NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

REPORT No. 496

GENERAL THEORY OF AERODYNAMIC INSTABILITY AND THE MECHANISM OF FLUTTER

By THEODORE THEOD

REPRINT OF R ... ORIGINALLY

1949

AERONAUTIC SYMBOLS

1. FUNDAMENTAL AND DERIVED UNITS

	Symbol	Metric		English	
		Unit	Abbreviation	Unit	Abbreviation
Length	l	meter	m	foot (or mile)	ft (or mi)
Time	t	second	s	second (or hour)	sec (or hr)
Force	F	weight of 1 kilogram	kg	weight of 1 pound	lb
Power	P	horsepower (metric)		horsepower	hp
Speed	V	kilometers per hour	kph	miles per hour	mph
		meters per second	mps	feet per second	fps

2. GENERAL SYMBOLS

W Weight $=mg$

g Standard acceleration of gravity $=9.80665$ m/s² or 32.1740 ft/sec²

m Mass $=\frac{W}{g}$

I Moment of inertia $=mk^2$. (Indicate axis of radius of gyration k by proper subscript.)

μ Coefficient of viscosity

ν Kinematic viscosity

ρ Density (mass per unit volume)

Standard density of dry air, 0.12497 kg-m^{-4}-s^2 at 15° C and 760 mm; or 0.002378 lb-ft^{-4} sec^2

Specific weight of "standard" air, 1.2255 kg/m³ or 0.07651 lb/cu ft

3. AERODYNAMIC SYMBOLS

S Area

S_w Area of wing

G Gap

b Span

c Chord

A Aspect ratio, $\frac{b^2}{S}$

V True air speed

q Dynamic pressure, $\frac{1}{2}\rho V^2$

L Lift, absolute coefficient $C_L=\frac{L}{qS}$

D Drag, absolute coefficient $C_D=\frac{D}{qS}$

D_0 Profile drag, absolute coefficient $C_{D_0}=\frac{D_0}{qS}$

D_i Induced drag, absolute coefficient $C_{D_i}=\frac{D_i}{qS}$

D_p Parasite drag, absolute coefficient $C_{D_p}=\frac{D_p}{qS}$

C Cross-wind force, absolute coefficient $C_C=\frac{C}{qS}$

i_w Angle of setting of wings (relative to thrust line)

i_t Angle of stabilizer setting (relative to thrust line)

Q Resultant moment

Ω Resultant angular velocity

R Reynolds number, $\rho\frac{Vl}{\mu}$ where l is a linear dimension (e.g., for an airfoil of 1.0 ft chord, 100 mph, standard pressure at 15° C, the corresponding Reynolds number is 935,400; or for an airfoil of 1.0 m chord, 100 mps, the corresponding Reynolds number is 6,865,000)

α Angle of attack

ϵ Angle of downwash

α_0 Angle of attack, infinite aspect ratio

α_i Angle of attack, induced

α_a Angle of attack, absolute (measured from zero-lift position)

γ Flight-path angle

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National Advisory Committee for Aeronautics

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SUMMARY

The aerodynamic forces on an oscillating airfoil or airfoil-aileron combination of three independent degrees of freedom have been determined. The problem resolves itself into the solution of certain definite integrals, which have been identified as Bessel functions of the first and second kind and of zero and first order. The theory, being based on potential flow and the Kutta condition, is fundamentally equivalent to the conventional wing-section theory relating to the steady case.

The air forces being known, the mechanism of aerodynamic instability has been analyzed in detail. An exact solution, involving potential flow and the adoption of the Kutta condition, has been arrived at. The solution is of a simple form and is expressed by means of an auxiliary parameter k. The mathematical treatment also provides a convenient cyclic arrangement permitting a uniform treatment of all subcases of two degrees of freedom. The flutter velocity, defined as the air velocity at which flutter starts, and which is treated as the unknown quantity, is determined as a function of a certain ratio of the frequencies in the separate degrees of freedom for any magnitudes and combinations of the airfoil-aileron parameters.

For those interested solely or particularly in the numerical solutions Appendix I has been prepared. The routine procedure in solving numerical examples is put down detached from the theoretical background of the paper. It first is necessary to determine a certain number of constants pertaining to the case, then to perform a few routine calculations as indicated. The result is readily obtained in the form of a plot of flutter velocity against frequency for any values of the other parameters chosen. The numerical work of calculating the constants is simplified by referring to a number of tables, which are included in Appendix I. A number of illustrative examples and experimental results are given in Appendix II.

