TECHNICAL NOTES

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No. 817

CORRECTION OF THE LIFTING-LINE THEORY FOR THE EFFECT OF THE CHORD

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SUMMARY

It is shown that a simple correction for the chord of a finite wing can be deduced from the three-dimensional potential flow around an elliptic plate. When this flow is compared with the flow around a section of an endless plate, it is found that the edge velocity is reduced by the factor 1/E, where E is the ratio of the semiperimeter to the span. Applying this correction to the circulation brings the theoretical lift into closer agreement with experiments.

INTRODUCTION

Although a number of simple, exact solutions exist for the two-dimensional potential flow around wing sections, there are no corresponding solutions that are directly applicable to the three-dimensional problem. It has consequently been necessary to employ approximate corrections to the two-dimensional theory in its application to a wing of finite aspect ratio. The correction commonly used is that introduced by Prandtl and is known as the lifting-line theory. In this theory, the finite span of the wing is taken into account but the lift and hence the chord are assumed to be concentrated along a single line.

In the present paper it is shown that a simple approximate correction for the chord can be deduced from the three-dimensional flow around an elliptic plate.

CORRECTION OF THE LIFTING-LINE THEORY

FOR THE EFFECT OF THE CHORD

In the wing-section theory, the relation between the lift and the normal velocity of the section is established by the Kutta condition, whereby the magnitude of the circulation, and hence of the lift, depends directly on the edge velocity induced by the relative normal motion of the section.

In the application of the lifting-line theory, the relative normal velocity of the wing section is corrected for the downflow induced by the wake. It is assumed that the effect of a given normal velocity of the wing section in producing lift is the same as that for a section of an infinite wing, as long as the corrected normal velocity computed from the downwash is used. This assumption is expressed by the equation:

$$C_L = 2\pi\ (\alpha - \alpha_i) \tag{1}$$

where C_L is the lift coefficient, 2π is the slope of the lift curve of a thin wing for infinite aspect ratio, α is the angle of attack of the section, and α_i is the induced angle of downflow.

It may be shown by the potential-flow theory that the velocity near the edge of a finite elliptic plate is less than the velocity near the edges of an endless plate. The ratio of the edge velocities in potential flow is found (see the appendix) to be $1/E$, where E is the ratio of the semiperimeter to the span. Hence, the circulation required by the finite elliptic plate is expected to be less than that required by the infinite plate in this ratio. The corrected formula for the lift is then

$$C_L = \frac{2\pi}{E}\ (\alpha - \alpha_i) \tag{2}$$

This correction may be given a physical interpretation by considering that with a finite plate the fluid has a longer edge around which to escape and that the velocity is less in inverse proportion to the length of the edge. The rule is not exact for plan forms other than the elliptical ones. Also, it will be noted that the form of the circulation in three dimensions is assumed similar to that in two dimensions. Because this similarity has not been proved for the elliptic plate, equation (2) must be considered simply as a first correction to the lifting-line theory for the effect of the chord.

According to the lifting-line theory, for elliptic loading,

$$\alpha_i = \frac{C_L}{\pi A}$$

where A is the aspect ratio. Substitution of this value into equation (2) gives:

$$\frac{dC_L}{d\alpha} = 2\pi \frac{A}{EA + 2}$$

The more accurate correction for the aspect ratio is therefore $A/(EA + 2)$ rather than $A/(A + 2)$, which has been used heretofore.

Figure 1 shows that this simple correction for the chord brings the lifting-line theory into closer agreement with the results obtained by Blenk for a rectangular wing (reference 1) and by Krienes (reference 2), who used the more complex analysis of lifting-surface theory.

It is found that the additional correction for the chord accounts for an appreciable fraction of the loss in lift that has been attributed to viscosity. Measurements made on wings of aspect ratio 6 and corrected by the lifting-line theory have led to the conclusion that the lift-curve slope in two-dimensional flow is about 5.7, a value that is only 85 percent of the slope predicted by the wing-section theory for moderately thick airfoils. At the same time, however, experiments in which two-dimensional flow has been closely simulated (reference 3) have shown slopes consistently closer to the theoretical than were obtained by applying the conventional correction to the values measured on wings of finite aspect ratio. The modified correction brings the standard wind-tunnel experiments (A = 6) into agreement with the tests reported in reference 3 and also accounts for about one-third of the discrepancy between them and the theory. The greater part of the loss of lift is presumably due to an imperfection of the flow in satisfying the Kutta condition.

