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TORSION AND TRANSVERSE BENDING OF CANTILEVER PLATES

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SUMMARY

The problem of combined bending and torsion of cantilever plates of variable thickness, such as might be considered for solid thin high-speed airplane or missile wings, is considered in this paper. The deflections of the plate are assumed to vary linearly across the chord; minimization of the potential energy by means of the calculus of variations then leads to two ordinary linear differential equations for the bending deflections and the twist of the plate. Because the cantilever is analyzed as a plate rather than as a beam, the effect of constraint against axial warping in torsion is inherently included. The application of this method to specific problems involving static deflection, vibration, and buckling of cantilever plates is presented. In the static-deflection problems, taper and sweep are considered.

INTRODUCTION

For analysis of thin solid wings of small aspect ratio such as might be utilized in high-speed airplanes and missiles, beam theory is no longer adequate. Wings of this type are more nearly plates than beams and should be analyzed by plate theory. However, solutions to the partial-differential equation of plate theory are not readily obtained, especially for plates of arbitrary shape and loading. In the present paper, therefore, a method is described for obtaining ordinary differential equations to replace the partial-differential equation. The method employs the minimum-potential-energy principle in conjunction with the assumption that the chordwise deflection shape may be represented by terms of a power series. The analysis of the present paper is limited to the first two terms of this series. The first term represents transverse displacement and the second represents twist. Together they permit linear chordwise deflection (the assumption usually made in wing design). Use of the first two terms in the series leads to two ordinary differential equations that define the spanwise variation of the transverse displacement and rotation. If results of greater accuracy are required, additional terms in the power series may be included (that is, quadratic, cubic, etc.) with a corresponding increase in the number of ordinary differential equations obtained.

Solution of the two ordinary differential equations, subject to boundary conditions which arise naturally in the minimization procedure, gives the bending deflection and twist at any cross section. The stresses may be obtained from the deflections by using the well-known equations of plate theory. The order of accuracy of the stresses will generally be less than that of the deflections. Because ordinary plate theory is employed, the applicability of the analysis is limited to plates in which the order of magnitude of the plan-form dimensions is greater than ten times the order of magnitude of the thickness. To extend the applicability of such an analysis to other dimensions, the effects of transverse shear deformation must be included.

The derivation of the differential equations and boundary conditions is presented and their application is discussed, and then specific problems involving static deflection, vibration, and buckling of cantilever plates are solved. In the static-deflection problems, taper and sweep are considered.

Since completion of the present paper, there has come to the attention of the authors a recent paper (reference 1) which presents essentially the same ordinary differential equations that are presented herein for static-deflection problems. The derivations of these ordinary differential equations were done by different methods, and different specific problems were solved.

The basic differential equations presented herein were obtained in June 1948 while the first-named author was temporarily at the Langley Laboratory, and the work was continued by correspondence.

SYMBOLS

a, b, i, j	parameters specifying taper variation
c	local chord of plate
h	local thickness of plate
l	length of plate measured perpendicular to root
m_A	mass per unit area
p	lateral load per unit length

r	parameter specifying twisting-moment distribution
t	constant applied twisting moment per unit length
w	transverse deflection, positive in z-direction
x, y, z	coordinates defined in figure 1
D	local flexural stiffness $\left(\frac{Eh^3}{12\left(1 - \mu^2\right)}\right)$
E	modulus of elasticity of material
P	tip shearing force
T	tip torque
W, θ	functions of x appearing in assumption $w = W(x) + y\theta(x)$
a_1, a_2, a_3; n_1, n_2, n_3; p_1, p_2; s_1, s_2, s_3	coefficients in differential equations
$c_1(x), c_2(x)$	functions defining plan form (see fig. 1)
x_1	variable obtained by transformation $x_1 = 1 - b\frac{x}{l}$
M_1	tip bending moment
M_2	higher-order moment of stresses
M_x	bending moment per unit width
$\overline{M}_x$	externally applied tip bending moment per unit width
M_{xy}	twisting moment per unit width
$\overline{M}_{xy}$	externally applied tip twisting moment per unit width

N_x	normal force per unit width
N_{x_0}, N_{x_1}	constants appearing in normal-force-distribution equation $N_x = N_{x_0} + \frac{2y}{c} N_{x_1}$
Q_x	shearing force per unit width
$\overline{Q}_x$	externally applied tip shearing force per unit width
I_ν, K_ν	modified Bessel functions of order ν
λ	aspect-ratio parameter $\left(\frac{l}{c}\sqrt{\frac{3}{2}(1 - \mu)}\right)$
μ	Poisson's ratio
$\varphi = \frac{d\theta}{dx_1}$	
ω	frequency of torsional vibration
Λ	angle of sweep
Π	energy
σ_x	normal stress
τ_{xy}, τ_{xz}	shear stresses

Subscripts:

St V	according to St. Venant torsion theory

Subscripts x and y on w denote partial differentiation with respect to x and y, respectively.

Superscripts:

h	homogeneous solution
pi	particular integral

Primes denote differentiation with respect to x.

ANALYSIS

The structure considered in the present paper is a thin, elastic, isotropic plate of gradually varying thickness and chord, as shown in figure 1. The loading may consist of distributed lateral forces and torques, spanwise normal forces acting in the midplane of the plate, and tip shears and torques. In addition, the plate may be undergoing simple harmonic motion. The potential-energy expression for such a plate in its deformed position will now be presented. The aforementioned assumption of linear chordwise deformation will then be incorporated, and finally the potential energy will be minimized by means of the calculus of variations.

The strain energy of bending is given by the following expression:

$$\Pi_{strain} = \frac{1}{2}\int_0^l \int_{c_1(x)}^{c_2(x)} D(x,y)\left[\left(w_{xx} + w_{yy}\right)^2 + 2(1 - \mu)\left(w_{xy}{}^2 - w_{xx}w_{yy}\right)\right]dx\ dy \tag{1}$$

where w is the transverse deflection and

$$D(x,y) = \frac{Eh^3}{12\left(1 - \mu^2\right)}$$

where h is the local thickness which is a function of x and y.

The potential energy of transverse loads of intensity p is

$$\Pi_p = -\int_0^l \int_{c_1(x)}^{c_2(x)} p(x,y)w\ dx\ dy \tag{2}$$

The potential energy of the middle-plane spanwise forces is

$$\Pi_{N_x} = -\frac{1}{2}\int_0^l \int_{c_1(x)}^{c_2(x)} N_x w_x{}^2 dx\ dy \tag{3}$$

The potential energy of the tip forces, moments, and torques is

$$\Pi_{tip} = \int_{c_1(l)}^{c_2(l)} \left(-\bar{Q}_x w + \bar{M}_x w_x - \bar{M}_{xy} w_y\right)_{x=l} dy \tag{4}$$

If the plate is undergoing simple harmonic motion of circular frequency ω, and $w(x,y)$ is the deflection shape at the time of maximum deflection, the potential energy due to inertia loading is

$$\Pi_\omega = -\frac{1}{2} \int_0^l \int_{c_1(x)}^{c_2(x)} m_A(x,y)\omega^2 w^2 dx\, dy \tag{5}$$

The total potential energy is defined as the sum of all the energies just listed or

$$\Pi_{total} = \Pi_{strain} + \Pi_p + \Pi_{N_x} + \Pi_{tip} + \Pi_\omega$$

If Π_{total} were minimized with respect to the deflection $w(x,y)$ by means of the calculus of variations, the well-known partial-differential equation of plate theory would result as the necessary condition the deflection shape must satisfy. However, if first the deflection w is assumed to be of the form

$$w = W(x) + y\theta(x) \tag{6}$$

and the potential energy is minimized with respect to $W(x)$ and $\theta(x)$, two ordinary differential equations are obtained for W and θ. The latter procedure is followed herein. It should be noticed that the right-hand side of equation (6) is merely the first two terms of a power series in y. If greater accuracy is desired, additional terms of the series such as $y^2\alpha(x)$ and $y^3\gamma(x)$ may be used. Substitution of expression (6) in the energy formulas (equations (1) to (5)) gives

$$\Pi_{strain} = \frac{1}{2} \int_0^l \left[a_1 W''^2 + 2a_2 W''\theta'' + a_3\theta''^2 + 2(1 - \mu)a_1\theta'^2\right]dx \tag{7}$$

$$\Pi_p = -\int_0^l \left(p_1 W + p_2\theta\right)dx \tag{8}$$

$$\Pi_{N_x} = -\frac{1}{2} \int_0^l \left(n_1 W'^2 + 2n_2 W'\theta' + n_3 \theta'^2 \right) dx \tag{9}$$

$$\Pi_{tip} = -PW(l) - T\theta(l) + M_1 W'(l) + M_2 \theta'(l) \tag{10}$$

$$\Pi_\omega = -\frac{\omega^2}{2} \int_0^l \left(s_1 W^2 + 2s_2 W\theta + s_3 \theta^2 \right) dx \tag{11}$$

where the coefficients a_n, p_n, n_n, P, T, M_n, and s_n are defined as follows:

$$a_n(x) = \int_{c_1(x)}^{c_2(x)} D(x,y) y^{n-1} dy$$

$$p_n(x) = \int_{c_1(x)}^{c_2(x)} p(x,y) y^{n-1} dy$$

$$n_n(x) = \int_{c_1(x)}^{c_2(x)} N_x y^{n-1} dy$$

$$P = \int_{c_1(l)}^{c_2(l)} \overline{Q}_x \, dy$$

$$T = \int_{c_1(l)}^{c_2(l)} \left(\overline{M}_{xy} + \overline{Q}_x y \right) dy$$