INTRODUCTION

It has been known that a wing or wing-aileron structurally restrained to a certain position of equilibrium becomes unstable under certain conditions. At least two degrees of freedom are required to create a condition of instability, as it can be shown that vibrations of a single degree of freedom would be damped out by the air forces. The air forces, defined as the forces due to the air pressure acting on the wing or wing-aileron in an arbitrary oscillatory motion of several degrees of freedom, are in this paper treated on the basis of the theory of nonstationary potential flow. A wing-section theory and, by analogy, a wing theory shall be thus developed that applies to the case of oscillatory motion, not only of the wing as a whole but also to that of an aileron. It is of considerable importance that large oscillations may be neglected; in fact, only infinitely small oscillations about the position of equilibrium need be considered. Large oscillations are of no interest since the sole attempt is to specify one or more conditions of instability. Indeed, no particular type or shape of airfoil shall be of concern, the treatment being restricted to primary effects. The differential equations for the several degrees of freedom will be put down. Each of these equations contains a statement regarding the equilibrium of a system of forces. The forces are of three kinds: (1) The inertia forces, (2) the restraining forces, and (3) the air forces.

There is presumably no necessity of solving a general case of damped or divergent motion, but only the border case of a pure sinusoidal motion, applying to the case of unstable equilibrium. This restriction is particularly important as the expressions for the air force developed for oscillatory motion can thus be employed. Imagine a case that is unstable to a very slight degree; the amplitudes will then increase very slowly and the expressions developed for the air forces will be applicable. It is of interest simply to know under what circumstances this condition may obtain and cases in which the amplitudes are decreasing or increasing at a finite rate need not be treated or specified. Although it is possible to treat the latter cases, they are of no concern in the present problem. Nor is the internal or solid friction of the structure of primary concern. The fortunate situation exists that the effect of the solid friction is *favorable*. Knowledge is desired concerning the condition as existing in the absence of the internal friction, as this case constitutes a sort of lower limit, which it is not always desirable to exceed.

Owing to the rather extensive field covered in the paper it has been considered necessary to omit many elementary proofs, it being left to the reader to verify certain specific statements. In the first part of the paper, the velocity potentials due to the flow around the airfoil-aileron are developed. These potentials are treated in two classes: The noncirculating flow potentials, and those due to the surface of discontinuity behind the wing, referred to as "circulatory" potentials. The magnitude of the circulation for an oscillating wing-aileron is determined next. The

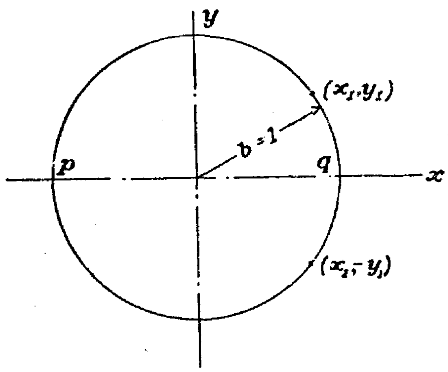

FIGURE 1.—Conformal representation of the wing profile by a circle.

forces and moments acting on the airfoil are then obtained by integration. In the latter part of the paper the differential equations of motion are put down and the particular and important case of unstable equilibrium is treated in detail. The solution of the problem of determining the flutter speed is finally given in the form of an equation expressing a relationship between the various parameters. The three subcases of two degrees of freedom are treated in detail.

The paper proposes to disclose the basic nature of the mechanism of flutter, leaving modifications of the primary results by secondary effects for future investigations.[1] Such secondary effects are: The effects of a finite span, of section shape, of deviations from potential flow, including also modifications of results to include twisting and bending of actual wing sections instead of pure torsion and deflection as considered in this paper.

The supplementary experimental work included in Appendix II similarly refers to well-defined elementary cases, the wing employed being of a large aspect ratio, nondeformable, and given definite degrees of freedom by a supporting mechanism, with external springs maintaining the equilibrium positions of wing or wing-aileron. The experimental work was carried on largely to verify the general shape of and the approximate magnitudes involved in the theoretically predicted response characteristics. As the present report is limited to the mathematical aspects of the flutter problem, specific recommendations in regard to practical applications are not given in this paper.

[1] The effect of internal friction is in some cases essential; this subject will be contained in a subsequent paper.

