\omega_r^2\xi_r^2$$

where

$$\frac{\partial E_i}{\partial q} = 0 \qquad (q \neq \xi_r)$$

$$\frac{\partial E_i}{\partial q} = M_r\omega_r^2\xi_r \qquad (q = \xi_r)$$

The potential energy of the external forces is

$$E_e = \int_S \left[z + \theta x + \phi y + \sum A_r(x,y)\xi_r\right] \Delta p(x,y,t)\, dy\, dx$$

The pressure difference acting on the aircraft $\Delta p(x,y,t)$ contains two distinct contributions. The first part is due to disturbances of the field caused directly by motions of the aircraft. This part may be correctly accounted for by solution of the wave equation with appropriate boundary conditions (ref. 12)

$$\nabla^2 P = P_{tt}$$

The other part is due to disturbances originating elsewhere in the fluid (usually at great distance from the aircraft) and, for incompressible flow, can be described by the Navier-Stokes equations (ref. 13, p. 215)

$$\rho\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j\, \partial x_j}$$

There may appear to be a contradiction in the above discussion, namely, that motion effects are determined from perfect-fluid considerations through the concept of a velocity potential, while the disturbance effects may be due to turbulence, which, in general, cannot be described by a velocity potential. However, the scale of the turbulence which is important for the aircraft gust problem is large enough so that viscous effects may be neglected; that is, viscosity is associated with the very small eddies in the fluid. With viscosity thus eliminated, there is no contradiction.

Solution of the problem as outlined for the case of a turbulent flow superposed on uniform potential flow is an extremely difficult undertaking even in incompressible flow. In order to gain further insight into the problem, therefore, it seems highly desirable to make some simplifying assumptions.

Of all the parameters which enter the Navier-Stokes equation, namely, pressure, density, velocity, and viscosity, it may be assumed that, so far as the present application is concerned, velocity plays the dominant role. Furthermore, since interest is centered upon aircraft response in the three rigid-body modes of heaving, pitching, and rolling and in a number of elastic modes, all components of fluid velocity except that which is parallel to the initial z-axis may be safely discarded.[1] Thus, the significant part of the problem reduces to that of finding a satisfactory description of one component of velocity, namely, w. For small perturbations one can consider an equivalent angle of attack $w/U = \alpha_i$. The reasoning here can be made general in the sense that it can be applied to incompressible and compressible flow alike. It should be mentioned that the elimination of the effect of the static-pressure influence in the disturbance field is not completely straightforward, but it simplifies the problem considerably.

The influence of the motion of the aircraft, as mentioned previously, can be handled as a straightforward problem in unsteady potential flow; this is very tedious, however, especially if all boundary conditions are correctly included in the analysis. It is usual to make some simplifying assumptions here also.

If it is decided that structural response is of primary interest, for example, bending moment or shear, it is convenient to consider simplifying the aerodynamics picture as much as is consistent with the desired accuracy. This is particularly desirable in a statistical analysis, which can become unwieldy for even the simplest of problems.

To show how the statistical picture can get complicated, consider the factors that would be introduced by including an additional component of velocity in the analysis, say the crossflow v. This means that instead of "one-component" turbulence as given by the perturbation w, another vector quantity v has been added and accordingly "two-component" turbulence must be considered. This means that additional statistical quantities are brought into the analysis. Important statistical parameters such as probability functions, correlation functions, and spectra would have to reflect this fact. In the general case of two-component turbulence, the correlation tensor $R_{ji}(\bar{s})$ would contain four terms. This may be compared with a single quantity $R_{11}(\bar{s})$ in the relatively simple case of one-component turbulence. The probability functions for two-component turbulence are correspondingly more difficult to determine, and the spectra become extremely difficult to calculate in the general case.

These reflections add considerable weight to the argument for simplification of the analysis. Indeed, it is a matter of practical interest that such simplifications are a necessary adjunct to keeping the mathematics within reasonable proportions.

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[1]Note that a small effect may be expected because of crossflow v arising on the vertical tail and a somewhat larger effect of crossflow for swept wings may be expected. However, these effects of crossflow are neglected.

The formulation of the equations from this point requires a specification with regard to the aerodynamic theory; namely, will the unperturbed (steady) flow conditions be incompressible or compressible and, if the latter, what Mach number characterizes the flow? Also, how does one include the aspect-ratio effects? There are a number of theories which afford an approximate description of $\Delta p(x,y,t)$. The theory which is used will depend largely upon the free-flight Mach number and the plan form of the wing. In order that the present analysis may have a definite framework within which to proceed, the following specifications are made regarding the aerodynamic theory:

(1) Aerodynamic strip theory is used for both motion effects and gust effects

(2) Quasi-steady aerodynamic theory for incompressible flow is used to predict all motion forces by resolving the angle of attack at 3/4 chord

(3) Unsteady aerodynamic theory for incompressible flow is used to predict gust forces (later in the analysis, this will be modified)

(4) For simplicity, only the forces on the wing and horizontal tail are considered in the analysis

According to the first statement, the analysis must be restricted to aircraft of "large" aspect ratio, but it may also include swept as well as straight wings. The second statement restricts the analysis to flight configurations where all free-body motions (including the vibratory modes) occur at low values of reduced frequency, namely, $k < 0.05$; and, of course, incompressible flow requires that the Mach number be low.

On the basis of these assumptions E_e may be written as follows:

$$E_e = \int_{-b}^{b} \left[z + \theta\tilde{x} + \phi y + \sum A_r(\tilde{x},y)\xi_r\right] L(y,\tilde{x},t)\ dy +$$

$$\int_{-b'}^{b'} \left[z + \theta\tilde{x}' + \phi y + \sum A_r(\tilde{x}',y)\xi_r\right] L'(y,\tilde{x}',t)\ dy$$

The lift forces on the wing $L(y,\tilde{x},t)\ dy$ and on the tail $L'(y,\tilde{x}',t)\ dy$ are assumed to be concentrated at the local quarter-chord points of those surfaces.

There are two parts to $L(y,\tilde{x},t)$ and $L'(y,\tilde{x}',t)$, corresponding to the forces due to motion and the forces due to turbulence. The forces due to motion may be calculated by resolving the angle of attack at the three-quarter-chord point as follows (ref. 12):[2]

$$L(y,\tilde{x},t)_m = \bar{q}c(y)C_{l_\alpha}\left[\theta + \sum \theta_r(\bar{x},y)\xi_r - \frac{\dot{z} + \dot{\theta}\bar{x} + \dot{\phi}y + \sum A_r(\bar{x},y)\dot{\xi}_r}{U}\right]$$

$$L'(y,\tilde{x}',t)_m = \bar{q}'c'(y)C_{l_\alpha}'\left[\theta + \sum \theta_r(\bar{x},y)\xi_r - \frac{\dot{z} + \dot{\theta}\bar{x}' + \dot{\phi}y + \sum A_r(\bar{x}',y)\dot{\xi}_r}{U'}\right]$$

The dynamic pressure is given the symbol $\bar{q}$ to distinguish it from the generalized coordinate q. It is important to keep account of the coordinates involved in the above expressions; unprimed quantities refer to wing coordinates while primed quantities refer to tail coordinates, $\tilde{x},y$ is the local coordinate of the wing quarter chord, and $\bar{x},y$ is the local coordinate of the wing three-quarter chord. Note that a new quantity is introduced in addition to the new coordinates, that is, the quantity Θ_r which gives the elastic twist of the wing (or tail) at the three-quarter-chord point.

The other part of the aerodynamic force, that part due to turbulence, may be calculated from the relations (ref. 12)

$$L(y,\tilde{x},t)_g = \bar{q}c(y)C_{l_\alpha}\int_{-\infty}^{\infty} \dot{\psi}(t - \tau,y)\alpha_i(\tau,y)\,d\tau$$

$$L'(y,\tilde{x}',t)_g = \bar{q}'c'(y)C_{l_\alpha}'\int_{-\infty}^{\infty} \dot{\psi}'(t - \tau,y)\alpha_i(\tau,y)\,d\tau$$

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[2]Note that effect of the wing wake vorticity on the horizontal tail is here neglected for simplicity since it is debatable whether much is to be gained by including this effect in an analysis of this complexity. Probably the only time it can be fully justified is when tail loads or tail stresses are of primary interest.

In these relations the specifications of the kernel function $\dot{\psi}$ are purposely left in general terms in order to avoid confusion. However, it may be said that this function must take into account sweep, taper, and the finite time required for the gust pattern to pass from wing to tail. In this connection it should be brought out that Taylor's hypothesis is assumed valid for this analysis; that is, the gust field is regarded as a "frozen" pattern which does not alter with time locally, but rather the time dependence is caused by the uniform motion (at velocity U) of the aircraft. This may be expressed in succinct terms by stating merely that $\partial/\partial t = 0$ in the fluid.

This completes the formulation of the problem sufficiently to allow derivation of the dynamical equations of motion for the aircraft. Direct substitution of the above results into Lagrange's equation yields the following matrix equations:

$$\left[m\right]\left\{\ddot{q}\right\} + \left[\overset{o}{\alpha}\right]\left\{\dot{q}\right\} + \left[\alpha\right]\left\{q\right\} + \left[e\right]\left\{q\right\} = \left[F_{jj}\right]\left\{1\right\}$$

The various square matrices are given in the appendix. The generalized coordinate matrices $\left\{q\right\}$ contain the generalized coordinates z, θ, ϕ, and $\xi_r (r = 1, 2, 3, \ldots)$. It is interesting to note in passing that the following aeroelastic problems may be dropped out of the above equation as special cases: Divergence,

$$\left[\alpha\right]\left\{q\right\} + \left[e\right]\left\{q\right\} = 0 \tag{1}$$

stability,

$$\left[m\right]\left\{\ddot{q}\right\} + \left[\overset{o}{\alpha}\right]\left\{\dot{q}\right\} + \left[\alpha\right]\left\{q\right\} + \left[e\right]\left\{q\right\} = 0 \tag{2}$$

control response,

$$\left[m\right]\left\{\ddot{q}\right\} + \left[\overset{o}{\alpha}\right]\left\{\dot{q}\right\} + \left[\alpha\right]\left\{q\right\} + \left[e\right]\left\{q\right\} = \left[\Delta_{ii}\right]\left\{1\right\}\delta_e(t) \tag{3}$$

and flutter,

$$-\omega^2\left[m'\right]\left\{q\right\} + i\omega C(k)\left[\overset{o}{\alpha}\right]\left\{q\right\} + C(k)\left[\alpha\right]\left\{q\right\} + \left[e\right]\left\{q\right\} = 0 \tag{4}$$

(The matrix $[m']$ includes apparent mass effects.) A slight revision is necessary to obtain the flutter case; namely, the Theodorsen function $C(k)$ must be included in the motion forces. For simplicity, $C(k)$ may be assumed to be independent of the span coordinate.

It is interesting to note that equation (3) provides the basis for studying a gust alleviator system. If $\left[F_{jj}\right]\{1\}$ is added to the right side of equation (3), an investigation can be made of the effectiveness of various systems whose behavior is governed by a relation of the form

$$\delta_e(t) = \int_{-\infty}^{\infty} h_s(t - \tau)w(\tau)\,d\tau$$

where $h_s(t)$ is the system function. An interesting problem may be posed by requiring that $h_s(t)$ be selected for minimum root-mean-square response of, say, pitching displacement. This is the so-called Wiener-Hopf type of problem.

