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A METHOD FOR THE CALCULATION OF THE LATERAL RESPONSE OF AIRPLANES TO RANDOM TURBULENCE

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A METHOD FOR THE CALCULATION OF THE LATERAL RESPONSE OF AIRPLANES TO RANDOM TURBULENCE

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SUMMARY

A relatively short method of calculating the lateral motions of an airplane due to random atmospheric turbulence is presented. The gust velocities are represented as equivalent rigid-body rotations of the airplanes; namely, rolling gusts, yawing gusts, and side gusts. Random distributions of gust velocities across the span are taken into account in defining the rolling and yawing gusts. Complex stability derivatives are used to account for the random distribution of side gusts along the fuselage and vertical tail and the lag effect incurred as the airplane penetrates the gusts. The suggested gust spectrum is based on a simple analytical expression which fits available measurements of atmospheric turbulence. A 45-step sample calculation procedure for obtaining the response of the airplane in each degree of freedom is presented in tabular form.

INTRODUCTION

Most calculations of the response of airplanes to gusts have been made on the assumption that the effect of the gust on the airplane is approximately equivalent to the effect of a rigid-body motion of the airplane producing a distribution of angle of attack similar to that caused by the gust. On the basis of this assumption, disturbances in the form of rolling gusts, yawing gusts, and side gusts have been employed in calculating lateral response to gust disturbances. This approach is convenient because the standard aerodynamic stability derivatives which are used in airplane stability calculations may also be used to determine moments caused by the gust velocities. This approach as usually applied neglects effects due to lag in penetration of the gusts by different parts of the airplane and, because linear gradients of the gust velocity along the span are assumed, it cannot account for the random spanwise distribution of gust velocities encountered in flight through atmospheric turbulence. Furthermore, the relations between the magnitudes of the rolling, yawing, and side gusts required to produce effects similar to actual atmospheric turbulence are not known beforehand.

In reference 1, a theoretical method for calculating the lateral response of an airplane to atmospheric turbulence has been proposed which accounts in a rather complete manner for the effects neglected or approximated in previous methods. This method uses an approach somewhat different from that described in the preceding paragraph in that the forces and moments applied to the airplane by gusts are determined in power-spectral form in terms of the horizontal, vertical, and side components of gust velocity.

In the present report, it is shown that the more conventional method of assuming gust-velocity distributions equivalent to rigid-body motions of the airplane may be refined to provide results equivalent to those given by the method of reference 1. The refinements consist in replacing some of the constant aerodynamic stability derivatives with complex quantities to account for the gust penetration effects and determining the correct relations between the spectra of rolling, yawing, and side gusts to yield results in agreement with the more exact analysis. The present method requires somewhat simpler calculations than the method of reference 1 and provides a clearer physical picture of the relations between the various sources of lateral gust disturbances.

SYMBOLS

b wing span

D nondimensional operator, $\frac{b}{U}\frac{d}{dt}$

G matrix containing stability derivatives relating airplane moments and forces to gust velocities

$\tilde{G}$ alternate form of G containing frequency-dependent stability derivatives

h height of center of pressure of vertical tail above X-axis of airplane

I imaginary part of a complex number

$i = \sqrt{-1}$

K_X nondimensional radius of gyration about X-axis

K_Z nondimensional radius of gyration about Z-axis

K_{XZ} nondimensional product of inertia

$k' = \omega L/U$

L integral scale of turbulence

l_t tail length between airplane center of gravity and quarter-chord point of mean aerodynamic chord of vertical tail

p rolling velocity, $d\phi/dt$

q dynamic pressure, $\frac{1}{2}\rho U^2$

r yawing velocity, $d\psi/dt$

S wing area

T time interval over which power spectrum is evaluated (eq. 18)

t time

U relative velocity between airplane and general air mass

u velocity along X-axis

v velocity along Y-axis

w velocity along Z-axis

X,Y,Z three orthogonal reference axes of airplane

C_L lift coefficient, $\frac{\text{Lift}}{qS}$

C_l rolling-moment coefficient, $\frac{\text{Rolling moment}}{qSb}$

C_n yawing-moment coefficient, $\frac{\text{Yawing moment}}{qSb}$

C_Y side-force coefficient, $\frac{\text{Side force}}{qS}$

α angle of attack

β angle of sideslip

$\beta' = b/L$

γ flight-path angle

Δ matrix containing airplane equations of motion in still air

λ wavelength, $\frac{2\pi U}{\omega}$

μ airplane relative density factor, $\frac{\text{Mass}}{\rho S b}$

ρ density of atmosphere

σ sidewash angle

Φ power spectral density

ϕ angle of roll

ψ angle of yaw

ω circular frequency

$\omega' = \omega b/U$

Stability derivatives of airplane are indicated by subscript notations; for example,

$$C_{l_p} = \frac{\partial C_l}{\partial\left(\frac{pb}{2U}\right)} \qquad C_{n_r} = \frac{\partial C_n}{\partial\left(\frac{rb}{2U}\right)} \qquad C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

Subscripts:

o general air mass

g gusts

W wing

F fuselage

T vertical tail

Matrix notation:

$\lvert\ \rvert$	determinant or absolute value of quantity
$[\ ]$	rectangular matrix
$\{\ \}$	column matrix
$\lfloor\ \rfloor$	row matrix

Bar over a quantity denotes mean value. Asterisk denotes complex conjugate.

THEORY

Equations of Motion

The equations of lateral motion for an airplane (ref. 2) are given by

$$\left.\begin{aligned}
&2\mu K_X{}^2D^2\phi_o - \frac{1}{2}C_{l_p}D\phi - 2\mu K_{XZ}D^2\psi_o - \frac{1}{2}C_{l_r}D\psi - C_{l_\beta}\beta = 0\\
&-2\mu K_{XZ}D^2\phi_o - \frac{1}{2}C_{n_p}D\phi + 2\mu K_Z{}^2D^2\psi_o - \frac{1}{2}C_{n_r}D\psi - C_{n_\beta}\beta = 0\\
&-\frac{1}{2}C_{Y_p}D\phi - C_L\phi_o + 2\mu D\psi_o - \frac{1}{2}C_{Y_r}D\psi - C_L \tan\gamma\,\psi_o + 2\mu D\beta_o - C_{Y_\beta}\beta = 0
\end{aligned}\right\} \tag{1}$$

where the subscript o appearing in the inertial and weight terms is used to denote angular displacement with respect to an absolute system of axes fixed in the general air mass. In calculations of the motion of an airplane in still air, the angular displacements and velocities appearing in the aerodynamic terms are identical with these values. When flying in turbulent air, however, the airplane is subjected to the motion of local air masses, generally referred to as gusts. The relative linear and angular velocities of the airplane with respect to the local air mass each may be considered as made up of two parts:

$$\left.\begin{aligned} D\phi &= D\phi_o + D\phi_g \\ D\psi &= D\psi_o + D\psi_g \\ \beta &= \beta_o + \beta_g \end{aligned}\right\} \tag{2}$$

where the subscript g is used to denote gust velocities. Substituting equations (2) into equations (1) and transposing the terms resulting from gust disturbances to the right-hand side of the equation gives the result written in the convenient matrix form:

$$\begin{bmatrix}\Delta\end{bmatrix}\begin{Bmatrix}\phi_o \\ \psi_o \\ \beta_o\end{Bmatrix} = \begin{bmatrix}G\end{bmatrix}\begin{Bmatrix}D\phi_g \\ D\psi_g \\ \beta_g\end{Bmatrix} \tag{3}$$

The matrix

$$\begin{bmatrix}\Delta\end{bmatrix} = \begin{bmatrix} 2\mu K_X{}^2D^2 - \frac{1}{2}C_{l_p}D & -2\mu K_{XZ}D^2 - \frac{1}{2}C_{l_r}D & -C_{l_\beta} \\ -2\mu K_{XZ}D^2 - \frac{1}{2}C_{n_p}D & 2\mu K_Z{}^2D^2 - \frac{1}{2}C_{n_r}D & -C_{n_\beta} \\ -\frac{1}{2}C_{Y_p}D - C_L & \left(2\mu - \frac{1}{2}C_{Y_r}\right)D - C_L \tan\gamma & 2\mu D - C_{Y_\beta} \end{bmatrix} \tag{4}$$

is the familiar "still air" rigid-airframe characteristic equation, and the matrix

$$\begin{bmatrix}G\end{bmatrix} = \begin{bmatrix} \frac{1}{2}C_{l_p} & \frac{1}{2}C_{l_r} & C_{l_\beta} \\ \frac{1}{2}C_{n_p} & \frac{1}{2}C_{n_r} & C_{n_\beta} \\ \frac{1}{2}C_{Y_p} & \frac{1}{2}C_{Y_r} & C_{Y_\beta} \end{bmatrix} \tag{5}$$

gives the relationship between the aerodynamic moments and forces resulting from gust velocities encountered.

In the classical method of treating simple isolated gust inputs (for example, ref. 3), the elements of the G matrix would be the "still air" stability derivatives, the values in the gust-velocity matrix $\left(D\phi_g,\ D\psi_g,\ \text{and}\ \beta_g\right)$ would be those appropriate to step or ramp functions, and the solution would be defined by

$$\begin{Bmatrix} \phi_o(\omega) \\ \psi_o(\omega) \\ \beta_o(\omega) \end{Bmatrix} = \left[\Delta(\omega)\right]^{-1}\left[G\right]\begin{Bmatrix} D\phi_g(\omega) \\ D\psi_g(\omega) \\ \beta_g(\omega) \end{Bmatrix} \tag{6}$$

where $\left[\Delta\right]^{-1}$ is the inverse of $\left[\Delta\right]$ with $D = i\frac{b}{U}\omega$. Unique time responses could then be obtained by taking the inverse Fourier transform of each solution.

