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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4229

STAGNATION-POINT HEAT TRANSFER TO BLUNT SHAPES IN HYPERSONIC FLIGHT, INCLUDING EFFECTS OF YAW

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SUMMARY

An approximate theory is developed for predicting the rate of heat transfer to the stagnation region of blunt bodies in hypersonic flight. Attention is focused on the case where wall temperature is small compared to stagnation temperature. The theoretical heat-transfer rate at the stagnation point of a hemispherical body is found to agree with available experimental data. The effect of yaw on heat transfer to a cylindrical stagnation region is treated at some length, and it is predicted that large yaw should cause sizable reductions in heat-transfer rate.

INTRODUCTION

It has been suggested (see refs. 1 and 2) that blunting or rounding the leading edges of wings and bodies might substantially alleviate aerodynamic heating of these regions in hypersonic flight. There is, of course, the added advantage that round leading edges are structurally more practical than sharp leading edges, especially when the problem of absorbing heat is considered. Another consequence of blunting may be increased pressure drag. In the case of ballistic vehicles, this consequence is often an advantage (see ref. 1). In the case of glide vehicles, however, or more generally any vehicles required to operate for sustained periods in more or less level hypersonic flight, increased drag may be viewed as a disadvantage.

Now, to be sure, rounding or blunting the nose of a body does not always increase drag. Indeed, small amounts of blunting may reduce the drag of a body (see, e.g., refs. 3 and 4). The same, however, cannot be said for blunting the leading edge of a wing. Even small blunting causes a sizable increase in drag. It is natural, then, to look for methods of minimizing this drag penalty, and the possibility of yawing or sweeping the leading edge comes to mind. Impact pressures should be, according to simple-sweep theory, decreased in proportion to the cosine squared of the

angle of sweep; hence, as is intuitively obvious, large sweep should substantially reduce the drag penalty due to blunting. In view of this possibility it is important to inquire of the effect of yaw or sweep on heat transfer to a blunt leading edge.

The purpose of this paper is to investigate theoretically the heat transfer to the stagnation regions of bodies in hypersonic flight, including the effects of yaw, by a simplified method which is suited to take account of real gas effects such as dissociation. This method, which was previously given limited distribution, is used along with recent estimates of transport properties for high temperature air, and the solutions are compared with some heat transfer results for blunt shapes.

SYMBOLS

$A,B,C,D,E,F,G,\ldots$	integration constants
C_p	specific heat at constant pressure, ft-lb/slug $^{\circ}R$
h	specific enthalpy, ft-lb/slug
k	coefficient of thermal conductivity, ft-lb/ft-sec $^{\circ}R$
M	Mach number, dimensionless
n	exponent of temperature in thermal conductivity and viscosity functions (see eqs. (37) and (38)), dimensionless
Nu	Nusselt number based on a length $2R_b$ and stagnation temperature conditions, dimensionless
p	static pressure, lb/ft^2 (unless otherwise specified)
Pr	Prandtl number, dimensionless
q	heat flux per unit area, ft-lb/ft^2-sec
$q(0)$	heat flux per unit area at zero yaw, ft-lb/ft^2-sec
$q(\lambda)$	heat flux per unit area at yaw angle λ, ft-lb/ft^2-sec
R	gas constant, ft-lb/slug $^{\circ}R$

R_b radius of curvature of body at the stagnation point, ft

R_s radius of curvature of the shock wave at the stagnation streamline, ft

Re Reynolds number, based on twice the radius of curvature of the body at the stagnation point, dimensionless

$\bar{r},\theta,\varphi$ spherical coordinates, feet, degrees, and degrees, respectively

T static temperature, $^{\circ}R$

T_b temperature of the body, $^{\circ}R$

T_o temperature at the interface, x = 0, with body at zero yaw, $^{\circ}R$

$T_o(\lambda)$ temperature at the interface, x = 0, with body at angle of yaw λ, $^{\circ}R$

T_r recovery temperature, $^{\circ}R$

T_t stagnation temperature, $^{\circ}R$

U_∞ stream velocity, ft/sec

u,v,w velocity components in the x, y, and z directions, respectively, ft/sec

u,v velocity components in the x and r directions, respectively, ft/sec

x,y,z Cartesian coordinates, ft

x,r cylindrical coordinates, ft

δ flow deflection angle, deg

ϵ dimensionless coordinate, $\frac{\bar{r} - R_b}{R_b}$ or $\frac{x_b - x}{R_b}$

γ ratio of specific heat at constant pressure to specific heat at constant volume, dimensionless

γ_s a function of density change across a shock wave, $\frac{(\rho_s/\rho_\infty) + 1}{(\rho_s/\rho_\infty) - 1}$, dimensionless

λ angle of yaw, deg

ρ density, slugs/cu ft

η $\int_0^T k\,dT$, a function of the coefficient of thermal conductivity and of temperature, ft-lb/ft-sec (unless otherwise specified)

σ acute angle of shock wave relative to stream velocity vector, deg

μ coefficient of viscosity, slugs/ft sec

μ_o coefficient of viscosity at temperature T_o, slugs/ft sec

$\mu_o(\lambda)$ coefficient of viscosity at temperature $T_o(\lambda)$, slugs/ft sec

Subscripts

s conditions just behind shock wave on the stagnation streamline

b conditions at the stagnation point of the body

o conditions at the interface between regions 1 and 2 on the stagnation streamline (see sketch (b))

∞ conditions in the free stream

Superscripts

'	first derivative with respect to the x coordinate
''	second derivative with respect to the x coordinate

THEORY

General Equations in Cartesian Coordinates

The analysis proceeds from the equations of momentum, continuity, energy, and state for continuum fluid flow. The x, y, and z momentum equations are, respectively,

$$\rho \frac{\partial u}{\partial t} + \rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}\right) = - \frac{\partial p}{\partial x} - \frac{2}{3} \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + 2 \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \tag{1}$$

$$\rho \frac{\partial v}{\partial t} + \rho\left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}\right) = - \frac{\partial p}{\partial y} - \frac{2}{3} \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] + 2 \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \tag{2}$$

$$\rho \frac{\partial w}{\partial t} + \rho\left(u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}\right) = - \frac{\partial p}{\partial z} - \frac{2}{3} \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] +$$

$$2 \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] +$$

$$\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \tag{3}$$

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{4}$$

and the energy equation is

$$\rho\left(u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w \frac{\partial h}{\partial z} + \frac{\partial h}{\partial t}\right) - \left(\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 +\right.$$

$$\left. 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right] \tag{5}$$

while the equation of state is taken in the form

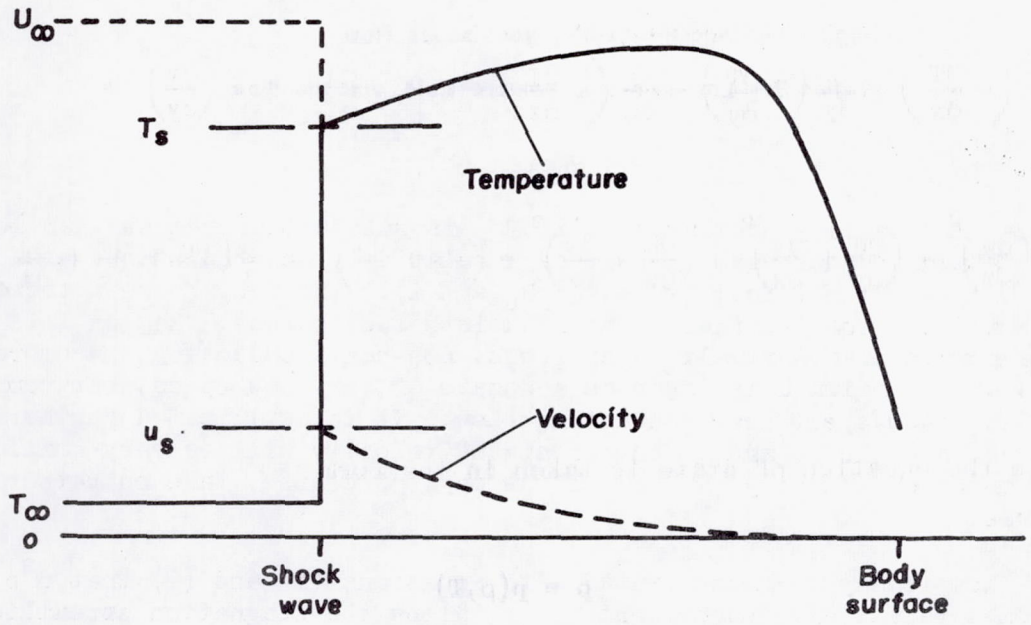
$$p = p(\rho, T) \tag{6}$$

Derivations of the momentum and energy equations are given in numerous sources (see, e.g., refs. 5, 6, and 7). Note that the coefficients of viscosity and thermal conductivity, and the heat capacity have been treated as variables. It is intended that by so doing a more accurate solution will be obtained for hypersonic flows with their characteristically large temperature and pressure gradients.

Let us now consider the particular flows of interest in this paper, namely, those in the region of a stagnation point.

Model of Flow and Method of Analysis

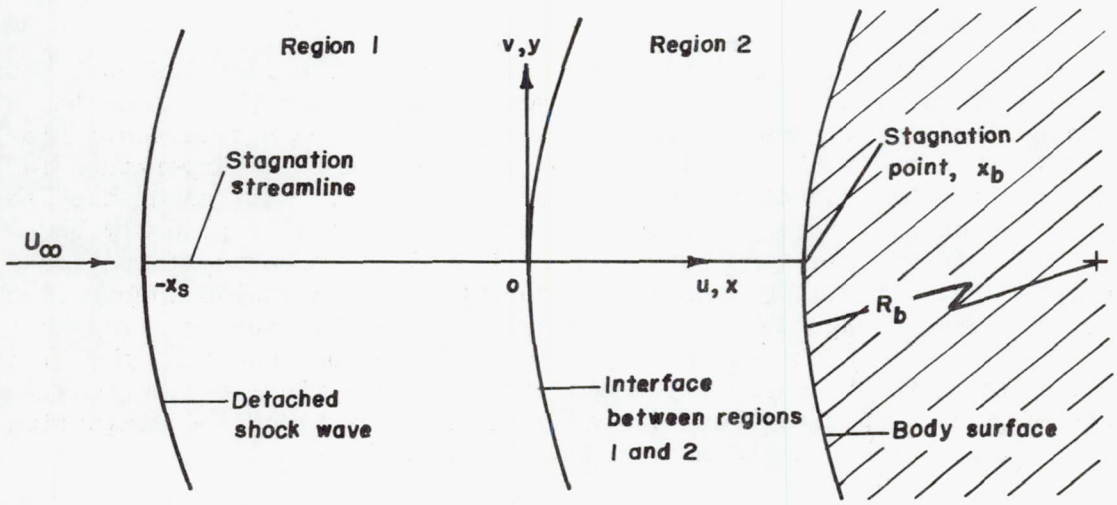
It is instructive in setting up the model to consider the qualitative aspects of temperature and velocity variations in the flow along the stagnation streamline. Restricting the analysis to steady hypersonic flow, that is $M_\infty \sin \delta >> 1$, we will assume that the surface temperature is low compared to the stagnation temperature of the air. This assumption seems quite reasonable since practical surface materials will probably be destroyed if surface temperatures are allowed to approach stagnation temperature. It will be assumed further that the Reynolds number of the flow is large enough so that heat conduction and viscous shearing in the shock process is distinct and separate from the corresponding phenomena occurring in the boundary layer adjacent to the surface of the body. Accordingly, temperature and velocity should vary along the stagnation streamline similar to the manner shown in sketch (a).



Sketch (a)

There is an abrupt and large increase in temperature and decrease in velocity of the air as it passes through the bow shock. Proceeding from the shock in the direction of the body, temperature continues to increase slowly while the velocity decreases slowly towards zero. Near the surface of the body, the air temperature ceases to increase and, in fact, begins to fall off steeply in the direction of the body temperature. The velocity of the flow must, of course, be close to zero in this region.

On the basis of these observations the following simplified model is proposed and employed throughout this study of heat transfer in a stagnation region.



Region 1– Incompressible, nonviscous flow

Region 2– Low-velocity, compressible, viscous flow

Sketch (b)

Since M_∞ is large compared to 1, M_s is substantially less than 1 and the detached shock wave is located a relatively short distance ahead of the body surface (i.e., $(x_s + x_b)/R_b << 1$). The flow between the shock wave and the body surface is divided into two regions. Region 1 is taken as a domain of essentially nonviscous, non-heat-conducting, incompressible flow while region 2 is taken as a domain of very low speed, but compressible, viscous, and heat-conducting flow. It is anticipated further that in region 2 the u and v components of velocity will be very small. The component of velocity w due to yaw may, of course, take on rather large values.