SOLUTION OF DYNAMICAL EQUATIONS OF MOTION AND DETERMINATION OF ROOT-MEAN-SQUARE RESPONSE

It was shown in the previous discussion that the equations of motion for an aircraft subject to turbulent velocity fluctuations in the xy-plane and perpendicular to the xy-plane are formulated as follows:

$$\left[m\right]\left\{\ddot{q}\right\} + \left[\overset{\circ}{\alpha}\right]\left\{\dot{q}\right\} + \left[\alpha\right]\left\{q\right\} + \left[e\right]\left\{q\right\} = \left[F_{jj}\right]\left\{1\right\}$$

where the various matrices are given in the appendix. It is shown that the diagonal matrix $\left[F_{jj}\right]$ is the forcing-function matrix and is of the form

$$\begin{bmatrix} f_{11}(\zeta) & & & 0 \\ & \ddots & & \\ & & f_{jj}(\zeta) & \\ & & & \ddots \\ 0 & & & f_{NN}(\zeta) \end{bmatrix}$$

Note that it is a function of the parameter $\zeta = Ut$. Various methods of solving this set of equations are now considered.

It is well to point out that in the present problem $f_{11}(\zeta)$, $f_{22}(\zeta)$, . . . are Stochastic functions of ζ, and solutions of the equations must be approached in light of this fact. This increases the complexity of the problem considerably.

For purposes of comparison, some more-or-less standard methods for solving linear equations are listed as follows: (1) By Laplace transforms, (2) by Fourier transforms, (3) by finite differences, and (4) by superposition of indicial solutions.

It is evident that direct application of Laplace transforms must fail because of the nature of F_{ij}; that is, the matrix is made up of random functions which are considered to have a beginning at $-\infty$ in time and the initial conditions are indeterminant. Direct application of the Fourier transform, even in the bilateral case, must also fail because, in general, the Fourier transform for the functions entering the right-hand side of the equation does not exist; that is, the integral may be divergent. Aside from these difficulties there is no generally consistent way of specifying the functions $f_{ij}(\zeta)$ because they depend upon the random variable $\alpha_i(x,y)$. Even if it were possible to specify these functions in detail, direct application of the two transform methods would yield nothing because of the aforementioned difficulties. Solutions by finite-difference techniques could, in principle, be carried out but, in practice, would be a formidable undertaking. This leaves the last method listed above as a logical choice, if not, indeed, the only choice. It will be shown presently that the method of superposition can be applied to determine a limited amount of information about the problem.

The method of superposition consists in building up a complete solution from certain special types of solutions which are relatively easy to obtain. There are two common methods available, either of which may be applied here, namely, superposition of simple-harmonic solutions through the use of the Fourier integral or superposition of unit-impulse-response solutions through the convolution integral. The latter seems to be the better method because it brings in the statistical parameters associated with the problem in a more straightforward manner. However, it is interesting to point out that the solutions obtained by the unit-impulse method are related to the solutions obtained by the simple-harmonic method through the Fourier transform, just as the unit-impulse response is related to the simple-harmonic response.

In considering a solution to the matrix equation by the method of superposition it is convenient to introduce the Dirac delta function to define a "primitive" solution, which corresponds to replacing the right-hand side of the equation by the following matrix:

$$\begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & \delta_j(\zeta) & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix} \left\{ 1 \right\}$$

The unit impulse $\delta(\zeta)$ is the forcing function in this instance and its position in the matrix is given by the subscript j. A solution for this case is called q_j and q may be any one of the several generalized coordinates z, θ, ϕ, or ξ_r. Clearly, then, the solution of interest may be written (ref. 12, p. 245)

$$q(l) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} q_j(l - \zeta) f_{jj}(\zeta)\, d\zeta$$

where N is the total number of generalized coordinates and l is in the same units as ζ. By definition $q_j(x) = 0$ for $x < 0$. The calculation concerning the quantities q_j may be carried out by any convenient method, including Laplace transforms and finite differences. However, these calculations are especially tedious if N is much larger than 3 or 4. For cases where a large number of generalized coordinates is used in the analysis, the use of high-speed computing machines is strongly recommended.

The problem is far from being solved, however, with the writing down of the above equation. At this point in the analysis it is realized that one must apply statistical methods to learn additional information about $q(l)$, which is a stochastic function simply because the quantities $f_{jj}(\zeta)$ are stochastic functions. Another question must also be settled at this point in the analysis; namely, what is the quantity of primary interest?

If the quantity of interest is one of the generalized coordinates or the nth derivative of one of these coordinates, then most of the information is available from the equation as it stands. On the other hand, if a quantity such as bending moment or shear at some point in the wing is of interest, then additional considerations must be given to the solution in its present form. This point may be made clear if one considers the following two typical cases, each one or both of which may be of practical interest: (1) Study of aircraft center-of-gravity acceleration, and (2) study of wing-root bending moment.

In the first case the solution to be studied takes the form

$$\ddot{z}(l) = \sum_{1}^{N} \int_{-\infty}^{\infty} \ddot{z}_j(l - \zeta) f_{jj}(\zeta)\, d\zeta$$

while in the second case the solution to be studied takes the following form:

$$M_B(l) = -\int_0^b y\, dm \left[y\ddot{\phi} + x\ddot{\theta} + \ddot{z} + \sum_r A(x,y)\ddot{\xi}_r \right] + \int_0^b dy \left[L(x,y,t)_m + L(x,y,t)_g \right] y$$

Notice that each of the quantities $\ddot{\phi}$, $\ddot{\theta}$, and so forth as well as $L(y,x,t)_m$ are complicated functions of the corresponding primitive solutions $\ddot{\phi}_j$, $\ddot{\theta}_j$, $\ddot{z}_j$, and so forth and the forcing functions f_{jj}.

It is apparent that the calculation of the bending moment in the manner indicated above is quite involved. An alternative means is available which is somewhat simpler in mathematical form; that is, the bending moment can be obtained by linear superposition of the normal modes as follows (ref. 12):

$$M_B(l) = \sum_{r=1}^{M} \lambda_r \xi_r(l)$$

where the quantity λ_r gives the root bending-moment coefficient due to unit deformation of the rth normal mode. It can be computed from the inertia loading on the wing corresponding to the rth normal vibration mode. The quantity M is the total number of modes that are used in the analysis; for this case, $M = N - 3$. The quantities ξ_r can be obtained by superposition of the primitive solutions ξ_{rj} discussed previously. Thus,

$$M_B(l) = \sum_{r=1}^{M} \sum_{j=1}^{N} \lambda_r \int_{-\infty}^{\infty} \xi_{rj}(l - \zeta) f_{jj}(\zeta)\, d\zeta$$

Consideration is given now to the determination of certain statistical quantities associated with the solutions of the dynamical equations of motion. Interest is here centered on the study of center-of-gravity acceleration and wing-root bending moment, respectively, as given by the relations

$$\ddot{z} = \sum_{j=1}^{N} \int_{-\infty}^{\infty} \ddot{z}_j(\zeta) f_{jj}(l - \zeta)\, d\zeta$$

$$M_B = \sum_{j=1}^{N} \sum_{r=1}^{M} \lambda_r \int_{-\infty}^{\infty} \xi_{rj}(\zeta) f_{jj}(l - \zeta)\, d\zeta$$

Both of the physical quantities on the left of the above equations are, of course, stochastic functions and the methods of statistical analysis must accordingly be applied to obtain additionally desired information. There are several questions one might pose at this point in regard to the statistical properties of $\ddot{z}$ and M_B. Some of these are worth noting as follows: (1) What is the probability density distribution? (2) What is the correlation function? (3) What is the power density spectrum?

The probability density distribution provides information on the amplitude of the stochastic function. It can be shown that both $\ddot{z}$ and M_B are Gaussianly distributed provided that f_{jj} is also Gaussianly distributed. This follows as a consequence of the central limit theorem and the properties of linear systems. It may be assumed that all of the quantities concerned here are Gaussian for convenience, but it is not a necessary assumption in what is to follow. There are, however, strong motivations for making this assumption; that is, (1) the Gaussian curve has been exhaustively studied and accordingly much is known about it, (2) it may be characterized by a single parameter, namely, the mean square or standard deviation, (3) the characterization by the mean-square parameter provides a link between amplitude spectra and power spectra, and (4) many physical quantities follow the Gaussian distribution.

The correlation function and power density spectrum are Fourier transform pairs. The correlation function provides information on the coherence of the stochastic function at adjacent intervals in time and/or space, while the power density spectrum gives information as to the distribution of power with frequency or wavelength. The use of the word power is rather generalized in that it need not refer to actual physical power. Perhaps a better terminology would be the mean-square density spectrum since the area under the power spectral density curve is equal to the mean square of the stochastic function. However, the term "power spectrum" seems to be rather firmly entrenched in the literature.

If the probability density of a stochastic function W is called $p(W)$, the autocorrelation function of W is called $\varphi_{WW}(x)$, and the power density spectrum of W is called $\Phi_{WW}(\omega)$, then one may write (ref. 14)

$$\overline{W^2} = \sigma^2 = \int_{-\infty}^{\infty} W^2 p(W)\ dW = \int_{-\infty}^{\infty} \Phi_{WW}(\omega)\ d\omega = \varphi_{WW}(0)$$

which means that the mean-square value is obtainable from all three of the statistical quantities discussed above. In the following discussion, it is assumed that $\overline{W} = 0$; that is, the mean value of all stochastic functions is assumed to be zero. Also, the analysis is directed toward the determination of autocorrelation functions for (1) center-of-gravity acceleration and (2) wing-root bending moment. From these results it is possible to compute the root-mean-square response.

Autocorrelation of Center-of-Gravity Acceleration

The autocorrelation of the center-of-gravity acceleration may be formed from the solutions to the dynamical equations of motion as follows:

$$\overline{\ddot{z}(l)\ddot{z}(l + x)} = \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \int_{-\infty}^{\infty} d\nu \; \ddot{z}_i(\nu) f_{ii}(l + x - \nu) \times \int_{-\infty}^{\infty} d\zeta \; \ddot{z}_j(\zeta) f_{jj}(l - \zeta)\right]_{av}$$

where the bar over the symbols on the left-hand side and the brackets $[\,]_{av}$ on the right-hand side indicate that the average is being taken with respect to the ensemble at a particular value of l.

With an appropriate change of independent variable and interchanging the order of integration and averaging operations, the autocorrelation of $\ddot{z}$ may be written

$$\overline{\ddot{z}(l)\ddot{z}(l + x)} = \sum_{i}^{N} \sum_{j}^{N} \iint_{-\infty}^{\infty} \ddot{z}_i(\zeta)\ddot{z}_j(\zeta - \nu)\overline{f_i f_j}(x - \nu) \; d\zeta \; d\nu$$

In order to reduce the autocorrelation of $\ddot{z}$ to this form it is also necessary to assume that the forcing functions f_i and f_j are <u>stationary</u> random functions; that is, the average product $\overline{f_i f_j}$ must be independent of parallel translations of these functions along the abscissa axis. Another way of saying the same thing in regard to $\ddot{z}$ is to say that the left-hand side of the equation is independent of l for a stationary process. (These remarks do not mean, however, that the functions are ergodic.)