However, when the responses of the airplane to continuous random gusts are to be considered, the gust velocities can be defined only in a statistical (power-spectral) manner and the resulting moments and forces will in turn be related only in a statistical sense. This means that the elements of matrix G cannot be evaluated in the usual sense and the effect of random distributions of gust velocities along the fuselage and across the wing span must be taken into account.

Forces and Moments Due to Turbulence

In the application of the method to random turbulence, the sources of the forces and the moments on the airplane must be considered. As shown in reference 4, yawing and rolling moments on the wing result from gradients of the horizontal and vertical gusts. Wing rolling moments and moments and forces on the fuselage and vertical tail are produced by side gusts. Although the wing moments cannot be determined at any instant as a function of the gust velocities measured at the center of gravity, the power spectra of these moments have been determined in reference 4 as a function of the power spectrum of the vertical gust velocity as measured at one point. In isotropic turbulence, the spectrum of the side gusts measured at a point is identical to that of the vertical gusts. For this reason, the spectrum of the yawing and rolling moments may be related equally well to that of the side gusts. This procedure is used in the present analysis.

Spanwise gradient of vertical gust.- In general, the rolling moment acting at any instant results from a random distribution of vertical-gust velocity across the span which has some average linear spanwise gradient. An effective gradient which produces the rolling moment due to this random spanwise distribution may be defined by the relation

$$D\phi_g = \frac{2}{\left(C_{l_p}\right)_W} C_l(w_g)$$

which is derived from the equation

$$C_l(w_g) = \frac{1}{2}\left(C_{l_p}\right)_W D\phi_g \tag{7}$$

where $C_l(w_g)$ is the rolling moment due to vertical gusts, $\left(C_{l_p}\right)_W$ is the wing damping-in-roll stability derivative, and $D\phi_g$ is the equivalent rolling-gust gradient. Likewise, the vertical-gust distribution at any given instant also produces a yawing moment. In reference 4, this moment was assumed to be in phase with the rolling moment and was determined from the formula

$$C_n(w_g) = \left(\frac{C_{n_p}}{C_{l_p}}\right)_W C_l(w_g) \tag{8}$$

Substituting equation (7) into equation (8) gives the yawing moment in terms of the effective gust gradient:

$$C_n(w_g) = \frac{1}{2}\left(C_{n_p}\right)_W D\phi_g \tag{9}$$

Spanwise gradient of horizontal gust.- Similar arguments may be used to show that a random distribution of horizontal gusts across the span of the wing produces both rolling and yawing moments which are assumed to be in phase and related by

$$C_n(u_g) = \left(\frac{C_{n_r}}{C_{l_r}}\right)_W C_l(u_g) \tag{10}$$

In terms of the equivalent yawing-gust gradient $D\psi_g$, these moments are given by

$$C_l(u_g) = \frac{1}{2}\left(C_{l_r}\right)_W D\psi_g \tag{11}$$

$$C_n(u_g) = \frac{1}{2}\left(C_{n_r}\right)_W D\psi_g \tag{12}$$

These formulas (eqs. (7) to (12)) are in a convenient form for application to the present analysis in which the gusts are considered as equivalent rigid-body motions of the airplane. Moreover, since only the wing moments are influenced by the horizontal- and vertical-gust distributions across the span of the wing, these formulas completely account for the effect of these gusts on the lateral motion.

Side gusts.- The only remaining source of gust disturbance is the side gust. In the present report this disturbance is expressed as an equivalent sideslip $\beta_g = v_g/U$, but the method of calculating its effect is essentially the same as that used in reference 1. Effects due to the difference in time at which a given gust encounters different sections of the airplane, known as gust-penetration effects, are accounted for in reference 1 by considering the relations between the aerodynamic force and moment coefficients and the gust inputs measured at the center of gravity as frequency-dependent transfer functions. These relations are herein converted to frequency-dependent stability derivatives C_{Y_β}, C_{n_β}, and C_{l_β} by multiplying the expressions used in reference 1 by the flight velocity U. Expressions for these stability derivatives are given in appendix A.

To an observer in the airplane the side-gust disturbance appears as a random velocity distribution which moves along the fuselage and vertical tail at the mean velocity U. If this speed in terms of body lengths per unit time is large (as is usually the case with most airplanes), these gust velocities will not change appreciably during their "time of exposure" to the fuselage. When the condition is satisfied, the forces and moments acting on the fuselage are uniquely, rather than statistically, related to the side-gust distribution along the flight path, and phase relations can be correctly accounted for by treating the frequency-dependent derivatives as complex quantities.

Matrix Solution to Equations of Motion

The effects of the gusts on the wing and the fuselage-tail combination may be incorporated into a matrix similar to the simplified G matrix of equation (5). If this matrix is denoted by $\tilde{G}(\omega)$ and if only those force and moment coefficients are included which will have a significant effect (for example, side forces due to rolling and yawing are usually negligible compared with side force due to sideslip), the modified G matrix is defined as

$$\left[\tilde{G}(\omega)\right] = \begin{bmatrix} \frac{1}{2}\left(C_{l_p}\right)_W & \frac{1}{2}\left(C_{l_r}\right)_W & \left[C_{l_\beta}(\omega)\right]_{WT} \\ \frac{1}{2}\left(C_{n_p}\right)_W & \frac{1}{2}\left(C_{n_r}\right)_W & \left[C_{n_\beta}(\omega)\right]_{FT} \\ 0 & 0 & \left[C_{Y_\beta}(\omega)\right]_{FT} \end{bmatrix} \tag{13}$$

In terms of this matrix, the solution to the equations of motion given by equation (3) is obtained formally by inversion:

$$\begin{Bmatrix} \phi_o(\omega) \\ \psi_o(\omega) \\ \beta_o(\omega) \end{Bmatrix} = \left[\Delta(\omega)\right]^{-1}\left[\tilde{G}(\omega)\right]\begin{Bmatrix} D\phi_g(\omega) \\ D\psi_g(\omega) \\ \beta_g(\omega) \end{Bmatrix} \tag{14}$$

The inverse of the Δ matrix given by equation (4) is shown in appendix B to consist of transfer functions relating the response in lateral angular displacements to a sinusoidal rolling moment, yawing moment, or side force of unit amplitude:

$$\left[\Delta(\omega)\right]^{-1} = \begin{bmatrix} \frac{\phi}{C_l}(\omega) & \frac{\phi}{C_n}(\omega) & \frac{\phi}{C_Y}(\omega) \\ \frac{\psi}{C_l}(\omega) & \frac{\psi}{C_n}(\omega) & \frac{\psi}{C_Y}(\omega) \\ \frac{\beta}{C_l}(\omega) & \frac{\beta}{C_n}(\omega) & \frac{\beta}{C_Y}(\omega) \end{bmatrix} \tag{15}$$

Inasmuch as the random nature of the moments resulting from random turbulence makes it necessary to place the result given by equation (14) in a power-spectral form, both sides of the equation must be squared. If equation (14) is squared in its present form, cross-power terms between all the forces and moments will appear in the final result. However, if the product of $\left[\Delta\right]^{-1}$ and $\left[\tilde{G}(\omega)\right]$ is taken beforehand, then when the product matrix is squared these cross-power relationships do not appear explicitly since their equivalent effect has been taken into account in the multiplication process. It is shown in appendix C that the result obtained by first multiplying these two matrices before squaring is the same as the result obtained by squaring each matrix separately and including cross-power terms.

Each element in the matrix product of $\left[\Delta\right]^{-1}$ and $\left[\tilde{G}(\omega)\right]$ has the form of a transfer function relating one of the airplane response quantities to one of the gust components. This relationship is indicated as follows:

$$\left[\Delta^{-1}\tilde{G}(\omega)\right] = \begin{bmatrix} \frac{\phi}{D\phi_g}(\omega) & \frac{\phi}{D\psi_g}(\omega) & \frac{\phi}{\beta_g}(\omega) \\ \frac{\psi}{D\phi_g}(\omega) & \frac{\psi}{D\psi_g}(\omega) & \frac{\psi}{\beta_g}(\omega) \\ \frac{\beta}{D\phi_g}(\omega) & \frac{\beta}{D\psi_g}(\omega) & \frac{\beta}{\beta_g}(\omega) \end{bmatrix} \tag{16}$$

In terms of this matrix product, the relations between the power spectra of the airplane response and the gust components are given in the form

$$\begin{Bmatrix} \Phi_{\phi} \\ \Phi_{\psi} \\ \Phi_{\beta} \end{Bmatrix} = \begin{bmatrix} \left|\frac{\phi}{D\phi_g}\right|^2 & \left|\frac{\phi}{D\psi_g}\right|^2 & \left|\frac{\phi}{\beta_g}\right|^2 \\ \left|\frac{\psi}{D\phi_g}\right|^2 & \left|\frac{\psi}{D\psi_g}\right|^2 & \left|\frac{\psi}{\beta_g}\right|^2 \\ \left|\frac{\beta}{D\phi_g}\right|^2 & \left|\frac{\beta}{D\psi_g}\right|^2 & \left|\frac{\beta}{\beta_g}\right|^2 \end{bmatrix} \begin{Bmatrix} \Phi_{D\phi_g} \\ \Phi_{D\psi_g} \\ \Phi_{\beta_g} \end{Bmatrix} \tag{17}$$

where the power spectrum of a quantity X is defined by

$$\Phi_X = \lim_{T \to \infty} \frac{1}{T}|X|^2 \tag{18}$$

The assumption has been made that any cross power between the three gust components is either zero or negligible. Justification for this assumption is made in a subsequent section.