Now it may be demonstrated with equations (1) and (2) that $\partial^2 p/\partial y^2$ becomes relatively independent of x along the stagnation streamline in

the limit as the disturbed flow extends only a short distance away from the body. Inasmuch as this is the type of flow of interest here, it will be assumed throughout this analysis that $\partial^2 p/\partial y^2$ is essentially constant along the stagnation streamline between the shock and the body.

With these assumptions, the derivative with respect to y of the y momentum equation yields a differential equation that becomes tractable, both in regions 1 and 2, when terms that vanish in the neighborhood of the stagnation streamline are dropped. Approximate solutions to these simplified y momentum equations are found for the u velocity along the stagnation streamline in region 1, and for the derivative of this velocity along the stagnation streamline in region 2. The constants appearing in these solutions are determined by matching the boundary conditions at the shock wave and at the surface of the body, and by matching flow conditions at the interface. This procedure fixes the locations of the shock wave and interface relative to the body.

The energy equation is simplified in an analogous manner, and solutions valid in the neighborhood of the stagnation streamline are found for regions 1 and 2. The rate of heat transfer per unit area to the stagnation region of the body follows from the solution to the energy equation for region 2.

Let us see how these thoughts apply in the case of a two-dimensional stagnation region.

Heat Transfer to a Cylindrical Stagnation Region

Zero yaw.- This problem has been treated for incompressible flow by Howarth (ref. 7) and more recently for the compressible flow by Cohen and Reshotko (ref. 8). One reason for re-investigating the matter here is to obtain compressible flow solutions which can be extended with relative ease to the case of a yawed cylinder. In addition it was desired to obtain solutions which may be better suited to account for real gas effects, such as dissociation.

To proceed, then, the stagnation streamlines are taken to lie in the x-z plane. The origin of the coordinate system is at the interface between regions 1 and 2, and the shock-wave and body-surface locations in this plane are $-x_s$ and x_b, respectively (see sketch (b)). For the case of zero yaw, the z component of velocity and all derivatives with respect to z are, of course, identically zero.

First a solution will be found to the steady-state y momentum equation near the stagnation streamline in region 1. Since the flow is assumed incompressible and nonviscous in this region, equation (2)

simplifies to

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial y} \tag{7}$$

Differentiating equation (7) with respect to y there is obtained

$$u \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2 + v \frac{\partial^2 v}{\partial y^2} = - \frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} \tag{8}$$

On the stagnation streamline v is identically zero and, therefore, $\partial v/\partial x$ is also zero. In addition, the continuity equation (eq. (4)) becomes, for incompressible, two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

Using this information with equation (8), one obtains

$$-u \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = - \frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} \tag{10}$$

Treating $\partial^2 p/\partial y^2$ as a function of y only, and noting that equation (10) becomes a total differential equation along a line y = constant, yields a general solution for velocity along the stagnation streamline

$$u = Ae^{Cx} + Be^{-Cx} \tag{11}$$

where the constants A, B, and C are related by

$$4ABC^2 = \frac{1}{\rho} \frac{\partial^2 p}{\partial y^2} \tag{12}$$

Note that the constants may be real or imaginary, depending on the boundary conditions.

Now it is anticipated that the velocity u will very nearly vanish at the interface $x = 0$ (i.e., in the sense that $u_o/u_s << 1$); hence B will be approximately $-A$, and the corresponding approximate solution for velocity is[1]

$$u = 2A \sinh Cx \tag{13}$$

To the same order of approximation, the second derivative of velocity at the interface, u_o'', also vanishes. The product $2AC$ is just the velocity derivative at the interface and can be evaluated from equations (10) and (13), thus

$$2AC = u_o' = \pm \sqrt{-\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2}} \tag{14}$$

Note that the negative root correctly describes the flow in the coordinate system of sketch (b), since velocity decreases with increasing x.

Consider next the steady-state y momentum equation near the stagnation streamline in region 2. In this domain viscous terms must, of course, be retained and thus the derivative of equation (2) with respect to y yields

$$\rho u \frac{\partial^2 v}{\partial x \partial y} + u \frac{\partial \rho}{\partial y}\frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \rho v \frac{\partial^2 v}{\partial y^2} + v \frac{\partial \rho}{\partial y}\frac{\partial v}{\partial y} + \rho \left(\frac{\partial v}{\partial y}\right)^2$$

$$= -\frac{\partial^2 p}{\partial y^2} - \frac{2}{3}\frac{\partial^2}{\partial y^2}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + 2\frac{\partial^2}{\partial y^2}\left(\mu \frac{\partial v}{\partial y}\right) + \frac{\partial^2}{\partial y \partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \tag{15}$$

Now close to the surface of the body the left-hand side of this expression is negligible and the right-hand side simplifies so that the equation may be written (see Appendix A)

$$\frac{\partial}{\partial x}\left(\mu \frac{\partial^2 u}{\partial x^2}\right) = -\frac{\partial^2 p}{\partial y^2} \tag{16}$$

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[1]In the limit of zero boundary-layer thickness, this solution is exactly the one to which equation (11) reduces.

Along the stagnation streamline this equation integrates to

$$\mu \frac{\partial^2 u}{\partial x^2} = - \frac{\partial^2 p}{\partial y^2} x + D \tag{17}$$

The constant D is zero since $\partial^2 u/\partial x^2 = u_o'' = 0$ at the interface ($x = 0$). Near the surface of the body, equation (17) can be integrated to obtain

$$\mu \frac{\partial u}{\partial x} = - \frac{\partial^2 p}{\partial y^2} \frac{x^2}{2} + \mu_o u_o' \tag{18}$$

In order to satisfy the boundary condition at the body surface $\left(\frac{\partial v}{\partial y}\right)_b = \left(\frac{\partial u}{\partial x}\right)_b = 0$, it follows from equations (18) and (14) that

$$x_b^2 = \frac{2\mu_o}{\sqrt{-\rho \frac{\partial^2 p}{\partial y^2}}} \tag{19}$$

Now ρ and $\partial^2 p/\partial y^2$ can be evaluated at the shock wave since both are considered constant throughout region 1. In Appendix B it is demonstrated that for two-dimensional flow

$$\left(\rho \frac{\partial^2 p}{\partial y^2}\right)_s = - \frac{6\rho_\infty^2 U_\infty^2}{(\gamma_s - 1) R_s^2} \tag{20}$$

where R_s is the radius of curvature of the shock wave in the stagnation region. Substituting equation (20) in equation (19) we obtain

$$\frac{x_b}{R_b} = \left(\frac{\gamma_s - 1}{6}\right)^{1/4} \left(\frac{\mu_o}{\mu_\infty} \frac{R_s}{R_b}\right)^{1/2} \frac{2}{Re_\infty^{1/2}} \tag{21}$$

where Re_∞ is the free-stream Reynolds number based on $2R_b$, twice the radius of curvature of the body at the stagnation point. Note also that the effective value of γ, the ratio of specific heats, at the shock wave is allowed to vary from the free-stream value. In this way, changes in internal molecular energy which are manifest at the high temperatures encountered in hypersonic flight can be considered.

There remains the problem of solving the energy equation. In region 1, the energy equation is simplified by neglecting all the viscous and heat-conduction terms. Then, for the two-dimensional problem considered here, equation (5) reduces to

$$u \frac{\partial u}{\partial x} + C_p \frac{\partial T}{\partial x} = 0 \tag{22}$$

for which the solution is

$$\int_T^{T_t} C_p dT = \frac{u^2}{2} \tag{23}$$

It can be seen from equation (23) that the interface temperature T_o is approximately the stagnation temperature T_t, since the velocity at the interface nearly vanishes. The stagnation temperature is, of course, given by the integral equation

$$\int_{T_\infty}^{T_t} C_p dT = \frac{\gamma_\infty R T_\infty M_\infty^2}{2} \tag{24}$$

where for very high velocity flow the lower limit of the integral will be neglected.

Next consider the energy equation in region 2. Proceeding in a manner analogous to that used in studying the y momentum equation in this region, we neglect the terms with the factors u, v, $\partial u/\partial x$, and $\partial v/\partial y$. Thus equation (5) becomes simply the heat-conduction equation

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) = 0 \tag{25}$$

The coefficient of thermal conductivity, k, is considered a known function of temperature (pressure is essentially constant). Thus a new function of temperature, η, may be defined such that

$$\eta = \int_o^T k \, dT \tag{26}$$

Then equation (25) may be expressed in terms of the function η

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0 \tag{27}$$

Inasmuch as the body boundary is cylindrical, it is convenient to use the general solution to equation (27) in terms of the polar coordinates $(\bar{r}, \theta)$. Thus

$$\eta = A + B \ln \bar{r} + \sum_{n=1}^{\infty} \left[\left(C_n \bar{r}^n + \frac{D_n}{\bar{r}^n} \right) \cos n\theta + \left(E_n \bar{r}^n + \frac{F_n}{\bar{r}^n} \right) \sin n\theta \right] \tag{28}$$

The origin of the coordinate system is now taken as the center of curvature of the body, and θ as the acute angle between the radius vector $\bar{r}$ and the stagnation streamline. If a surface temperature is assumed independent of the angle θ,[2] the solution on the stagnation streamline $(\theta = 0)$ reduces to

$$\eta = \eta_b + B \ln \frac{\bar{r}}{R_b} + \sum_{n=1}^{\infty} C_n \bar{r}^n \left[1 - \left(\frac{R_b}{\bar{r}} \right)^{2n} \right] \tag{29}$$

Letting $\frac{\bar{r}}{R_b} = 1 + \epsilon$, where ϵ is very small compared to unity, and expanding equation (29) in a series of ascending powers of ϵ, we obtain

$$\eta = \eta_b + G \left(\epsilon - \frac{\epsilon^2}{2} \right) + O(\epsilon^3) \tag{30}$$

where G is the constant $\left(B + \sum_{n=1}^{\infty} 2nR_b{}^n C_n \right)$. It is indicated by this equation that η varies essentially linearly with ϵ, since $\epsilon^2/2$ is negligible compared to ϵ and terms of higher order in ϵ should be very small indeed.[3] Since $\epsilon = (x_b - x)/R_b << 1$, equation (30) can be written

$$\eta = \eta_o - (\eta_o - \eta_b) \frac{x}{x_b} \tag{31}$$

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[2]The dependence of surface temperature on θ should be small in the stagnation region.

[3]It should be pointed out that this argument hinges implicitly on the assumption that η is a weak function of θ near the stagnation streamline.

According to this expression, the rate of heat transfer per unit area to the stagnation region of the body is

$$q = -\eta_b' = \frac{\eta_o - \eta_b}{x_b} = \frac{1}{x_b} \int_{T_b}^{T_o} k \, dT \tag{32}$$

The stagnation-line coordinate x_b is substituted from equation (21), and the rate of heat transfer becomes

$$q = \left[\frac{3}{8(\gamma_s - 1)}\right]^{1/4} \left(\frac{\mu_\infty}{\mu_o} \frac{R_b}{R_s}\right)^{1/2} \frac{Re_\infty^{1/2}}{R_b} \int_{T_b}^{T_o} k \, dT \tag{33}$$

A Nusselt number is defined for interface temperature conditions using a characteristic length equal to twice the radius of curvature of the body and a temperature potential of $(T_o - T_b)$; thus

$$Nu = -\frac{2\eta_b' R_b}{k_o(T_o - T_b)} \tag{34}$$

or, substituting from equation (33) into (34)

$$Nu = \left(\frac{6}{\gamma_s - 1}\right)^{1/4} \left(\frac{\mu_\infty}{\mu_o} \frac{R_b}{R_s}\right)^{1/2} \frac{Re_\infty^{1/2}}{k_o(T_o - T_b)} \int_{T_b}^{T_o} k \, dT \tag{35}$$

For a relatively cool body in hypersonic flight, it is possible to disregard the lower limit of the integral and the value of body temperature T_b compared to the interface temperature T_o.