The double integral on the right may be reduced to a single integral by defining the new quantity

$$\varphi_{ij}^{(\ddot{z})}(-\nu) = \int_{-\infty}^{\infty} \ddot{z}_i(\zeta)\ddot{z}_j(\zeta - \nu) \; d\zeta$$

which is the cross correlation between the ith primitive solution and the jth primitive solution corresponding to $\ddot{z}$. In general, there will be N^2 such cross-correlation functions needed in the analysis. These quantities depend on the primitive solutions obtained by successively substituting unit-impulse functions on the right-hand side of the dynamical equations of motion. These cross-correlation functions are referred to as the primitive cross-correlation functions corresponding to $\ddot{z}$. After substitution, the autocorrelation for $\ddot{z}$ reads

$$\overline{\ddot{z}z}(x) = \sum_{i}^{N} \sum_{j}^{N} \int_{-\infty}^{\infty} \varphi_{ij}^{(\ddot{z})}(-\nu)\overline{f_i f_j}(x - \nu)\, d\nu$$

There remains the problem of computing $\overline{f_i f_j}$, but before this is considered the autocorrelation of wing-root bending moment will be discussed.

Autocorrelation of Wing-Root Bending Moment

The autocorrelation of wing-root bending moment may be formulated as follows:

$$\overline{M_B(l)M_B(l + x)} = \sum_{i}^{N} \sum_{j}^{N} \sum_{r}^{M} \sum_{s}^{M} \lambda_r\lambda_s \iint_{-\infty}^{\infty} \xi_{ri}(\zeta)\xi_{sj}(\zeta - \nu)\overline{f_i f_j}(\bar{x} - \nu)\, d\nu\, d\zeta$$

In a manner entirely analogous to the formulation of $\ddot{z}$ autocorrelation, a new cross-correlation function may be defined as

$$\varphi_{ij}^{r,s}(-\nu) = \int_{-\infty}^{\infty} \xi_{ri}(\zeta)\xi_{sj}(\zeta - \nu)\, d\zeta$$

In general, there will be $(NM)^2$ such cross-correlation functions needed in the analysis for bending moment. These quantities may be referred to as primitive cross-correlation functions corresponding to ξ_r and ξ_s. It is of utmost importance to keep the subscripts and superscripts in their proper order to avoid error. After substitution, the autocorrelation of bending moment reads

$$\overline{M_B M_B}(x) = \sum_{i}^{N} \sum_{j}^{N} \sum_{r}^{M} \sum_{s}^{M} \lambda_r \lambda_s \int_{-\infty}^{\infty} \varphi_{ij}^{r,s}(-\nu)\overline{f_i f_j}(x - \nu)\, d\nu$$

Notice that the cross-correlation function $\overline{f_i f_j}$ occurs in the expression for bending-moment autocorrelation in the same form as in the acceleration autocorrelation. Before proceeding to a discussion of this quantity in detail some remarks should be made about the analysis at its present stage of development. First of all, it is quite evident that the analysis will reach vast proportions if the numbers N and M are very large. For example, if three rigid-body modes are used in conjunction with three elastic modes then 36 primitive cross correlations are needed for the acceleration autocorrelations, while 324 primitive cross correlations are needed for the bending-moment autocorrelation. Some idea of how this goes may be seen in table I.

Cross-Correlation Function $\overline{f_i f_j}$

Consideration is now given to the determination of the cross-correlation function $\overline{f_i f_j}$ keeping in mind that the analysis is becoming extremely complicated and that some simplifying assumptions may be made in the interest of retaining an analytical procedure which is useful for practical applications. It is shown in the appendix that a typical term of the forcing-function matrix corresponding to the jth row can be written

$$f_{jj} = \int_{-b}^{b} \Lambda_j(y)\, dy \int_{-\infty}^{\infty} \overset{\odot}{\psi}\left[\frac{\zeta - x_o - x}{c(y)}\right]\alpha_i(x,y)\, dx\ +$$

$$\int_{-b'}^{b'} \Lambda_j'(y)\, dy \int_{-\infty}^{\infty} \overset{\odot}{\psi}\left[\frac{\zeta - x_o' - x}{c'(y)}\right]'\alpha_i(x,y)\, dx$$

The function ψ may be identified as Küssner's function. The argument of the function as above written takes into account the effects of sweep and taper. The Küssner function, as is well known, takes into account the unsteady aerodynamic effect of the lift buildup on the airfoil due to variations in α_i. If the variations in α_i are slow compared with the variation in $\overset{\odot}{\psi}$, over, say, 10 chords, then quasi-steady aerodynamics may be used, and the quantity $\overset{\odot}{\psi}$ may be replaced by a unit impulse.[3] In the following discussion it is assumed that α_i variations are slow enough to warrant this simplification. Under this simplifying assumption the forcing function reads

$$f_{jj} = \int_{-b}^{b} dy\ \Lambda_j(y)\alpha_i(\zeta - x_o, y) + \int_{-b'}^{b'} dy\ \Lambda_j'(y)\alpha_i(\zeta - x_o', y)$$

Notice that the effects of sweep and taper are still included. The cross correlation $\overline{f_i f_j}(\chi)$ may be computed in a straightforward calculation using the above relation; namely,

$$\begin{aligned}\overline{f_i f_j}(\chi) = &\iint_{-b}^{b} dy\ d\eta\ \Lambda_i(y)\Lambda_j(\eta)\ \overline{\alpha_i(\zeta - x_o, y)\alpha_i(\zeta + \chi - x_o, \eta)} + \\ &\int_{-b}^{b}\int_{-b'}^{b'} dy\ d\eta\ \Lambda_i(y)\Lambda_j'(\eta)\ \overline{\alpha_i(\zeta - x_o, y)\alpha_i(\zeta + \chi - x_o', \eta)} + \\ &\int_{-b}^{b}\int_{-b'}^{b'} dy\ d\eta\ \Lambda_i'(y)\Lambda_j(\eta)\ \overline{\alpha_i(\zeta - x_o', y)\alpha_i(\zeta + \chi - x_o, \eta)} + \\ &\iint_{-b'}^{b'} dy\ d\eta\ \Lambda_i'(y)\Lambda_j'(\eta)\ \overline{\alpha_i(\zeta - x_o', y)\alpha_i(\zeta + \chi - x_o', \eta)}\end{aligned}$$

If the turbulence is homogeneous, the quantity $\overline{\alpha_i\alpha_i}$ may be identified as a spatial autocorrelation function in two dimensions in which the spatial dependence is given in terms of the differences $|y - \eta|$, $|x_o - x_o'|$, and so forth. It is important to notice that in the case of a swept wing (or tail) x_o is a function of the spanwise coordinate. In particular, if Γ is the sweep angle at midchord then

$$x_o(y) - x_o(\eta) = \sin \Gamma, (y - \eta)$$

[3]Since a "frozen" turbulence field has been postulated, it may be stated equivalently that unsteady aerodynamic effects may be neglected in the cases where the turbulence integral scale is large compared with a characteristic dimension of the wing, in this case the semichord. An argument which leads to this simplification is given in reference 3.

Also, the last term on the right-hand side of the relation for $\overline{f_i f_j}(X)$ is small compared with the other terms, being approximately in the ratio of tail area to wing area squared. Therefore, this last term may be neglected with little error. Finally, if the autocorrelation for α_i is given the symbol $\varphi_{\alpha\alpha}$, the relation reads

$$
\begin{aligned}
\overline{f_i f_j}(X) = & \iint_{-b}^{b} dy\, d\eta\, \Lambda_i(y)\Lambda_j(\eta)\varphi_{\alpha\alpha}\left[X + \sin\Gamma(y - \eta), \left|y - \eta\right|\right] + \\
& \int_{-b}^{b}\int_{-b'}^{b'} dy\, d\eta\, \Lambda_i(y)\Lambda_j'(\eta)\varphi_{\alpha\alpha}\left[X + x_o - x_o', \left|y - \eta\right|\right] + \\
& \int_{-b}^{b}\int_{-b'}^{b'} dy\, d\eta\, \Lambda_i'(y)\Lambda_j(\eta)\varphi_{\alpha\alpha}\left[X + x_o' - x_o, \left|y - \eta\right|\right]
\end{aligned}
$$

Notice that the effect of sweep is taken into account by including spanwise dependence in both coordinates of $\varphi_{\alpha\alpha}(x,y)$. For the first integral this dependence is rather simply included, but in the last two integrals a more general relation must be used, namely,

$$x_o(y) - x_o'(\eta) = l_o - \sin\Gamma,y + \sin\Gamma,\eta$$

where l_o is the distance between the midchord of wing and the midchord of the tail at the plane of symmetry. For most aircraft wherein both the wing and the tail employ moderate sweepback, the right-hand side of the above relation may be replaced by l_o. This affords considerable simplification in the last two integrals affecting $\overline{f_i f_j}$.

Certain special cases will now be considered in light of these results. The first case studied is for straight or moderately swept wings. Another special case will be considered which leads to somewhat simpler results as will be seen, that is, the case when $\varphi_{\alpha\alpha}(x,y)$ may be replaced by $\varphi_{\alpha\alpha}(x)$.

Special Cases

Straight wings or wings with moderate sweep.- In order to avoid mathematical encumbrances, only one term will be considered on the right-hand side of the expression for $\overline{f_i f_j}$ remembering, however, that this is only part of the total contribution. The remaining two terms should contribute in the order of 10 to 20 percent so that they should not be ignored completely. For this case, therefore, one may write

$$\overline{f_i f_j}(X) = \iint_{-b}^{b} dy\, d\eta\, \Lambda_i(y)\Lambda_j(\eta)\varphi_{\alpha\alpha}(X,|y - \eta|) \ldots$$

Considering this expression in light of the definition of Λ (see appendix), it is evident that an important special case is obtained when $i = j = 1$, that is, the case of the total lift on a rigid unswept (or moderately swept) wing restrained against motion. For this case,

$$\Lambda_1 = \bar{q}c(y)c_{l_\alpha}(y)$$

Other special cases (e.g., rolling moment on a rigid restrained wing) may be obtained in a similar manner from Λ_3. In order to evaluate the expression for $\overline{f_i f_j}$, a specification is needed with respect to $\varphi_{\alpha\alpha}$. In other words, the "space-time" autocorrelation (space-time correlation is directly proportional to space-space correlation under Taylor's hypothesis) for the turbulence is needed. Here, indeed, is the crux of the matter, but it is this quantity about which the least is known so far as atmospheric turbulence is concerned. The space-time correlation is extremely difficult to measure even under ideal experimental test conditions. Furthermore, very limited information is available which affords a rational basis for estimating this quantity theoretically.

If the turbulence is assumed to be isotropic, then theoretical expressions could be used, but it would be difficult to justify their use. For example, an expression which is based upon isotropic turbulence considerations and which is not too difficult to handle mathematically is the Gaussian curve

$$\frac{\sqrt{\pi}}{2} U^2\varphi_{\alpha\alpha}(x,y) = \overline{w^2}\left[\exp -(x^2 + y^2)/L^2\right]$$

where $\overline{w^2}$ is mean-squared intensity and L is integral scale of turbulence. From the remarks it is clear that something more definitive is needed in order to complete a gust-load analysis of a full-scale aircraft, especially when spanwise partial correlation effects are to be included.