Finally, the gust inputs in isotropic turbulence are specified in terms of a single quantity. The spectrum of β_g is selected for this purpose inasmuch as β_g is directly related to the linear gust component v_g for which the spectrum is available from turbulence theory. Therefore, the final form of the equations is given by

$$\begin{Bmatrix} \Phi_\phi \\ \Phi_\psi \\ \Phi_\beta \end{Bmatrix} = \begin{bmatrix} \left|\frac{\phi}{D\phi_g}\right|^2 & \left|\frac{\phi}{D\psi_g}\right|^2 & \left|\frac{\phi}{\beta_g}\right|^2 \\ \left|\frac{\psi}{D\phi_g}\right|^2 & \left|\frac{\psi}{D\psi_g}\right|^2 & \left|\frac{\psi}{\beta_g}\right|^2 \\ \left|\frac{\beta}{D\phi_g}\right|^2 & \left|\frac{\beta}{D\psi_g}\right|^2 & \left|\frac{\beta}{\beta_g}\right|^2 \end{bmatrix} \begin{Bmatrix} \left|\frac{D\phi_g}{\beta_g}\right|^2 \\ \left|\frac{D\psi_g}{\beta_g}\right|^2 \\ 1 \end{Bmatrix} \Phi_{\beta_g} \tag{19}$$

Gust Spectra and Their Relationships

The following relationships for the gust spectra and gust velocities are based on the assumptions of homogeneous isotropic turbulence:

$$\left.\begin{aligned} \Phi_{w_g}(\omega) &= \Phi_{v_g}(\omega) = U^2\Phi_{\beta_g}(\omega) \\ \overline{u_g^2} &= \overline{v_g^2} = \overline{w_g^2} \end{aligned}\right\} \tag{20}$$

Physically, these relations state that, regardless of the motions or the direction of travel of the airplane, the vertical- and side-gust components measured at the same point on the airplane have the same spectrum and that the mean-square value of all three gust components is the same.

For the purposes of equation (19), the power spectra of gust gradients as a function of the gust spectra measured at one point are required in the form of the ratios of the power spectra of $D\phi_g$ and $D\psi_g$ to the power spectrum of β_g when the latter is known. Taking the square of the absolute value of equation (7) and considering the equation to apply at any given frequency yields

$$\left.\begin{aligned} \left|D\phi_g\right|^2 &= \frac{4}{\left(C_{l_p}\right)_W^{\,2}}\left|C_l(w_g)\right|^2 \\ \Phi_{D\phi_g} &= \frac{4}{\left(C_{l_p}\right)_W^{\,2}}\Phi_{C_l(w_g)} \end{aligned}\right\} \qquad (21)$$

Dividing both sides of equations (21) by the power spectrum of the side gust or vertical gust (see eqs. (20)) as measured at the center of gravity gives

$$\frac{\Phi_{D\phi_g}}{\Phi_{\beta_g}} = \frac{4U^2}{\left(C_{l_p}\right)_W^{\,2}}\frac{\Phi_{C_l(w_g)}}{\Phi_{w_g}}$$

At any given frequency, then

$$\left|\frac{D\phi_g}{\beta_g}\right|^2 = \frac{4U^2}{\left(C_{l_p}\right)_W^{\,2}}\frac{\Phi_{C_l(w_g)}}{\Phi_{w_g}} \qquad (22)$$

Expressions for both power spectra on the right-hand side of equation (22) are available in the literature. An expression based on measurements of turbulence in wind tunnels (ref. 5) which appears to fit well most of the available data from measurements of atmospheric turbulence (for example, refs. 6 and 7) is given in terms of β_g by

$$U^2\Phi_{\beta_g} = \Phi_{w_g} = \frac{\overline{w_g^{\,2}}L}{\pi U}\frac{1 + 3k'^2}{\left(1 + k'^2\right)^2} \qquad (23)$$

where $k' = \omega L/U$ and L is the so-called scale of turbulence. A plot of equation (23) is given in figure 1.

Calculations based on analytical expressions (similar to eq. (23)) for the gust spectra measured at one point are given in reference 4 for the power spectra of the coefficients of the rolling and yawing moments on wings of arbitrary span which are subject to continuous isotropic turbulence. These spectra, which take into account the random distributions of gusts across the span and along the flight path, are given in reference 4 for various values of $\beta' = b/L$ and for four spanwise lift distributions on the wing. Since the effect of different lift distributions was small, only one distribution (the rectangular distribution) is considered and the power spectrum of rolling-moment coefficient due to vertical gusts on the wing is denoted here by

$$\Phi_{C_l(w_g)} = \frac{\overline{w_g^2}L\left(C_{l_p}\right)_W^2}{\pi U^3} \times \text{Quantity plotted in fig. 7(a), ref. 4}$$

The quantity plotted in figure 7(a) of reference 4 is $\dfrac{\Phi_{C_l}(k')}{\overline{w_g^2}LC_{l_p}^2/U^3\pi}$.

Dividing the above equation by equation (23) and substituting the result into equation (22) yields

$$\left|\frac{D\phi_g}{\beta_g}\right|^2 = 4 \times \frac{\text{Quantity plotted in fig. 7(a), ref. 4}}{\dfrac{1 + 3k'^2}{(1 + k'^2)^2}} \tag{24}$$

This relationship is plotted in figure 2 for the various values of β' used in reference 4 and as a function of reduced frequency ω', where

$$\omega' = \frac{\omega b}{U} = \beta' k' \tag{25}$$

In a like manner the relationship between the power spectra of yawing gusts and side gusts may be determined. From equation (11),

$$\Phi_{D\psi_g} = \frac{4}{\left(C_{l_r}\right)_W^2}\Phi_{C_l(u_g)} \tag{26}$$

Dividing through by the power spectrum of the side gusts or vertical gusts as measured at the center of gravity gives

$$\frac{\Phi_{D\psi_g}}{\Phi_{\beta_g}} = \frac{4U^2}{\left(C_{l_r}\right)_W^2}\frac{\Phi_{C_l(u_g)}}{\Phi_{w_g}}$$

where at any given frequency

$$\left|\frac{D\psi_g}{\beta_g}\right|^2 = \frac{4U^2}{\left(C_{l_r}\right)_W^2}\frac{\Phi_{C_l(u_g)}}{\Phi_{w_g}} \tag{27}$$

Again, from reference 4, the power spectrum of the rolling-moment coefficient due to horizontal gusts on the wing is given by

$$\Phi_{C_l}(u_g) = \frac{\overline{u_g^2} L \left(C_{l_p}\right)_W^2 \alpha_W^2}{\pi U^3} \times \text{Quantity plotted in fig. 9(a), ref. 4}$$

The quantity plotted in figure 9(a) of reference 4 is $\dfrac{\Phi_{C_l}(k')}{\overline{u_g^2} L C_{l_p} \alpha_o^2 / U^3 \pi}$,

where α_o is trim angle of attack. Dividing the above equation by equation (23) and substituting the result into equation (27) yields the required relationship between the yawing- and side-gust spectra:

$$\left|\frac{D\psi_g}{\beta_g}\right|^2 \left(\frac{\alpha C_{l_p}}{C_{l_r}}\right)_W^{-2} = 4 \times \frac{\text{Quantity plotted in fig. 9(a), ref. 4}}{\dfrac{1 + 3k'^2}{(1 + k'^2)^2}} \tag{28}$$

This relationship is plotted in figure 3 as a function of ω' for a range of values of β'.

Presentation of the data of figures 2 and 3 in the form of ratios between the gust spectra is not meant to imply that these data are independent of the approximate spectra chosen for β_g. These ratios represent the filtering effect of the wing on gusts having the frequency spectrum given by equation (23), and, if other approximations to this frequency spectrum were used, the filtering effect would not necessarily be the same.

DISCUSSION

The foregoing development leads to a set of equations which hold for small disturbances about some trimmed flight condition. In the application of the method, it is necessary to know the flight conditions, the stability derivatives of the airplane at those flight conditions, and some basic physical dimensions of the airframe. Such quantities are required for any dynamics study of an airplane and, aside from the equations and figures given herein, no additional information is required to obtain the lateral response of the airplane in power-spectral form to continuous atmospheric turbulence.

As an aid in setting up the required calculations, a list of columns and steps which may be followed in the calculation of Φ_ϕ is given in table I. The column numbers and headings are listed vertically in this table; however, they would be arranged horizontally across the top of an actual calculation sheet. Although the calculation of Φ_ϕ is used as an example, the same steps are followed for the response of the airplane in any angular displacement, rate, or acceleration. For a given airplane and flight condition the first 12 steps (part (a)) may be tabulated from the equations and plots of this report. Steps 13 to 45 (part (b)) are used for the calculation of the response of the airplane in any lateral angular displacement (for the example, ϕ is used). It is necessary to tabulate the real and imaginary parts of three transfer functions relating C_l, C_n, and C_Y to that response parameter (columns (13) to (18)) and to follow the indicated steps. For each of the other response motions or their derivatives it is only necessary to substitute the appropriate transfer functions into columns (13) to (18) and to repeat the process.

In calculating the frequency-dependent stability derivatives (columns (4) to (9)), care should be taken to insure that, when $\omega = 0$, these derivatives agree with the steady-state aerodynamic derivatives used in calculating the transfer functions of equation (15).