VELOCITY POTENTIALS, FORCES, AND MOMENTS OF THE NONCIRCULATORY FLOW

We shall proceed to calculate the various velocity potentials due to position and velocity of the individual parts in the whole of the wing-aileron system. Let us temporarily represent the wing by a circle (fig. 1). The potential of a source ϵ at the origin is given by

$$\varphi=\frac{\epsilon}{4\pi}\log\,(x^2+y^2)$$

For a source ϵ at (x_1,y_1) on the circle

$$\varphi=\frac{\epsilon}{4\pi}\log\,\{(x-x_1)^2+(y-y_1)^2\}$$

Putting a double source 2ϵ at (x_1,y_1) and a double negative source -2ϵ at $(x_1,-y_1)$ we obtain for the flow around the circle

$$\varphi=\frac{\epsilon}{2\pi}\log\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}$$

The function φ on the circle gives directly the surface potential of a straight line pq, the projection of the circle on the horizontal diameter. (See fig. 1.) In this case $y=\sqrt{1-x^2}$ and φ is a function of x only.

We shall need the integrals:

$$\int_c^1\log\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}dx_1=2(x-c)\log N-2\sqrt{1-x^2}\cos^{-1}c$$

and

$$\int_c^1\log\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}(x_1-c)dx_1=-\sqrt{1-c^2}\sqrt{1-x^2}$$
$$-\cos^{-1}c(x-2c)\sqrt{1-x^2}+(x-c)^2\log N$$

where
$$N=\frac{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}{x-c}$$

The location of the center of gravity of the wing-aileron x_α is measured from a, the coordinate of the axis of rotation (fig. 2); x_β the location of the center

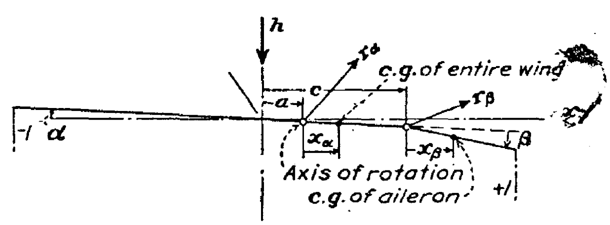

FIGURE 2.—Parameters of the airfoil-aileron combination.

of gravity of the aileron is measured from c, the coordinate of the hinge; and r_α and r_β are the radii of gyration of the wing-aileron referred to a, and of the aileron referred to the hinge. The quantities x_β and r_β are "reduced" values, as defined later in the paper. The quantities a, x_α, c, and x_β are positive toward the rear (right), h is the vertical coordinate of the axis of rotation at a with respect to a fixed reference frame and is positive downward. The angles α and β are positive clockwise (right-hand turn). The wind velocity v is to the right and horizontal. The angle (of attack) α refers to the direction of v, the aileron angle β refers to the undeflected position and *not* to the wind direction. The quantities r_α and r_β always occur as squares. Observe that the leading edge is located at -1, the trailing edge at $+1$. The quantities a, c, x_α, x_β, r_α, and r_β, which are repeatedly used in the following treatment, are all dimensionless with the half chord b as reference unit.

The effect of a flap bent down at an angle β (see fig. 2) is seen to give rise to a function φ obtained by substituting $-v\beta b$ for ϵ; hence

$$\varphi_\beta=\frac{v\beta b}{\pi}[\sqrt{1-x^2}\cos^{-1}c-(x-c)\log N]$$

To obtain the effect of the flap going down at an angular velocity $\dot{\beta}$, we put $\epsilon=-(x_1-c)\dot{\beta}b^2$ and get

$$\varphi_{\dot\beta}=\frac{\dot\beta b^2}{2\pi}[\sqrt{1-c^2}\sqrt{1-x^2}+\cos^{-1}c(x-2c)\sqrt{1-x^2}$$
$$-(x-c)^2\log N]$$

To obtain the effect of an angle α of the entire airfoil, we put $c=-1$ in the expression for φ_β, hence

$$\varphi_\alpha=v\alpha b\sqrt{1-x^2}$$

To depict the airfoil in downward motion with a velocity $\dot{h}$ (+ down), we need only introduce $\frac{\dot{h}}{v}$ instead of α. Thus

$$\varphi_{\dot h}=\dot{h}b\sqrt{1-x^2}$$

Finally, to describe a rotation around point a at an angular velocity $\dot\alpha$, we notice that this motion may be taken to consist of a rotation around the leading edge $c=-1$ at an angular velocity $\dot\alpha$ plus a vertical motion with a velocity $-\dot\alpha(1+a)b$. Then

$$\varphi_{\dot\alpha}=\frac{\dot\alpha b^2}{2\pi}\pi(x+2)\sqrt{1-x^2}-\dot\alpha(1+a)b^2\sqrt{1-x^2}$$
$$=\dot\alpha b^2\left(\frac{1}{2}x-a\right)\sqrt{1-x^2}$$

The following tables give in succession the velocity potentials and a set of integrals [2] with associated constants, which we will need in the calculation of the air forces and moments.