Note that the solutions given by equations (33) and (35) can be used for the case where viscosity, thermal conductivity, and specific heat are arbitrary functions of temperature. For instance, these functions can be calculated to include the effects of vibrational and dissociational molecular energy if the extent to which these energy modes are excited is known throughout the flow. It is also useful to consider the case where the specific heat is treated as a constant and the viscosity and thermal

conductivity as proportional to the nth power of temperature. In this case from equation (24) $\left(\text{note } \frac{T_o}{T_t} \approx 1, \frac{T_o}{T_\infty} >> 1\right)$

$$\frac{T_o}{T_\infty} = \left(\frac{\gamma_\infty R}{2C_p}\right) M_\infty^2 \tag{36}$$

Noting that

$$\frac{1}{k_o(T_o - T_b)} \int_{T_b}^{T_o} k \, dT = \frac{1}{T_o - T_b} \int_{T_b}^{T_o} \left(\frac{T}{T_o}\right)^n dT = \frac{1}{n+1} \frac{1 - \left(T_b/T_o\right)^{n+1}}{1 - T_b/T_o} \tag{37}$$

and that

$$\left(\frac{\mu_\infty}{\mu_o}\right)^{1/2} = \left(\frac{T_\infty}{T_o}\right)^{n/2} = \left(\frac{2C_p}{\gamma_\infty R}\right)^{n/2} M_\infty^{-n} \tag{38}$$

it is seen that the expression for Nusselt number (eq. (35)) becomes

$$Nu = \frac{1}{n+1} \left(\frac{6}{\gamma_s - 1}\right)^{1/4} \left(\frac{R_b}{R_s}\right)^{1/2} \left(\frac{2C_p}{\gamma_\infty R}\right)^{n/2} \frac{Re_\infty^{1/2}}{M_\infty^n} \frac{1 - \left(T_b/T_o\right)^{n+1}}{1 - (T_b/T_o)} \tag{39}$$

and the rate of heat transfer per unit area to the stagnation region of the body is, in terms of free-stream conditions,

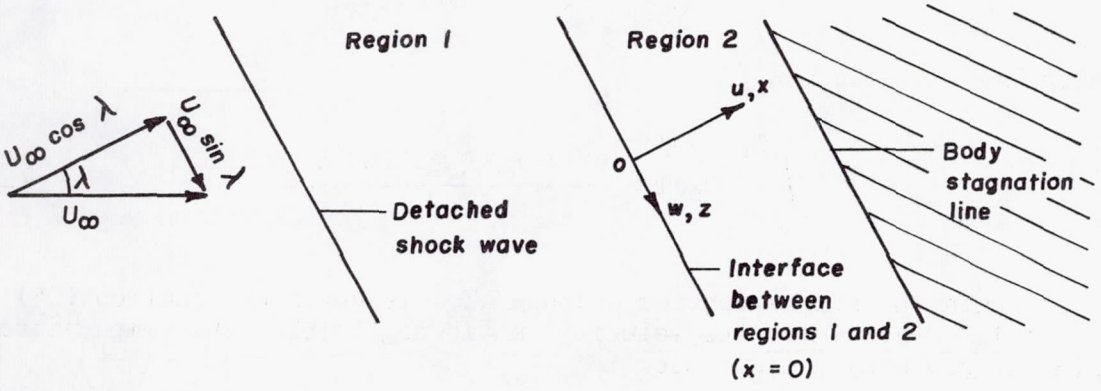
$$q = \left(\frac{k_\infty T_\infty}{R_b}\right) \left[\frac{3}{8(\gamma_s - 1)}\right]^{1/4} \left(\frac{R_b}{R_s}\right)^{1/2} \left(\frac{\gamma_\infty R}{2C_p}\right)^{\frac{n}{2}+1} \frac{Re_\infty^{1/2} M_\infty^{n+2}}{n+1} \left[1 - \left(\frac{T_b}{T_o}\right)^{n+1}\right] \tag{40}$$

These considerations complete the zero-yaw analysis. However, before undertaking the study of effects of yaw on heat transfer it is appropriate to make a few remarks. There is the general question of the legitimacy of

the several assumptions underlying the present treatment of stagnation-point flows. In order to shed some light on this matter it is undertaken later in the report to examine the solutions obtained to see whether they are consistent with these assumptions and with pertinent results obtained by others. In this regard it is shown that the presumption of a constant second derivative of pressure normal to the stagnation streamline yields solutions for the distance between shock wave and body which are quite close to observed values. Next, it is demonstrated that, as assumed, the velocity u is negligibly small throughout region 2 under continuum flow conditions. Then it is shown that the largest of the viscous dissipation terms neglected in the energy equation for region 2 is indeed small compared to the heat-conduction terms. It is found too that the analysis predicts an amount of heat convected into region 2 which is the proper order of magnitude to account for the heat transferred to the body. Finally, it is shown that under comparable conditions equation (35) of this paper predicts essentially the same heat transfer as references 7 and 8.

In view of these results it would seem that the simplified analysis presented here for stagnation-region flows is, while on the one hand certainly approximate, on the other hand quite capable of predicting useful information. Accordingly, we proceed to the study of effects of yaw on heat transfer.

Yaw.- In this case the x direction is normal to and the z direction is parallel to the stagnation line of the body (see plan view, sketch (c)). Then the z component of velocity has a finite value, but all z derivatives are again zero.



Region 1— Incompressible, nonviscous flow

Region 2— Low velocity, compressible, viscous flow

Sketch (c)

The y momentum equation in region 1, differentiated with respect to y, takes the same form as equation (10) on the stagnation streamline. Thus the velocity u is again given by the solution

$$u = -\frac{1}{C}\sqrt{-\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2}}\sinh Cx \tag{41}$$

The z momentum equation for the stagnation streamline in region 1 becomes, on dropping the negligible terms from equation (3),

$$u\frac{\partial w}{\partial x} = 0 \tag{42}$$

which has the solution

$$w^2 = \gamma_\infty R T_\infty M_\infty{}^2 \sin^2\lambda \tag{43}$$

since the transverse component of velocity is unchanged on passing through the shock wave.

The energy equation for the stagnation streamline in region 1 reduces to a form similar to equation (22)

$$u\frac{\partial u}{\partial x} + w\frac{\partial w}{\partial x} + C_p\frac{\partial T}{\partial x} = 0 \tag{44}$$

which has the solution

$$\int_T^{T_t} C_p dT = \frac{u^2 + \gamma_\infty R T_\infty M_\infty{}^2 \sin^2\lambda}{2} \tag{45}$$

where again the stagnation temperature T_t is given by equation (24). At the interface where the velocity u is negligible, the temperature $T_o(\lambda)$ is given by the solution to

$$\int_{T_o(\lambda)}^{T_t} C_p dT = \frac{\gamma_\infty R T_\infty M_\infty{}^2 \sin^2\lambda}{2} \tag{46}$$

which, for a constant heat capacity, C_p, is

$$\frac{T_o(\lambda)}{T_\infty} = \left(\frac{\gamma_\infty R}{2C_p}\right) M_\infty^2 \cos^2\lambda \tag{47}$$

The differentiated y momentum equation for region 2 takes on the same form on the stagnation streamline as equation (16). Hence, the solution is

$$\mu \frac{\partial u}{\partial x} = - \frac{\partial^2 p}{\partial y^2} \frac{x^2}{2} - \mu_o(\lambda) \sqrt{- \frac{1}{\rho} \frac{\partial^2 p}{\partial y^2}} \tag{48}$$

and the body stagnation point coordinate is

$$[x_b(\lambda)]^2 = \frac{2\mu_o(\lambda)}{\sqrt{- \rho \frac{\partial^2 p}{\partial y^2}}} \tag{49}$$

Now, however, the second derivative of pressure is a function of the angle of yaw (see Appendix B),

$$\left(\rho \frac{\partial^2 p}{\partial y^2}\right)_s = - \frac{6\rho_\infty^2 U_\infty^2 \cos^2\lambda}{(\gamma_s - 1)R_s^2} \tag{50}$$

so the stagnation-point coordinate is given by

$$\frac{x_b(\lambda)}{R_b} = \left(\frac{\gamma_s - 1}{6}\right)^{1/4} \left(\frac{\mu_o(\lambda)}{\mu_\infty} \frac{R_s}{R_b}\right)^{1/2} \frac{2}{Re_\infty^{1/2} \cos^{1/2}\lambda} \tag{51}$$

In region 2, the solutions to the z momentum equation and the energy equation are considered simultaneously. The z momentum equation simplifies (to the order of this analysis), in the region of the stagnation streamline, to

$$\frac{\partial}{\partial x}\left(\mu \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial w}{\partial y}\right) = 0 \tag{52}$$

Similarly, the energy equation near the stagnation streamline in region 2 may be written (note that $\partial w/\partial y$ is zero by symmetry)

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \mu \left(\frac{\partial w}{\partial x}\right)^2 = 0 \tag{53}$$

In order to facilitate the solution of equation (52), it is helpful to observe that the yawed boundary layer, identified with the w component of velocity, resembles the boundary layer on a flat plate. It might be anticipated then that, just as in the case of the flat plate, the variation of w with x is relatively insensitive to variations of μ with x. In this event equation (52) has the approximate form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \tag{54}$$

The solution is taken in polar coordinates in order to conveniently fit the boundary condition that w is identically zero at the body surface. Then following the same arguments used in deriving equations (29) and (30), one obtains on the stagnation streamline

$$w = B \ln \frac{\bar{r}}{R_b} + \sum_{n=1}^{\infty} C_n \bar{r}^n \left[1 - \left(\frac{R_b}{\bar{r}}\right)^{2n}\right] = I\left(\epsilon - \frac{\epsilon^2}{2}\right) + O(\epsilon^3) \tag{55}$$

where again $\epsilon = (\bar{r}/R_b) - 1 << 1$. If second order and higher terms in ϵ are neglected, the z component of velocity on the stagnation streamline becomes, in terms of x/x_b,

$$w = w_o \left(1 - \frac{x}{x_b}\right) \tag{56}$$

whence

$$\frac{\partial w}{\partial x} = - \frac{w_o}{x_b} \tag{57}$$

If this result is substituted for the last term in equation (53), the energy equation becomes

$$\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + \mu\left(\frac{w_o}{x_b}\right)^2 = 0 \tag{58}$$

A solution for equation (58) which satisfies symmetry conditions on the stagnation streamline and also the boundary conditions that η and μ are constant along the surface of the body is

$$\eta(\lambda) = \eta_b + B \ln\frac{\bar{r}}{R_b} + \sum_{n=1}^{\infty} C_n \bar{r}^n\left[1 - \left(\frac{R_b}{\bar{r}}\right)^{2n}\right]\cos n\theta - \frac{\bar{\mu}}{4}\left(\frac{w_o}{x_b}\right)^2(\bar{r}^2 - R_b^2) \tag{59}$$

where $\bar{\mu}$ is a mean value of μ in the stagnation region. If equation (59) is expanded in terms of ϵ, η takes the following form on the stagnation streamline

$$\eta(\lambda) = \eta_b + J\left(\epsilon - \frac{\epsilon^2}{2}\right) - \frac{\bar{\mu}w_o^2}{2\epsilon_o^2}\left(\epsilon + \frac{\epsilon^2}{2}\right) + O(\epsilon^3) \tag{60}$$

The constant J is evaluated by letting η be η_o when ϵ is $\epsilon_o = x_b/R_b$ and is given by the relation

$$J = \frac{\eta_o - \eta_b}{\epsilon_o}\left(1 + \frac{\epsilon_o}{2}\right) + \frac{\bar{\mu}w_o^2}{2\epsilon_o^2}(1 + \epsilon_o) + \cdots \tag{61}$$

The rate of heat transfer per unit area to the stagnation region of the body at angle of yaw λ is, from equation (60),

$$q(\lambda) = -\left.\frac{\partial\eta(\lambda)}{\partial x}\right|_b = \frac{1}{R_b}\left.\frac{\partial\eta(\lambda)}{\partial\epsilon}\right|_b = \frac{1}{R_b}\left(J - \frac{\bar{\mu}w_o^2}{2\epsilon_o^2}\right) \tag{62}$$

Substituting equations (26) and (61) into this expression and neglecting terms of the order ϵ_o compared to 1, one obtains

$$q(\lambda) = \frac{1}{x_b(\lambda)}\left(\int_{T_b}^{T_o(\lambda)} k\,dT + \frac{\bar{\mu}w_o^2}{2}\right) \tag{63}$$

Multiplying by $\frac{2R_b}{k_o(T_o - T_b)}$ and substituting from equation (51) yields

$$-\frac{2q(\lambda)R_b}{k_o(T_o - T_b)} = \left(\frac{6}{\gamma_s - 1}\right)^{1/4}\left(\frac{\mu_\infty}{\mu_o(\lambda)}\frac{R_b}{R_s}\right)^{1/2}\frac{Re_\infty^{1/2}\cos^{1/2}\lambda}{k_o(T_o - T_b)}$$

$$\left(\int_{T_b}^{T_o(\lambda)} k\,dT + \frac{\bar{\mu}w_o^2}{2}\right) \tag{64}$$

For a constant heat capacity it follows from equations (23), (43), and (47) that

$$\left.\begin{aligned}\frac{T_o(\lambda)}{T_o} &= \cos^2\lambda \\ \frac{w_o^2}{T_o} &= 2C_p\sin^2\lambda\end{aligned}\right\} \tag{65}$$

If, in addition, the thermal conductivity is proportional to the nth power of temperature, then

$$\frac{1}{k_o(T_o - T_b)}\int_{T_b}^{T_o(\lambda)} k\,dT = \frac{\cos^{2n+2}\lambda}{(n+1)(1 - T_b/T_o)}\left[1 - \left(\frac{T_b}{T_o\cos^2\lambda}\right)^{n+1}\right] \tag{66}$$

and

$$\frac{\bar{\mu}w_o^2}{2k_o(T_o - T_b)} = \left(\frac{\bar{\mu}}{\mu_o}\right)\frac{Pr\sin^2\lambda}{1 - T_b/T_o} = \frac{\bar{\mu}}{\mu_o(\lambda)}\frac{Pr\cos^{2n}\lambda\sin^2\lambda}{1 - T_b/T_o} \tag{67}$$