$$M_n = \int_{c_1(l)}^{c_2(l)} \overline{M}_x y^{n-1} dy$$

$$s_n(x) = \int_{c_1(x)}^{c_2(x)} m_A(x,y) y^{n-1} dy$$

If the following variational condition is imposed

$$\delta\Pi_{total} = \delta\left(\Pi_{strain} + \Pi_p + \Pi_{N_x} + \Pi_{tip} + \Pi_\omega\right) = 0$$

then

$$\int_0^l \Bigg\{\Big[\left(a_1W''\right)'' + \left(a_2\theta''\right)'' - p_1 - \omega^2\left(s_1W + s_2\theta\right) + \left(n_1W'\right)' + \left(n_2\theta'\right)'\Big]\delta W +$$

$$\Big[\left(a_3\theta''\right)'' + \left(a_2W''\right)'' - 2(1 - \mu)\left(a_1\theta'\right)' - p_2 - \omega^2\left(s_3\theta + s_2W\right) + \left(n_3\theta'\right)' +$$

$$\left(n_2W'\right)'\Big]\delta\theta\Bigg\}dx - \Big[\Big[\left(a_1W''\right)' + \left(a_2\theta''\right)' + n_1W' + n_2\theta'\Big]\delta W\Big]_0^l -$$

$$\Big[\Big[\left(a_3\theta''\right)' + \left(a_2W''\right)' - 2(1 - \mu)a_1\theta' + n_3\theta' + n_2W'\Big]\delta\theta\Big]_0^l +$$

$$\Big[\left(a_1W'' + a_2\theta''\right)\delta W'\Big]_0^l + \Big[\left(a_3\theta'' + a_2W''\right)\delta\theta'\Big]_0^l -$$

$$P\ \delta W(l) - T\ \delta\theta(l) + M_1\ \delta W'(l) + M_2\ \delta\theta'(l) = 0 \tag{12}$$

At the root $(x = 0)$, the following clamped-edge conditions are imposed:

$$w(0,y) = w_x(0,y) = 0$$

It follows that

$$W(0) = W'(0) = \theta(0) = \theta'(0) = 0 \tag{13}$$

and, consequently, the variations of these quantities ($\delta W(0)$, $\delta W'(0)$, etc.) also vanish.

At the tip $(x = l)$, δW, $\delta\theta$, $\delta W'$, and $\delta\theta'$ are taken to be arbitrary and, consequently, the tip boundary conditions follow from equation (12) in the form

$$\Big[\left(a_1W''\right)' + \left(a_2\theta''\right)' + n_1W' + n_2\theta'\Big]_{x=l} = -P \tag{14}$$

$$\left[\left(a_3\theta''\right)' + \left(a_2W''\right)' - 2(1 - \mu)a_1\theta' + n_3\theta' + n_2W'\right]_{x=l} = -T \tag{15}$$

$$\left(a_1W'' + a_2\theta''\right)_{x=l} = -M_1 \tag{16}$$

$$\left(a_3\theta'' + a_2W''\right)_{x=l} = -M_2 \tag{17}$$

The differential equations for W and θ follow from the variational equation (12) in the form

$$\left(a_1W''\right)'' + \left(a_2\theta''\right)'' - p_1 - \omega^2\left(s_1W + s_2\theta\right) + \left(n_1W'\right)' + \left(n_2\theta'\right)' = 0 \tag{18}$$

$$\begin{aligned}\left(a_3\theta''\right)'' + \left(a_2W''\right)'' - 2(1 - \mu)\left(a_1\theta'\right)' - p_2 - \\ \omega^2\left(s_3\theta + s_2W\right) + \left(n_3\theta'\right)' + \left(n_2W'\right)' = 0\end{aligned} \tag{19}$$

The problem is to solve equations (18) and (19) subject to the eight conditions given by equations (13) to (17). Solution of equations (18) and (19) results in expressions for $W(x)$ and $\theta(x)$. The deflection w of the plate is then given by equation (6).

The stresses may then be calculated by taking appropriate derivatives of the deflections. The order of accuracy of the stresses will be less than that of the deflections since successive derivatives of approximate expressions become more and more in error. For this reason only those stresses are given which are least subject to error resulting from restrictions placed on the deflection function. From known results of plate theory

$$\sigma_x(x,y,z) = \frac{6M_x}{h^2}\frac{z}{h/2} + \frac{N_x}{h} \tag{20}$$

$$\tau_{xy}(x,y,z) = -\frac{6M_{xy}}{h^2}\frac{z}{h/2} \tag{21}$$

$$\tau_{xz}(x,y,z) = \frac{3Q_x}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{22}$$

where

$$M_x = -D\left(w_{xx} + \mu w_{yy}\right) = -D(W'' + y\theta'')$$

$$M_{xy} = (1 - \mu)Dw_{xy} = (1 - \mu)D\theta'$$

$$Q_x = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = -D(W''' + y\theta''') - \frac{\partial D}{\partial x}(W'' + y\theta'') - (1 - \mu)\frac{\partial D}{\partial y}\theta'$$

The stresses σ_x and τ_{xy} are numerically largest when $z = \pm\frac{h}{2}$; τ_{xz} is numerically largest when $z = 0$.

SOLUTIONS OF SPECIFIC PROBLEMS

Outline of Problems Solved

Solutions are presented for a number of problems involving cantilever plates of various shapes under various loadings. The problems may be grouped as follows:

(A) Rectangular plate of constant thickness

(1) Tip torque

(2) Uniform distribution of applied twisting moments

(3) Torsional vibrations

(4) Lateral buckling

(B) Symmetrical-plan-form plate with chord and thickness variation

(1) Symmetric cross section with spanwise variation of chord and thickness according to a power law

(a) Linearly varying chord; tip torque and various spanwise distributions of twisting moments

(b) Chord variations other than linear (solution corresponding to arbitrary torque loadings left in a formal state for a class of chord variations)

(2) Rectangular cross section with constant chord and exponential spanwise variation in thickness; tip torque

(C) Skewed plate of constant thickness and chord under tip loading and uniform lateral loading

The problems of group A were selected because they are simple fundamental problems for which solutions obtained by the present method can be readily compared with solutions obtained by other methods. Comparison with elementary beam theory is shown for these problems. The problems in group B involve tapered plates and are therefore presented for their possible application to wing analysis. Problem C was selected to show the applicability of the method to swept wings.

Generally, the plan forms and loadings in the problems chosen are those for which the assumption of linear chordwise deflections might be expected to hold - namely, problems involving unswept plates under torque loading. A single exception is problem C for which the solution presented must be regarded as only a first approximation.

Rectangular Plate of Constant Thickness

For a rectangular plate of constant thickness, the flexural stiffness D is independent of x and y. With the chord of the plate denoted by c and with the origin of coordinates at the center of the root, the differential equations (18) and (19) become

$$DcW^{IV} - p_1 - \omega^2\left(s_1W + s_2\theta\right) + \left(n_1W'\right)' + \left(n_2\theta'\right)' = 0 \tag{23}$$

$$\frac{Dc^3}{12}\theta^{IV} - 2(1 - \mu)Dc\theta'' - p_2 - \omega^2\left(s_3\theta + s_2W\right) + \left(n_3\theta'\right)' + \left(n_2W'\right)' = 0 \tag{24}$$

and the boundary conditions, from equations (13) to (17), become

$$W(0) = W'(0) = \theta(0) = \theta'(0) = 0 \tag{25}$$

$$\left(DcW''' + n_1W' + n_2\theta'\right)_{x=l} = -P \tag{26}$$

$$\left[\frac{Dc^3}{12}\theta''' - 2(1 - \mu)Dc\theta' + n_3\theta' + n_2W'\right]_{x=l} = -T \tag{27}$$

$$DcW''(l) = -M_1 \tag{28}$$

$$\frac{Dc^3}{12}\theta''(l) = -M_2 \tag{29}$$

For each loading condition the differential equations are solved and solutions that satisfy the boundary conditions are obtained in closed form.

Tip torque.- For a plate with a torque T applied at $x = l$, the differential equations (23) and (24) become

$$DcW^{IV} = 0 \tag{30}$$

$$\frac{Dc^3}{12}\theta^{IV} - 2(1 - \mu)Dc\theta'' = 0 \tag{31}$$

with the following boundary conditions (from equations (25) to (29)):

$$W(0) = W'(0) = \theta(0) = \theta'(0) = 0 \tag{32}$$

$$DcW'''(l) = 0 \tag{33}$$

$$\frac{Dc^3}{12}\theta'''(l) - 2(1 - \mu)Dc\theta'(l) = -T \tag{34}$$

$$DcW''(l) = 0 \tag{35}$$

$$\frac{Dc^3}{12}\theta''(l) = 0 \tag{36}$$

The differential equations (30) and (31) have the following solutions which satisfy the boundary conditions:

$$W = 0$$

$$\theta = \frac{Tl}{2(1 - \mu)Dc}\left[\frac{x}{l} - \frac{\sinh \frac{4\lambda x}{l}}{4\lambda} - \frac{\tanh 4\lambda}{4\lambda}\left(1 - \cosh \frac{4\lambda x}{l}\right)\right] \qquad (37)$$

From this equation it follows that θ' is not a constant as in the St. Venant torsion theory, in which no constraint against axial warping is assumed, but is equal to

$$\theta' = \frac{T}{2(1 - \mu)Dc}\left(1 - \cosh \frac{4\lambda x}{l} + \tanh 4\lambda \sinh \frac{4\lambda x}{l}\right)$$

The twist at the tip $(x = l)$ is

$$\theta(l) = \frac{Tl}{2(1 - \mu)Dc}\left(1 - \frac{\tanh 4\lambda}{4\lambda}\right) \qquad (38)$$

For infinite aspect ratio $(\lambda \longrightarrow \infty)$ equation (38) gives the tip twist corresponding to the St. Venant torsion theory

$$\theta(l)_{St\ V} = \frac{Tl}{2(1 - \mu)Dc}$$

A comparison of the tip twist given by the present theory with the tip twist given by the St. Venant torsion theory is presented in figure 2 and shows that, for aspect ratios lower than 3, the tip twists given by the present theory are appreciably lower.