The method requires a choice of values for the scale of turbulence L and the mean square of the turbulence $\overline{w_g^2}$. Very little information is available at present on the proper magnitude for L; however, it appears to be within the range of 1,000 to 2,000 feet and probably closer to the lower figure. The mean square of the turbulence depends on the severity of the turbulence to be considered. As an approximation, a value of $\overline{w_g^2}$ of $(3\ \text{ft/sec})^2$ for light turbulence, $(6\ \text{ft/sec})^2$ for moderate turbulence, and $(10\ \text{ft/sec})^2$ for thunderstorms may be used. (See ref. 7.)

In the present method the effects of turbulence of the airplane have been separated into the equivalent effects of rolling, yawing, and sideslip of the airplane in still air. The application of this concept, however, has been made in such a way that the effects of the u-, v-, and w-components of the gusts are taken into account as was done in reference 1. Although the procedure of this report differs from that of reference 1, exactly the same effects are treated in both methods and both methods yield the same results. This agreement has been verified by using the present method to calculate responses for the example airplanes of reference 1. Since the responses of three airplanes are presented in reference 1, no numerical examples are presented herein. Those interested in the trends and relative effects of the different gust components over certain frequency ranges will find these effects discussed therein.

One distinction should be noted between the present method and the more conventional treatment as contained in equation (3). In the conventional treatment, yawing gusts are assumed to include both the rotational effects introduced by gradients of side gust along the fuselage and by gradients of horizontal gusts across the wing span. Thus, the values of C_{l_r}, C_{n_r}, and C_{Y_r} of equation (5) would be those for the entire airplane. In the present method, the yawing gusts are assumed to include only the effects of horizontal gusts on the wing, and the values of $\left(C_{l_r}\right)_W$ and $\left(C_{n_r}\right)_W$ used in equation (13) are those for the wing alone. Rotational effects of the side gusts along the fuselage are completely accounted for by the complex stability derivatives $\left(C_{l_\beta}\right)_{WT}$, $\left(C_{n_\beta}\right)_{FT}$, and $\left(C_{Y_\beta}\right)_{FT}$. This method is used because it allows the utilization of the more accurate calculation of the fuselage penetration effects given in reference 8 and because the separation of the effects of the wing and fuselage allows the random distribution of horizontal gusts across the span to be taken into account.

With most current airplane configurations in flight at low angles of attack a simplification of equations may be obtained by neglecting the yawing gusts on the wing. In the numerical examples of reference 1 it was observed that this gust component had a negligible effect in all degrees of freedom for the airplanes investigated. This result is the basis for the assumption made in the derivation of the method used herein that the cross power between the gust-velocity components may be neglected. By the theory of isotropic turbulence only the cross power between the components herein referred to as yawing gusts and side gusts exists. However, when the yawing-gust contribution to the motion of the airplane is small, neglecting this cross power appears to be justified.

It was also found in reference 1 that the further simplification of neglecting the contribution of both rolling and yawing gusts in calculating the response in sideslip is justified.

The plots of the ratios of rolling gusts and yawing gusts to side gusts derived in this report give a physical picture of the relative importance of these gust disturbances at various frequencies. For calculation purposes, the plots of the quantities $\left|\frac{D\phi_g}{\beta_g}\right|^2$ and $\left|\frac{D\psi_g}{\beta_g}\right|^2$ as functions of frequency on log-log paper (figs. 2 and 3) are convenient, but a plot of this type gives a somewhat distorted picture of the true variations of these quantities. For this reason, plots of $\left|\frac{D\phi_g}{\beta_g}\right|$ and $\left|\frac{D\psi_g}{\beta_g}\right|$ as functions of frequency on linear scales are given in figure 4. Frequency is plotted in terms of the ratio of wing span to gust wavelength b/λ. These curves show that, for the small values of β' (ratios of wing span to scale of turbulence) ordinarily encountered, the ratio $\left|\frac{D\phi_g}{\beta_g}\right|$ or $\left|\frac{D\psi_g}{\beta_g}\right|$ falls on a single curve as a function of b/λ, except at low values of b/λ. The curves reach a peak in the neighborhood of $b/\lambda = 0.5$ to 1.0, as might be expected on the basis that the spanwise averaging effects would become important when the gust wavelength is shorter than the span. The hypothesis, expressed in reference 9, that the gust gradient measured along the flight path would give an approximation to the effective spanwise gust gradient is indicated by the dashed line drawn on the figure (fig. 4(a)). The agreement in the trends of the curves with this dashed line shows that this hypothesis may be a reasonable explanation of the mechanism of the rolling and yawing effects of turbulence over an intermediate range of wavelengths. At shorter wavelengths, the spanwise averaging causes a decrease in the effects; whereas at long wavelengths, or low frequencies, another mechanism apparently comes into play to increase the rolling and yawing effects. This mechanism is believed to be the chance encounter of the wing with rolling and yawing gusts distributed along the flight path at relatively long intervals. This effect increases with the ratio of wing span to scale of turbulence. Possibly this result occurs because a wing of larger span effectively samples a larger portion of the atmosphere and is therefore more likely to encounter rolling and yawing gusts. This effect might be important in explaining increased lateral-control difficulties of large airplanes during landing approaches in rough air, if it can be shown that the scale of turbulence decreases at low altitudes.

CONCLUDING REMARKS

A procedure is presented for calculating in power-spectral form the lateral response of airplanes to random atmospheric turbulence. By following the tabulated sample calculation procedure, these calculations may be made in a routine manner without detailed knowledge of the derivation of the method. It has been verified by using the present method to calculate the responses of the example airplanes of NACA Technical Note 3954 that the present method gives results identical to those of the method therein. The present method requires simpler calculations and provides a clearer physical picture of the relations between the various sources of lateral gust disturbances.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., October 15, 1957.

APPENDIX A

FREQUENCY-DEPENDENT STABILITY DERIVATIVES

DUE TO SIDE GUSTS

All derivatives are referred to the center of gravity of the airplane. The coefficient of rolling moment due to side gusts on the wing and vertical tail is given by

$$\left[C_{l_\beta}(\omega)\right]_{WT} = \left(C_{l_\beta}\right)_W + \left(C_{Y_\beta}\right)_T \frac{h}{b}\left(1 + \frac{\partial\sigma}{\partial\beta}\right)e^{-i\left(\frac{l_t\omega}{U}\right)} \tag{A1}$$

The coefficients of side force and yawing moment due to side gusts along the fuselage and vertical tail (see ref. 8) are expressed by

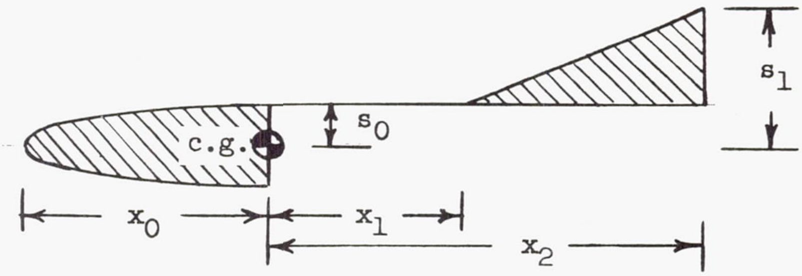
$$\left[C_{Y_\beta}(\omega)\right]_{FT} = \frac{2\pi}{S}\left\{\frac{2s_0^{\,2}}{k_0^{\,2}}\left[1 - \left(1 - ik_0\right)e^{ik_0}\right] + \left(\frac{s_1 - s_0}{k_2 - k_1}\right)^2\left[e^{-ik_1} - \left(1 - ik_1 + ik_2\right)e^{-ik_2}\right]\right\} \tag{A2}$$

$$\left[C_{n_\beta}(\omega)\right]_{FT} = \frac{2\pi}{bS}\left\{\frac{-2x_0 s_0^{\,2}}{k_0^{\,3}}\left[\left(2k_0 - ik_0^{\,2} + 2i\right)e^{ik_0} - 2i\right] + \frac{\left(x_2 - x_1\right)\left(s_1 - s_0\right)^2}{\left(k_2 - k_1\right)^3}\left[\left(2k_2 - k_1 - 2i - ik_1k_2 + ik_2^{\,2}\right)e^{-ik_2} - \left(k_1 - 2i\right)e^{-ik_1}\right]\right\} \tag{A3}$$

where

$$k_n = \frac{\omega x_n}{U} \qquad (n=0,1,2)$$

and where x_0, x_1, x_2, s_0, and s_1 are the profile dimensions of the fuselage and vertical tail. These profile dimensions, as given in the derivation in reference 8, are illustrated in the following sketch:



The fuselage is an ellipse truncated at the center of gravity, $2s_0$ being the length of the minor axis and $2x_0$ being the length of the major axis. The vertical tail is a right triangle of base $x_2 - x_1$ and height $s_1 - s_0$.

At $\omega = 0$, equations (A1), (A2), and (A3) become, respectively,

$$\left[C_{l_\beta}(\omega=0)\right]_{WT} = \left(C_{l_\beta}\right)_W + \left(C_{l_\beta}\right)_T \tag{A4}$$

$$\left[C_{Y_\beta}(\omega=0)\right]_{FT} = -\frac{\pi}{S}\left[2s_0^2 + (s_1 - s_0)^2\right] \tag{A5}$$

$$\left[C_{n_\beta}(\omega=0)\right]_{FT} = \frac{2\pi}{3Sb}\left[-2s_0^2x_0 + (s_1 - s_0)^2\left(x_2 + \frac{x_1}{2}\right)\right] \tag{A6}$$

These quantities should be made numerically equal to the steady-state stability derivatives as obtained from flight-test measurements or more exact theories by suitably adjusting the dimensions of the assumed fuselage-tail profile.

