TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 455

NOTE ON RESEARCH WORK BY HELMHOLTZ AND WIEN RELATING TO THE FORM OF WAVES PROPAGATED ALONG THE SURFACE OF SEPARATION OF TWO LIQUIDS

By J. M. Burgers

Reprinted from
"Rendiconti della R. Accademia Nazionale dei Lincei"
Volume V, No. 5, March, 1927

Washington
March, 1928.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL MEMORANDUM NO. 455.

NOTE ON RESEARCH WORK BY HELMHOLTZ AND WIEN RELATING TO THE FORM OF WAVES PROPAGATED ALONG THE SURFACE OF SEPARATION OF TWO LIQUIDS.*

By J. M. Burgers.

In 1889 Helmholtz developed, for the calculation of waves along the surface of separation of two different liquids, formulas which rendered it possible to obtain a little closer approximation than that given by the primitive theory. His researches were subsequently continued by W. Wien, who has given several different approximations.** In the dissertations of Helmholtz and Wien, the arguments applied, for the most part, to developments in series. It is not always easy to see how they arrived at these developments and, on the other hand, we do not obtain any idea of the degree of approximation of their results.

Although the importance of these researches has diminished, due to the development of strict analytical methods of Levi Civita, Struik, Weinstein and others, I wish to take up again

*"Sur quelques recherches de Helmholtz et de Wien relatives à la forme des ondes se propageant à la surface de separation de deux liquides." Reprinted from "Rendiconti della R. Accademia Nazionale dei Lincei," Vol. V, 6th Series, 1st Semester, No. 5, pp. 333-338, Rome, March, 1927.

** Compare H. Von Helmholtz, "Ueber atmosphärische Bewegungen" II ("Zur Theorie von Wind und Wellen"), "Sitz. Ber. Preuss. Akad.," 1889, p.761 ("Wissenschaftliche Abhandlungen," Vol. III, p.309); W. Wien, "Sitz. Ber. Preuss. Akad.," 1894, p.509, and "Lehrbuch der Hydrodynamik" (Leipzig, 1900), p.169.

the considerations of Helmholtz, in order to present them in a little clearer form which, while not affecting the order of approximation, renders possible a clearer perception of their principles and of what they can give.

Let us adopt a system of coordinates Oxy constantly connected with the waves, so that the motion of the liquids is stationary with respect to this system (Fig. 1). Let ABC represent the profile of the wave, which we assume to be symmetrical with respect to the crests (and hence also with respect to the troughs). Let λ represent the length of the wave. For brevity, let $n = 2\pi/\lambda$. Let U_1 represent the velocity of the upper liquid in the direction of the positive x's, and U_2 the velocity of the lower liquid in the opposite direction.

Let $z = x + iy$. Let $\chi_1 = \varphi_1 + i\psi_1$, the complex potential of the motion of the upper liquid, and $\chi_2 = \varphi_2 + i\psi_2$, the similar function for the lower liquid. The first of these two functions gives us a conformal representation of the part situated above the profile of the wave ABC of a band of the width λ of the plane of the z's on a semi-band of the width $U_1\lambda$ of the plane of the function χ_1. The second function gives us the conformal representation of the lower part of the same band of the plane of the z's on a semi-band of the width $U_2\lambda$ of the plane χ_2 (Compare Figs. 2a and 2b. In all the figures, the corresponding domains are designated by similar hatchings). The immediate determination of these two correspondences presents difficulties due to the unknown shape of the profile ABC. The methods of Helmholts and Wien consist in replacing the true profile by another curve of such shape that it is possible to make it correspond to a straight line by means of a simple auxiliary function.

The function introduced by the abovementioned authors through the formula

$$v = \sigma + i\tau = e^{-inz} = e^{ny-inx} \tag{1}$$

gives a transformation of a band of the plane of the z's in the whole domain of the plane of the variable v. The profile of the wave is transformed into a closed curve (Fig. 3). For $y = +\infty$, v becomes equal to ∞. For $y = -\infty$, v becomes equal to zero (the point F in Fig. 3). In the case of waves of infinitely small amplitude, the curve ABC in the plane of the v's becomes a circle. In order to arrive at a more general case, Helmholtz substituted an ellipse for the true curve. The degree of approximation obtainable in this way depends on two parameters: the eccentricity of the ellipse and the location of the center. (The dimensions of the ellipse are of no importance, because a homographic expansion with reference to the center of the plane of the v's only produces a displacement of the plane of the z's parallel to the axis of the y's.)