The normal stresses, or so-called bending stresses due to torsion, are obtained from equation (20) for the value of θ given by equation (37). These stresses are

$$\sigma_x(x,y,z) = \frac{12Tyz}{c^2h^3}\sqrt{\frac{6}{1 - \mu}}\left(\sinh \frac{4\lambda x}{l} - \tanh 4\lambda \cosh \frac{4\lambda x}{l}\right)$$

Figure 3 shows the spanwise distribution of the normal stress as estimated by the present theory for rectangular cantilever plates for values of the aspect-ratio parameter λ of 1, 2, and 4. This figure shows that the normal stress is 0 at the tip (l,y,z) and maximum at the extreme fiber at the root $\left(0,\frac{c}{2},\frac{h}{2}\right)$. This maximum value of the normal stress is

$$\sigma_x\left(0,\frac{c}{2},\frac{h}{2}\right) = \frac{-3T}{ch^2}\sqrt{\frac{6}{1 - \mu}} \tanh 4\lambda$$

<u>Uniform spanwise distribution of applied twisting moments</u>.- For a plate with uniform spanwise distribution of applied twisting moments (that is, $p_2 = t = \text{Constant}$), the differential equations (23) and (24) become

$$DcW^{IV} = 0 \tag{39}$$

$$\frac{Dc^3}{12}\theta^{IV} - 2(1 - \mu)Dc\theta'' = t \tag{40}$$

The boundary conditions are the same as those for tip torque, equations (32) to (36), except that equation (34) is replaced by

$$\frac{Dc^3}{12}\theta'''(l) - 2(1 - \mu)Dc\theta'(l) = 0 \tag{41}$$

The differential equations (39) and (40) have the following solutions which satisfy the boundary conditions:

$$W = 0$$

$$\theta = \frac{tl^2}{2(1 - \mu)Dc}\left[\frac{x}{l} - \frac{x^2}{2l^2} - \frac{\sinh \frac{4\lambda x}{l}}{4\lambda} + \frac{1}{4\lambda}\left(\tanh 4\lambda + \frac{1}{4\lambda \cosh 4\lambda}\right)\left(\cosh \frac{4\lambda x}{l} - 1\right)\right] \tag{42}$$

At $x = l$,

$$\theta(l) = \frac{tl^2}{4(1 - \mu)Dc}\left[1 - \frac{\tanh 4\lambda}{2\lambda} - \frac{1}{8\lambda^2}\left(\frac{1}{\cosh 4\lambda} - 1\right)\right]$$

For infinite aspect ratio ($\lambda \rightarrow \infty$) this equation gives the tip twist corresponding to the St. Venant torsion theory

$$\theta(l)_{St\ V} = \frac{tl^2}{4(1 - \mu)Dc}$$

A comparison of the tip twist given by the present theory with the tip twist given by the St. Venant torsion theory is presented in figure 4 and shows that, for aspect ratios lower than 5, the tip twists given by the present theory are appreciably lower.

The normal stress at any point and the maximum normal stress are, respectively,

$$\sigma_x(x,y,z) = \frac{12tlyz}{c^2h^3}\sqrt{\frac{6}{1 - \mu}}\left[\sinh\frac{4\lambda x}{l} - \left(\tanh 4\lambda + \frac{1}{4\lambda \cosh 4\lambda}\right)\cosh\frac{4\lambda x}{l} + \frac{1}{4\lambda}\right]$$

and

$$\sigma_x\left(0,\frac{c}{2},\frac{h}{2}\right) = \frac{-3tl}{ch^2}\sqrt{\frac{6}{1 - \mu}}\left(\tanh 4\lambda + \frac{1}{4\lambda \cosh 4\lambda} - \frac{1}{4\lambda}\right)$$

The spanwise variation of the normal stress as estimated by the present theory for rectangular cantilever plates for values of the aspect-ratio parameter λ of 1, 2, and 4 is shown in figure 5.

Torsional vibrations.- For a plate undergoing torsional vibration the differential equation for θ (equation (24)) becomes

$$\frac{Dc^3}{12}\theta^{IV} - 2(1 - \mu)Dc\theta'' - m_A\omega^2\frac{c^3}{12}\theta = 0 \tag{43}$$

with equations (32), (36), and (41) as boundary conditions. The solution to the differential equation (43) is

$$\theta = A_1 \sinh \frac{\beta x}{l} + A_2 \cosh \frac{\beta x}{l} + A_3 \sin \frac{\gamma x}{l} + A_4 \cos \frac{\gamma x}{l}$$

where

$$\beta^2 = 8\lambda^2\left[\sqrt{1 + \frac{\pi^2}{16\lambda^2}\left(\frac{\omega}{\omega_{St\ V}}\right)^2} + 1\right]$$

$$\gamma^2 = 8\lambda^2\left[\sqrt{1 + \frac{\pi^2}{16\lambda^2}\left(\frac{\omega}{\omega_{St\ V}}\right)^2} - 1\right]$$

and

$$\omega_{St\ V}^2 = \frac{6(1 - \mu)\pi^2 D}{m_A c^2 l^2}$$

From the boundary conditions $\theta(0) = \theta'(0) = 0$, which are included in equation (32),

$$A_3 = \frac{-\beta}{\gamma} A_1$$

$$A_4 = -A_2$$

From the remaining boundary conditions, equations (36) and (41), the following criterion is obtained:

$$1 + \frac{4\lambda}{\pi}\frac{\omega_{St\ V}}{\omega}\ \sinh\beta\ \sin\gamma + \left(1 + \frac{32\lambda^2}{\pi^2}\frac{\omega_{St\ V}}{\omega}\right)\cosh\beta\ \cos\gamma = 0$$

This equation is solved for the fundamental frequency by finding the lowest value of the frequency ratio $\omega/\omega_{St\ V}$ that satisfies it for a given value of the aspect-ratio parameter λ. A comparison of the fundamental frequency of torsional vibration given by the present theory with that given by the St. Venant torsion theory is presented in figure 6 and shows that, for aspect ratios lower than 3, the fundamental frequencies given by the present theory are appreciably higher.

Lateral buckling.- For a cantilever plate loaded by a spanwise force,

$$N_x = N_{x_0} + \frac{2y}{c}N_{x_1}$$

which is a combination of an axial force N_{x_0} and a bending force in the plane of the plate N_{x_1}. The differential equations for this case are

$$DcW^{IV} + N_{x_0}cW'' + N_{x_1}\frac{c^2}{6}\theta'' = 0 \tag{44}$$

$$\frac{Dc^3}{12}\theta^{IV} - 2(1 - \mu)Dc\theta'' + N_{x_0}\frac{c^3}{12}\theta'' + N_{x_1}\frac{c^2}{6}W'' = 0 \tag{45}$$

with the following boundary conditions:

$$W(0) = W'(0) = \theta(0) = \theta'(0) = 0 \tag{46a}$$

$$\left(DcW''' + N_{x_0}cW' + N_{x_1}\frac{c^2}{6}\theta'\right)_{x=l} = 0 \tag{46b}$$

$$\left[\frac{Dc^3}{12}\theta''' - 2(1 - \mu)Dc\theta' + N_{x_0}\frac{c^3}{12}\theta' + N_{x_1}\frac{c^2}{6}W'\right]_{x=l} = 0 \tag{46c}$$

$$W''(l) = \theta''(l) = 0 \tag{46d}$$

Integration of each differential equation and use of the boundary conditions (46b) and (46c) lead to

$$DcW''' + N_{x_0}cW' + N_{x_1}\frac{c^2}{6}\theta' = 0 \tag{47}$$

$$\frac{Dc^3}{12}\theta''' - 2(1 - \mu)Dc\theta' + N_{x_0}\frac{c^3}{12}\theta' + N_{x_1}\frac{c^2}{6}W' = 0 \tag{48}$$

The other boundary conditions are satisfied by taking

$$W = A\left(1 - \cos\frac{n\pi x}{2l}\right)$$

$$\theta = B\left(1 - \cos\frac{n\pi x}{2l}\right)$$

where n is an integer which represents the number of spanwise buckles. Equations (47) and (48) are also satisfied by these expressions for W and θ if the following stability criterion is satisfied:

$$\left(\frac{n^2}{4} - \frac{N_{x_0}l^2}{\pi^2 D}\right)\left(\frac{n^2}{4} + \frac{16\lambda^2}{\pi^2} - \frac{N_{x_0}l^2}{\pi^2 D}\right) - \frac{1}{3}\left(\frac{N_{x_1}l^2}{\pi^2 D}\right)^2 = 0 \tag{49}$$

Equation (49) gives the critical combinations of N_{x_0} and N_{x_1} for a given value of the aspect-ratio parameter λ. For each value of λ it is necessary to use the value of n that gives the lowest value of N_{x_0} for a given value of N_{x_1}, or vice versa. For the present problem, $n = 1$ always gives the lowest values.