Since the Kutta condition plays such an important part in the wing theory and since its applicability depends on the existence of a sharp trailing edge, it is of interest to note the actual effect of changes in the slope of the wing near the trailing edge. This effect is illustrated in figure 2. As pointed out by Munk, the airfoil with the thick, wedge-shaped trailing edge (NACA 0012-65, reference 4) shows a considerably smaller lift than do conventional wings. On the other hand, the lift of the wing with a thin, sharp edge (NACA 0012-F_0, reference 5) attains nearly its theoretical value and if corrected to infinite span, would exceed the value 2π for the flat plate.

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APPENDIX

The problem of the fluid motion produced by translation of a solid ellipsoid was first solved by Green in an investigation of the vibration of pendulums. Formulas for this problem given in textbooks on hydrodynamics become indeterminate when applied to the case of an elliptic disk. The following short discussion is therefore presented to show the application to the present problem.

As explained in reference 6, the surface potential of an ellipsoid can be given by a very simple formula. For motion along a principal axis, the potential at any point on the surface is proportional to the coordinate of the point in the direction of motion. Thus, if the ellipsoid with semiaxes $a > b > c$ along x, y, and z, respectively, is moving with unit velocity in the direction of z, the surface potential is simply

$$\phi = Cz \tag{3}$$

The equipotential lines are the similar ellipses formed by the intersection of the ellipsoidal surface with a series of equidistant parallel planes perpendicular to z. The constant C depends on the axis ratio and its evaluation involves a special class of transcendental function known as Green's Integrals. The solution for the surface potential appears in the form:

$$\phi = \frac{\gamma_o}{2 - \gamma_o} z \tag{4}$$

where

$$\gamma_o = abc \int_0^\infty \frac{d\lambda}{(c^2 + \lambda)\sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}}$$

is Green's integral.

The reduction of these integrals to the standard elliptic functions is given in reference 6. Following equation (6.1) of reference 6 and substituting $\lambda = 0$, in order to restrict the solution to the surface of the ellipsoid, will give

$$\gamma_o = \frac{2abc}{\sqrt{(a^2 - c^2)^3}} \frac{a^2 - c^2}{b^2 - c^2} \left(\frac{b}{ac}\sqrt{a^2 - c^2} - E\right)$$

where E is the complete elliptic integral with the modulus $k = \sqrt{\frac{a^2 - b^2}{a^2}}$. The integral E is equal to the perimeter of a quadrant of the ellipse ab divided by the semiaxis a. (See reference 7.)

As $c \longrightarrow 0$

$$\gamma_o \longrightarrow 2\left(1 - \frac{c}{b} E\right)$$

Since equation (4) becomes indeterminate ($z \longrightarrow 0$ and $\gamma_o \longrightarrow 2$), it is necessary to express the solution in terms of x and y, which are related to z through the equation of the ellipsoidal surface:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{5}$$

or

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{6}$$

Substitution for γ_o and z in equation (4) gives

$$\phi = \frac{b}{E}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \tag{7}$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\phi^2}{(b/E)^2} = 1 \tag{8}$$

Hence the distribution of the surface potential over the disk may be represented by the ordinates of a circumscribed ellipsoid having the vertical axis $\frac{2b}{E}$. For infinite axis ratio, $E = 1$ and the chordwise cross sections of the potential distribution are circles of radius b.

In order to illustrate the analogy with two-dimensional flow, it is convenient to introduce the angle θ defined, at a particular value of x, by

$$\cos\theta = \frac{y}{y_e} \tag{9}$$

where $y_e = b\sqrt{1 - \frac{x^2}{a^2}}$ is the ordinate of the edge of the disk. Then, from equation (8),

$$\phi = \frac{y_e}{E}\sin\theta \tag{10}$$

which is the potential function of the two-dimensional case except for the factor $1/E$. It follows then that the edge velocity is also reduced from that in two-dimensional flow by the factor $1/E$.