In agreement with Helmholtz and Wien, let us put

$$v = \cos\zeta - E \tag{2}$$

in which $\zeta = \xi + i\eta$, while E is a real constant, which we can represent by $\cos\epsilon$. Hence the equation η = constant = h produces an ellipse in the plane of the v's. The coordinates of the points A and B determine the height of the waves in the plane of the z's and we readily find

$$y_B - y_A = \frac{\lambda}{2\pi} \lg \frac{\cosh h + \cos\epsilon}{\cosh h - \cos\epsilon} .$$

Formula (2) gives a conformal representation of the plane of the v's and hence of a band of the plane of the z's on a semi-band of the plane of the ζ's (Fig. 4). On recalling the relations existing between z and the functions χ_1 and χ_2, we find that the part of the semi-band in the plane of the ζ's situated above the straight line ABC corresponds to a semi-band of the plane of the χ_1's, while the rectangular part A'B'C'CBAA' of the plane of the ζ corresponds to a semi-band of the plane of the χ_2's.

It is easily found that the first of these two correspondences is given by the homographic transformation

$$\chi_1 = \frac{U_1}{n} \zeta + \text{constant}. \qquad (3)$$

The correspondence between the domain A'B'C'CBAA' of the plane of the ζ's and the semi-band of the plane of the χ_2's can be formulated by means of an auxiliary variable t, by employing the method of Schwarz-Christoffel. As Levi Civita has kindly shown me, it is advantageous to consider only half of each of these two figures, obtained by dividing each one by a vertical line passing through the points B and B'. Let us take the left half A'B'BAA' of the domain in the plane of the ζ's, corresponding to the right half of the semi-band in the plane of the χ_2's. Let us make these two parts correspond to a semi-plane of the variable t (Fig. 5), so that the point A' corresponds to $t = 0$, B' to $t = 1$, B to $t = k^{-2}$ and A to $t = \infty$. On taking k as the modulus of an elliptical function, and on designating by K and K' the complete integrals of the first form, the corresponding parameter being $q = e^{-h}$, we have

$$t = \mathrm{sn}^2 \left(\frac{K\zeta}{\pi}\right) \tag{4}$$

On the other hand, if $t = c^2$ represents the point which must correspond to the point $F(\psi_2 = +\infty)$ of the plane of the χ_2's, we find

$$\frac{d\chi_2}{dt} = \frac{\text{const.}}{(t - c^2)\sqrt{t - k^{-2}}}$$

whence, on suitably determining the multiplicative constant

$$\cos \frac{n\chi_2}{U_2} = \frac{t + c^2 - 2k^{-2}}{t - c^2} \tag{5}$$

The relation between c^2 and $E = \cos\epsilon$ is given by the equation

$$c = \mathrm{sn}\left(\frac{K\epsilon}{\pi}\right).$$

In the figures, it is assumed that $E < 1$. Then ϵ is real.

The relations between the variable z and the potentials χ_1 and χ_2, which are given by the developments in series by

Helmholtz and Wien, now find themselves expressed in finite form.

The equation which expresses the equality of the pressures on both sides of the line ABC may be written

$$(\rho_2 - \rho_1)gy + \frac{1}{2}\rho_2 \left|\frac{d\chi_2}{dz}\right|^2 - \frac{1}{2}\rho_1 \left|\frac{d\chi_1}{dz}\right|^2 = \text{constant} \quad (6)$$

Let us abbreviate by making

$$nx = X, \quad ny = Y, \quad nz = Z; \quad P = \frac{n\rho_1 U_1^2}{2(\rho_2 - \rho_1)g}; \quad Q = \frac{n\rho_2 U_2^2}{2(\rho_2 - \rho_1)g}.$$

Then, on introducing the variable ζ, equation (6) takes the form

$$Y + C = \left\{P\left(\frac{n}{U_1}\ \frac{d\chi_1}{d\zeta}\right)^2 - Q\left(\frac{n}{U_2}\ \frac{d\chi_2}{d\zeta}\right)^2\right\}\left|\frac{d\zeta}{dZ}\right|^2 \quad (6a)$$

The derivatives $\frac{d\chi_1}{d\zeta}, \frac{d\chi_2}{d\zeta}$ are real on the line ABC. We can make all the necessary differentiations and, after eliminating t, obtain

$$\frac{1}{2}\lg\left\{(\cos\xi\cosh h - E)^2 + \sin^2\xi\sinh^2 h\right\} + C =$$

$$= \left\{P - Q\left(\frac{2K}{\pi}\right)^2 \frac{(1-c^2k^2)\left(1-k^2\ \mathrm{sn}^2\frac{K\xi}{\pi}\right)}{\left(1-c^2k^2\ \mathrm{sn}^2\frac{K\xi}{\pi}\right)^2}\right\}\frac{(\cos\xi\cosh h-E)^2+\sin^2\xi\sinh^2 h}{\sin\xi\cosh^2 h+\cos^2\xi\sinh^2 h}. \quad (7)$$

Equation (7) contains five parameters: k, c, P, Q, and C. It is evident that we cannot satisfy this equation identically, i.e., at all the points of the line ABC. We can only demand that it be satisfied in a limited number of points (five at most).