Timoshenko, in reference 2, has presented a solution for the lateral buckling of a strip bent by two equal and opposite eccentrically applied forces in its plane. As would be expected, the present solution differs from that by Timoshenko in that it includes (a) the effect of constraint against axial warping and (b) the effect of the uniform axial force in reducing the torsional stiffness of the beam.

Symmetrical Plate with Chord and Thickness Variation

A class of explicit solutions are presented for torsion of symmetrical cantilever plates with chord and thickness variation. With the origin of coordinates at the center of the root the differential equations (18) and (19) are independent of each other for a symmetrical plate in tip torque and distributed twisting moments ($a_2 = 0$). Only equation (19) need be considered; this equation becomes

$$\left(a_3\theta''\right)'' - 2(1 - \mu)\left(a_1\theta'\right)' - p_2 = 0 \tag{50}$$

with the boundary conditions

$$\theta(0) = \theta'(0) = \theta''(l) = 0$$

$$\left[\left(a_3\theta''\right)' - 2(1 - \mu)a_1\theta'\right]_{x=l} = -T$$

where a_3, a_1, and p_2 are functions of x defined in the section entitled "Analysis." Integration of equation (50) for tip torque alone and use of the bracketed boundary condition lead to

$$\left(a_3\theta''\right)' - 2(1 - \mu)a_1\theta' = -T \tag{51}$$

For applied twisting moments alone, after integration equation (50) becomes

$$\left(a_3\theta''\right)' - 2(1 - \mu)a_1\theta' = -\int_x^l p_2(\xi)d\xi \tag{52}$$

Symmetric cross section with algebraic spanwise variation of chord and thickness according to a power law.- Equations (51) and (52) can be solved in closed form when the stiffness D (which is proportional to the third power of the thickness) and the chord c vary according to the laws

$$D = D_0\left(1 - \frac{bx}{l}\right)^i K\left(\frac{y}{c}\right)$$

$$c = c_0\left(1 - \frac{bx}{l}\right)^j$$

where $D_0 = \dfrac{Eh^3(0,0)}{12\left(1 - \mu^2\right)}$, K is a symmetric function of y/c, and c_0 is the root chord. From the definition of a_1 and a_3

$$a_1 = D_0\left(1 - \frac{bx}{l}\right)^i \int_{-c/2}^{c/2} K\left(\frac{y}{c}\right)dy$$

$$a_3 = D_0\left(1 - \frac{bx}{l}\right)^i \int_{-c/2}^{c/2} K\left(\frac{y}{c}\right)y^2 dy$$

Setting $\eta = \frac{y}{c}$ leads to

$$a_1 = D_0 c_0\left(1 - \frac{bx}{l}\right)^{i+j} \int_{-1/2}^{1/2} K(\eta)\,d\eta$$

$$a_3 = \frac{D_0 c_0^3}{12}\left(1 - \frac{bx}{l}\right)^{i+3j}\ 12\int_{-1/2}^{1/2} K(\eta)\eta^2 d\eta$$

By use of

$$x_1 = 1 - \frac{bx}{l}$$

$$\xi_1 = 1 - \frac{b\xi}{l}$$

$$\varphi = \frac{d\theta}{dx_1}$$

$$p = i + 3j$$

$$q = i + j$$

$$D_0^{\,1} = k_D D_0$$

$$\lambda_0^{\,2} = k_\lambda \frac{l^2}{c_0^{\,2}} \frac{3}{2}(1 - \mu)$$

$$k_D = \int_{-1/2}^{1/2} K(\eta)\,d\eta \tag{53}$$

$$k_\lambda = \frac{\int_{-1/2}^{1/2} K(\eta)\,d\eta}{12\int_{-1/2}^{1/2} K(\eta)\eta^2\,d\eta} \tag{54}$$

equation (51) becomes

$$\frac{d}{dx_1}\left(x_1^{\,p} \frac{d\varphi}{dx_1}\right) - \frac{16\lambda_0^{\,2}}{b^2} x_1^{\,q}\varphi = \frac{12Tl^3}{D_0^{\,1}c_0^{\,3}b^3} \tag{55}$$

and equation (52) becomes

$$\frac{d}{dx_1}\left(x_1^{p} \frac{d\varphi}{dx_1}\right) - \frac{16\lambda_0^2}{b^2} x_1^{q}\varphi = -\frac{12l^4}{D_0{}^1c_0{}^3b^4} \int_{x_1}^{1-b} p_2(\xi_1)d\xi_1 \tag{56}$$

It may be seen that $K = k_D = k_\lambda = 1$ for a plate of rectangular cross section. Values of k_D and k_λ are given in the following table for some typical cross sections:

Cross section	k_D	k_λ
Rectangular	1	1
Elliptical	$\frac{3\pi}{32}$	2
Parabolic-arc	$\frac{16}{105}$	3
Diamond	$\frac{1}{20}$	5

For symmetric sections not given in this table, equations (53) and (54) may be used.

This solution is divided into two cases: (a) linearly varying chord and (b) chord variations other than linear.

(a) Linearly varying chord; tip torque and various spanwise distributions of twisting moments: In the special case of linearly varying chord ($j = 1$ or $p = q + 2$), the solution to the homogeneous part of equations (55) and (56) can be expressed in the form

$$\varphi^h = A_1 x_1{}^{\beta_1} + A_2 x_1{}^{\beta_2}$$

where A_1 and A_2 are arbitrary constants and

$$\beta_1 = -\frac{q + 1}{2} + \sqrt{\frac{16\lambda_0^2}{b^2} + \left(\frac{q + 1}{2}\right)^2}$$

$$\beta_2 = -\frac{q + 1}{2} - \sqrt{\frac{16\lambda_0^2}{b^2} + \left(\frac{q + 1}{2}\right)^2}$$

The particular solution of equation (55) which applies to tip-torque loading is

$$\varphi^{pi} = -\frac{12Tl^3}{D_0{}^1 c_0{}^3 b^3\left(q + \frac{16\lambda_0^2}{b^2}\right)} x_1^{-q}$$

The complete solution to equation (55) therefore is

$$\varphi = \varphi^h + \varphi^{pi} = A_1 x_1^{\beta_1} + A_2 x_1^{\beta_2} - \frac{12Tl^3}{D_0{}^1 c_0{}^3 b^3\left(q + \frac{16\lambda_0^2}{b^2}\right)} x_1^{-q}$$

Since $\varphi = \dfrac{d\theta}{dx_1}$,

$$\theta = \frac{A_1}{\beta_1 + 1} x_1^{\beta_1+1} + \frac{A_2}{\beta_2 + 1} x_1^{\beta_2+1} + A_3 - \frac{12Tl^3}{D_0{}^1 c_0{}^3 b^3\left(q + \frac{16\lambda_0^2}{b^2}\right)} \frac{x_1^{1-q}}{1 - q}$$

and A_1, A_2, and A_3 are determined by use of the boundary conditions

$$\theta(1) = \frac{d\theta}{dx_1}(1) = \frac{d^2\theta}{dx_1{}^2}(1 - b) = 0$$

The resulting expression for the angle of twist is

$$\theta = \frac{12Tl^3}{D_0{}^1c_0{}^3b^3\left(q + \frac{16\lambda_0{}^2}{b^2}\right)}\left(\frac{1}{\beta_2(1 - b)^{\beta_1} - \beta_1(1 - b)^{\beta_2}}\left\{\left[\beta_2(1 - b)^{\beta_2} + q(1 - b)^{-q}\right]\frac{1 - x_1{}^{\beta_1+1}}{\beta_1 + 1} + \left[\beta_1(1 - b)^{\beta_1} + q(1 - b)^{-q}\right]\frac{x_1{}^{\beta_2+1} - 1}{\beta_2 + 1}\right\} - \frac{x_1{}^{1-q} - 1}{1 - q}\right)$$

The particular solutions for equation (56) can be found in a similar manner when the applied twisting moments p_2 are known. If p_2 has the form

$$p_2 = p_{20}\left(1 - \frac{bx}{l}\right)^r$$

where the value assigned to r defines the distribution of applied twisting moments, then

$$\varphi^{pi} = \frac{12p_{20}l^4}{D_0{}^1c_0{}^3b^4(r + 1)}\left[\frac{x_1{}^{r+1-q}}{(r + 2)(r + 1 - q) - \frac{16\lambda_0{}^2}{b^2}} + \frac{(1 - b)^{r+1}x_1{}^{-q}}{q + \frac{16\lambda_0{}^2}{b^2}}\right]$$

As is done for tip-torque loading, the angle of twist can be found to be

$$\theta = \frac{-12p_{20}l^4}{D_0^{\,1}c_0^{\,3}b^4(r + 1)}\left(\frac{1}{\beta_2(1 - b)^{\beta_2} - \beta_1(1 - b)^{\beta_1}}\left\{\left[\left(\gamma_1 + \gamma_2\right)\beta_2(1 - b)^{\beta_2} - \right.\right.\right.$$

$$\left.\gamma_1(r + 1 - q)(1 - b)^{r+1-q} + \gamma_2 q(1 - b)^{-q}\right]\frac{x_1^{\beta_1+1} - 1}{\beta_1 + 1} -$$

$$\left[\left(\gamma_1 + \gamma_2\right)\beta_1(1 - b)^{\beta_1} - \gamma_1(r + 1 - q)(1 - b)^{r+1-q} + \right.$$

$$\left.\left.\left.\gamma_2 q(1 - b)^{-q}\right]\frac{x_1^{\beta_2+1} - 1}{\beta^2 + 1}\right\} - \gamma_1\,\frac{x_1^{r+2-q} - 1}{r + 2 - q} - \gamma_2\,\frac{x_1^{1-q} - 1}{1 - q}\right) \tag{57}$$

where

$$\gamma_1 = \frac{1}{(r + 2)(r + 1 - q) - \dfrac{16\lambda_0^{\,2}}{b^2}}$$

and

$$\gamma_2 = \frac{(1 - b)^{r+1}}{q + \dfrac{16\lambda_0^{\,2}}{b^2}}$$

Equation (57) can be used for finding the angle of twist due to distributed twisting moment which varies as the rth power of the local chord.