VELOCITY POTENTIALS

$$\varphi_\alpha=v\alpha b\sqrt{1-x^2}$$

$$\varphi_{\dot h}=\dot{h}b\sqrt{1-x^2}$$

$$\varphi_{\dot\alpha}=\dot\alpha b^2\left(\frac{1}{2}x-a\right)\sqrt{1-x^2}$$

$$\varphi_\beta=\frac{1}{\pi}v\beta b[\sqrt{1-x^2}\cos^{-1}c-(x-c)\log N]$$

$$\varphi_{\dot\beta}=\frac{1}{2\pi}\dot\beta b^2[\sqrt{1-c^2}\sqrt{1-x^2}+(x-2c)\sqrt{1-x^2}\cos^{-1}c$$
$$-(x-c)^2\log N]$$

where
$$N=\frac{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}{x-c}$$

[2] Some of the more difficult integral evaluations are given in Appendix III.

INTEGRALS

$$\int_c^1\varphi_\alpha dx=-\frac{b}{2}v\alpha T_4 \qquad \int_{-1}^{+1}\varphi_\alpha dx=\frac{b}{2}v\alpha\pi$$

$$\int_c^1\varphi_{\dot h}dx=-\frac{b}{2}\dot{h}T_4 \qquad \int_{-1}^{+1}\varphi_{\dot h}dx=\frac{b}{2}\dot{h}\pi$$

$$\int_c^1\varphi_{\dot\alpha}dx=\dot\alpha b^2T_9 \qquad \int_{-1}^{+1}\varphi_{\dot\alpha}dx=-\dot\alpha b^2\frac{\pi a}{2}$$

$$\int_c^1\varphi_\beta dx=-\frac{b}{2\pi}v\beta T_5 \qquad \int_{-1}^{+1}\varphi_\beta dx=-\frac{b}{2}v\beta T_4$$

$$\int_c^1\varphi_{\dot\beta}dx=-\frac{b^2}{2\pi}\dot\beta T_2 \qquad \int_{-1}^{+1}\varphi_{\dot\beta}dx=-\frac{b^2}{2}\dot\beta T_1$$

$$\int_c^1\varphi_\alpha(x-c)dx=-\frac{b}{2}v\alpha T_1 \qquad \int_{-1}^{+1}\varphi_\alpha(x-c)dx=-\frac{b}{2}v\alpha c\pi$$

$$\int_c^1\varphi_{\dot h}(x-c)dx=-\frac{b}{2}\dot{h}T_1 \qquad \int_{-1}^{+1}\varphi_{\dot h}(x-c)dx=-\frac{b}{2}\dot{h}c\pi$$

$$\int_c^1\varphi_{\dot\alpha}(x-c)dx=\dot\alpha b^2T_{13} \qquad \int_{-1}^{+1}\varphi_{\dot\alpha}(x-c)dx=\dot\alpha b^2T_{14}\pi$$

$$\int_c^1\varphi_\beta(x-c)dx=-\frac{b}{2\pi}v\beta T_2 \qquad \int_{-1}^{+1}\varphi_\beta(x-c)dx=-\frac{b}{2}v\beta T_8$$

$$\int_c^1\varphi_{\dot\beta}(x-c)dx=-\frac{b^2}{2\pi}\dot\beta T_3 \qquad \int_{-1}^{+1}\varphi_{\dot\beta}(x-c)dx=-\frac{b^2\dot\beta T_7}{2}$$