<u>One-dimensional gust structure</u>.- The case of the one-dimensional gust structure affords considerable simplification in two distinct areas of the problem, namely, (1) the determination of the primitive solution and (2) the determination of the autocorrelation. If the gust structure is one-dimensional the terms entering the right-hand side of the dynamical equations of motion include the angle-of-attack fluctuations α_i as a common linear factor. The matrix F_{jj} may be replaced by

$$\left[F_{jj}\right] = \alpha_i(\zeta)\begin{bmatrix} f_{11}^* & & & 0 \\ & f_{22}^* & & \\ & & \ddots & \\ 0 & & & f_{NN}^* \end{bmatrix}$$

where the quantities f_{jj}^* and f_{jj} are identical except that the angle-of-attack fluctuation α_i does not enter into the spanwise integration. For this case, only one primitive solution is required for any particular quantity of interest. This is true, for example, in computing either center-of-gravity acceleration or bending moment or any other quantity which may be of interest. All that is required to arrive at a solution to a particular problem is to replace $\alpha_i(\zeta)$ by the unit impulse, to compute the desired primitive solution, and to use this primitive solution in a superposition on $\alpha_i(\zeta)$ to obtain the final desired result. Suppose, for example, it is desired to determine the center-of-gravity acceleration and the autocorrelation function of the wing bending moment. The primitive solutions may be designated $\ddot{z}^*$ and M_B^*, respectively. Thus,

$$\ddot{z}(l) = \int_{-\infty}^{\infty} \ddot{z}^*(l - \zeta)\alpha_i(\zeta)\, d\zeta$$

$$M_B(l) = \int_{-\infty}^{\infty} M_B^*(l - \zeta)\alpha_i(\zeta)\, d\zeta$$

It is evident that this is a much simpler means of determining the final results. The analysis for this case is in all respects similar to the analysis of an electrical circuit with stochastic inputs. Accordingly, the relations which are developed in the analysis of such systems may be directly applied here. It is interesting to compare the above scheme with the previous one in terms of relative complexity. Such a comparison is best brought out by considering the number of primitive cross-correlation functions needed to complete the above analyses. In the case of the center-of-gravity acceleration it is necessary to compute only one primitive correlation which is, of course, an autocorrelation rather than a cross correlation. In the case of the bending moment, it is similarly necessary to compute only one primitive autocorrelation. However, for bending moment, it is important to notice that the primitive solution depends upon all normal modes ξ_r^*, and the total bending moment must be obtained from the relation

$$M_B^* = \sum_r \lambda_r \xi_r^*$$

where the values of ξ_r^* are obtained by replacing $\alpha_1(\zeta)$ by the unit impulse.

The primitive-solution requirements and associated primitive-correlation-function requirements may be compared with those in table I. It is clear that the one-dimensional case is several orders of magnitude simpler to apply. One may generalize this result in the following way. For the one-dimensional case, let μ represent some physical quantity of interest which depends upon the solution of the dynamical equations of motions. Let μ^* be the solution of the equations for $\alpha_1(\zeta) = \delta =$ Unit impulse. Then, the autocorrelation of μ is given by

$$\varphi_{\mu\mu}(x) = \int_{-\infty}^{\infty} \varphi_{\mu^*\mu^*}(x - \zeta)\varphi_{\alpha\alpha}(\zeta)\, d\zeta$$

where $\varphi_{\mu^*\mu^*}$ is the autocorrelation of μ^* and $\varphi_{\alpha\alpha}$ is the autocorrelation of angle of attack α.

If the power density spectrum of μ is of interest, then (ref. 7, p. 322)

$$\Phi_{\mu\mu}(\omega) = \left|M_{\mu}^{*}(\omega)\right|^{2}\Phi_{\alpha\alpha}(\omega)$$

where Φ is mean-power density spectrum and $M^{*}(\omega)$ is obtained by letting $\alpha_1 = \exp i\omega t$ in the dynamical equations of motion.

This relation is a familiar one which is basic to the study of stationary time series and linear passive systems. It may be applied to any linear system with constant parameters provided the input quantity is a stationary random function of one independent variable. For a linear passive electric circuit these conditions are satisfied. For an airplane, these conditions will, in general, be satisfied only when the turbulence scale is "large." In this case the turbulent velocity fluctuations which influence the airplane response at any instant may be considered constant in the lateral (spanwise) direction. The analysis in these cases of large turbulence scale is a great deal simpler than the general analysis. Flight measurements of the power density spectrum of the gust considered as a one-dimensional process (refs. 3, 16, and 17) may be used in a one-dimensional analysis of the type discussed herein.

The function $M^{*}(\omega)$ appearing in the above relation may be calculated from the dynamical equations of motion. For example, if 70 points in the power density spectrum are to be calculated and if the analysis is for six degrees of freedom, it will be necessary to perform 70 inversions of sixth-order matrices to obtain $M^{*}(\omega)$.

SOME SIMPLE ILLUSTRATIVE EXAMPLES

The general formulation of the problem of airplane gust response has been examined. This problem becomes exceedingly complex and difficult to handle analytically when instantaneous spanwise variations of the gust field occur. Several simplifying assumptions were introduced into the formulation. But the assumption which achieves the greatest simplification in the analysis is the postulation of a one-dimensional gust structure.

For this case, the response may be characterized by the relation

$$\Phi_{\mu\mu}(\omega) = \left|M_{\mu}^{*}(\omega)\right|^{2}\Phi_{\alpha\alpha}(\omega)$$

where

μ generalized dependent (response) variable

$\Phi_{\mu\mu}(\omega)$ power density spectrum of μ

$M_{\mu}^{*}(\omega)$ transfer function of μ corresponding to sinusoidal gust input (obtained by solving dynamical equations of motion for airplane)

The quantity $\Phi_{\alpha\alpha}(\omega)$ is the power spectrum of w/U and may be represented by the one-dimensional isotropic spectrum (ref. 7, p. 322) as

$$U^2\Phi_{\alpha\alpha}(\omega) = \overline{w^2}\,\frac{1}{\pi U}\,\frac{1 + 3\left(\frac{\omega L}{U}\right)^2}{\left[1 + \left(\frac{\omega L}{U}\right)^2\right]^2}$$

where L is the integral scale of turbulence, and $\overline{w^2}$ is the mean-square gust velocity of turbulent velocity fluctuations normal to the airplane flight direction and lying in the plane of symmetry. The mean-square lift response as a function of the parameters of the problem will now be investigated. In general, the transfer function $M^{*}(\omega)$ is a function of reduced frequency $\omega b_c/U$ and certain nondimensional airplane parameters, for example, airplane relative density, pitch radius of gyration, tail length, center-of-gravity position, tail effectiveness factor, and so forth.

Using that information, one can write the following relation for the mean-square lift coefficient (ref. 18):

$$\overline{C_L{}^2} = \int_0^\infty \Phi_{C_LC_L}(\omega)\;d\omega = \int_0^\infty \left|M_{C_L}{}^{*}(\omega)\right|^2\Phi_{\alpha\alpha}(\omega)\;d\omega$$

$$= \int_0^\infty \left|M_{C_L}{}^{*}(\xi b_c/L,\ \kappa,\ l,\ R,\ .\ .\ .)\right|^2\Phi_{\alpha\alpha}(\xi)\;d\xi$$

where $\xi = \frac{\omega L}{U}$. Thus, it is seen that the mean-square lift coefficient is a function of a number of nondimensional airplane parameters and the parameter L/b_c, which is the ratio of turbulence scale to wing semichord. The lift coefficient is also proportional to the turbulence mean-square intensity. Any other response variable of interest, for example, bending moment at wing root, would similarly depend upon these numerous parameters, though not in the same way, of course.

A complete parametric investigation (see, e.g., ref. 15) of the problem as above formulated is beyond the present scope of study. However, it is desired to consider the following important questions:

(1) What is the importance of unsteady aerodynamics in the lift buildup due to the gust?

(2) What is the importance of the pitching degree of freedom?

(3) How do the results compare with results obtained by the sharp-edged-gust formula?

Importance of Unsteady Aerodynamics

In a paper by Fung (ref. 7), the problem of unsteady aerodynamics was considered in the framework of a single-degree-of-freedom vertical translation. Results were obtained which included the effect of unsteady aerodynamics and also apparent mass effects but which neglected contributions of the horizontal tail. In the general formulation presented here the unsteady aerodynamic effects and apparent mass effects have been neglected, but the contribution of the horizontal tail has been included. It is desired to isolate the unsteady aerodynamic effect. To do this, consider the integral

$$\frac{\overline{C_L{}^2}}{(C_L{}^2)_{ref}} = \frac{1}{\pi} \int_0^\infty \frac{\xi^2(1 + 3\xi^2)}{\left[\left(\frac{2L}{b_c \kappa}\right)^2 + \xi^2\right](1 + \xi^2)^2} d\xi$$

where $\kappa = 2m/\pi\rho S b_c$ is the airplane relative density. This may be compared with the expression given in Fung's paper.

The reference lift coefficient is based upon the sharp-edged-gust formula

$$(C_L)_{ref} = 2\pi w'/U$$

where w' is the root-mean-square gust intensity. Figure 2 shows a direct comparison of results. It can be seen that unsteady aerodynamic effects become less important as the integral turbulence scale becomes large relative to the wing semichord. Also, for large values of airplane relative density, the effect of unsteady aerodynamics decreases for given values of L/b_c. This latter influence is much less apparent than the scale effect.

Importance of Pitching Degree of Freedom

A parametric study of gust loads in two degrees of freedom was made by Brenner and Isaksen (ref. 18). In that study, the gust structure was greatly simplified to the case of (1) a uniform step-function gust and (2) a uniform ramp-function gust. In what is perhaps the most exhaustive parametric treatment of the problem so far, charts and numerical data are given for more than 225 case studies. Certain effects on the gust response were investigated, for example, effects of airplane relative density, effects of pitch radius of gyration, effects of tail length and tail effectiveness, and so forth. A parametric study of such scope is outside the present investigation. However, to settle the question of the importance of the pitching degree of freedom, a similar analytical study for stochastic gusts is warranted. Such a program could make use of results obtained by Brenner and Isaksen if sufficient data on the asymptotic behavior of their responses are available.

Only peak values of the responses were of interest in their study, so the responses for later times are not given in the report. To make use of step-function response data for computing responses to stochastic gusts, a Fourier integral operation is involved, and this integration extends over the range $0 \leqq t \leqq \infty$. Unfortunately, the data of reference 18 do not cover the required range on t.