(b) Chord variations other than linear: The solution to the homogeneous part of equations (55) and (56) when $p \neq q + 2$ and therefore $j \neq 1$ is

$$\varphi^h = x_1^{\nu/\alpha}\left[A_1 I_\nu\left(\frac{4\lambda_0\alpha}{b} x_1^{1/\alpha}\right) + A_2 K_\nu\left(\frac{4\lambda_0\alpha}{b} x_1^{1/\alpha}\right)\right]$$

where I_ν is the modified Bessel function of the first kind of order ν and K_ν is the modified Bessel function of the second kind of order ν and

$$\nu = \frac{1 - p}{q - p + 2} = \frac{1 - i - 3j}{2(1 - j)}$$

$$\frac{1}{\alpha} = \frac{q - p + 2}{2} = 1 - j$$

The next step is to find the particular integral φ^{pi} for the torsion load considered from equation (55) or (56). The complete expression for φ is the sum of the homogeneous solution φ^h and the particular integral φ^{pi}. It is then necessary to integrate φ and to use the boundary conditions to get the final expression for the twist θ.

Solution for the twist by means of Bessel functions is straightforward for many values of ν. Solutions for cases in which $j \neq 1$ are not carried beyond this point since they involve tabular functions and therefore must be worked out separately for any set of values of the parameters.

Rectangular cross section with constant chord and exponential spanwise variation in thickness; tip torque.- A case that may be of interest is the constant-chord cantilever plate with exponentially decreasing stiffness (stiffness is proportional to the third power of the thickness)

$$D = D_0 e^{-ax}$$

and

$$a_1 = D_0 c e^{-ax}$$

$$a_3 = \frac{D_0 c^3}{12} e^{-ax}$$

For a plate subjected to tip torque the differential equation (51) is

$$\left(e^{-ax}\theta''\right)' - \frac{16\lambda^2}{l^2} e^{-ax}\theta' = \frac{-12T}{D_0 c^3} \tag{58}$$

The solution of equation (58) is

$$\theta = A_1 e^{b_1 x} + A_2 e^{b_2 x} + A_3 + \frac{3Tl^2}{4D_0 c^3 a\lambda^2} e^{ax}$$

where

$$b_1 = \frac{a + \sqrt{a^2 + \frac{64\lambda^2}{l^2}}}{2}$$

$$b_2 = \frac{a - \sqrt{a^2 + \frac{64\lambda^2}{l^2}}}{2}$$

For the boundary conditions $\theta(0) = \theta'(0) = \theta''(l) = 0$, the final solution is

$$\theta = \frac{3Tl^2}{4D_0 c^3 \lambda^2}\left\{\frac{1}{b_2 e^{b_2 l} - b_1 e^{b_1 l}}\left[\frac{1}{b_1}\left(-b_2 e^{b_2 l} + a e^{al}\right)\left(e^{b_1 x} - 1\right) + \frac{1}{b_2}\left(b_1 e^{b_1 l} - a e^{al}\right)\left(e^{b_2 x} - 1\right)\right] + \frac{1}{a}\left(e^{ax} - 1\right)\right\}$$

Skewed Uniform Plate with Tip and Lateral Loading

In this section the differential equations for a skewed uniform plate under tip and lateral loading are solved, and equations are obtained for the twisting and bending deflections of a cantilever plate under tip load, tip torque, tip bending moment, and uniform lateral load.

For a skewed plate of chord c (measured parallel to the root as shown in fig. 7) and with sweepback angle Λ

$$c_1(x) = (l - x) \tan \Lambda - \frac{c}{2}$$

$$c_2(x) = (l - x) \tan \Lambda + \frac{c}{2}$$

and for uniform thickness

$$a_1 = Dc$$

$$a_2 = Dc(l - x) \tan \Lambda$$

$$a_3 = Dc\left[\frac{c^2}{12} + (l - x)^2 \tan^2\Lambda\right]$$

Equations (18) and (19) become

$$Dc\left\{W^{IV} + \left[(l - x) \tan \Lambda \, \theta''\right]''\right\} = p_1 \tag{59}$$

$$Dc\left(\left\{\left[\frac{c^2}{12} + (l - x)^2 \tan^2\Lambda\right]\theta''\right\}'' + \left[(l - x) \tan \Lambda \, W''\right]'' - 2(1 - \mu)\theta''\right) = p_2 \tag{60}$$

and the corresponding tip boundary conditions from equations (14) to (17) become

$$Dc\left\{W''' + \left[(l - x) \tan \Lambda\, \theta''\right]'\right\}_{x=l} = -P \tag{61}$$

$$Dc\left(\left\{\left[\frac{c^2}{12} + (l - x)^2 \tan^2 \Lambda\right]\theta''\right\}' + \left[(l - x) \tan \Lambda\, W''\right]' - 2(1 - \mu)\theta'\right)_{x=l} = -T \tag{62}$$

$$DcW''(l) = -M_1 \tag{63}$$

$$\frac{Dc^3}{12}\, \theta''(l) = -M_2 \tag{64}$$

Two integrations of equation (59) with respect to x and use of boundary conditions (61) and (63) give

$$Dc\left[W'' + (l - x) \tan \Lambda\, \theta''\right] = \int_x^l \int_\eta^l p_1(\xi)d\xi\, d\eta + P(l - x) - M_1 \tag{65}$$

Integration of equation (60) with respect to x and use of boundary condition (62) give

$$Dc\left(\left\{\left[\frac{c^2}{12} + (l - x)^2 \tan^2\Lambda\right]\theta''\right\}' + \left[(l - x) \tan \Lambda\, W''\right]' - 2(1 - \mu)\theta'\right) = -\int_x^l p_2(\xi)d\xi - T \tag{66}$$

If W'' from equation (65) is substituted into equation (66), the following equation in θ alone results:

$$Dc\left[\frac{c^2}{12}\theta''' - 2(1 - \mu)\theta'\right] = -\int_x^l p_2(\xi)d\xi - T -$$

$$\tan\Lambda\left\{(l - x)\left[\int_x^l \int_\eta^l p_1(\xi)d\xi\,d\eta + P(l - x) - M_1\right]\right\}' \qquad (67)$$

The solution of equation (66) for the case of uniform lateral load $\left(p_1 = pc,\ \ p_2 = pc(l - x)\tan\Lambda\right)$ which satisfies the tip boundary condition (64) and the root boundary conditions $\theta(0) = \theta'(0) = 0$ is

$$\theta = A_1\left(\cosh\frac{4\lambda x}{l} - 1\right) + A_2\sinh\frac{4\lambda x}{l} + \frac{p\tan\Lambda}{6D(1 - \mu)}\left[(l - x)^3 - l^3 + \frac{3l^2x}{8\lambda^2}\right] +$$

$$\frac{P\tan\Lambda}{2Dc(1 - \mu)}\left[(l - x)^2 - l^2\right] + \frac{T}{2Dc(1 - \mu)}x + \frac{M_1\tan\Lambda}{2Dc(1 - \mu)}x \qquad (68)$$

where

$$A_1 = -A_2\tanh 4\lambda - \frac{l^2}{16\lambda^2\cosh 4\lambda}\left[\frac{12M_2}{Dc^3} + \frac{P\tan\Lambda}{Dc(1 - \mu)}\right]$$

$$A_2 = \frac{l}{4\lambda}\left[\frac{pl^2\tan\Lambda}{2D(1 - \mu)}\left(1 + \frac{1}{8\lambda^2}\right) + \frac{Pl\tan\Lambda}{Dc(1 - \mu)} - \frac{T}{2Dc(1 - \mu)} - \frac{M_1\tan\Lambda}{2Dc(1 - \mu)}\right]$$

Equation (68) gives the angle of twist θ of a skewed uniform cantilever plate under tip load P, tip torque T, tip bending moments M_1 and M_2, and uniform lateral load p. (If, for example, bending moments are distributed symmetrically over the tip of magnitude M, $M_1 = M$ and $M_2 = 0$.) From equation (65) with the root boundary conditions $W(0) = W'(0) = 0$, the corresponding bending deflection W is found to be

$$W = \frac{p}{24D}\left[\left(6l^2x^2 - 4lx^3 + x^4\right)\left(1 + \frac{2\tan^2\Lambda}{1-\mu}\right) + \frac{3\tan^2\Lambda}{2(1-\mu)}\frac{l^2x^2}{\lambda^2}\right] +$$

$$\frac{P}{6Dc}\left(3lx^2 - x^3\right)\left(1 + \frac{2\tan^2\Lambda}{1-\mu}\right) - \frac{M_1x^2}{2Dc}\left(1 + \frac{\tan^2\Lambda}{1-\mu}\right) - \frac{T\tan\Lambda}{2Dc(1-\mu)}x^2 -$$

$$\tan\Lambda(l - x)\theta - \tan\Lambda\frac{l}{2\lambda}\left[A_1\left(\sinh\frac{4\lambda x}{l} - \frac{4\lambda x}{l}\right) + A_2\left(\cosh\frac{4\lambda x}{l} - 1\right)\right]$$

With W and θ completely determined, the deflection at any point can now be found directly from equation (1) and the stresses can be found from equations (20) to (22).