APPENDIX B

AIRFRAME TRANSFER FUNCTIONS

In coefficient form, the lateral moments and forces on an airplane are related to the angles which define the lateral motion by

$$\left[\Delta\right]\begin{Bmatrix}\phi\\ \psi\\ \beta\end{Bmatrix} = \begin{Bmatrix}C_l\\ C_n\\ C_Y\end{Bmatrix}$$

where $\left[\Delta\right]$ is defined in equation (4) for the case where the X-axis is initially alined with the relative wind (stability axes). The change in the angles due to changes in the moment and force coefficients is then expressed as

$$\begin{Bmatrix}\phi\\ \psi\\ \beta\end{Bmatrix} = \left[\Delta\right]^{-1}\begin{Bmatrix}C_l\\ C_n\\ C_Y\end{Bmatrix}$$

One method of obtaining the inverse of a matrix $\left[\Delta\right]$ is given by the relationship

$$\left[\Delta\right]^{-1} = \frac{\text{Adjoint }\left[\Delta'\right]}{\left|\Delta\right|}$$

where the prime denotes the transpose of $\left[\Delta\right]$. It may be seen that, when $\left[\Delta\right]$ is defined by equation (4), $\left[\Delta\right]^{-1}$ will be made up of transfer functions as given in equation (15). The transfer functions are defined by

$$\frac{\phi}{C_l} = \left[4\mu^2 K_Z{}^2 D^3 - \mu\left(2K_Z{}^2 C_{Y_\beta} + C_{n_r}\right)D^2 + \left(\frac{1}{2}C_{n_r}C_{Y_\beta} + 2\mu C_{n_\beta} - \frac{1}{2}C_{n_\beta}C_{Y_r}\right)D - C_{n_\beta}C_L \tan\gamma\right]\frac{1}{\left|\Delta\right|}$$

$$\frac{\psi}{C_l} = \left[4\mu^2 K_{XZ} D^3 + \mu\left(C_{n_p} - 2K_{XZ}C_{Y_\beta}\right)D^2 + \frac{1}{2}\left(C_{n_\beta}C_{Y_p} - C_{Y_\beta}C_{n_p}\right)D + C_L C_{n_\beta}\right]\frac{1}{|\Delta|}$$

$$\frac{\beta}{C_l} = \left\{\mu\left[-2K_{XZ}\left(2\mu - \frac{1}{2}C_{Y_r}\right) + K_Z{}^2 C_{Y_p}\right]D^3 - \left[-2\mu C_L\left(K_{XZ}\tan\gamma + K_Z{}^2\right) + \frac{1}{2}C_{n_p}\left(2\mu - \frac{1}{2}C_{Y_r}\right) + \frac{1}{4}C_{Y_p}C_{n_r}\right]D^2 + \frac{C_L}{2}\left(C_{n_p}\tan\gamma - C_{n_r}\right)D\right\}\frac{1}{|\Delta|}$$

$$\frac{\phi}{C_n} = \left[4\mu^2 K_{XZ} D^3 + \mu\left(C_{l_r} - 2K_{XZ}C_{Y_\beta}\right)D^2 - \left(\frac{1}{2}C_{l_r}C_{Y_\beta} + 2\mu C_{l_\beta} - \frac{1}{2}C_{Y_r}C_{l_\beta}\right)D + C_{l_\beta}C_L\tan\gamma\right]\frac{1}{|\Delta|}$$

$$\frac{\psi}{C_n} = \left[4\mu^2 K_X{}^2 D^3 - \mu\left(C_{l_p} + 2K_X{}^2 C_{Y_\beta}\right)D^2 + \frac{1}{2}\left(C_{l_p}C_{Y_\beta} - C_{l_\beta}C_{Y_p}\right)D - C_{l_\beta}C_L\right]\frac{1}{|\Delta|}$$

$$\frac{\beta}{C_n} = \left\{\mu\left[K_{XZ}C_{Y_p} - 2K_X{}^2\left(2\mu - \frac{1}{2}C_{Y_r}\right)\right]D^3 + \left(\frac{1}{4}C_{l_r}C_{Y_p} + 2\mu K_{XZ}C_L + \mu C_{l_p} - \frac{1}{4}C_{l_p}C_{Y_r} + 2\mu K_X{}^2 C_L\tan\gamma\right)D^2 + \frac{C_L}{2}\left(C_{l_r} - C_{l_p}\tan\gamma\right)D\right\}\frac{1}{|\Delta|}$$

$$\frac{\phi}{C_Y} = \left[2\mu\left(K_Z{}^2 C_{l_\beta} + K_{XZ}C_{n_\beta}\right)D^2 + \frac{1}{2}\left(C_{l_r}C_{n_\beta} - C_{n_r}C_{l_\beta}\right)D\right]\frac{1}{|\Delta|}$$

$$\frac{\psi}{C_Y} = \left[2\mu\left(K_X{}^2 C_{n_\beta} + K_{XZ}C_{l_\beta}\right)D^2 + \frac{1}{2}\left(C_{l_\beta}C_{n_p} - C_{n_\beta}C_{l_p}\right)D\right]\frac{1}{|\Delta|}$$

$$\frac{\beta}{C_Y} = \left\{4\mu^2\left(K_X{}^2 K_Z{}^2 - K_{XZ}{}^2\right)D^4 - \mu\left[K_X{}^2 C_{n_r} + K_{XZ}\left(C_{l_r} + C_{n_p}\right) + K_Z{}^2 C_{l_p}\right]D^3 + \frac{1}{4}\left(C_{l_p}C_{n_r} - C_{l_r}C_{n_p}\right)D^2\right\}\frac{1}{|\Delta|}$$

$$
\begin{aligned}
|\Delta| = {} & 8\mu^3\left(K_X{}^2K_Z{}^2 - K_{XZ}{}^2\right)D^5 - 2\mu^2\left[K_X{}^2\left(2K_Z{}^2C_{Y_\beta} + C_{n_r}\right) + K_Z{}^2C_{l_p} + \right. \\
& \left. K_{XZ}\left(C_{n_p} + C_{l_r} - 2K_{XZ}C_{Y_\beta}\right)\right]D^4 + \mu\left[K_Z{}^2\left(C_{Y_\beta}C_{l_p} - C_{Y_p}C_{l_\beta}\right) + \right. \\
& K_X{}^2\left(C_{Y_\beta}C_{n_r} + 4\mu C_{n_\beta} - C_{Y_r}C_{n_\beta}\right) + \frac{1}{2}\left(C_{l_p}C_{n_r} - C_{n_p}C_{l_r}\right) + \\
& \left. K_{XZ}\left(C_{Y_\beta}C_{l_r} + C_{Y_\beta}C_{n_p} + 4\mu C_{l_\beta} - C_{Y_r}C_{l_\beta} - C_{Y_p}C_{n_\beta}\right)\right]D^3 + \\
& \left\{-2\mu C_L\left[\tan\gamma\left(K_X{}^2C_{n_\beta} + K_{XZ}C_{l_\beta}\right) + K_Z{}^2C_{l_\beta} + K_{XZ}C_{n_\beta}\right] + \right. \\
& \frac{1}{4}\left[C_{Y_\beta}\left(C_{n_p}C_{l_r} - C_{l_p}C_{n_r}\right) + C_{l_\beta}\left(C_{n_r}C_{Y_p} - C_{Y_r}C_{n_p}\right) + \right. \\
& \left.\left. C_{n_\beta}\left(C_{Y_r}C_{l_p} - C_{Y_p}C_{l_r}\right)\right] + \mu\left(C_{l_\beta}C_{n_p} - C_{n_\beta}C_{l_p}\right)\right\}D^2 + \\
& \frac{C_L}{2}\left[\tan\gamma\left(C_{n_\beta}C_{l_p} - C_{l_\beta}C_{n_p}\right) + C_{l_\beta}C_{n_r} - C_{n_\beta}C_{l_r}\right]D
\end{aligned}
$$

APPENDIX C

TWO EQUIVALENT WAYS OF EXPRESSING THE POWER SPECTRAL RELATIONSHIP BETWEEN GUSTS AND AIRPLANE MOTIONS

The statement is made in the main text that, in the derivation of the power spectral relationships between the airplane motions and the gust-velocity inputs, the same result is obtained whether the matrix relationships are treated in the form

$$\left|\Delta^{-1}(\omega)\right|^2 \left|\tilde{G}(\omega)\right|^2 \tag{C1}$$

or in the form

$$\left|\Delta^{-1}\tilde{G}(\omega)\right|^2 \tag{C2}$$

In the former case, cross-power terms between the elements of the matrices (i.e., between the moments and forces on the airplane due to each gust component acting on the various parts of the airplane simultaneously) will appear, whereas in the latter case, these quantities do not appear explicitly but are taken into account when the matrices are first multiplied together. The former approach is used in the method of reference 1; the latter and simpler approach is used in the method of this paper. In order to show the equivalence of the two approaches it is necessary only to prove equality between equations (C1) and (C2), that is, that the square of the absolute value of the product of two matrices is equal to the product of the squares of their absolute values. Since matrix equations may be treated as linear algebraic quantities, the matrix equality defined in the main text

$$\left[\Delta(\omega)\right]^{-1}\left[\tilde{G}(\omega)\right] = \left[\Delta^{-1}\tilde{G}(\omega)\right] \tag{C3}$$

will be denoted by the complex quantities

$$AB = C \tag{C4}$$

The complex conjugate of this equality may be shown to be

$$A^*B^* = C^* \tag{C5}$$

Multiplying equations (C4) and (C5) yields

$$AA^*BB^* = CC^*$$

which is equivalent to

$$|A|^2 |B|^2 = |C|^2$$

Hence,

$$\left|\Delta^{-1}(\omega)\right|^2 \left|\tilde{G}(\omega)\right|^2 = \left|\Delta^{-1}\tilde{G}(\omega)\right|^2 \tag{C6}$$

where the elements of these matrices are likewise complex quantities.