Thus equation (64) becomes

$$-\frac{2q(\lambda)R_b}{k_o(T_o - T_b)} = \frac{1}{n+1}\left(\frac{6}{\gamma_s - 1}\right)^{1/4}\left(\frac{R_b}{R_s}\right)^{1/2}\left(\frac{2C_p}{\gamma_\infty \bar{R}}\right)^{n/2}\frac{Re_\infty^{1/2}\cos^{n+1/2}\lambda}{M_\infty^{n}(1 - T_b/T_o)}$$

$$\left\{\cos^2\lambda\left[1 - \left(\frac{T_b}{T_o\cos^2\lambda}\right)^{n+1}\right] + (n+1)Pr\frac{\bar{\mu}}{\mu_o(\lambda)}\sin^2\lambda\right\} \tag{68}$$

The ratio of equation (68) to equation (39) is the ratio of the rate of heat transfer to the stagnation region of a yawed body to the rate of heat transfer to the stagnation region of the same body at zero yaw. This ratio is

$$\frac{q(\lambda)}{q(0)} = \frac{\cos^{n+1/2}\lambda}{1 - (T_b/T_o)^{n+1}}\left\{\cos^2\lambda\left[1 - \left(\frac{T_b}{T_o\cos^2\lambda}\right)^{n+1}\right] + (n+1)Pr\frac{\bar{\mu}}{\mu_o(\lambda)}\sin^2\lambda\right\} \tag{69}$$

An analogous expression can be obtained for the ratio of Nusselt numbers, thus,

$$\frac{Nu(\lambda)}{Nu(0)} = \frac{q(\lambda)}{q(0)}\left(\frac{T_o}{T_r}\right)^{n}\frac{T_o - T_b}{T_r - T_b} \tag{70}$$

where from equation (63) the recovery temperature, T_r, is the solution to

$$\int_{T_r}^{T_o(\lambda)} k\,dT = -\frac{\bar{\mu}w_o^2}{2} \tag{71}$$

However, it should be noted that the assumptions used in the analysis tend to be violated when the body temperature approaches recovery conditions. Therefore it should not be expected that equation (71) will yield accurate values for recovery temperature.

There remains, of course, the problem of determining $\bar{\mu}$. For the purposes of this report $\bar{\mu}$ will be taken as the arithmetic average between $\mu_o(\lambda)$ and μ_b, that is, $\bar{\mu} = [\mu_o(\lambda) + \mu_b]/2$.[4] In this event equation (69) can be written

$$\frac{q(\lambda)}{q(0)} = \frac{\cos^{n+1/2}\lambda}{1 - (T_b/T_o)^{n+1}}\left\{\cos^2\lambda\left[1 - \left(\frac{T_b}{T_o\cos^2\lambda}\right)^{n+1}\right] + \frac{n+1}{2}\,Pr\left[1 + \left(\frac{T_b}{T_o\cos^2\lambda}\right)^{n}\right]\sin^2\lambda\right\} \tag{72}$$

which in the case of a relatively cool surface (i.e., $T_b/T_o\cos^2\lambda << 1$) becomes

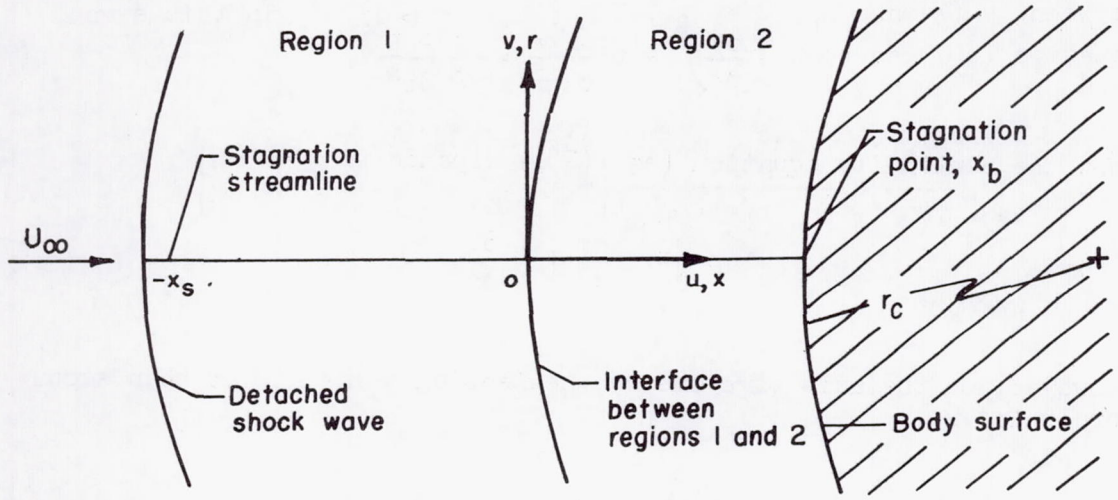
$$\frac{q(\lambda)}{q(0)} = \cos^{n+1/2}\lambda\left(\cos^2\lambda + \frac{n+1}{2}\,Pr\,\sin^2\lambda\right) \tag{73}$$

Heat Transfer to an Axially Symmetric Stagnation Region

The methods used to calculate the rate of heat transfer to a cylindrical stagnation region can also be applied to the stagnation region of a spherical body. This analysis is parallel to that for the cylinder at zero yaw and thus the x axis is taken as the stagnation streamline and the origin of the coordinate system is placed at the interface between the assumed incompressible nonviscous region 1 and the viscous, low-velocity, compressible region 2. For the purpose of obtaining the solutions for velocity in regions 1 and 2 on the stagnation streamline, it

[4]Actually this procedure might better be considered the first step in an iteration method where $\bar{\mu}$ is recalculated on the basis of the preceding calculation of T as a function of x. This refinement is not considered warranted here where only the gross effects of yaw for angles of yaw well less than 90° are of principal interest. As the angle of yaw approaches 90°, the analysis as a whole tends to break down due to the violation of the several assumptions predicated on the flow being hypersonic normal to the axis of the cylinder.

is most convenient to consider the momentum and continuity equations in cylindrical coordinates (sketch (d)). Because of axial symmetry, all



Region 1 – Incompressible, nonviscous flow

Region 2 – Low-velocity, compressible, viscous flow

Sketch (d)

properties are independent of the angular coordinate and, accordingly, the r direction momentum equation becomes

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}\right)=-\frac{\partial p}{\partial r}-\frac{2}{3}\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}\right)\right]+2\frac{\partial}{\partial r}\left(\mu\frac{\partial v}{\partial r}\right)+$$

$$2\frac{\mu}{r}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)\right] \tag{74}$$

While the continuity equation is

$$\frac{\partial}{\partial x}(\rho u)+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v)=0 \tag{75}$$

In region 1 where the viscous terms are considered identically zero, the r momentum equation (eq. (74)) reduces to

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r}\right)=-\frac{\partial p}{\partial r} \tag{76}$$

Differentiating equation (76) with respect to r and dropping terms with factors v and $\partial v/\partial x$, which vanish on the stagnation streamline, gives

$$\left(\frac{\partial v}{\partial r}\right)^2 + u\frac{\partial^2 v}{\partial x \partial r} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial r^2} \tag{77}$$

Now the continuity equation (eq. (75)) expands to

$$\rho\frac{\partial u}{\partial x} + \frac{\rho v}{r} + \rho\frac{\partial v}{\partial r} = 0 \tag{78}$$

however, on the axis of symmetry, neglecting terms higher than second order in r,

$$\frac{\partial v}{\partial r} = \frac{v}{r} \tag{79}$$

Thus, for incompressible flow, the continuity equation on the stagnation streamline reduces to

$$\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial r} = 0 \tag{80}$$

Substituting equation (80) into equation (77) yields

$$\frac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{u}{2}\frac{\partial^2 u}{\partial x^2} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial r^2} \tag{81}$$

which, upon differentiating with respect to x, and assuming $\frac{1}{\rho}\frac{\partial^2 p}{\partial r^2}$ = constant, becomes

$$u\frac{\partial^3 u}{\partial x^3} = 0 \tag{82}$$

For nonzero values of the velocity u, this differential equation has as a solution

$$u = \frac{u_o''}{2}x^2 + u_o'x + u_o \tag{83}$$

The value of velocity at the interface, u_o, is again considered very small. Thus, from equation (81), the first derivative of velocity at the interface is, approximately,

$$u_o' = -\sqrt{-\frac{4}{\rho}\frac{\partial^2 p}{\partial r^2}} \tag{84}$$

As can be seen from the solution for velocity, equation (83), the second derivative of velocity is constant. Therefore the second derivative of velocity may be evaluated from equation (81) using conditions just behind the shock wave, thus,

$$u_o'' = \frac{u_s'^2}{2u_s} + \frac{2}{u_s\rho_s}\left(\frac{\partial^2 p}{\partial r^2}\right)_s \tag{85}$$

Substituting for the values of velocity, velocity derivative, and second pressure derivative behind the shock wave (see Appendix B) yields

$$u_o'' = \frac{8(3 - 2\gamma_s)}{\gamma_s^2 - 1}\frac{U_\infty}{R_s^2} \tag{86}$$

and

$$u_o' = -\frac{4\sqrt{2(\gamma_s - 1)}}{\gamma_s + 1}\frac{U_\infty}{R_s} \tag{87}$$

Now in region 2, the viscous terms are retained in equation (74). Following the procedure used in studying two-dimensional flow (see Appendix A), the r momentum equation, differentiated with respect to r, is simplified to

$$\frac{\partial}{\partial x}\left(\frac{\mu}{2}\frac{\partial^2 u}{\partial x^2}\right) = -\frac{\partial^2 p}{\partial r^2} \tag{88}$$

which integrates to

$$\frac{\mu}{2}\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 p}{\partial r^2}x + A \tag{89}$$

and, as in the case of the two-dimensional flow,

$$\frac{\mu}{2}\frac{\partial u}{\partial x} = -\frac{\partial^2 p}{\partial r^2}\frac{x^2}{2} + Ax + B \tag{90}$$

The constants A and B are again determined by matching the first and second derivatives of velocity at the interface. Thus

$$\begin{aligned} A &= \frac{\mu_o u_o''}{2} \\ B &= \frac{\mu_o u_o'}{2} \end{aligned} \tag{91}$$

At the body $\partial u/\partial x$ vanishes, and solving for the coordinate x_b from equation (90) results in

$$x_b = +\frac{\mu_o u_o''}{2\left(\frac{\partial^2 p}{\partial r^2}\right)}\left(1 - \sqrt{1 + \frac{4\partial^2 p}{\partial r^2}\frac{u_o'}{\mu_o u_o''}}\right) \tag{92}$$

In Appendix B it is shown that

$$\frac{\partial^2 p}{\partial r^2} = -\frac{8}{\gamma_s + 1}\frac{\rho_\infty U_\infty^2}{R_s^2} \tag{93}$$

Thus from equations (86), (87), and (93), it can be shown that

$$4\frac{\partial^2 p}{\partial r^2}\frac{u_o'}{\mu_o u_o''^2} = \frac{(\gamma - 1)^2\sqrt{2(\gamma - 1)}}{(3 - 2\gamma)^2}\frac{\mu_\infty}{\mu_o}\frac{R_s}{R_b}\,Re_\infty \tag{94}$$

which is large compared to unity for any reasonably large value of Reynolds number (of the order of hundreds or greater). Therefore, if

quantities of the order of unity are neglected in equation (92), the stagnation point coordinate reduces to

$$x_b^2 = \frac{2\mu_o}{\sqrt{-\rho \frac{\partial^2 p}{\partial r^2}}} \tag{95}$$

which is identical in form to the relation for body surface coordinate in the two-dimensional flow (eq. (19)).

Next, in region 1 the viscous dissipation and heat-conduction terms are again neglected in the energy equation, and terms that vanish by reasons of symmetry along the stagnation streamline are dropped. Thus the energy equation for region 1 takes the same form as equation (23) for the two-dimensional problem and, since the interface velocity is small, the interface temperature T_o is again approximately the stagnation temperature T_t.