CONCLUDING REMARKS

A simplified plate theory applicable to thin cantilever plates of arbitrary shape and thickness variation and with arbitrary load has been presented. The theory, as presented, is based on the assumption of linear deformations in the chordwise direction. Therefore, it is to be expected that good results would be obtained for unswept wings in torsion. For pure bending, the deflections obtained in this manner would be off by a factor of as much as $1 - \mu^2$ as a result of the artificial restraint against anticlastic curvature. For problems involving principally bending, therefore, it may be desired to extend the present theory to give more accurate results by considering a more general assumption for the deflection w. The expression

$$w = W(x) + \theta(x)y + \alpha(x)y^2$$

which includes the quadratic term $\alpha(x)y^2$ in addition to the linear terms would give more accurate results. This more general expression for w, when used with the energy method, would lead to three linear, fourth-order, differential equations for the quantities W, θ, and α and a complete set of boundary conditions.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., March 6, 1951

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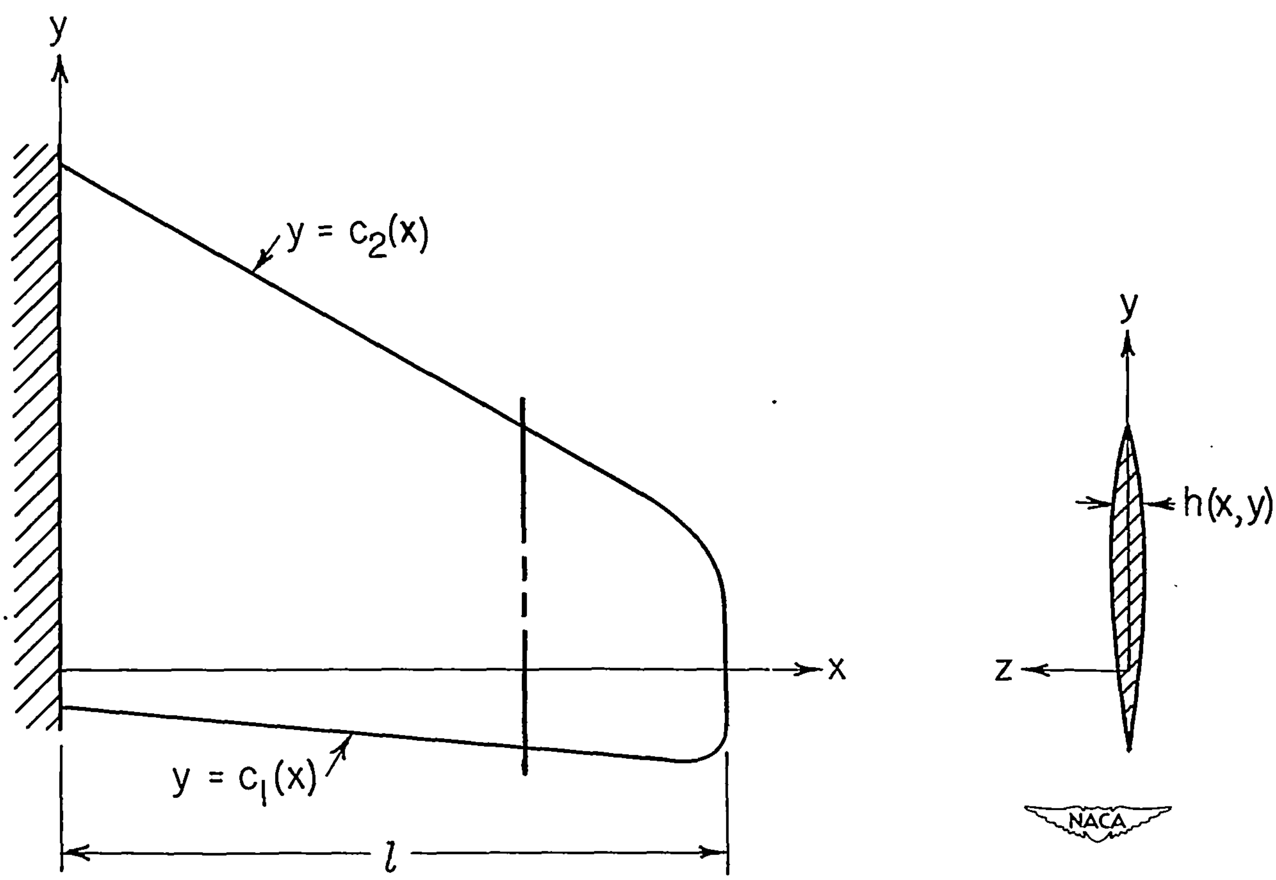

Figure 1.- Coordinate system used in the present analysis for a cantilever plate of arbitrary shape with arbitrary thickness variation.

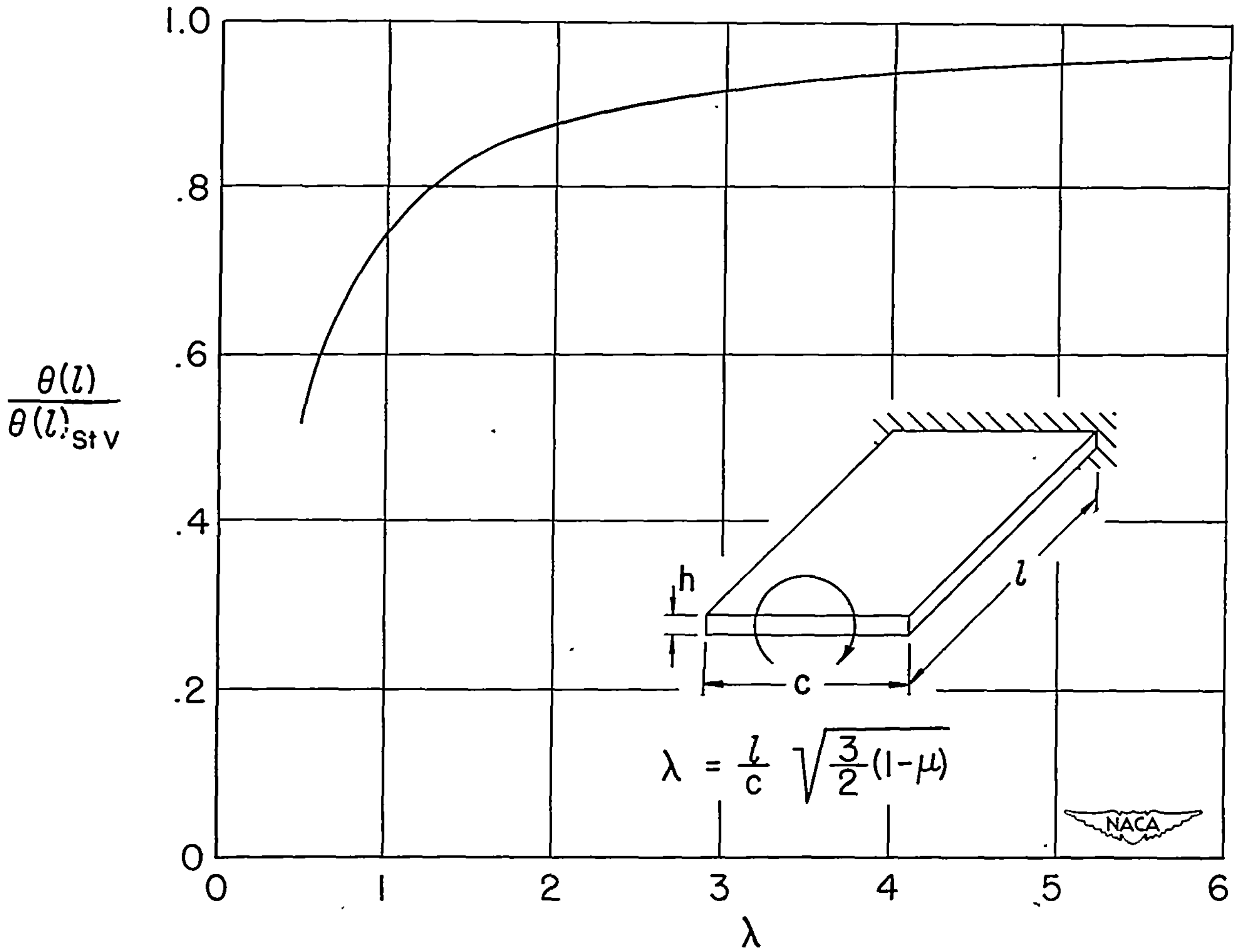

Figure 2.- Comparison of the tip twist given by present theory with that given by St. Venant torsion theory for a cantilever plate subjected to tip torque.