For the sake of simplicity, two special cases are considered which are characterized by two extreme values of the pitch radius of gyration, namely, $J \longrightarrow 0$ and $J \longrightarrow \infty$. The case for infinite radius of gyration is essentially the one-degree-of-freedom case of vertical translation, since with infinite moment of inertia about the pitch axis, the airplane cannot respond in pitch. For the other case, the pitch radius of gyration (moment of inertia in pitch) is so small that the airplane offers effectively no inertial restraint about the pitch axis. Motion about the pitch axis is thus entirely the result of aerodynamic moments about the center of gravity. Any actual case would lie between limits given by these two simple cases. To establish these limits, the dynamic equations of motion must be solved corresponding to the two stated limits of pitch radius of gyration. This procedure leads to an expression for the transfer function in terms of the airplane parameters and the reduced frequency. If, in addition, one considers that the center of gravity of the airplane is subject to a narrow range of variation, say between 25 and 50 percent of the chord, some additional simplifications in the expression for the transfer function are possible. An expression which closely approximates the transfer function for small pitch radius of gyration is included in the following relations for mean-square lift coefficient with center of gravity at 50 percent of the wing chord:

$$\left[\frac{\overline{C_L{}^2}}{\left(\overline{C_L{}^2}\right)_{ref}}\right]_{J\to 0} = \frac{1}{\pi}\int_0^\infty \frac{\xi^2(1 + 3\xi^2)\,d\xi}{\left\{\left[\frac{2L}{b_c\kappa l'(1 + R)}\right]^2 + \xi^2\right\}(1 + \xi^2)^2}$$

Here, two new parameters have been introduced, the nondimensional tail length l', referred to wing semichord, and the ratio of tail area to wing area R.

The ratio of $\overline{C_L{}^2}$ for $J\to 0$ to $\overline{C_L{}^2}$ for $J\to\infty$ may be computed by referring to previous results, namely,

$$\frac{\left(\overline{C_L{}^2}\right)_{J\to 0}}{\left(\overline{C_L{}^2}\right)_{J\to\infty}} = \frac{\displaystyle\int_0^\infty \frac{\xi^2(1 + 3\xi^2)}{\left[2L/b_c\kappa l(1 + R)\right]^2 + \xi^2}\times\frac{d\xi}{(1 + \xi^2)^2}}{\displaystyle\int_0^\infty \frac{\xi^2(1 + 3\xi^2)}{(2L/b_c\kappa)^2 + \xi^2}\times\frac{d\xi}{(1 + \xi^2)^2}}$$

This relation is plotted in figure 3 for the value $l(1 + R) = 4.0$. For the given parameters, the relative effect of the pitch degree of freedom becomes more important as the turbulence scale becomes large relative to the wing semichord. Also, the effect of the airplane relative density is to decrease the effect of the pitch degree of freedom as the relative density increases; all other parameters remain fixed. An increase in tail length will have the same relative effect as an increase in airplane relative density. A larger tail will have a mitigating effect on the gust loads.

Comparison of Results With Sharp-Edged-Gust Formula

To see how these results compare with the sharp-edged-gust formula, some representative curves are shown in figure 4. These curves were calculated using the formula

$$\frac{\overline{C_L{}^2}}{\left(C_L{}^2\right)_{ref}} = \frac{1}{\pi}\int_0^\infty \frac{\xi^2\left(1 + 3\xi^2\right)}{\left(2L/b_c\kappa\right)^2 + \xi^2}\,\frac{d\xi}{\left(1 + \xi^2\right)^2}$$

These results show a decrease in mean-square lift response with increasing L/b_c. Also, the effect of decreasing the airplane relative density is to mitigate the gust load (compare this with the relative effect of κ on the loads when pitching is included); all other parameters remain fixed. This latter effect is in accord with the results of Brenner and Isaksen for step-function gusts. Perhaps the effect of increasing the turbulence scale is tantamount to increasing the gust-gradient distance for ramp-type gusts. The overall effect on the lift response to stochastic gusts is in qualitative agreement with ramp-gust responses if one uses this reasoning. It must be remembered that this is a statistical process, and, if a root-mean-square load factor of, say, 0.8 is predicted, this load is actually exceeded many times, as evidenced by the distribution of the load factor in the probability sense (ref. 19). A load factor of 2.0, for example, is not impossible. In fact, if the distribution is Gaussian, the probability of this happening is 1.24 percent, and the odds against the load factor equaling or exceeding 2.0 are only about 80 to 1. Thus, while the sharp-edged-gust formula gives conservative results for root-mean-square load predictions, it may be perfectly adequate for design.

CONCLUDING REMARKS

An analytical scheme has been derived which can be used to estimate autocorrelation functions and, by use of a Fourier transformation, power density functions of airplane responses to atmospheric gusts. The analysis takes into account the principle degrees of freedom of the airplane and includes effects of spanwise (as well as chordwise) variations in the instantaneous gust pattern. The final results depend upon the solution of dynamic equations of motion for the airplane for a set of relatively simple forcing functions, namely, a set of delta functions. These solutions are superposed by convolution with the actual gust pattern to obtain the desired results. Two typical responses of interest are discussed, the center-of-gravity acceleration response and the wing-root bending-moment response. The autocorrelation functions of these two quantities are expressed in terms of a multiple summation, extending over the total number of degrees of freedom. Each term in the summation involves a correlation of the primitive solution cross correlation with a cross-correlation function. This cross-correlation function can be calculated by postulating that the turbulence field is stationary and homogeneous; also, the autocorrelation function in two dimensions must be known for the gust pattern. This quantity is defined by $\varphi_{\alpha\alpha}(x,y)$; it is by means of this function that the lateral gust field is brought into the analysis. Unfortunately, little is known about this function for the atmosphere, indicating that practical application of these results must await the determination of $\varphi_{\alpha\alpha}(x,y)$ from atmospheric-turbulence research investigations.

A special case which is vastly simpler in principle follows from neglecting the lateral variations in the gust pattern. This case has been labeled "one-dimensional" gust structure. The simplification of the analysis is such that the amount of work can be reduced by a factor of at least N^2, if N is the number of degrees of freedom. For this case, only one-dimensional autocorrelations enter; for example, $\varphi_{\alpha\alpha}(x,y)$ may be replaced by $\varphi_{\alpha\alpha}(x)$. Flight measurements of the power density spectrum of the gust considered as a one-dimensional process may be used in a one-dimensional analysis of the type discussed here.

Some simple illustrative examples are treated for the case of a one-dimensional gust structure. The validity of neglecting the effect of unsteady aerodynamics is investigated by comparing calculations for two cases of one degree of freedom. The results show that this assumption may be justified only when the integral scale of turbulence is large compared with the wing semichord. A corollary to this conclusion may be stated as follows: When conditions are such that unsteady aerodynamic effects are significant, then the lateral gust effects may become important also, and it does not seem consistent to include the one and neglect the other. Both should be either included or neglected in a rational analysis.

In a one-dimensional analysis, it is conservative (i.e., higher loads are predicted) to neglect unsteady aerodynamic effects. It is not possible to state at this time the comparison for a two-dimensional analysis.

The importance of the pitching degree of freedom is investigated by solving the two equations of motion (for heaving and pitching) under two extreme values of the pitch moment of inertia, namely, for a pitch radius of gyration of 0 and one of ∞. The latter case effectively includes only the plunging degree of freedom. The results of this simplified study show that the pitching becomes relatively more important as the turbulence integral scale becomes large relative to the wing semichord. Before one can draw general conclusions, however, an extensive parameter investigation must be made.

The results of a single-degree-of-freedom analysis (neglecting unsteady aerodynamic effects) are compared with the results of the sharp-edged gust formula. For this comparison, the root-mean-square gust intensities are set equal. These results show that the root-mean-square loads are always less than those predicted by the sharp-edged-gust formula and become even smaller as the integral scale of turbulence increases.

One final point is worthy of mention, namely, that this is a statistical process, and if a root-mean-square load factor of, say, 0.8 is predicted, it should be remembered that this load is actually exceeded many times, as evidenced by the distribution of the load factor in the probability sense. A load factor of 2.0, for example, is not impossible. In fact, if the distribution is Gaussian, the probability of this happening is 1.24 percent, and the odds against the load factor equaling or exceeding 2.0 are only about 80 to 1. Thus, while the sharp-edged-gust formula gives conservative results for root-mean-square load predictions, it may be perfectly adequate for design.

Extensive research investigations should be carried out (1) to determine much needed information on the atmospheric gust structure, particularly on two-dimensional turbulence spectra so that a direct comparison can be made with data now available and (2) to study methods of applying atmospheric turbulence data to practical problems of flight.

Massachusetts Institute of Technology,
Cambridge, Mass., July 1, 1955.

APPENDIX

DERIVATION OF MATRICES ENTERING INTO DYNAMICAL EQUATIONS OF MOTION

The equations of motion for an aircraft flying at mean velocity $\bar{U}$ through turbulent air are represented by the matrix equation

$$\left[m\right]\left\{\ddot{q}\right\} + \left[\overset{\circ}{\alpha}\right]\left\{\dot{q}\right\} + \left[\alpha\right]\left\{q\right\} + \left[e\right]\left\{q\right\} = \left[F_{jj}\right]\left\{1\right\}$$

The various square matrices will now be considered in detail.

Inertia Matrix $\left[m\right]$

The inertia matrix may be derived by noting that the column matrix $\left[m\right]\left\{\ddot{q}\right\}$ follows from

$$\frac{d}{dt}\int_S V \frac{\partial V}{\partial q}\, dm$$

for $\dot{q} = \dot{z},\ \dot{\theta},\ \dot{\phi}$, and $\dot{\xi}_r (r = 1, 2, 3, \ldots$ (in this order)) where

$$V = \dot{z} + x\dot{\theta} + y\dot{\phi} + \sum_r A_r(x,y)\dot{\xi}_r$$

For example, one may consider the terms in the third row of $\left[m\right]\left\{q\right\}$. These may be written

$$\left\lfloor m_{3j}\right\rfloor\left\{\ddot{q}\right\} = \frac{d}{dt}\int_S V\frac{\partial V}{\partial \phi}\, dm = \int_S dm \frac{d}{dt}\left[\dot{z} + x\dot{\theta} + y\dot{\phi} + \sum A_r(x,y)\dot{\xi}_r\right]y$$

$$= \ddot{z}\int_S dm\ y + \ddot{\theta}\int_S dm\ xy + \ddot{\phi}\int_S dm\ y^2 + \sum \ddot{\xi}_r \int_S dm\ A_r(x,y)y$$

whence

$$\lfloor m_{3j} \rfloor = S_x, \quad I_{xy}, \quad I_x, \quad \alpha_{\phi 1}, \quad \alpha_{\phi 2}, \ . \ . \ .$$

where

S_x static moment at x-axis

I_{xy} product of inertia at xy-axis

I_x moment of inertia at x-axis

$$\alpha_{\phi 1} = \int_S dm\ A_1(x,y)y$$

$$\alpha_{\phi 2} = \int_S dm\ A_2(x,y)y$$

and so forth. The determination of the remaining rows of $\lceil m \rceil$ follows from an obvious extension of the above equations as

$$\lfloor m_{1j} \rfloor \left\{\ddot{q}\right\} = \ddot{z}M_A + \ddot{\theta}S_y + \ddot{\phi}S_x + \sum \ddot{\xi}_r\alpha_{zr}$$

$$\lfloor m_{2j} \rfloor \left\{\ddot{q}\right\} = \ddot{z}S_y + \ddot{\theta}I_y + \ddot{\phi}I_{xy} + \sum \ddot{\xi}_r\alpha_{\theta r}$$