In region 2, the heat-conduction terms in the energy equation predominate, and the equation reduces to the three-dimensional Laplace equation in the variable η

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial^2 \eta}{\partial z^2} = 0 \tag{96}$$

In order to fit the boundary conditions on a spherical surface, the solution is given in terms of spherical coordinates ($\bar{r}$, θ, and φ). The general solution which preserves symmetry about the x axis (i.e., which is independent of φ) is

$$\eta = A + \frac{B}{\bar{r}} + \sum_{n=1}^{\infty} \left(C_n \bar{r}^n + \frac{D_n}{\bar{r}^{n+1}} \right) P_n(\cos \theta) \tag{97}$$

where $P_n(\cos \theta)$ is the nth order Legendre polynomial in $\cos \theta$. If it is required that η be a constant, η_b, on the surface of the body, equation (97) can be reduced, on the stagnation streamline, to

$$\eta = \eta_b - \frac{B}{R_b} \left(1 - \frac{R_b}{\bar{r}} \right) + \sum_{n=1}^{\infty} C_n \bar{r}^n \left[1 - \left(\frac{R_b}{\bar{r}} \right)^{2n+1} \right] \tag{98}$$

then expanding in terms of $\epsilon = \frac{\bar{r}}{R_b} - 1 << 1$, results in

$$\eta = \eta_b + L(\epsilon - \epsilon^2) + \ldots \tag{99}$$

where L is $-\frac{B}{R_b} + \sum_{n=1}^{\infty} C_n(2n + 1)R_b^n$. Neglecting the quadratic term in ϵ, evaluating η at the interface, and transforming to the variable x, one obtains for equation (99) on the stagnation streamline

$$\eta = \eta_o - (\eta_o - \eta_b)\frac{x}{x_b} \tag{100}$$

Then the rate of heat transfer to the stagnation point is

$$q = -\eta_b' = \frac{\eta_o - \eta_b}{x_b} = \frac{1}{x_b}\int_{T_b}^{T_o} k\, dT \tag{101}$$

which is identical in form with the zero-yaw solution for the two-dimensional-flow problem (eq. (31)). Note that in Appendix B the second derivative of pressure given by equation (B18) is larger by a factor of 4/3 than it is for the corresponding two-dimensional-flow case with the same shock-wave curvature (eq. (B17)). Thus x_b given by equation (95) is changed by the factor $(3/4)^{1/4}$ and the rate of heat transfer to an axially symmetric region becomes

$$q = \left[\frac{1}{2(\gamma_s - 1)}\right]^{1/4}\left(\frac{\mu_\infty}{\mu_o}\frac{R_b}{R_s}\right)^{1/2}\frac{Re_\infty^{1/2}}{R_b}\int_{T_b}^{T_o} k\, dT \tag{102}$$

while the corresponding expression for Nusselt number is

$$Nu = \left(\frac{8}{\gamma_s - 1}\right)^{1/4}\left(\frac{\mu_\infty}{\mu_o}\frac{R_b}{R_s}\right)^{1/2}\frac{Re_\infty^{1/2}}{k_o(T_o - T_b)}\int_{T_b}^{T_o} k\, dT \tag{103}$$

Examination of Analysis and Assumptions

A number of assumptions have been made in the theoretical analysis, and it is desirable now to show that the solutions obtained are both realistic and consistent with these assumptions. In particular, it will be shown that the presumption of a constant second derivative of pressure normal to the stagnation streamline yields solutions for the distance between shock wave and body which are reasonably close to observed values. Secondly, it will be demonstrated that the u velocity throughout region 2 is indeed small, as assumed in the analysis, if the Reynolds number is large enough for continuum flow conditions. In addition, it will be shown that for region 2 the viscous-dissipation terms due to the u and v component velocity derivatives are small compared to the heat-conduction terms in the energy equation, again provided the Reynolds number is not too small. These findings, then, help to justify the manner in which the momentum and energy equations were treated in the analysis.

Now it is obvious that the assumption of an abrupt transition from nonviscous, convective flow to viscous, conductive flow is a substantial idealization of the actual flow.[5] It is possible, however, to make a gross check on the self-consistency of this model by comparing the amount of heat convected across the interface with the amount conducted to the body surface. When this is done it is found that from a heat-flow point of view, the model is self-consistent (i.e., heat convected provides for heat conducted).

As a final point, a comparison will be made between the analysis of this paper and the heat-transfer solutions for low-velocity flow given by Howarth (ref. 7) and Cohen and Reshotko (ref. 8).

Distance between shock wave and body.- Consider first axially symmetric flow. The velocity in region 1 was found to be (eq. (83))

$$u = u_o + u_o' x + \frac{u_o''}{2} x^2 \tag{104}$$

[5]Strictly speaking, this idealized model should be considered simply a first approximation to the correct situation. A second approximation would be to divide the domain between the body and shock wave into three regions rather than two as was done here.

Then, the shock-wave coordinate must be

$$x_s = \frac{u_s' - u_o'}{u_o''} \tag{105}$$

It can be shown from the relations in Appendix B and equations (84) and (85) that

$$\left.\begin{aligned} u_s' &= -\frac{4U_\infty}{(\gamma_s + 1)R_s} \\ u_o' &= -\sqrt{-\frac{4}{\rho}\frac{\partial^2 p}{\partial r^2}} = -\frac{4U_\infty}{(\gamma_s + 1)R_s}\sqrt{2(\gamma_s - 1)} \\ u_o'' &= \frac{8(3 - 2\gamma)}{\gamma_s^2 - 1}\frac{U_\infty}{R_s^2} \end{aligned}\right\} \tag{106}$$

Substituting these relations into equation (105) yields

$$\frac{x_s}{R_s} = -\frac{(\gamma_s - 1)\left[1 - \sqrt{2(\gamma_s - 1)}\right]}{2(3 - 2\gamma_s)} \tag{107}$$

Note that for $\gamma_s = 1.5$, u_o'' vanishes and the velocity profile becomes linear. For this case x_s/R_s reduces to $(\gamma_s - 1)/4$.

The actual distance between the body and the shock wave is, of course, the sum of x_s and x_b. However, it can be shown from equations (95) and (107) that x_b is small compared to x_s for reasonably large Reynolds numbers, and x_b will therefore be neglected. The ratio x_s/R_s calculated from equation (107) for γ_s equal 1.4 is 0.105. Measurements of x_s/R_b taken from spark photographs of high-velocity spheres presented by Charters and Thomas (ref. 9) and Dugundji (ref. 10) approach this value closely at high Mach numbers (i.e., x_s/R_b about 0.11 at Mach number 4). Heybey (ref. 11) has developed a theory which fits the data of references 9 and 10 closely and, for the limit of infinite Mach number, predicts x_s/R_b about 0.12. Thus it is seen that at high Mach numbers, the assumption that the second derivative of pressure is constant and that the ratio R_b/R_s is near unity yields results which are consistent with experimentally observed distances between the shock wave and a spherical body, as well as with the theory of Heybey.

It is of interest to calculate the shock-wave coordinate for two-dimensional flow as well. Recall that the solution for velocity in region 1 for this case is

$$u = \frac{u_o'}{C} \sinh Cx \tag{108}$$

and thus

$$\frac{\partial u}{\partial x} = u_o' \cosh Cx \tag{109}$$

The velocity derivatives at the shock wave and at the interface, given in Appendix B, are, respectively,

$$\left.\begin{aligned} u_s' &= -\frac{2U_\infty}{(\gamma_s + 1)R_s} \\ u_o' &= -\sqrt{-\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2}} = -\frac{U_\infty\sqrt{6(\gamma_s - 1)}}{(\gamma_s + 1)R_s} \end{aligned}\right\} \tag{110}$$

Then the product Cx_s is given by

$$Cx_s = \operatorname{arc\,cosh} \frac{2}{\sqrt{6(\gamma_s - 1)}} \tag{111}$$

With Cx_s known and the velocity at the shock wave

$$u_s = \frac{\gamma_s - 1}{\gamma_s + 1} U_\infty \tag{112}$$

The shock-wave coordinate becomes

$$\frac{x_s}{R_s} = -\sqrt{\frac{\gamma_s - 1}{6}} \frac{Cx_s}{\sinh Cx_s} \tag{113}$$

For a γ_s of 1.4, Cx_s takes the value 0.75 and since the sinh function is very nearly linear over this range, rather close bounds on the shock-wave coordinate are imposed by

$$\left|\frac{u_s}{u_s'}\right| < \left|x_s\right| < \left|\frac{u_s}{u_o'}\right| \tag{114}$$

or

$$\frac{\gamma_s - 1}{2} < \left|\frac{x_s}{R_s}\right| < \sqrt{\frac{\gamma_s - 1}{6}} \tag{115}$$

The exact theoretical solution for x_s/R_s at $\gamma_s = 1.4$ is 0.236. According to the theory then, a shock wave with given radius of curvature should be detached from a cylindrical body about twice as far as from a sphere, assuming $R_s/R_b \approx 1$.

Magnitude of velocity in region 2.- The y momentum equation in region 2 was reduced to

$$\mu \frac{\partial u}{\partial x} = - \frac{\partial^2 p}{\partial y^2} \frac{x^2}{2} + \mu_o u_o' \tag{116}$$

The left side of this equation may be approximated by $\frac{\partial}{\partial x}(\mu u)$ with the presumption that velocity in region 2 is small. Then equation (116) may be integrated to

$$\mu u = - \frac{\partial^2 p}{\partial y^2} \frac{x^3}{6} + \mu_o u_o' x + \mu_o u_o \tag{117}$$

Solving for u_o, noting that velocity vanishes at x_b, and substituting from equations (14) and (19), one obtains

$$u_o = - \frac{2}{3} u_o' x_b = \frac{2}{3} \sqrt{- \frac{2\mu_o u_o'}{\rho_s}} \tag{118}$$

It follows that the ratio of interface velocity to the velocity at the shock wave is given by

$$\left(\frac{u_o}{u_s}\right)^2 = -\frac{8}{9}\frac{\mu_o}{\rho_s u_s}\frac{u_o'}{u_s} = -\frac{16}{9}\left(\frac{\mu_o}{\mu_\infty}\right)\frac{R_b}{Re_\infty}\frac{u_o'}{u_s} \tag{119}$$

which on substituting the relations given in Appendix B for u_o' and u_s becomes

$$\left(\frac{u_o}{u_s}\right)^2 = \frac{16}{9}\sqrt{\frac{6}{\gamma_s - 1}}\left(\frac{\mu_o}{\mu_\infty}\frac{R_b}{R_s}\right)\frac{1}{Re_\infty} \tag{120}$$

It can be seen that for large Reynolds numbers, of the order of hundreds or greater, the velocity at the interface is small compared to the velocity at the shock wave. Since the velocity in region 2 is everywhere less than at the interface (see eq. (117)), the solutions obtained for velocity are consistent with the assumption that velocity is small throughout region 2.

Viscous dissipation in region 2.- Although the derivative of velocity vanishes at the body surface, it increases parabolically (see eq. (116)) to u_o' at the interface. Since viscous dissipation terms due to this velocity shear were neglected in solving the energy equation, it will be shown that the maximum value of these terms, which occurs at the interface, is small compared to the heat-conduction terms like $\partial^2\eta/\partial x^2$ (note that by continuity $\partial v/\partial y$ contributes a dissipation term of the same magnitude as $\partial u/\partial x$). From equation (30) it can be seen that the term $\partial^2\eta/\partial x^2$ is nearly constant everywhere along the stagnation streamline in region 2. Then the ratio of differential terms in the energy equation is, by equations (30) and (118),

$$\frac{4\mu_o(u_o')^2}{(\partial^2\eta/\partial x^2)_b} = \frac{9\mu_o u_o^2}{x_b} R_b \left(\int_{T_b}^{T_o} k\, dT\right)^{-1} \tag{121}$$

If equation (121) is evaluated for constant heat capacity and thermal conductivity proportional to the nth power of temperature, there is obtained

$$\frac{4\mu_o(u_o')^2}{(\partial^2\eta/\partial x^2)_b} = 18(n+1)\left(\frac{u_o}{U_\infty}\right)^2 Pr\left(\frac{R_b}{x_b}\right)$$

Substituting for velocity ratio u_o/U_∞ from equations (120) and (B4) and for the ratio R_b/x_b from equation (21), there results

$$\frac{4\mu_o(u_o')^2}{(\partial^2\eta/\partial x^2)_b} = 16(n+1)\mathrm{Pr}\left(\frac{\gamma-1}{\gamma+1}\right)^2\left(\frac{6}{\gamma-1}\right)^{3/4}\left(\frac{\mu_o}{\mu_\infty}\right)^{1/2}\left(\frac{R_b}{R_s}\right)^{3/2}\mathrm{Re}_\infty^{-1/2} \tag{123}$$

Once again the square root of Reynolds number is the predominant term for conditions of continuum flow and thus the viscous dissipation terms in the energy equation are small compared to the conduction terms in region 2.