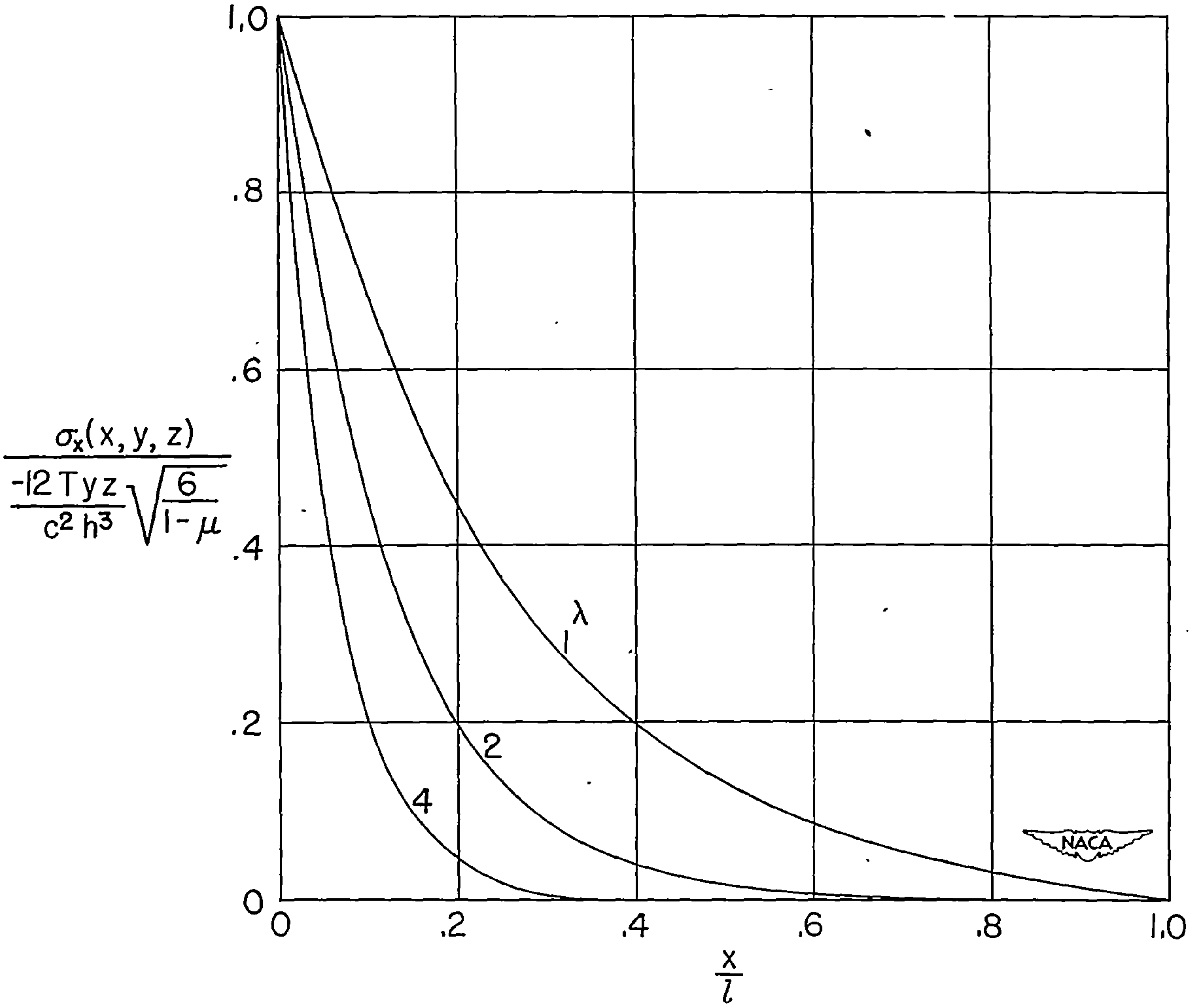

Figure 3.- Spanwise distribution of the normal stress as estimated by the present theory for a cantilever plate subjected to tip torque.

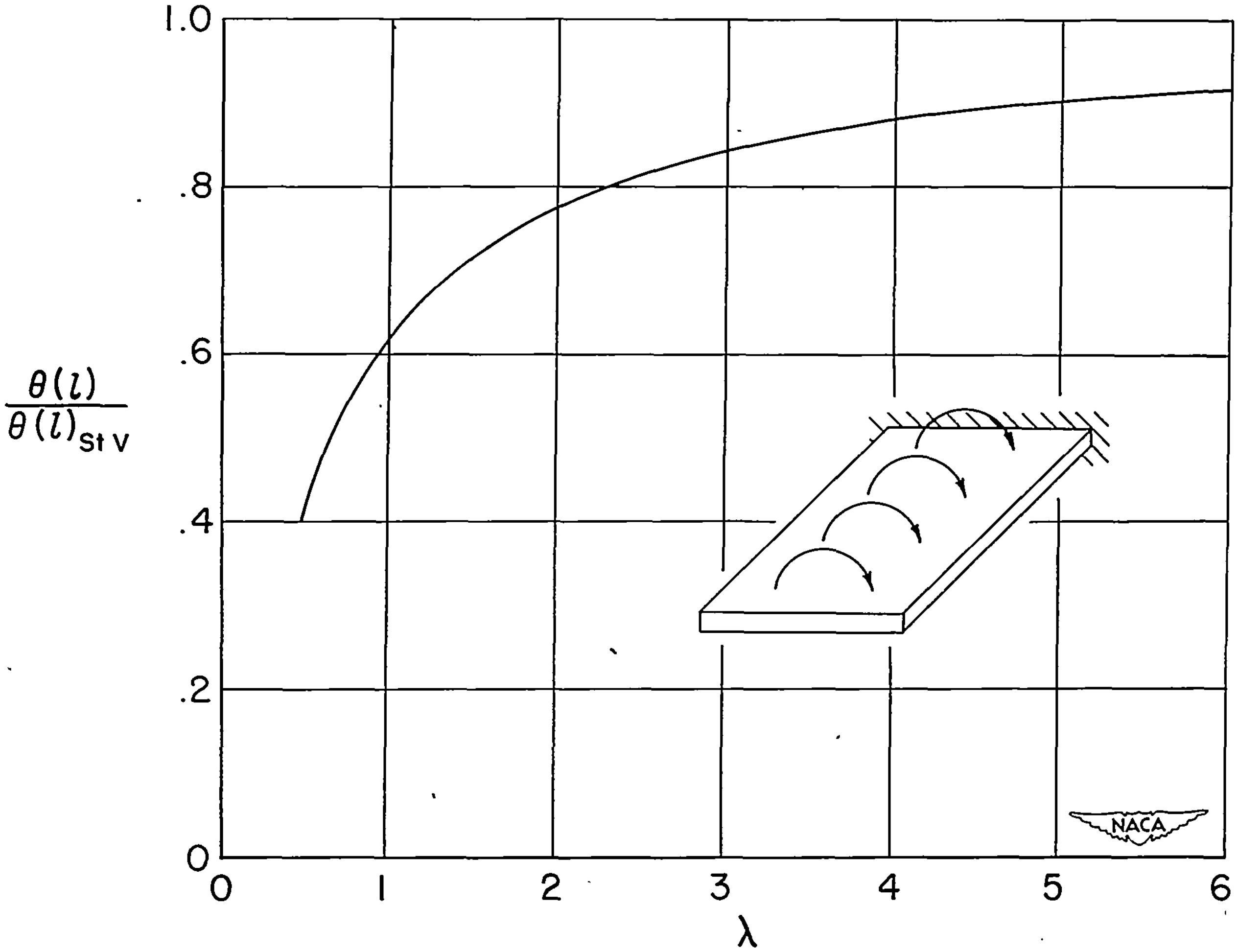

Figure 4.- Comparison of tip twist given by present theory with that given by St. Venant torsion theory for a cantilever plate with a uniform distribution of applied twisting moments.