CONSTANTS

$$T_1=-\frac{1}{3}\sqrt{1-c^2}(2+c^2)+c\cos^{-1}c$$

$$T_2=c(1-c^2)-\sqrt{1-c^2}(1+c^2)\cos^{-1}c+c(\cos^{-1}c)^2$$

$$T_3=-\left(\frac{1}{8}+c^2\right)(\cos^{-1}c)^2+\frac{1}{4}c\sqrt{1-c^2}\cos^{-1}c(7+2c^2)$$
$$-\frac{1}{8}(1-c^2)(5c^2+4)$$

$$T_4=-\cos^{-1}c+c\sqrt{1-c^2}$$

$$T_5=-(1-c^2)-(\cos^{-1}c)^2+2c\sqrt{1-c^2}\cos^{-1}c$$

$$T_6=T_2$$

$$T_7=-\left(\frac{1}{8}+c^2\right)\cos^{-1}c+\frac{1}{8}c\sqrt{1-c^2}(7+2c^2)$$

$$T_8=-\frac{1}{3}\sqrt{1-c^2}(2c^2+1)+c\cos^{-1}c$$

$$T_9=\frac{1}{2}\left[\frac{1}{3}\left(\sqrt{1-c^2}\right)+aT_4\right]=\frac{1}{2}(-p+aT_4)$$

where $p=-\frac{1}{3}\left(\sqrt{1-c^2}\right)^3$

$$T_{10}=\sqrt{1-c^2}+\cos^{-1}c$$

$$T_{11}=\cos^{-1}c(1-2c)+\sqrt{1-c^2}(2-c)$$

$$T_{12}=\sqrt{1-c^2}(2+c)-\cos^{-1}c(2c+1)$$

$$T_{13}=\frac{1}{2}[-T_7-(c-a)T_1]$$

$$T_{14}=\frac{1}{16}+\frac{1}{2}ac$$

FORCES AND MOMENTS

The velocity potentials being known, we are able to calculate local pressures and by integration to obtain the forces and moments acting on the airfoil and aileron.

Employing the extended Bernoulli Theorem for unsteady flow, the local pressure is, except for a constant

$$p_h = -\rho\left(\frac{w^2}{2} + \frac{\partial\varphi}{\partial t}\right)$$

where w is the local velocity and φ the velocity potential at the point. Substituting $w = v + \frac{\partial\varphi}{\partial x}$ we obtain ultimately for the pressure difference between the upper and lower surface at x

$$p = -2\rho\left(v\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial t}\right)$$

where v is the constant velocity of the fluid relative to the airfoil at infinity. Putting down the integrals for the force on the entire airfoil, the moment on the flap around the hinge, and the moment on the entire airfoil, we obtain by means of partial integrations

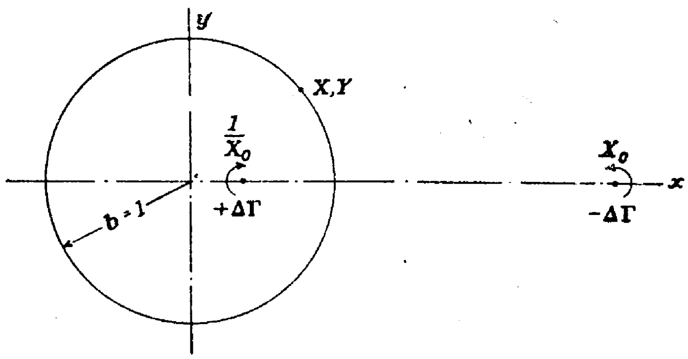

FIGURE 3.—Conformal representation of the wing profile with reference to the circulatory flow.

$$P = -2\rho b\int_{-1}^{+1}\dot{\varphi}\,dx$$

$$M_\beta = -2\rho b^2\int_c^1\dot{\varphi}(x-c)\,dx + 2\rho v b\int_c^1\varphi\,dx$$

$$M_\alpha = -2\rho b^2\int_{-1}^{+1}\dot{\varphi}\,(x-c)\,dx + 2\rho v b\int_{-1}^{+1}\varphi\,dx - 2\rho b^2\int_{-1}^{+1}\dot{\varphi}\,(c-a)\,dx$$