Heat convection across the interface, x = 0.- Next consider the ratio of the heat convected across the interface, $\rho u_o \int_o^{T_o} C_p dT$, to the heat transfer at the stagnation point of the body, $-\eta_b'$. The value of u_o given by equation (118) and $-\eta_b'$ from equation (32) yields

$$-\frac{\rho u_o}{\eta_b'}\int_o^{T_o} C_p dT = -\frac{2u_o' x_b^2 \rho}{3}\frac{\int_o^{T_o} C_p dT}{\int_{T_b}^{T_o} k\, dT} \tag{124}$$

Again evaluating for constant heat capacity and the nth power temperature function for thermal conductivity, and noting from equations (14) and (19) that $u_o' x_b^2$ reduces to $-(2\mu_o/\rho)$, one obtains

$$-\frac{\rho u_o C_p T_o}{\eta_b'} = \frac{4}{3}\mathrm{Pr}(n+1) \tag{125}$$

This ratio is the order of unity, and thus the right magnitude of heat is convected across the interface to balance the heat conducted to the body. The above result also provides a check on the value of x_b which was obtained by matching u_o' as a boundary condition of the y momentum equation.

Low-velocity heat transfer.- For hypersonic velocities it was found that taking shock-wave curvature equal to body curvature on the stagnation streamline gave approximately the correct answer for the distance between the body and the shock wave, so presumably the ratio R_b/R_s should be taken near unity when calculating the heat transfer as well. Undoubtedly this ratio will be somewhat less than unity for low Mach number supersonic flow, and it is of interest to see what the solutions developed in this paper will predict for this case (even though the assumptions made in the analysis are not expected to hold as well for the low-velocity flow conditions). For this purpose it is convenient to express the body coordinate x_b in terms of $(\partial v/\partial y)_o$ which by continuity equals $-u_o'$. From equations (9), (14), and (19)

$$x_b = \sqrt{\frac{2\mu_o}{\rho(\partial v/\partial y)_o}} \tag{126}$$

then solving for Nusselt number from equations (32) and (34) for the case of the cool wall ($T_b/T_o << 1$, and $n = 1/2$) one obtains

$$Nu = \frac{2}{3}\frac{D_b}{x_b} = 0.47\, D_b \sqrt{\frac{\rho(\partial v/\partial y)_o}{\mu_o}} \tag{127}$$

The method of boundary-layer solution for low-velocity flow about a cylinder given in reference 7, yields for the derivative of velocity component normal to the stagnation streamline at the edge of the boundary layer, in the notation of this paper,

$$\left(\frac{\partial v}{\partial y}\right) = \frac{3.8U_s}{D_b} \tag{128}$$

Substituting in equation (127) results in

$$Nu = 0.92\, Re_s^{1/2}$$

where the small differences between μ_s and μ_o are neglected. The constant 0.92 compares favorably with the value 0.95 given by Howarth for $Pr = 0.72$. This agreement is especially remarkable in the light of the fact that the analysis of reference 7 is for constant thermal properties, while variation in thermal properties is an essential feature of this analysis.

Cohen and Reshotko (ref. 8) find that the solution for a compressible boundary layer gives the following relation at the stagnation point of an axially symmetric body

$$\frac{Nu\ y}{D_b} = 0.440\sqrt{\frac{\rho v y}{\mu}} \tag{129}$$

for the case of a cool wall and a Prandtl number 0.7. If the radial component of velocity v is taken proportional to y, the ordinate can be eliminated and equation (129) reduces to

$$Nu = 0.440\ D_b\sqrt{\frac{\rho(\partial v/\partial y)_o}{\mu}} \tag{130}$$

The factor 0.440 given by Cohen and Reshotko compares favorably with the factor 0.47 given in equation (127).

HEAT-TRANSFER RESULTS FOR BLUNT SHAPES IN HYPERSONIC FLIGHT

Temperatures in the disturbed flow about vehicles in hypersonic flight may be sufficiently large to dissociate air molecules into atoms or even to ionize the atoms. At present the chemical reaction rates for these processes are not known with certainty. Available experimental evidence (ref. 12) indicates that air will be in equilibrium throughout the stagnation region flow for vehicles in flight at velocities up to 26,000 feet per second, and at altitudes up to about 200,000 feet. At much greater altitudes, the atmosphere is so rarefied that the chemical reactions will probably be frozen and the air will behave essentially as a gas with constant specific heat. These two limiting cases, at least, can be treated within the framework of the present analytical results. For this purpose it will be convenient to consider the heat-transfer rate expressed in the form of a parameter which is relatively independent of scale size or density. From equation (102), such a parameter is given by

$$\frac{qR_b}{\sqrt{Re_\infty/2}} = \left(\frac{\rho_s}{\rho_\infty} - 1\right)^{1/4}\left(\frac{\mu_\infty}{\mu_o}\right)^{1/2}\int_o^{T_o} k\ dT \tag{131}$$

where it has been assumed that the surface temperature is negligible compared to T_o and that the shock-wave curvature equals the body curvature in the stagnation region. Equation (131) applies in the

spherical case; the rate of heat transfer to a cylindrical stagnation region may be smaller by the factor $(3/4)^{1/4}$ according to equation (33). Note that the integral may be evaluated with the thermal conductivity coefficients taken at constant pressure, since the pressure is relatively invariant along a stagnation streamline. The integrals have been calculated graphically using the data given in reference 13 and the results are shown in figure 1.

For the case where chemical reactions are frozen, all translational, rotational, and vibrational modes of energy are considered fully excited, C_p/R is taken a constant at 9/2, and the coefficients of viscosity and thermal conductivity are taken proportional to the half power of temperature. The heat-transfer parameter given by equation (131) for these conditions is shown in figure 2 for flight velocities from 5,000 to 30,000 feet per second.

For the case of chemical equilibrium, Feldman (ref. 14) has calculated the densities and stagnation temperatures which occur behind shock waves, and reference 13 gives values for the coefficients of viscosity and thermal conductivity. The chemical reactions, which keep the flow in equilibrium, cause the thermal conductivity to be much larger than in the frozen flow, but this effect is compensated for by the large decrease in stagnation temperature due to the strong heat sinks created by the reactions. Incidentally Kuo (ref. 15) finds similar compensation for the case of heat transmitted through the boundary layer along a flat plate. Because of the compensating effects, it is not immediately apparent whether the integral in equation (131) will be increased or decreased by the dissociation and ionization reactions. In all the cases calculated it is found that the integral is slightly greater under equilibrium conditions. In addition, both of the other factors in equation (131) are increased slightly by the chemical reactions leading to equilibrium. The density ratio across a normal shock may increase more than a factor of 2 (see ref. 14), but the heat-transfer rate varies only as the fourth root of this ratio and is not strongly influenced. Reference 13 finds that the coefficient of viscosity is increased somewhat at equilibrium, but this also is compensated by the decrease in stagnation temperature. The resulting ratio μ_∞/μ_0 is increased slightly, but again the effect on heat transfer is weakened by the square-root dependence on this factor. The total result of increases in all factors is that the parameter $qR_b Re_\infty^{-1/2}$ is the order of 30 percent greater for stagnation region flow in equilibrium than for such flow in which the chemical reactions are frozen. The difference is indicated by the two curves in figure 2.

The heat transfer calculated for the equilibrium flow is in satisfactory agreement with the experimental results reported by Rose and Riddell (ref. 16) as indicated in figure 2. It may be noted that there is a few percent change in the heat-transfer parameter due to different ambient temperature and pressure conditions which occur at different altitudes, but in view of the order of the approximations inherent in the theory and of the ±20-percent variation in experimental results, the

change is not significant enough to be shown in figure 2. The theoretical results for equilibrium flow also agree with numerical integrations of more complete boundary-layer equations, including chemical reaction terms, which have been made by Fay and Riddell (ref. 17). Thus it is concluded that the approximate theory presented in this report retains the essential relationships which influence stagnation-region heat transfer.

In view of the foregoing results, it seems reasonable that the present theory would also yield approximately correct values for the effects of yaw. Figure 3 shows the product of the secant of the yaw angle and the ratio of the stagnation-region heat flux at yaw to the flux at zero yaw, for the case where the wall temperature is negligible compared to the stagnation temperature. This quantity, $q(\lambda)/q(0)\cos\lambda$, equals the ratio of the heat flux per unit of span normal to the stream velocity, to the same heat flux at zero yaw. The ratio of the heat flux per unit area is just $q(\lambda)/q(0)$, of course. The frozen flow case was calculated from equation (73) where the Prandtl number was taken equal to 0.75, and this result is independent of velocity. The equilibrium flow heat transfer was calculated for flight at 26,000 feet per second at 100,000 and 150,000 feet altitude from the relation

$$\frac{q(\lambda)}{q(0)\cos\lambda} = \left[\frac{\frac{\rho_s(\lambda)}{\rho_\infty} - 1}{\frac{\rho_s(0)}{\rho_\infty} - 1}\right]^{1/4} \left[\frac{\mu_o(0)}{\mu_o(\lambda)\cos\lambda}\right]^{1/2} \frac{\int_0^{T_o(\lambda)} k\,dT + \frac{\mu_o(0)U_\infty^2\sin^2\lambda}{4}}{\int_0^{T_o(0)} k\,dT} \tag{132}$$

which is derived from equations (33) and (64). At small angles of yaw, the effect of yaw is to reduce heat flux slightly more in the chemically frozen flow than in the two equilibrium flow cases shown. This is due primarily to particular variations in the integral of thermal conductivity with stagnation temperature in the equilibrium flow (fig. 1) and is not necessarily typical. At larger angles of yaw, the reduction in heat transfer is about the same in either case. As shown in figure 3, the stagnation-region heat flux per unit span is reduced approximately by the factor $(\cos\lambda)^{1/2}$ at large angles of yaw up to 70°. The corresponding heat flux per unit area is reduced by about the factor $(\cos\lambda)^{3/2}$.

The effect of wall temperature on the reduction in heat flux caused by yaw is shown in figure 4. The heat-transfer rates are graphed for wall temperature to stagnation temperature ratios of 0.2, 0.1, 0.05, 0.02, 0.01, and 0 for flow in which the Prandtl number is equal to 0.75 and the dissociation reactions are frozen in a state of no dissociation (note that vibrational energy may be excited, however, without appreciable influence on the ratio, $q(\lambda)/q(0)$). At high yaw angles, the viscous crossflow is the predominant factor contributing to the stagnation-region heat transfer. The principal effect of high wall temperature is to maintain sizable air temperature, and therefore sizable viscosity and viscous

dissipation (see eq. (58)) throughout the crossflow boundary layer. As a consequence, the stagnation-region heat flux per unit span does not decrease monotonically with increasing yaw angle, but goes through minima as shown in figure 4. As the wall temperature is reduced, the viscosity near the body gradually becomes negligible compared to the viscosity near the edge of the boundary layer (i.e., at the interface $x = 0$). The results are not strongly influenced until the wall temperature is depressed to the order of 0.1 the stagnation temperature. Then as wall temperature is further decreased, the heat flux rapidly approaches the limiting value given by equation (73). Because of strong compensating effects, similar to those which occur in the cold-wall case at zero yaw, it is likely that the effect of wall temperature on heat transfer to yawed shapes in equilibrium flow will be quite similar to that shown in figure 4.

CONCLUDING REMARKS

The theory for heat flux to the stagnation region of blunt axially symmetric shapes in hypersonic flight, which is developed in this report, is found to agree favorably with other theoretical results and with available experimental evidence. It is concluded that this theory, though approximate, preserves the essential functional relationships which influence stagnation-region heat transfer. A similar analysis is made for the heat flux to a cylindrical stagnation region at angle of yaw. It is deduced that wing sweepback should reduce the heat flux per unit area at the leading edge approximately by the factor $(\cos \lambda)^{3/2}$, if the wall temperature is held relatively cool. This will reduce the cooling required to alleviate hot spots and the thermal-stress concentrations induced by heating in the stagnation region at very high-speed flight. The total stagnation-region cooling required for a given wing span will also be reduced in this case, since the heat flux per unit span decreases approximately as $(\cos \lambda)^{1/2}$.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., May 2, 1955

APPENDIX A

SIMPLIFICATION OF THE y MOMENTUM EQUATION IN REGION 2

The steady-state, two-dimensional y momentum equation (see eq. (2)), differentiated with respect to y, yields

$$\rho u \frac{\partial^2 v}{\partial x \partial y} + u \frac{\partial \rho}{\partial y} \frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \rho v \frac{\partial^2 v}{\partial y^2} + v \frac{\partial \rho}{\partial y} \frac{\partial v}{\partial y} + \rho \left(\frac{\partial v}{\partial y} \right)^2$$

$$= - \frac{\partial^2 p}{\partial y^2} - \frac{2}{3} \frac{\partial^2}{\partial y^2} \left[\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + 2 \frac{\partial^2}{\partial y^2} \left(\mu \frac{\partial v}{\partial y} \right) + \frac{\partial^2}{\partial y \partial x} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \quad \text{(A1)}$$

Now on the stagnation streamline the velocity v is identically zero and therefore all x derivatives of v are zero. Also, all odd order y derivatives of functions like density ρ, viscosity μ, pressure p, and velocity u vanish since, by symmetry, these functions are even. In addition, it is assumed that near the stagnation streamline the velocity u is so small throughout region 2 that terms with this factor may be neglected. With this assumption an additional useful relation can be deduced from the continuity equation

$$\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} = 0 \quad \text{(A2)}$$

Eliminating the terms with factors u, v, or $\partial\rho/\partial y$ from equation (A2) there results, as for incompressible flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(A3)}$$

Note that all derivatives of the sum $\partial u/\partial x + \partial v/\partial y$ are also zero in the regions where equation (A3) will hold.