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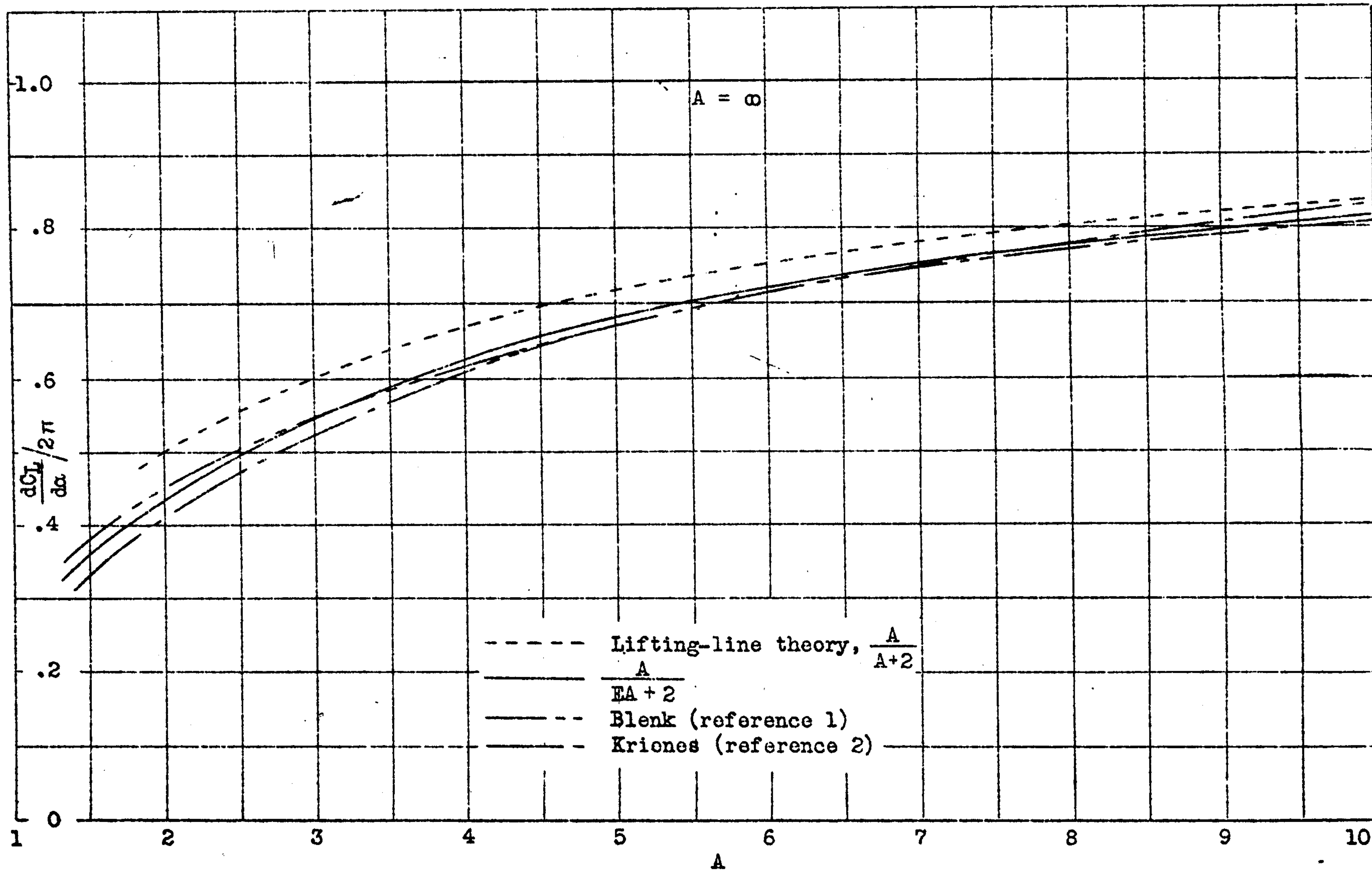

Figure 1.- Theoretical variations of lift-curve slope with aspect ratio.