$$. \ . \ . \ . \ . \ . \ .$$

$$\lfloor m_{4j} \rfloor \left\{\ddot{q}\right\} = \ddot{z}\alpha_{z1} + \ddot{\theta}\alpha_{\theta 1} + \ddot{\phi}\alpha_{\phi 1} + \xi_1 M_1$$

$$. \ . \ . \ . \ . \ . \ .$$

where

M_A mass of airplane

S_y static moment about y-axis

I_y moment of inertia about y-axis

$$\alpha_{zr} = \int_S dm\ A_r(x,y) \quad \text{with} \quad r = 1, 2, 3, \ldots$$

$$\alpha_{\theta r} = \int_S dm\ A_r(x,y)x \quad \text{with} \quad r = 1, 2, 3, \ldots$$

$$M_r = \int_S dm\ A_r^2(x,y)$$

It should be noted that use has been made of the orthogonality relation between normal modes in the above development. Also, depending on the use of either symmetrical or antisymmetrical normal modes, certain terms will be zero. Furthermore, since the origin of the principle-axis coordinate system is here taken at the virtual center of gravity of the aircraft, all static moments S_x, S_y, and I_{xy} will be zero. For normal modes it may be shown that the terms α_{zr}, $\alpha_{\theta r}$, and $\alpha_{\phi r}$ are zero also. However, they are retained here for generality. Thus, the mass matrix may be written

$$[m] = \begin{bmatrix} M_A & S_y & S_x & \alpha_{z1} & \alpha_{z2} & \cdots \\ S_y & I_y & I_{xy} & \alpha_{\theta 1} & \alpha_{\theta 2} & \cdots \\ S_x & I_{xy} & I_x & \alpha_{\phi 1} & \alpha_{\phi 2} & \cdots \\ \alpha_{z1} & \alpha_{\theta 1} & \alpha_{\phi 1} & M_1 & 0 & \cdots \\ \alpha_{z2} & \alpha_{\theta 2} & \alpha_{\phi 2} & 0 & M_2 & \cdots \end{bmatrix}$$

which is symmetric about the main diagonal.

Aerodynamic Matrix $\left[\overset{\circ}{\alpha}\right]$

The aerodynamic matrix $\left[\overset{\circ}{\alpha}\right]$ derives from those terms in the virtual work expression due to $\dot{q}$. These can be traced to the aerodynamic forces due to motion of the aircraft. A typical row from this matrix, say $\left[\overset{\circ}{\alpha}_{3j}\right]$, as obtained from $\partial U_e/\partial\dot{\phi}$ follows:

$$\underset{j=1,2,3,\ldots}{\left[\overset{\circ}{\alpha}_{3j}\right]}\left\{\dot{q}\right\} = \int_{-b}^{b} y\,dy\left\{\bar{q}c(y)C_{l_\alpha}(y)\left[\frac{\dot{z} + \dot{\theta}\bar{x} + \dot{\phi}y + \sum A_r(\bar{x},y)\dot{\xi}_r}{U}\right]\right\} +$$

$$\int_{-b'}^{b'} y\,dy\left\{\overline{q'}c'(y)C_{l_\alpha}{}'(y)\left[\frac{\dot{z} + \dot{\theta}\overline{x'} + \dot{\phi}y + \sum A_r(\overline{x'},y)\dot{\xi}_r}{U'}\right]\right\}$$

$$= \dot{z}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ yc(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ yc'(y)C_{l_\alpha}{}'(y)\right] +$$

$$\dot{\theta}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ y\bar{x}c(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ y\overline{x'}c'(y)C_{l_\alpha}{}'(y)\right] +$$

$$\dot{\phi}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ y^2c(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ y^2c'(y)C_{l_\alpha}{}'(y)\right] +$$

$$\sum_r \dot{\xi}_r\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ A_r(\bar{x},y)yc(y)C_{l_\alpha}(y) +\right.$$

$$\left.\frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ A_r(\overline{x'},y)yc'(y)C_{l_\alpha}{}'(y)\right]$$

$$= \dot{z}\overset{\circ}{\alpha}_{31} + \dot{\theta}\overset{\circ}{\alpha}_{32} + \dot{\phi}\overset{\circ}{\alpha}_{33} + \sum_r \dot{\xi}_r\overset{\circ}{\alpha}_{3r}$$

where the definition of the elements $\overset{\circ}{a}_{31}$, $\overset{\circ}{a}_{32}$, . . . is obvious. Continuation of this development yields the remaining matrix elements:

$$\underset{j=1,2,3,\ldots}{\left\lfloor \overset{\circ}{a}_{1j} \right\rfloor} \left\{\dot{q}\right\} = \dot{z}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ c(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ c'(y)C_{l_\alpha}{}'(y)\right] + \dot{\theta}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ c(y)C_{l_\alpha}(y)\bar{x} + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ c'(y)C_{l_\alpha}{}'(y)\overline{x'}\right] +$$

$$\dot{\phi}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ yc(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ yc'(y)C_{l_\alpha}{}'(y)\right] + \sum_r \dot{\xi}_r\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ A_r(\bar{x},y)c(y)C_{l_\alpha}(y) +\right.$$

$$\left.\frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ A_r(\overline{x'},y)c(y)C_{l_\alpha}(y)\right]$$

. . . .

$$\underset{j=1,2,3,\ldots}{\left\lfloor a_{2j} \right\rfloor} \left\{\dot{q}\right\} = \dot{z}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ \tilde{x}c(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ \tilde{x}c'(y)C_{l_\alpha}{}'(y)\right] + \dot{\theta}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ \tilde{x}\bar{x}c(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ \tilde{x}'\overline{x'}c'(y)C_{l_\alpha}{}'(y)\right] +$$

$$\dot{\phi}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ \tilde{x}yc(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ \tilde{x}'yc'(y)C_{l_\alpha}{}'(y)\right] + \sum_r \dot{\xi}_r\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ \tilde{x}A_r(\bar{x},y)c(y)C_{l_\alpha}(y) +\right.$$

$$\left.\frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ \tilde{x}'A_r(\overline{x'},y)c'(y)C_{l_\alpha}{}'(y)\right]$$

. . . .

$$\underset{j=1,2,3,\ldots}{\left\lfloor a_{4j} \right\rfloor} \left\{\dot{q}\right\} = \dot{z}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ A_1(\tilde{x},y)c(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ A_1(\tilde{x}',y)c'(y)C_{l_\alpha}{}'(y)\right] + \dot{\theta}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ \bar{x}A_1(\tilde{x},y)c'(y)C_{l_\alpha}(y) +\right.$$

$$\left.\frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ \overline{x'}A_1(\tilde{x},y)c'(y)C_{l_\alpha}{}'(y)\right] + \dot{\phi}\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ yA_1(\tilde{x},y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U}\int_{-b'}^{b'} dy\ yA_1(\tilde{x}',y)c'(y)C_{l_\alpha}{}'(y)\right] +$$

$$\sum_r \dot{\xi}_r\left[\frac{\bar{q}}{U}\int_{-b}^{b} dy\ A_1(\tilde{x},y)A_r(\tilde{x},y)c(y)C_{l_\alpha}(y) + \frac{\overline{q'}}{U'}\int_{-b'}^{b'} dy\ A_1(\tilde{x}',y)A_r(\overline{x'},y)c'(y)C_{l_\alpha}{}'(y)\right]$$

. . . .

This matrix is not symmetric since $\tilde{x} \neq \bar{x}$.

Aerodynamic Matrix $\left[\alpha\right]$

The aerodynamic matrix $\left[\alpha\right]$ may be derived in the same way as the matrix $\left[\overset{o}{\alpha}\right]$ except that one considers only the static terms $\left\{q\right\}$, which contribute to the "generalized force" $\partial E_e/\partial q$. For example, the third row of $\left[\alpha\right]$ is given by the expression

$$\left\lfloor \alpha_{3j} \right\rfloor_{j=2,4,5,\ldots} \left\{q\right\} = \int_{-b}^{b} -y \, dy \left\{ \bar{q} c(y) C_{l_\alpha}(y) \left[\theta + \sum \Theta_r(\bar{x},y)\xi_r \right] \right\} +$$

$$\int_{-b'}^{b'} -y \, dy \left\{ \overline{q'} c'(y) C_{l_\alpha}'(y) \left[\theta + \sum \Theta_r(\overline{x'},y)\xi_r \right] \right\}$$

$$= \theta \left[\bar{q} \int_{-b}^{b} -y \, dy \; c(y) C_{l_\alpha}(y) + \overline{q'} \int_{-b'}^{b'} -y \, dy \; c'(y) C_{l_\alpha}'(y) \right] +$$

$$\sum \xi_r \left[\bar{q} \int_{-b}^{b} -y \, dy \; \Theta_r(\bar{x},y) c(y) C_{l_\alpha}(y) + \right.$$

$$\left. \overline{q'} \int_{-b'}^{b'} -y \, dy \; \Theta_r(\overline{x'},y) c'(y) C_{l_\alpha}'(y) \right]$$

and so on. The matrix $\left[\alpha\right]$ is not symmetric.

Elasticity Matrix $\left[e\right]$

The elasticity matrix $\left[e\right]$ may be derived by noting that it derives from $\partial E_i / \partial q$ where $q = \xi_r$ and where

$$E_i = 1/2 \sum M_r \omega_r^2 \xi_r^2$$

Thus, this matrix is a diagonal matrix of the form

$$\left[e\right] = \begin{bmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & M_1\omega_1^2 & \\ 0 & & & & M_2\omega_2^2 \end{bmatrix}$$

Forcing-Function Matrix $\left[F_{jj}\right]$

The forcing-function matrix does not depend in any way upon the generalized coordinates. However, as will be presently shown, it is a diagonal matrix whose elements depend upon the result of forming $\partial E_e / \partial q$ for $q = z, \theta, \ldots$.

Consideration may be given to that part of $\partial E_e / \partial q$ due to turbulence as follows:

$$\frac{\partial}{\partial q} \int_{-b}^{b} dy \left[z + \theta\tilde{x} + \phi y + \sum A_r(\tilde{x},y)\xi_r\right] \bar{q}c(y)C_{l_\alpha}(y) \int_{-\infty}^{\infty} \dot{\psi}(t - \tau,y)\alpha_i(\tau,y)\, d\tau +$$

$$\frac{\partial}{\partial q} \int_{-b'}^{b'} dy \left[z + \theta\tilde{x}' + \phi y + \sum A_r(\tilde{x}',y)\xi_r\right] \overline{q'}c'(y)C_{l_\alpha}'(y) \int_{-\infty}^{\infty} \dot{\psi}'(t - \tau,y)\alpha_i(\tau,y)\, d\tau$$

If the partials are formed, one obtains

$$\frac{\partial}{\partial z} = \bar{q} \int_{-b}^{b} dy\ c(y)C_{l_\alpha}(y) \int_{-\infty}^{\infty} \dot{\psi}(t - \tau,y)\alpha_i(\tau,y)\ d\tau + \ldots$$

$$\frac{\partial}{\partial \theta} = \bar{q} \int_{-b}^{b} dy\ \tilde{x}c(y)C_{l_\alpha}(y) \int_{-\infty}^{\infty} \dot{\psi}(t - \tau,y)\alpha_i(\tau,y)\ d\tau + \ldots$$

$$\frac{\partial}{\partial \phi} = \bar{q} \int_{-b}^{b} dy\ yc(y)C_{l_\alpha}(y) \int_{-\infty}^{\infty} \dot{\psi}(t - \tau,y)\alpha_i(\tau,y)\ d\tau + \ldots$$

$$\frac{\partial}{\partial \xi_r} = \bar{q} \int_{-b}^{b} dy\ A_r(x,y)c(y)C_{l_\alpha}(y) \int_{-\infty}^{\infty} \dot{\psi}(t - \tau,y)\alpha_i(\tau,y)\ d\tau + \ldots$$

Using Taylor's hypothesis, it may be shown that a typical term from the above ensemble may be written

$$\int_{-b}^{b} dy\ \Lambda_j(y) \int_{-\infty}^{\infty} \overset{\odot}{\psi}\left[\frac{\zeta - x_o - x}{c(y)}\right]\alpha_i(x,y)\ dx$$

where $x_o = x_o(y)$ = Distance parallel to x-axis from center of gravity to midchord and $\overset{\odot}{\psi} \equiv d\psi/dx$.