Insight into the difference between the two approaches may be shown by considering one element of the $\left[\Delta^{-1}\tilde{G}\right]$ matrix. Expanding equation (C3) by means of equations (13) and (15) of the main text gives

$$\left\lfloor \frac{\phi}{C_l} \quad \frac{\phi}{C_n} \quad \frac{\phi}{C_Y} \right\rfloor \begin{Bmatrix} \frac{1}{2}\left(C_{l_p}\right)_W \\ \frac{1}{2}\left(C_{n_p}\right)_W \\ 0 \end{Bmatrix} = \frac{\phi}{D\phi_g}$$

or, in its expanded form,

$$\frac{\phi}{D\phi_g} = \frac{1}{2}\left(C_{l_p}\right)_W \frac{\phi}{C_l} + \frac{1}{2}\left(C_{n_p}\right)_W \frac{\phi}{C_n} \tag{C7}$$

If the real and imaginary parts of equation (C7) are grouped, then the absolute value squared of equation (C7) becomes

$$\left|\frac{\phi}{D\phi_g}\right|^2 = \left\{R\left[\frac{1}{2}\left(C_{l_p}\right)_W \frac{\phi}{C_l} + \frac{1}{2}\left(C_{n_p}\right)_W \frac{\phi}{C_n}\right]\right\}^2 + \left\{I\left[\frac{1}{2}\left(C_{l_p}\right)_W \frac{\phi}{C_l} + \frac{1}{2}\left(C_{n_p}\right)_W \frac{\phi}{C_n}\right]\right\}^2 \tag{C8}$$

A different form involving cross-power terms may be obtained by writing the conjugate of equation (C7)

$$\frac{\phi}{D\phi_g} = \frac{1}{2}\left(C_{l_p}\right)_W\left(\frac{\phi}{C_l}\right)^* + \frac{1}{2}\left(C_{n_p}\right)_W\left(\frac{\phi}{C_n}\right)^* \tag{C9}$$

and obtaining the product of equations (C7) and (C9):

$$\frac{\phi}{D\phi_g}\left(\frac{\phi}{D\phi_g}\right)^* = \frac{1}{4}\left(C_{l_p}\right)_W^2 \frac{\phi}{C_l}\left(\frac{\phi}{C_l}\right)^* + \frac{1}{4}\left(C_{n_p}\right)_W^2 \frac{\phi}{C_n}\left(\frac{\phi}{C_n}\right)^* +$$

$$\frac{1}{4}\left(C_{l_p}\right)_W\left(C_{n_p}\right)_W\left[\frac{\phi}{C_l}\left(\frac{\phi}{C_n}\right)^* + \frac{\phi}{C_n}\left(\frac{\phi}{C_l}\right)^*\right] \tag{C10}$$

Since

$$\frac{\phi}{C_n}\left(\frac{\phi}{C_l}\right)^* = \left[\left(\frac{\phi}{C_n}\right)^*\frac{\phi}{C_l}\right]^*$$

and

$$\frac{\phi}{C_l}\left(\frac{\phi}{C_n}\right)^* + \left[\frac{\phi}{C_l}\left(\frac{\phi}{C_n}\right)^*\right]^* = 2R\left[\frac{\phi}{C_l}\left(\frac{\phi}{C_n}\right)^*\right]$$

equation (C10) may be written in the form

$$\left|\frac{\phi}{D\phi_g}\right|^2 = \frac{1}{4}\left(C_{l_p}\right)_W^2\left|\frac{\phi}{C_l}\right|^2 + \frac{1}{4}\left(C_{n_p}\right)_W^2\left|\frac{\phi}{C_n}\right|^2 + 2R\left[\frac{1}{4}\left(C_{l_p}\right)_W\left(C_{n_p}\right)_W \frac{\phi}{C_l}\left(\frac{\phi}{C_n}\right)^*\right] \tag{C11}$$

The term $2R\left[\frac{1}{4}\left(C_{l_p}\right)_W\left(C_{n_p}\right)_W \frac{\phi}{C_l}\left(\frac{\phi}{C_n}\right)^*\right]$ is the cross-power term between the rolling and yawing moments on the wing, and had the coefficient $\left(C_{Y_p}\right)_W$ not been zero two other cross-power terms would have appeared.

The expressions of equations (C8) and (C11) are equivalent. Either form may be used, but, for the purposes of this report, equation (C8) is more useful and the illustrative tabulation process has been set up on this basis.

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TABLE I.- SAMPLE CALCULATION PROCEDURE

(a) Calculation of frequency-dependent derivatives and gust spectra

Column number	Column heading	Instructions
1	ω	
2	ω'	$= \omega \frac{b}{U}$
3	k'	$= \omega'/\beta'$
4	$R\left[\left(C_{l_\beta}\right)_{WT}\right]$	From equation (A1)
5	$I\left[\left(C_{l_\beta}\right)_{WT}\right]$	
6	$R\left[\left(C_{n_\beta}\right)_{FT}\right]$	From equation (A3)
7	$I\left[\left(C_{n_\beta}\right)_{FT}\right]$	
8	$R\left[\left(C_{Y_\beta}\right)_{FT}\right]$	From equation (A2)
9	$I\left[\left(C_{Y_\beta}\right)_{FT}\right]$	
10	$\left\lvert D\phi_g/\beta_g \right\rvert^2$	Quantity plotted in fig. 2
11	$\left\lvert D\psi_g/\beta_g \right\rvert^2$	Quantity plotted in fig. 3 $\times \left(\frac{\alpha C_{l_p}}{C_{l_r}}\right)_W^2$
12	Φ_{β_g}	Quantity plotted in fig. 1 $\times \frac{\overline{w_g^2} L}{\pi U^3}$

(b) Calculation of roll response

Column number	Column heading	Instructions
13	$R(\phi/C_l)$	Evaluation of airframe transfer functions (appendix B) at specified values of ω'
14	$I(\phi/C_l)$	
15	$R(\phi/C_n)$	
16	$I(\phi/C_n)$	
17	$R(\phi/C_Y)$	
18	$I(\phi/C_Y)$	
19		4 × 13 - 5 × 14
20		4 × 14 + 5 × 13
21		6 × 15 - 7 × 16

TABLE I.- SAMPLE CALCULATION PROCEDURE - Concluded

(b) Calculation of roll response - Concluded

Column number	Column heading	Instructions
22		6 × 16 + 7 × 15
23		8 × 17 - 9 × 18
24		8 × 18 + 9 × 17
25		19 + 21 + 23
26		20 + 22 + 24
27	$\left\|\phi/\beta_g\right\|^2$	25 ^{2} + 26 ^{2}
28		$\frac{1}{2}\left(C_{l_p}\right)_W$ × 13
29		$\frac{1}{2}\left(C_{l_p}\right)_W$ × 14
30		$\frac{1}{2}\left(C_{n_p}\right)_W$ × 15
31		$\frac{1}{2}\left(C_{n_p}\right)_W$ × 16
32		28 + 30
33		29 + 31
34	$\left\|\phi/D\phi_g\right\|^2$	32 ^{2} + 33 ^{2}
35	$\left\|\phi/D\phi_g\right\|^2 \left\|D\phi_g/\beta_g\right\|^2$	34 × 10
36		$\frac{1}{2}\left(C_{l_r}\right)_W$ × 13
37		$\frac{1}{2}\left(C_{l_r}\right)_W$ × 14
38		$\frac{1}{2}\left(C_{n_r}\right)_W$ × 15
39		$\frac{1}{2}\left(C_{n_r}\right)_W$ × 16
40		36 + 38
41		37 + 39
42	$\left\|\phi/D\psi_g\right\|^2$	40 ^{2} + 41 ^{2}
43	$\left\|\phi/D\psi_g\right\|^2 \left\|D\psi_g/\beta_g\right\|^2$	42 × 11
44		27 + 35 + 43
45	Φ_ϕ	44 × 12