Applying the above considerations simplifies equation (A1) to

$$\rho \left(\frac{\partial v}{\partial y} \right)^2 = - \frac{\partial^2 p}{\partial y^2} + 2 \frac{\partial^2}{\partial y^2} \left(\mu \frac{\partial v}{\partial y} \right) + \frac{\partial^2}{\partial y \partial x} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \quad \text{(A4)}$$

Now it will be assumed, as is usual, that the viscous flow in the region of the stagnation point of a blunt body is similar to viscous flow at the stagnation point of a body with infinite radius of curvature insofar as the velocity derivatives are concerned (i.e., the principle effect of the body curvature is to determine the magnitude of the pressure derivatives). Accordingly, $\partial^3 v/\partial y^3$ and $\partial^2 u/\partial y^2$ will be supposed to vanish in the stagnation region. Then expansion of the second member of the right side of equation (A4) yields

$$2\left(\frac{\partial^2 \mu}{\partial y^2}\frac{\partial v}{\partial y} + 2\frac{\partial \mu}{\partial y}\frac{\partial^2 v}{\partial y^2} + \mu\frac{\partial^3 v}{\partial y^3}\right)$$

in which the only term retained is $2(\partial^2\mu/\partial y^2)(\partial v/\partial y)$. Similar expansion of the last member of equation (A4) gives

$$\frac{\partial^2 \mu}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial \mu}{\partial y}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + \frac{\partial \mu}{\partial x}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \mu\left(\frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial y \partial x^2}\right)$$

Note that from equation (A3), $\partial^3 u/\partial y^2 \partial x$ is equivalent to $-(\partial^3 v/\partial y^3)$ and will therefore be neglected. The terms retained in this equation, then, are $(\partial\mu/\partial x)(\partial^2 v/\partial x \partial y) + \mu(\partial^3 v/\partial y \partial x^2)$. These terms can be combined into $-\frac{\partial}{\partial x}\left(\mu\frac{\partial^2 u}{\partial x^2}\right)$. Equation (A4) thus is reduced to

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial^2 u}{\partial x^2}\right) = -\frac{\partial^2 p}{\partial y^2} + 2\frac{\partial^2 \mu}{\partial y^2}\frac{\partial v}{\partial y} - \rho\left(\frac{\partial v}{\partial y}\right)^2 \tag{A5}$$

The derivative $\partial v/\partial y$ vanishes at the surface of the body, so that in the immediate region of the stagnation point, equation (A5) takes on the approximate from

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial^2 u}{\partial x^2}\right) = -\frac{\partial^2 p}{\partial y^2} \tag{A6}$$

This expression will be taken to hold near the stagnation streamline throughout region 2.

APPENDIX B

BOUNDARY VELOCITIES AND PRESSURE DERIVATIVES

For hypersonic Mach numbers, the density ratio across an oblique shock wave is

$$\frac{\rho_s}{\rho_\infty} = \frac{\gamma_s + 1}{\gamma_s - 1} \tag{B1}$$

then the pressure just downstream of the shock is

$$p_s = \frac{2\rho_\infty U_\infty{}^2 \cos^2\sigma}{\gamma_s + 1} \tag{B2}$$

where σ is the acute angle between the shock wave and the normal to the free-stream velocity vector (see ref. 18). It can also be shown that the v component of velocity just downstream of the shock is

$$v_s = \frac{2}{\gamma_s + 1} U_\infty \sin\sigma \cos\sigma \tag{B3}$$

while the u component on the stagnation streamline is

$$u_s = \frac{\gamma_s - 1}{\gamma_s + 1} U_\infty \tag{B4}$$

In evaluating the derivatives, consider a shock wave with radius of curvature R_s. Let s be the distance along this profile measured from the stagnation streamline and $x(s)$ and $y(s)$ be the equations for the shock-wave coordinates. Then

$$\frac{dv}{ds} = \frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds} \tag{B5}$$

while

$$\frac{d^2p}{ds^2} = \frac{\partial^2 p}{\partial x^2}\left(\frac{dx}{ds}\right)^2 + 2\frac{\partial^2 p}{\partial x \partial y}\frac{dx}{ds}\frac{dy}{ds} + \frac{\partial^2 p}{\partial y^2}\left(\frac{dy}{ds}\right)^2 + \frac{\partial p}{\partial x}\frac{d^2x}{ds^2} + \frac{\partial p}{\partial y}\frac{d^2y}{ds^2} \tag{B6}$$

In terms of the radius of curvature R_s, the differential equations for $x(s)$ and $y(s)$ are

$$\left.\begin{aligned} dx &= R_s\left(1 - \cos\frac{ds}{R_s}\right) \\ dy &= R_s \sin\frac{ds}{R_s} \end{aligned}\right\} \tag{B7}$$

and at the stagnation streamline ($ds = 0$) the following conditions hold

$$\left.\begin{aligned} \frac{dy}{ds} &= 1 \\ \frac{dx}{ds} &= 0 \\ \frac{d^2y}{ds^2} &= 0 \\ \frac{d^2x}{ds^2} &= \frac{1}{R_s} \end{aligned}\right\} \tag{B8}$$

Then, at the stagnation streamline, equations (B5) and (B6) become

$$\left(\frac{\partial v}{\partial y}\right)_s = \frac{dv}{ds} \tag{B9}$$

and

$$\left(\frac{\partial^2 p}{\partial y^2}\right) = \frac{d^2p}{ds^2} - \frac{1}{R_s}\left(\frac{\partial p}{\partial x}\right)_s \tag{B10}$$

Now by continuity and equations (B3) and (B9)

$$\left(\frac{\partial u}{\partial x}\right)_s = -\left(\frac{\partial v}{\partial y}\right)_s = -\frac{2U_\infty}{(\gamma_s + 1)R_s} \tag{B11}$$

for two-dimensional flow. For axially symmetric flow the corresponding relation is

$$\left(\frac{\partial u}{\partial x}\right)_s = -2\left(\frac{\partial v}{\partial r}\right)_s = -\frac{4U_\infty}{(\gamma_s + 1)R_s} \tag{B12}$$

According to equation (B2) the first right-hand term of equation (B10) is

$$\frac{d^2p}{ds^2} = -\frac{4\rho_\infty U_\infty^2}{(\gamma_s + 1)R_s^2} \tag{B13}$$

while the next term, $-\frac{1}{R_s}\left(\frac{\partial p}{\partial x}\right)_s$, is evaluated using the x momentum equation (eq. (1)) which for the nonviscous incompressible flow region on the stagnation streamline reduces to

$$\frac{\partial p}{\partial x} = -\rho u \frac{\partial u}{\partial x} \tag{B14}$$

According to equations (B11) and (B12), equation (B14) becomes

$$\frac{\partial p}{\partial x} = \frac{2(\gamma_s - 1)}{(\gamma_s + 1)^2}\frac{\rho_s U_\infty^2}{R_s} \tag{B15}$$

and

$$\frac{\partial p}{\partial x} = \frac{4(\gamma_s - 1)}{(\gamma_s + 1)^2}\frac{\rho_s U_\infty^2}{R_s} \tag{B16}$$

for the two-dimensional and the axially symmetric flow cases, respectively. Then the corresponding second partial derivatives of pressure are

$$\frac{\partial^2 p}{\partial y^2} = -\frac{6(\gamma_s - 1)\rho_s U_\infty^2}{(\gamma_s + 1)^2 R_s^2} \tag{B17}$$

and

$$\frac{\partial^2 p}{\partial r^2} = - \frac{8(\gamma_s - 1)\rho_s U_\infty^2}{(\gamma_s + 1)^2 R_s^2} \tag{B18}$$

Note that γ_s can have values somewhat different than 1.4 if vibrational and dissociational energies are excited at the shock wave. The results of this appendix are consistent if γ_s is defined by equation (B1) from the ratio of densities across the shock wave. When additional energy modes are excited at the shock wave, this effective value of γ_s is not exactly the ratio of specific heats.

It can be seen that for the case of a yawed two-dimensional body, the same relations hold as for the body at zero yaw except that the velocity U_∞ is replaced by the normal component of velocity, $U_\infty \cos \lambda$. Thus the yawed two-dimensional body has a second derivative of pressure

$$\frac{\partial^2 p}{\partial y^2} = - \frac{6(\gamma_s - 1)\rho_s U_\infty^2 \cos^2 \lambda}{(\gamma_s + 1)^2 R_s^2} \tag{B19}$$

In the above relations the radius of curvature of the shock wave R_s is yet undetermined. In the limit of infinite free-stream Mach number, the ratio of shock wave to body curvature, R_s/R_b, might be expected to approach unity as an upper bound. On the other hand, a value of R_s/R_b consistent with incompressible boundary-layer solutions may be a reasonable lower bound. In this regard Howarth (ref. 7) reports that for two-dimensional flow

$$\frac{\partial v}{\partial y} = \frac{2u_s}{R_b} \tag{B20}$$

which, according to equations (B4) and (B11), corresponds to a ratio

$$\frac{R_s}{R_b} = \frac{1}{\gamma_s - 1} \tag{B21}$$

Sibulkin (ref. 19), using a similar analysis finds that

$$\frac{\partial v}{\partial y} = \frac{3U_s}{2R_b} \tag{B22}$$

for axially symmetric flow. This corresponds to the ratio

$$\frac{R_s}{R_b} = \frac{4}{3(\gamma_s - 1)} \tag{B23}$$

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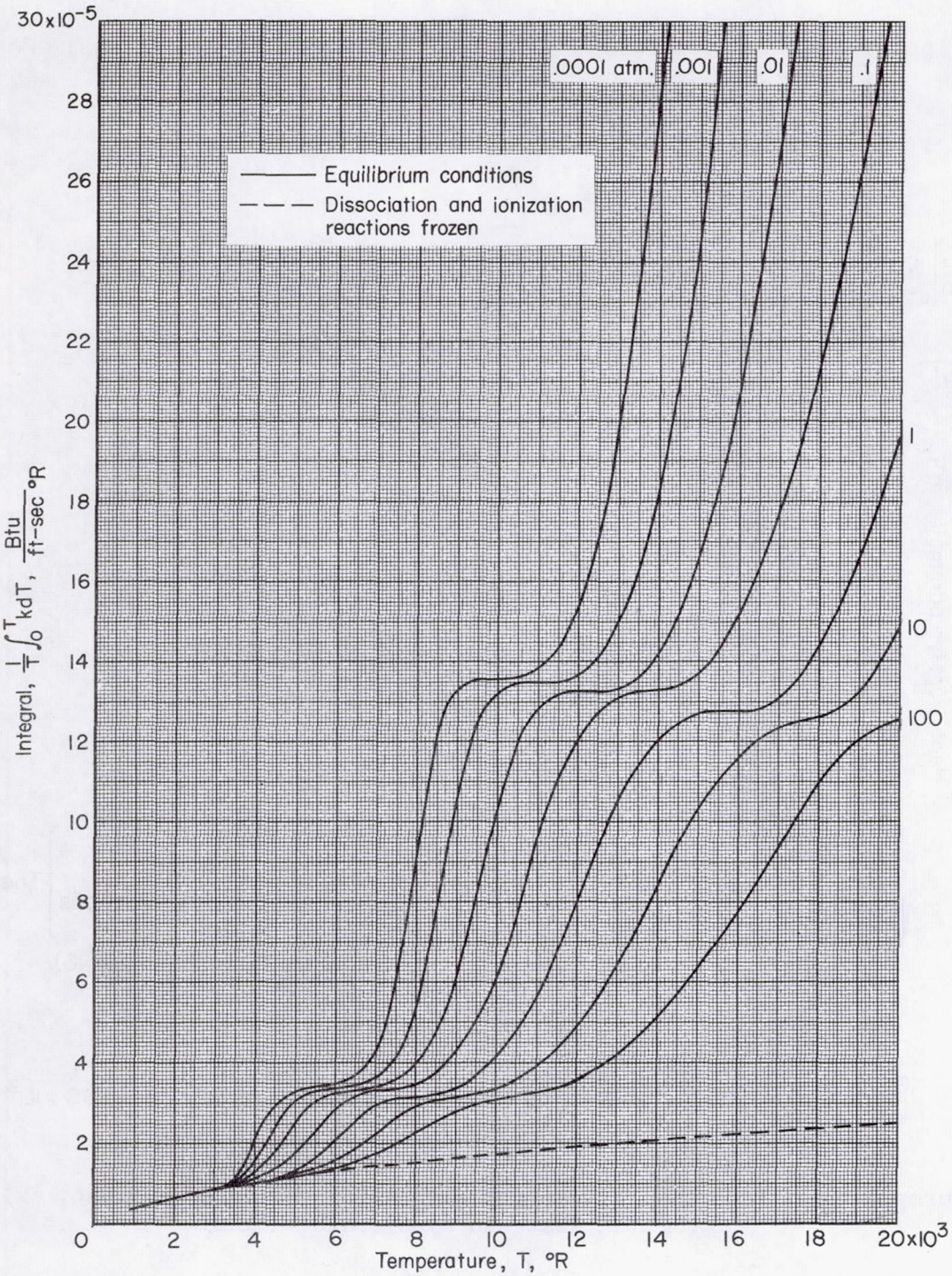


Figure 1.- Integral of thermal conductivity as a function of temperature.