Note that $Ut = \zeta$ and also that the turbulence $\alpha_i(x,y)$ is phased with respect to the aircraft center of gravity. Accordingly, the elements of F_{jj} are

$$f_{jj} = \int_{-b}^{b} dy\ \Lambda_j(y) \int_{-\infty}^{\infty} \overset{\odot}{\psi}\left[\frac{\zeta - x_o - x}{c(y)}\right]\alpha_i(x,y)\ dx +$$

$$\int_{-b'}^{b'} dy\ \Lambda'_j(y) \int_{-\infty}^{\infty} \overset{\odot}{\psi}\left[\frac{\zeta - x_o' - x}{c'(y)}\right]'\alpha_i(x,y)\ dx$$

The quantity $\Lambda_j(y)$ must be chosen in accordance with the relations:

$$\Lambda_1(y) = \bar{q}c(y)C_{l_\alpha}(y)$$

$$\Lambda_2(y) = \bar{q}\tilde{x}c(y)C_{l_\alpha}(y)$$

$$\Lambda_3(y) = \bar{q}yc(y)C_{l_\alpha}(y)$$

$$\Lambda_4(y) = \bar{q}A_1(\tilde{x},y)c(y)C_{l_\alpha}(y)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Lambda_1'(y) = \bar{q}'c'(y)C_{l_\alpha}'(y)$$

$$\Lambda_2'(y) = \bar{q}'\tilde{x}c'(y)C_{l_\alpha}'(y)$$

REFERENCES

1. Rhode, Richard V., and Lundquist, Eugene E.: Preliminary Study of Applied Load Factors in Bumpy Air. NACA TN 374, 1931.

2. Donely, Philip: Summary of Information Relating to Gust Loads on Airplanes. NACA Rep. 997, 1950. (Formerly NACA TN 1976.)

3. Clementson, Gerhardt C.: An Investigation of the Power Spectral Density of Atmospheric Turbulence. Sc. D. Thesis, M.I.T., May 1950.

4. Press, Harry, and Houbolt, John C.: Some Applications of Generalized Harmonic Analysis to Gust Loads on Airplanes. Preprint No. 449, Inst. Aero. Sci., Jan. 1954.

5. Liepmann, H. W.: On the Application of Statistical Concepts to the Buffeting Problem. Jour. Aero. Sci., vol. 19, no. 12, Dec. 1952, pp. 793-800, 822.

6. Liepmann, H. W.: Extension of the Statistical Approach to Buffeting and Gust Response of Wings of Finite Span. Jour. Aero. Sci., vol. 22, no. 3, Mar. 1955, pp. 197-200.

7. Fung, Y. C.: Statistical Aspects of Dynamic Loads. Jour. Aero. Sci., vol. 20, no. 5, May 1953, pp. 317-330.

8. Diederich, Franklin W.: The Response of an Airplane to Random Atmospheric Disturbances. Ph. D. Thesis, C.I.T., 1954.

9. Hakkinen, Raimo J., and Richardson, A. S., Jr.: Theoretical and Experimental Investigation of Random Gust Loads. Part I - Aerodynamic Transfer Function of a Simple Wing Configuration in Incompressible Flow. NACA TN 3878, 1956.

10. Lamson, P.: Measurements of Lift Fluctuations Due to Turbulence. NACA TN 3880, 1956.

11. V. Kármán, Theodore, and Biot, Maurice A.: Mathematical Methods in Engineering. McGraw-Hill Book Co., Inc., 1940.

12. Bisplinghoff, Raymond L., Ashley, Holt, and Halfman, Robert L.: Aeroelasticity. Addison-Wesley Pub. Co., Inc. (Cambridge), 1955.

13. Kuethe, A. M., and Schetzer, J. D.: Foundations of Aerodynamics, John Wiley & Sons, Inc., 1950.

14. Rice, S. O.: Mathematical Analysis of Random Noise. Bell System Tech. Jour., vol. 23, no. 3, July 1944, pp. 282-332, and vol. 24, no. 1, Jan. 1945, pp. 46-108.

15. Zbrozek, J. K.: A Study of the Longitudinal Response of Aircraft to Turbulent Air. Rep. No. Aero. 2530, British R.A.E., Jan. 1955.

16. Summers, Robert A.: A Statistical Description of Large-Scale Atmospheric Turbulence. Sc. D. Thesis, M.I.T., May 1954.

17. Notess, C. B., and Eakin, G. J.: Flight Test Investigation of Turbulence Spectra at Low Altitudes Using a Direct Method for Measuring Gust Velocities. Contract AF-33(616)-174, WADC and Cornell Aero. Lab., Inc., July 1, 1954.

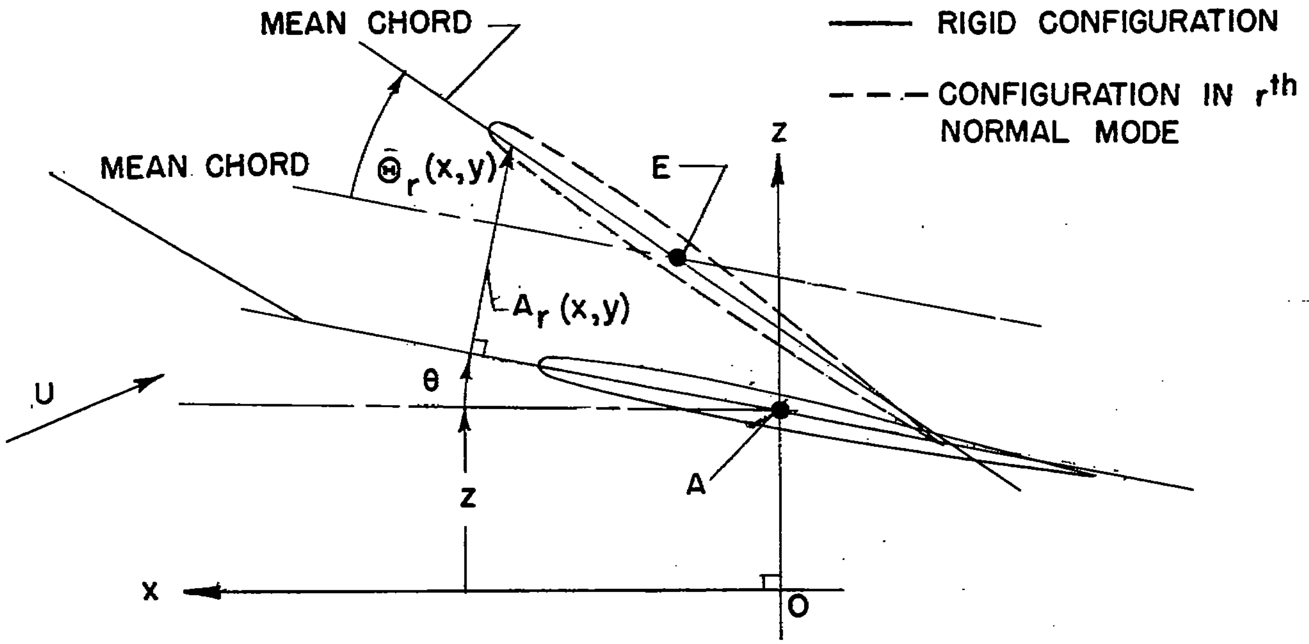
18. Brenner, Claude W., and Isaksen, Gabriel: A Parametric Investigation of Gust Loads and Airplanes in Two Degrees of Freedom. Contract NOa(s)51-183-c, Bur. Aero., Navy Dept., and Aeroelastic and Structures Res. Lab., M.I.T., Oct. 6, 1952.

19. Press, Harry: An Approach to the Prediction of the Frequency Distribution of Gust Loads on Airplanes in Normal Operations. NACA TN 2660, 1952.

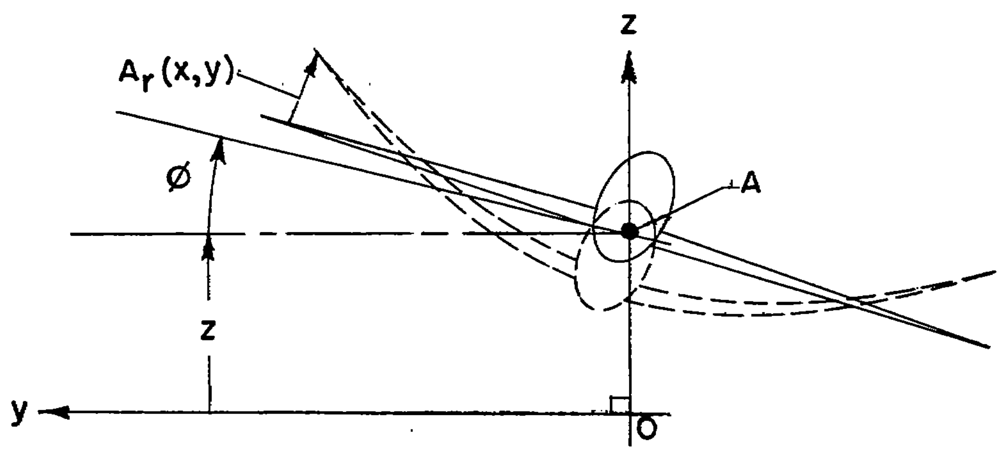
TABLE I.- DETERMINATION OF NUMBER OF PRIMITIVE CROSS

CORRELATIONS NEEDED IN ANALYSIS

Number of primitive cross correlations for -						
Center-of-gravity acceleration		Wing-root bending moment				
N	M	N	M = 1	M = 2	M = 3	M = 4
2	4	2	4	16	36	64
3	9	3	9	36	81	144
4	16	4	16	64	144	256
5	25	5	25	100	255	400
6	36	6	36	144	324	576

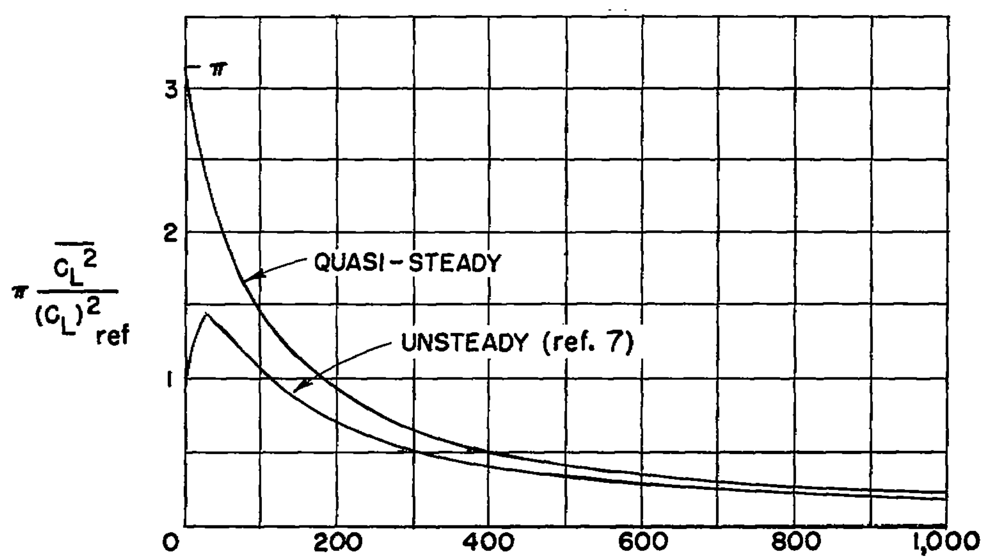


(a) xz-plane, left side view.