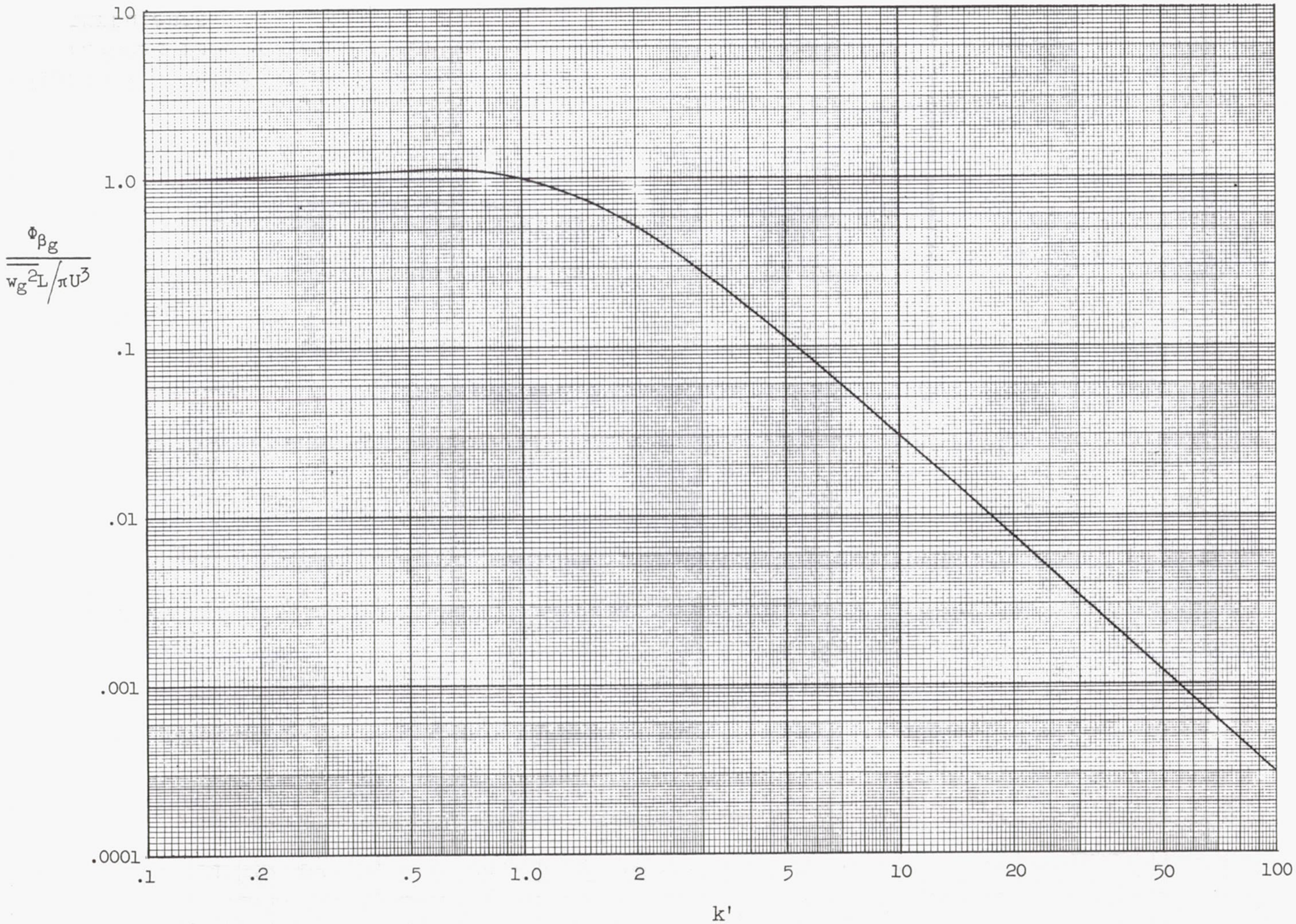


Figure 1.- Power spectral density of the side-gust component of isotropic turbulence. $k' = \frac{\omega L}{U}$.

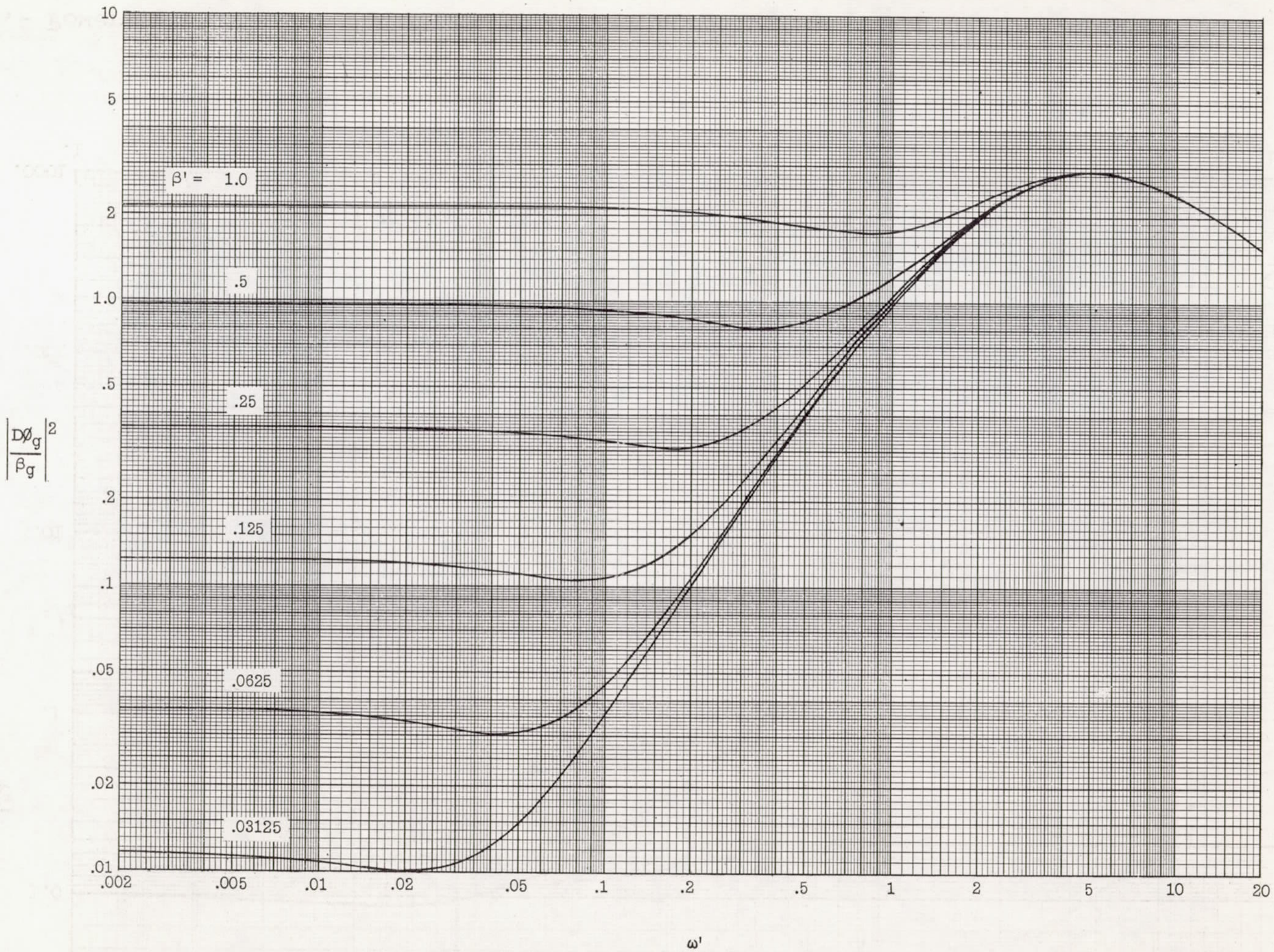


Figure 2.- Ratio of rolling-gust power spectrum to side-gust power spectrum as a function of reduced frequency ω' for various values of β', the ratio of wing span to scale of turbulence.