Or, on introducing the individual velocity potentials from page 5,

$$P = -\rho b^2[v\pi\dot{\alpha} + \pi\ddot{h} - b\pi a\ddot{\alpha} - vT_4\dot{\beta} - bT_1\ddot{\beta}] \qquad \text{(I)}$$

$$M_\beta = -\rho b^3\left[-vT_1\dot{\alpha} - T_1\ddot{h} + 2T_{13}b\ddot{\alpha} - \frac{1}{\pi}vT_2\dot{\beta} - \frac{1}{\pi}T_3b\ddot{\beta}\right] + \rho v b^2\left[-vT_4\alpha - T_4\dot{h} + 2T_9b\dot{\alpha} - \frac{1}{\pi}vT_5\beta - \frac{1}{\pi}T_2b\dot{\beta}\right]$$

$$= -\rho b^2\left[T_4v^2\alpha - (2T_9 + T_1)\,bv\dot{\alpha} + 2T_{13}b^2\dot{\alpha} + \frac{1}{\pi}T_5v^2\beta + \left(\frac{1}{\pi}T_2 - \frac{1}{\pi}T_2\right)bv\dot{\beta} - \frac{1}{\pi}b^2T_3\ddot{\beta} + T_4v\dot{h} - T_1b\ddot{h}\right] \qquad \text{(II)}$$

$$M_\alpha = -\rho b^2\left[-\pi v^2\alpha + \pi\left(\frac{1}{8} + a^2\right)b^2\ddot{\alpha} + v^2T_4\beta + \{T_1 - T - (c-a)\,T_4\}\,b\dot{\beta}v + \{-T_7 - (c-a)\,T_1\}\,b^2\ddot{\beta} - ba\pi\ddot{h} - \pi v\dot{h}\right] \qquad \text{(III)}$$

VELOCITY POTENTIALS, FORCES, AND MOMENTS OF THE CIRCULATORY FLOW

In the following we shall determine the velocity potentials and associated forces and moments due to a surface of discontinuity of strength U extending along the positive x axis from the wing to infinity. The velocity potential of the flow around the circle (fig. 3) resulting from the vortex element $-\Delta\Gamma$ at (X_0, O) is

$$\varphi_\Gamma = \frac{\Delta\Gamma}{2\pi}\left[\tan^{-1}\frac{Y}{X - X_0} - \tan^{-1}\frac{Y}{X - \frac{1}{X_0}}\right]$$

$$= \frac{\Delta\Gamma}{2\pi}\tan^{-1}\frac{\left(-\frac{1}{X_0} + X_0\right)Y}{X^2 - \left(X_0 + \frac{1}{X_0}\right)X + Y^2 + 1}$$

where (X, Y) are the coordinates of the variable and X_0 is the coordinate of $-\Delta\Gamma$ on the x axis.

Introducing $X_0 + \frac{1}{X_0} = 2x_0$

or $X_0 = x_0 + \sqrt{x_0^2 - 1}$ on the x axis

and $X = x$ and $Y = \sqrt{1 - x^2}$ on the circle

the equation becomes

$$\varphi_{xx_0} = -\frac{\Delta\Gamma}{2\pi}\tan^{-1}\frac{\sqrt{1-x^2}\sqrt{x_0^2-1}}{1 - xx_0}$$

This expression gives the clockwise circulation around the airfoil due to the element $-\Delta\Gamma$ at x_0.

We have: $p = -2\rho\left(\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x}\right)$

But, since the element $-\Delta\Gamma$ will now be regarded as moving to the right relative to the airfoil with a velocity v

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\varphi}{\partial x_0}v$$

Hence, $p = -2\rho v\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial x_0}\right)$

Further

$$\frac{2\pi}{\Delta\Gamma}\frac{\partial\varphi}{\partial x} = \sqrt{x_0^2 - 1}\,\frac{\frac{x}{(1 - xx_0)\sqrt{1-x^2}} + \frac{x_0\sqrt{1-x^2}}{(1-xx_0)^2}}{1 + \frac{(1-x^2)(x_0^2-1)}{(1-xx_0)^2}}$$

$$= \frac{\sqrt{x_0^2-1}}{\sqrt{1-x^2}}\,\frac{1}{(x_0 - x)}$$

and

$$\frac{2\pi}{\Delta\Gamma}\frac{\partial\varphi}{\partial x_0} = \sqrt{1-x^2}\,\frac{\frac{1}{(1-xx_0)}\,\frac{x_0}{\sqrt{x_0^2-1}} + \sqrt{x_0^2-1}\,\frac{x}{(1-xx_0)^2}}{1 + \frac{(1-x^2)(x_0^2-1)}{(1-xx_0)^2}}$$

$$= \frac{\sqrt{1-x^2}}{\sqrt{x_0^2-1}}\,\frac{1}{(x_0-x)}$$

By addition:

$$\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial x_0} = \frac{\Delta\Gamma}{2\pi}\,\frac{x_0 + x}{\sqrt{1-x^2}\sqrt{x_0^2-1}}$$