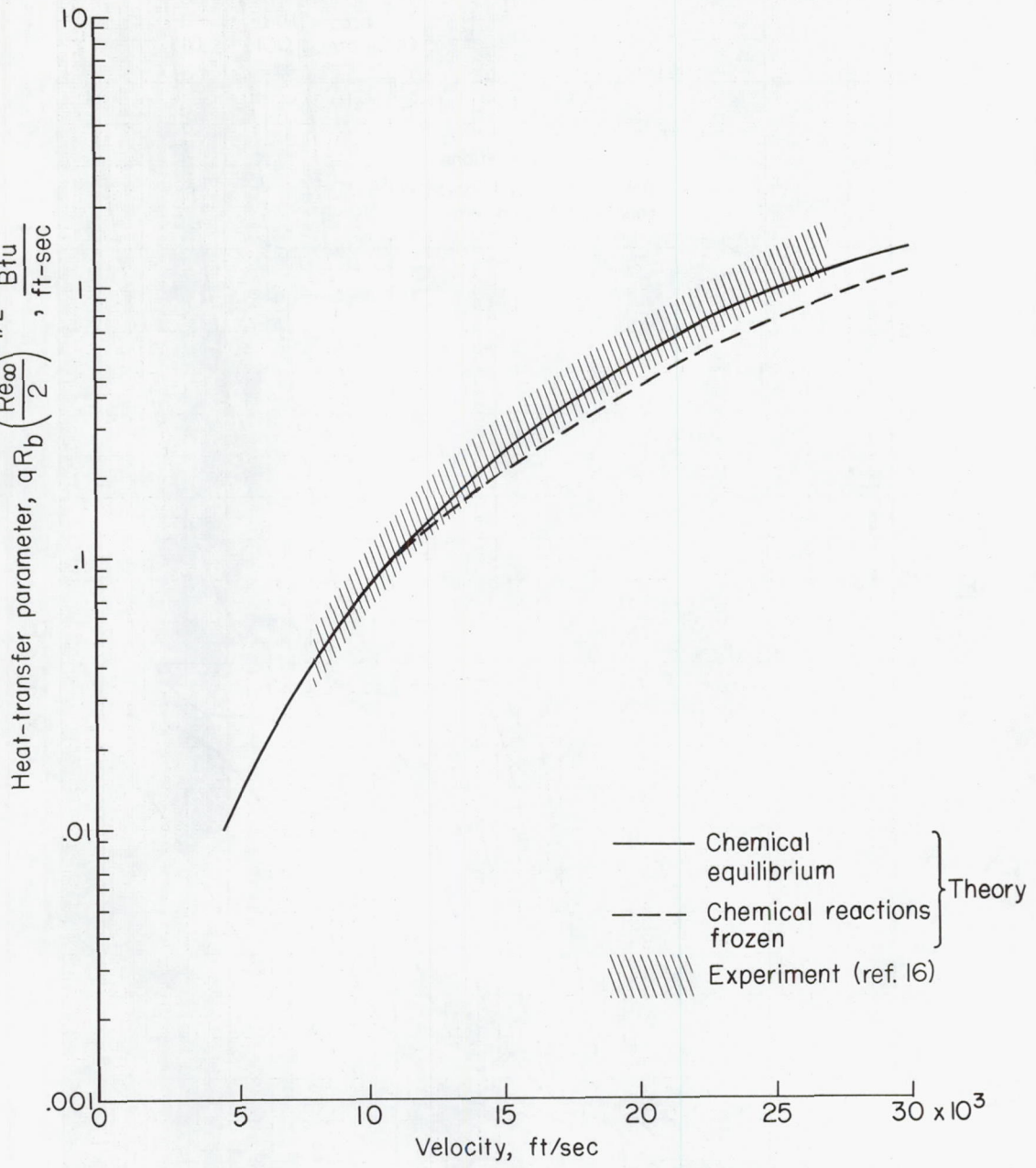


Figure 2.- Heat-transfer rate to the stagnation region of a blunt, axially symmetric shape in hypersonic flight.

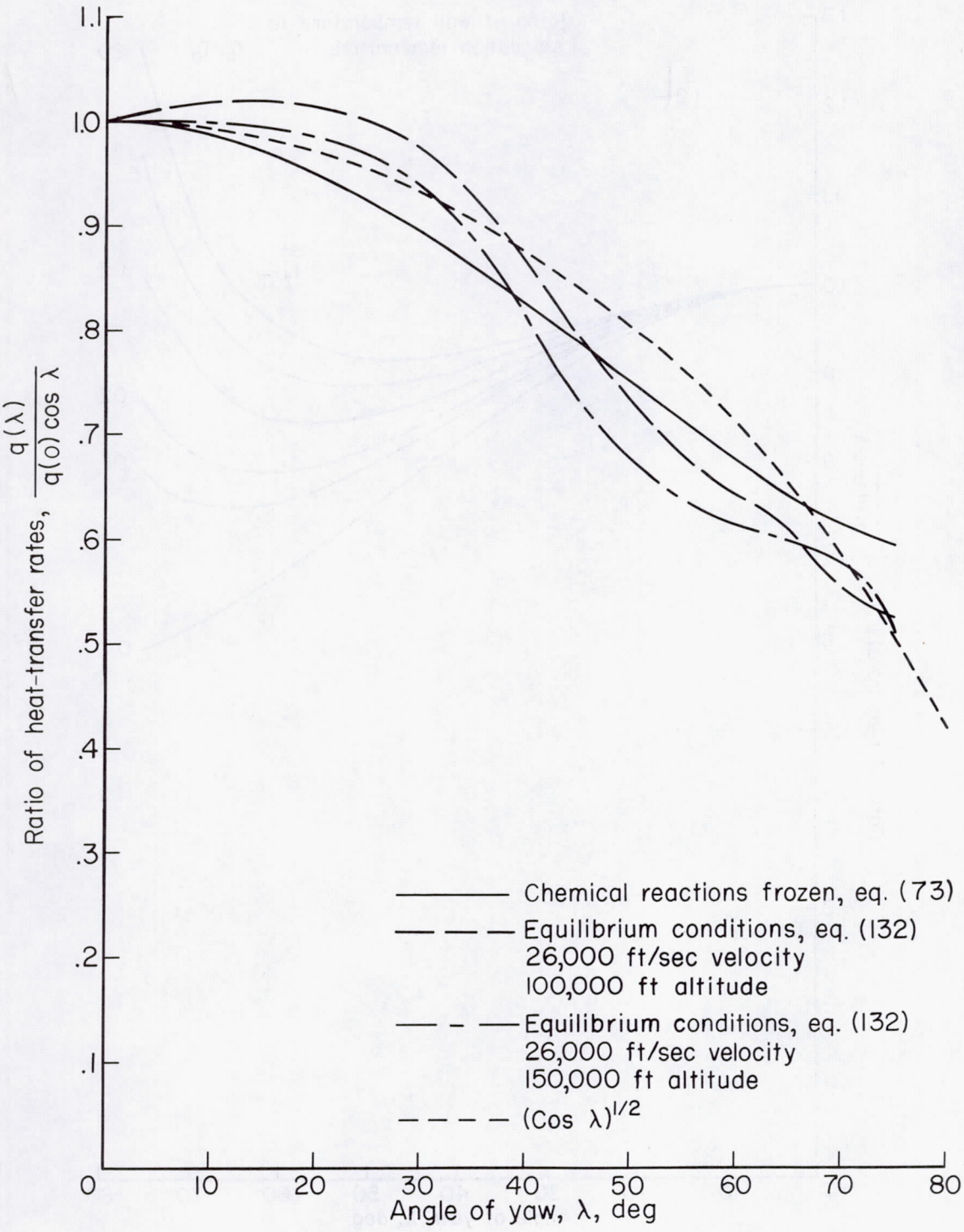


Figure 3.- Effect of yaw on heat-transfer rates to the stagnation region of a blunt cylindrical shape with a cool wall.

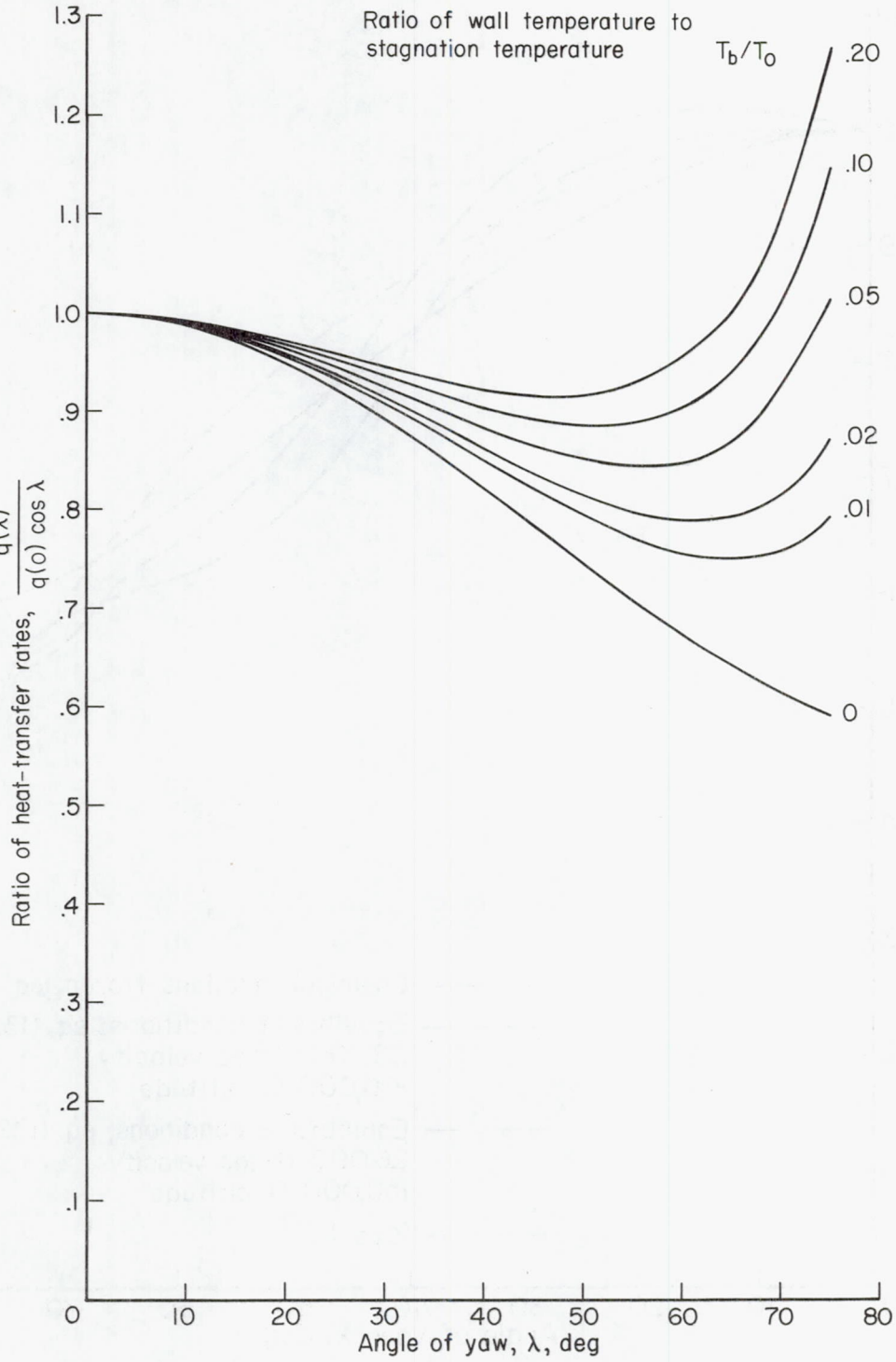


Figure 4.- Influence of wall temperature on the reduction in stagnation region heat flux due to yaw; chemical reactions frozen.