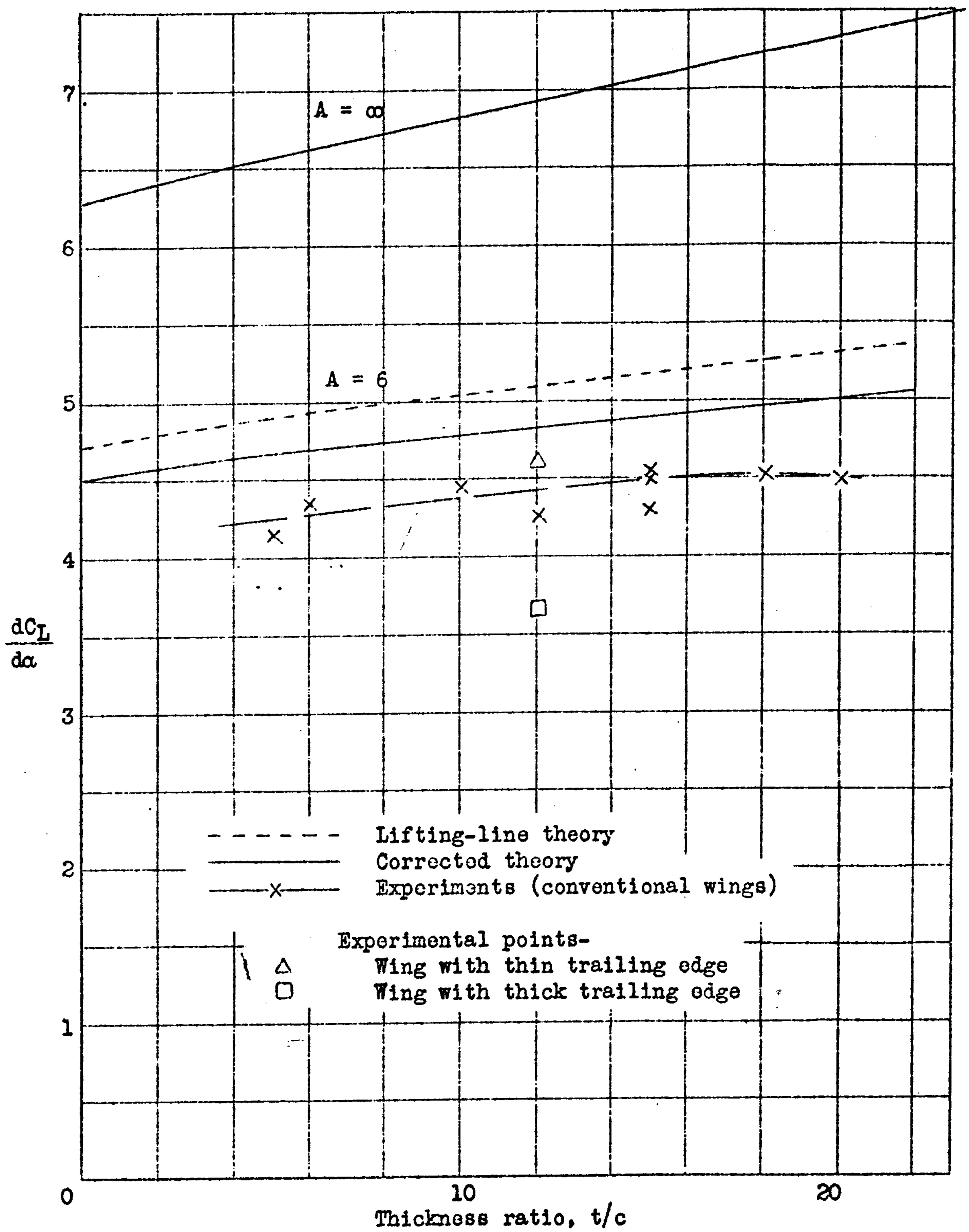

Figure 2.- Experimental and theoretical lift-curve slopes. Aspect ratio, 6.