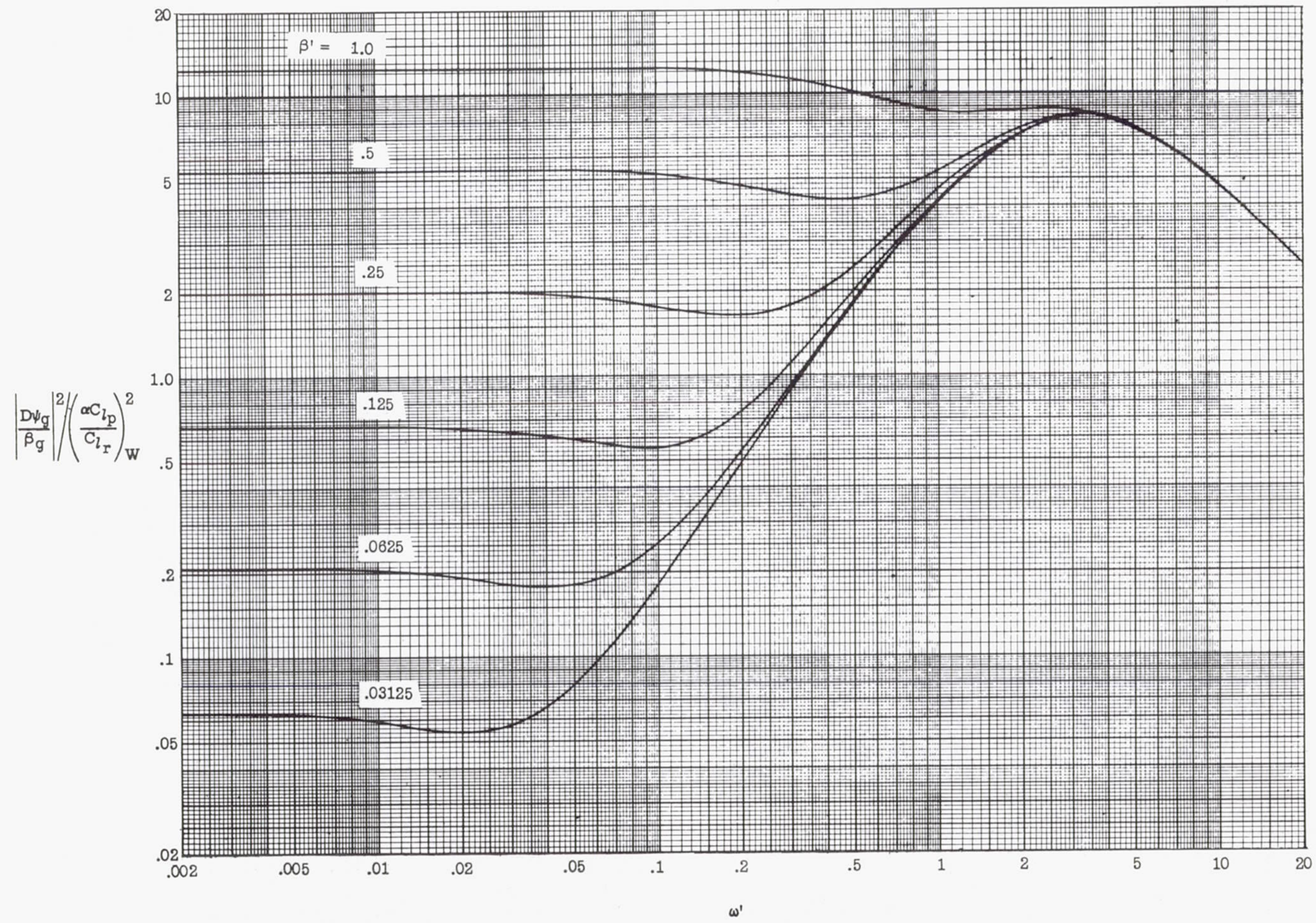
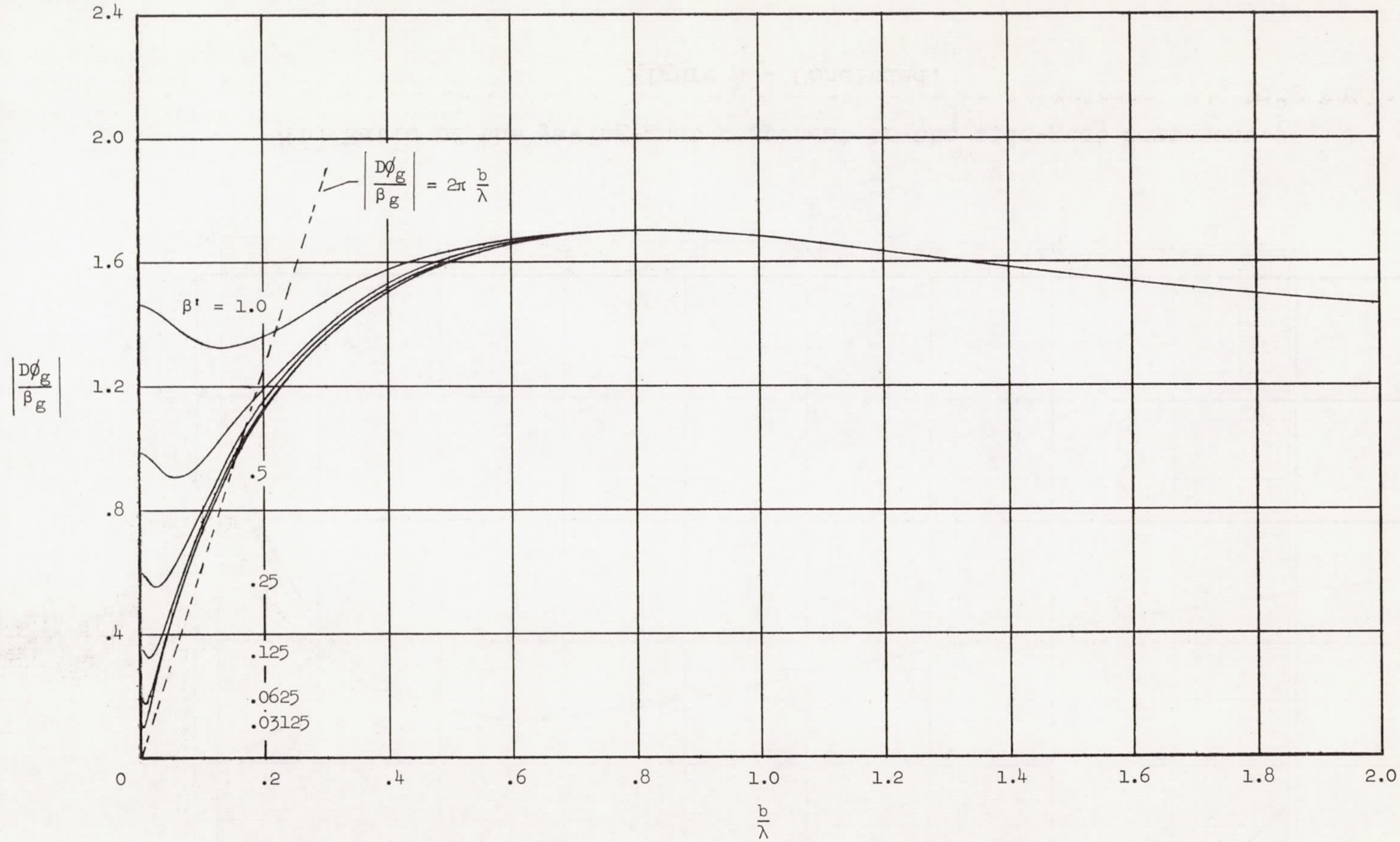
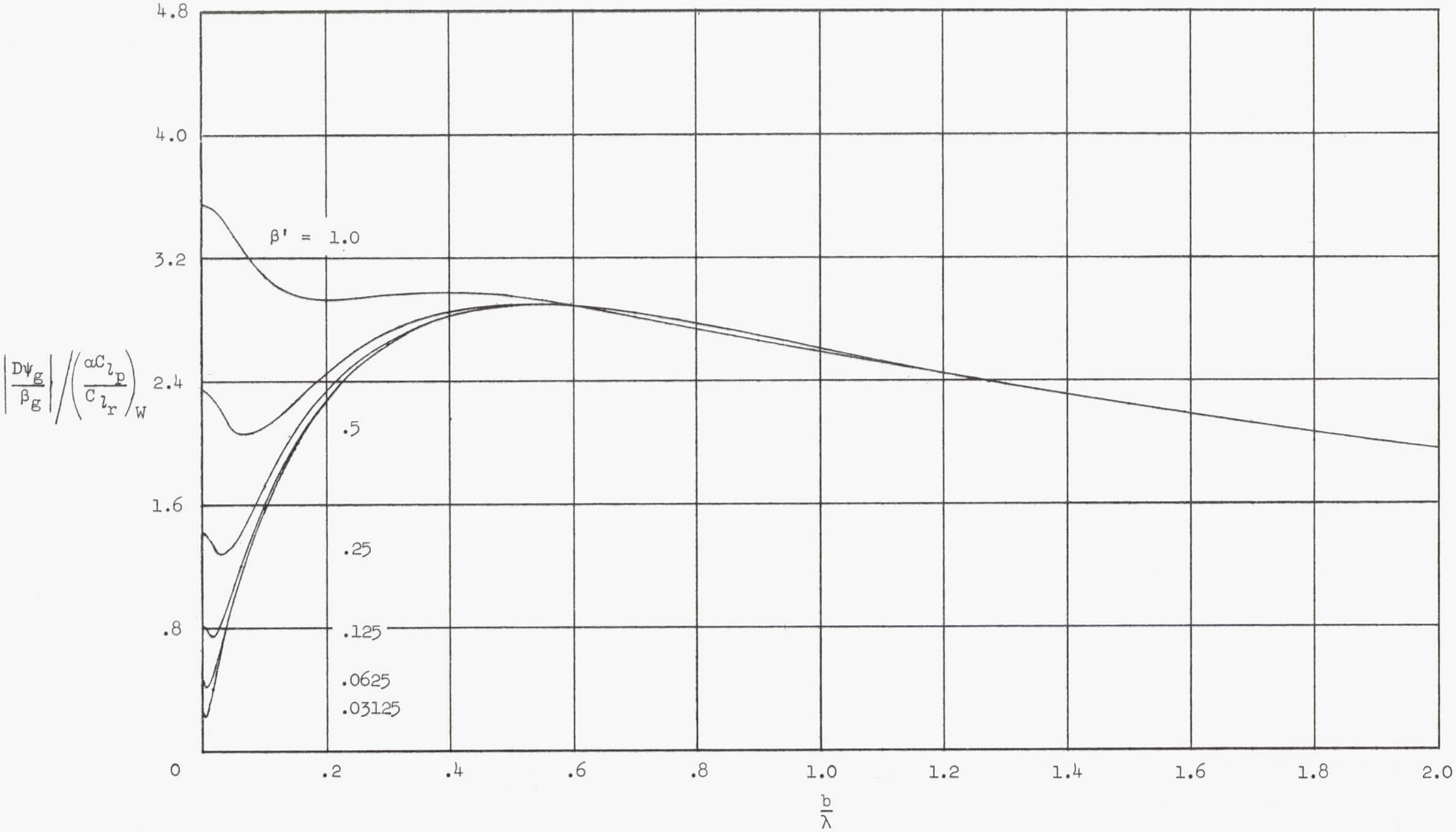


Figure 3.- Curves for determining the ratio of yawing-gust power spectrum to side-gust power spectrum for various values of the ratio of wing span to scale of turbulence β', wing angle of attack α_W, and wing stability derivatives C_{l_r} and C_{l_p}.



(a) Ratio of the rolling-gust component to the side-gust component. The approximation based on a constant antisymmetric gust-velocity gradient over the wing span is indicated by the dashed line.

Figure 4.- Ratio of the absolute amplitudes of the rolling- and yawing-gust components to the side-gust component as a function of the ratio of wing span to gust wavelength.



(b) Ratio of the yawing-gust component to the side-gust component.

Figure 4.- Concluded.

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