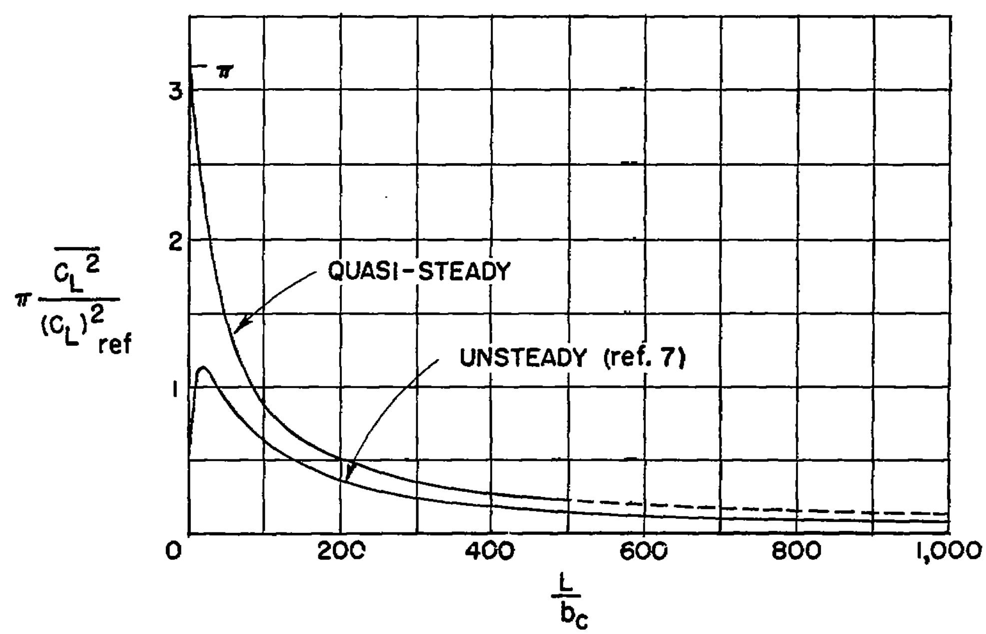


(b) yz-plane, front view.

Figure 1.- Coordinate system for dynamic analysis. Point A is center of gravity of rigid configuration; point E is center of elastic twist; $\Theta_r(x,y) = \Theta_r(y)$; point O is average center-of-gravity position of rigid configuration.



(a) $\kappa = 100$.



(b) $\kappa = 50$.

Figure 2.- Comparison of mean-square lift response using unsteady and quasi-steady aerodynamic theory.

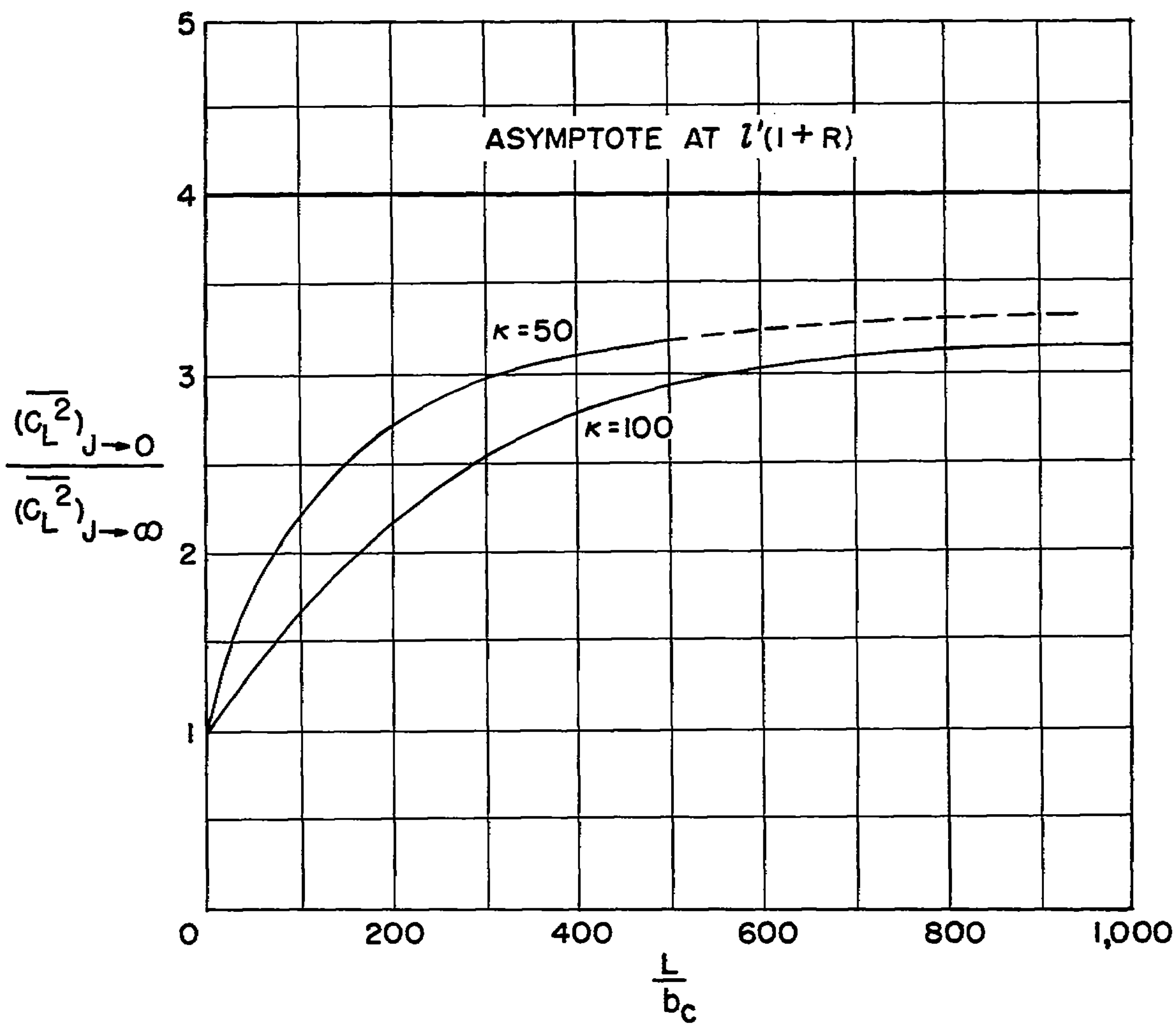


Figure 3.- Effect of pitching degree of freedom on mean-square lift response. $l'(1 + R) = 4.0$; center of gravity at 50 percent chord.

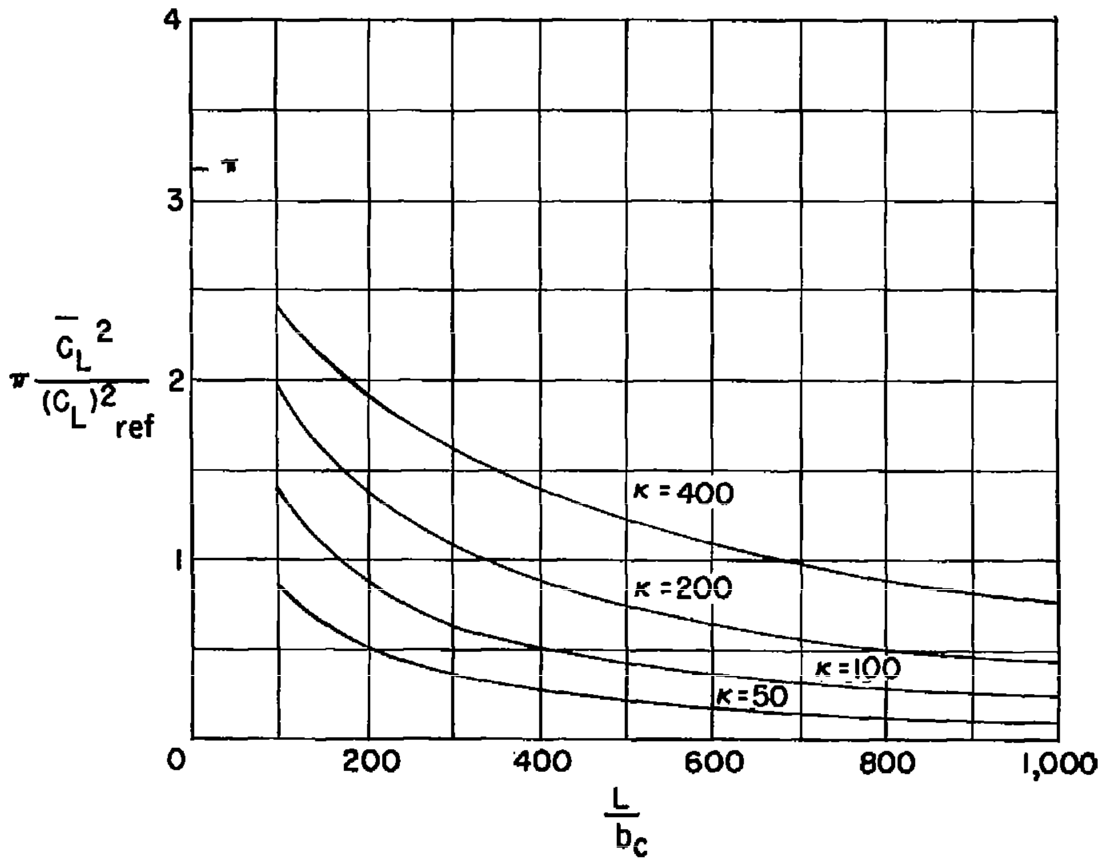


Figure 4.- Comparison of mean-square lift response with sharp-edged-gust formula.