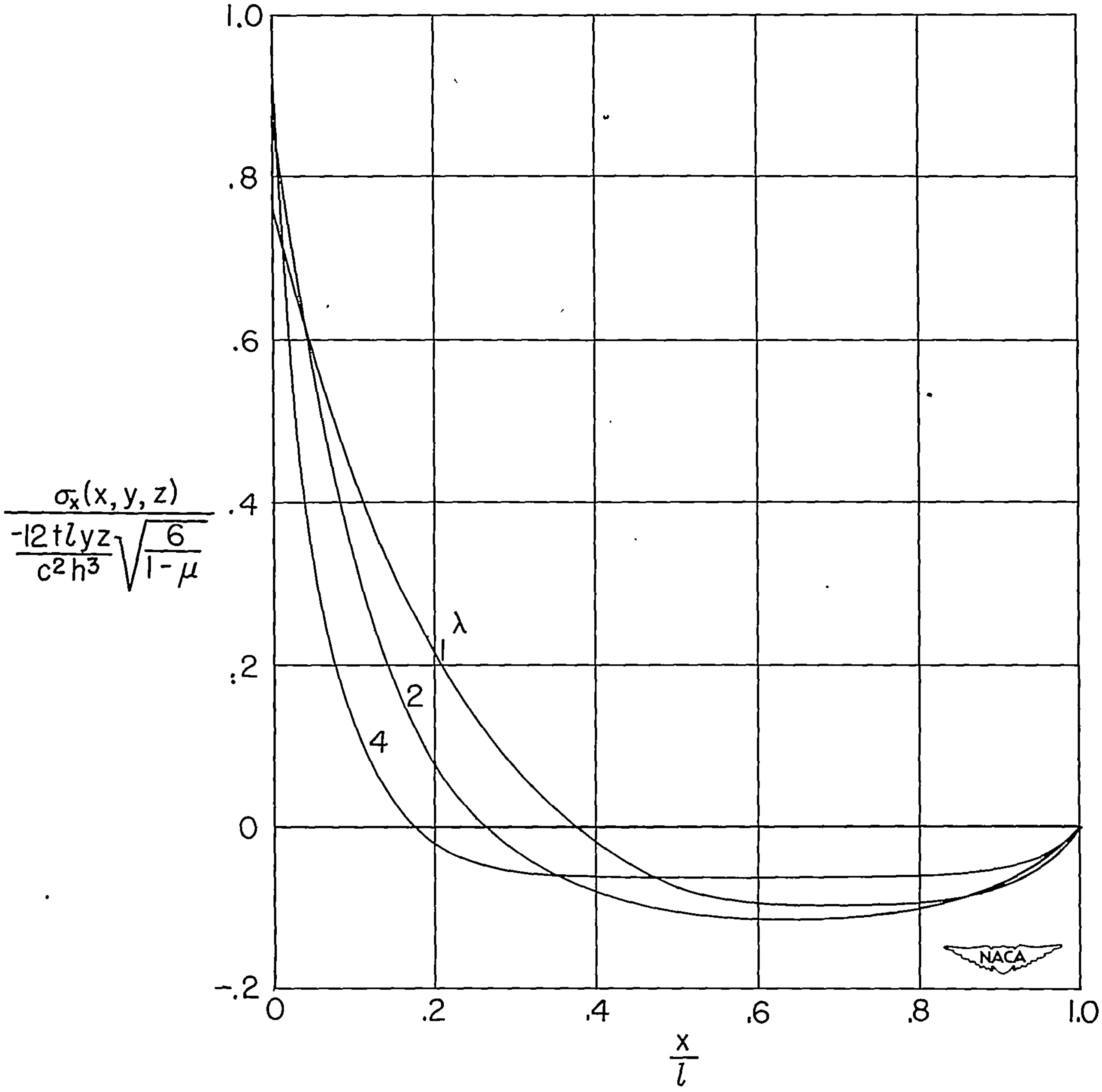

Figure 5.- Spanwise distribution of the normal stress as estimated by the present theory for a cantilever plate with uniform distribution of applied twisting moments.

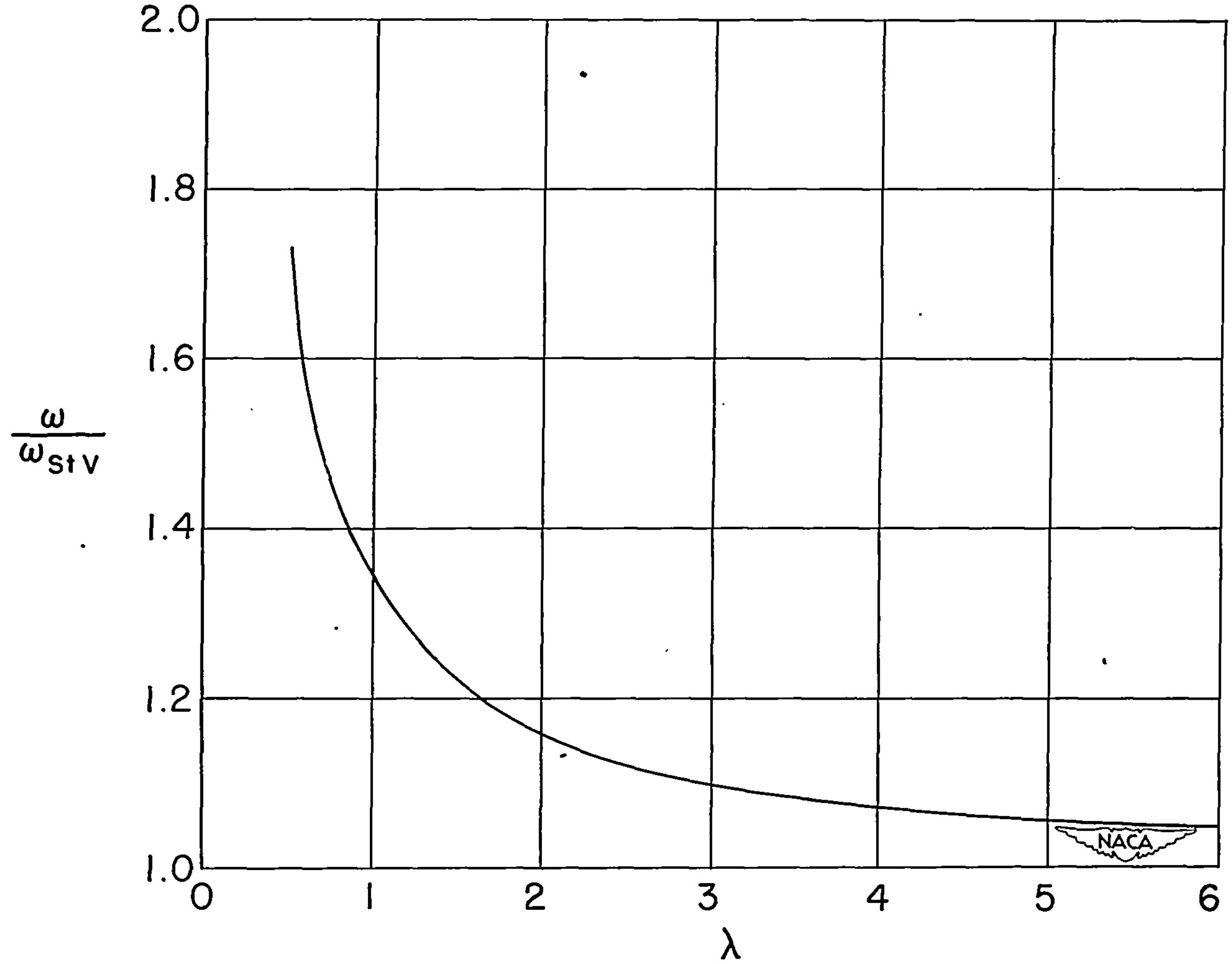

Figure 6.- Comparison of the fundamental frequency of torsional vibrations given by the present theory with that given by St. Venant torsion theory for a cantilever plate.

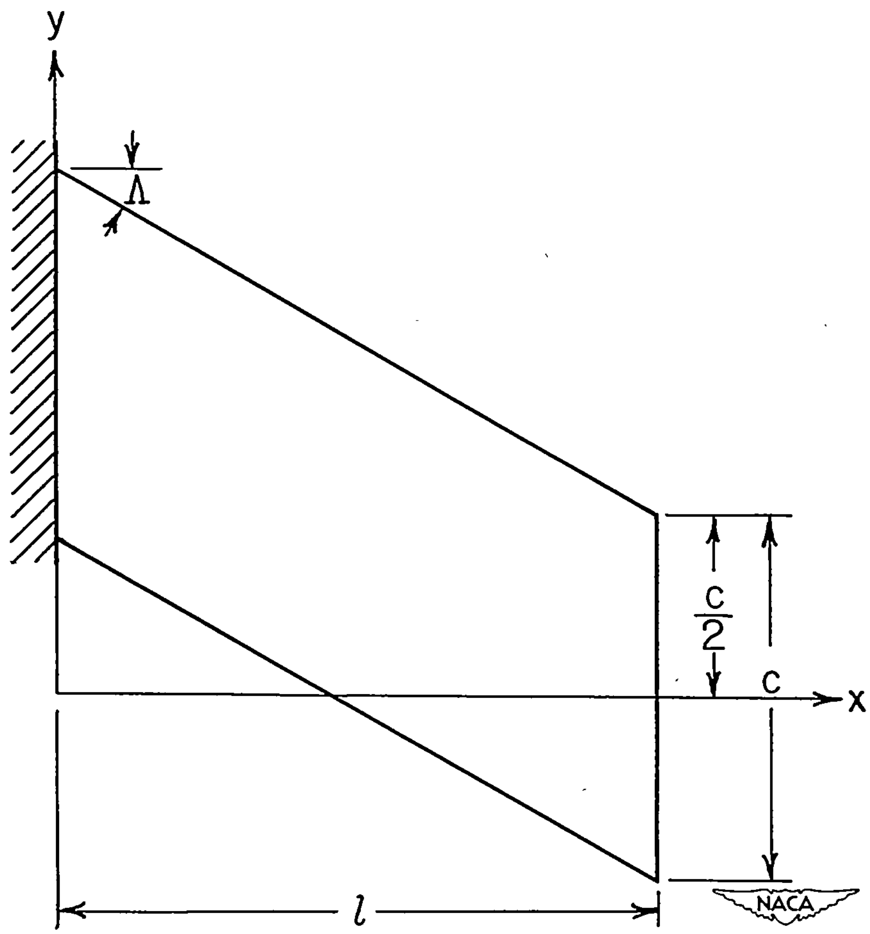

Figure 7.- Skewed uniform cantilever plate considered.