In order to retain the possibility of considering different types of waves, this number can be further diminished to four. Then one of the parameters remains arbitrary. For these points, I have taken those which correspond to the values of ξ, namely, o, $\pi/3$, $2\pi/3$ and π. For a series of values of k, or what amounts to the same thing, of the parameter q, I have made numerical calculations, whose results are recorded in the following table:

q	$E - \cos\epsilon$	P	Q	$\frac{Y_{max} - Y_{min}}{2\pi}$	Δ
0	0.828	0.0678	0.4322	0	--
0.207	0.816	0.0657	0.480	0.107	0.008
0.307	0.807	0.0613	0.547	0.156	0.021
0.403	0.800	0.0538	0.657	0.199	0.050
0.490	0.795	0.0418	0.831	0.235	0.088
(0.600)	(0.794)	--	--	(0.279)	--

The Δ column gives the difference between the values of the two members of equation (7) at the point $\xi = \pi/2$, expressed in fractions of the value of $Y_{max} - Y_{min}$. The profiles of the corresponding waves are represented in Figure 6, in which we have marked the points where ξ has one of the values o, $\pi/3$, $\pi/2$, $2\pi/3$, π.

As soon as $\cos\epsilon \cos h$ becomes smaller than unity, which occurs for q a little larger than 0.49, the profile of the wave, given by formulas (1) and (2), presents a secondary maximum in the troughs, accompanied by a minimum on each side (Compare the bottom curve in Figure 6, in which the minimum has also been marked). In this case the approximation method seems to be no longer practicable.

For small values of q we can develop in series the magnitudes which figure in equation (7). These series can be arranged according to the cosines of the multiples of ξ. If we retain only the terms of the order q^3, the series terminate at $\cos 3\xi$ and we can demand, as was done by Helmholtz, that the coefficients of $\cos o\,\xi$, $\cos \xi$, $\cos 2\,\xi$ and $\cos 3\xi$ be equal in both members of equation (7). In this way we again obtain four equations between the five parameters, which renders it possible to express them in terms of one of them alone. It was in this way that the values of E, P and Q for $q = o$ were found. I found that the equations thus obtained were the same as the ones given by Helmholtz in his memoir (excepting for one of the coefficients which figure in the development of Y).

/

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

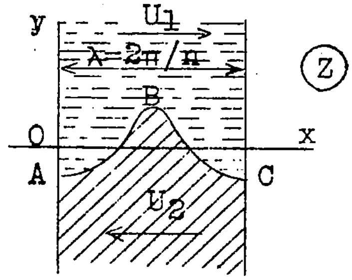

Fig.1

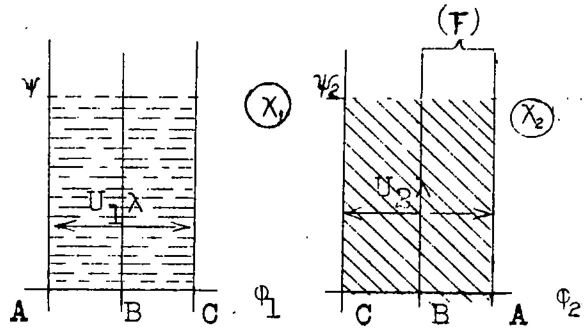

Fig.2a Fig.2b

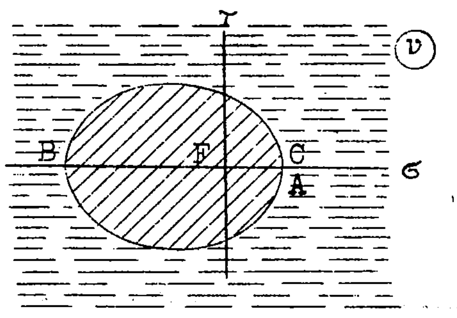

Fig.3

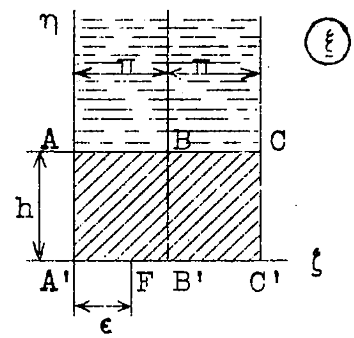

Fig.4

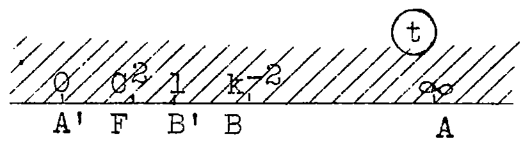

Fig.5

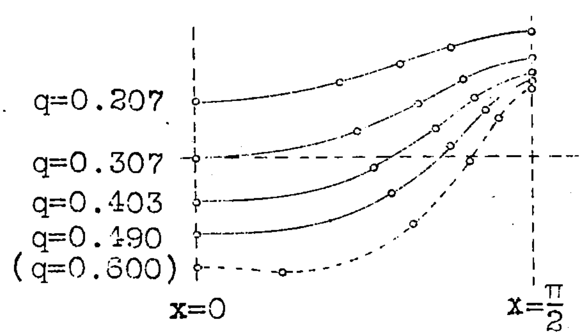

Fig.6