To obtain the force on the aileron, we need the integral

$$\int_c^1\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial x_0}\right)\mathrm{d}x=\frac{\Delta\Gamma}{2\pi}\int_c^1\frac{x_0+x}{\sqrt{x_0^2-1}\sqrt{1-x^2}}\mathrm{d}x$$

$$=-\frac{\Delta\Gamma}{2\pi}\left[\frac{x_0}{\sqrt{x_0^2-1}}\cos^{-1}x+\frac{\sqrt{1-x^2}}{\sqrt{x_0^2-1}}\right]_c^1$$

$$=\frac{\Delta\Gamma}{2\pi}\left[\frac{x_0}{\sqrt{x_0^2-1}}\cos^{-1}c+\frac{\sqrt{1-c^2}}{\sqrt{x_0^2-1}}\right]$$

Thus, for the force on the aileron

$$\Delta P_{c1}=-\rho vb\frac{\Delta\Gamma}{\pi}\left(\frac{x_0}{\sqrt{x_0^2-1}}\cos^{-1}c+\frac{1}{\sqrt{x_0^2-1}}\sqrt{1-c^2}\right)\quad\text{or}$$

$$\Delta P_{c1}=-\rho vb\frac{\Delta\Gamma}{\pi}\left[\frac{x_0}{\sqrt{x_0^2-1}}\left(\cos^{-1}c-\sqrt{1-c^2}\right)+\sqrt{\frac{x_0+1}{x_0-1}}\sqrt{1-c^2}\right]$$

Integrated, with $\Delta\Gamma=U\mathrm{d}x_0$

$$P_{c1}=-\frac{\rho vb}{\pi}\left[\left(\cos^{-1}c-\sqrt{1-c^2}\right)\int_1^\infty\frac{x_0}{\sqrt{x_0^2-1}}U\mathrm{d}x_0+\sqrt{1-c^2}\int_1^\infty\sqrt{\frac{x_0+1}{x_0-1}}U\mathrm{d}x_0\right]$$

for $c=-1$ we obtain the expression for P, the force on the *whole* airfoil

$$P=-\rho vb\int_1^\infty\frac{x_0}{\sqrt{x_0^2-1}}U\mathrm{d}x_0 \qquad \text{(IV)}$$

Since U is considered stationary with respect to the fluid elements

$$U=\mathrm{f}(vt-x_0)$$

where t is the time since the beginning of the motion. U is thus a function of the *distance* from the location of the *first* vortex element or, referred to a system moving with the fluid, U is *stationary* in value.

Similarly we obtain for the moment on the *aileron*

$$\int_c^1\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial x_0}\right)(x-c)\mathrm{d}x=\frac{\Delta\Gamma}{2\pi}\int_c^1\frac{(x-c)\ (x_0+x)}{\sqrt{1-x^2}\sqrt{x_0^2-1}}\mathrm{d}x$$

$$=-\frac{\Delta\Gamma}{2\pi}\frac{1}{\sqrt{x_0^2-1}}\left[x_0\sqrt{1-x^2}+\frac{x\sqrt{1-x^2}}{2}-c\sqrt{1-x^2}+\left(\frac{1}{2}-x_0c\right)\cos^{-1}x\right]_c^1$$

$$=+\frac{\Delta\Gamma}{2\pi}\frac{1}{\sqrt{x_0^2-1}}\left[\left(x_0+\frac{c}{2}-c\right)\sqrt{1-c^2}+\frac{1}{2}\left(1-2x_0c\right)\cos^{-1}c\right]$$

$$=+\frac{\Delta\Gamma}{2\pi}\left[\frac{x_0}{\sqrt{x_0^2-1}}\left(\sqrt{1-c^2}-c\cos^{-1}c\right)+\frac{\frac{1}{2}}{\sqrt{x_0^2-1}}\left(\cos^{-1}c-c\sqrt{1-c^2}\right)\right]$$

Finally

$$\Delta M_\beta=-\rho vb^2\frac{\Delta\Gamma}{\pi}\left[\frac{x_0}{\sqrt{x_0^2-1}}\left\{\sqrt{1-c^2}\left(1+\frac{c}{2}\right)-\cos^{-1}c\left(c+\frac{1}{2}\right)\right\}+\frac{1}{2}\sqrt{\frac{x_0+1}{x_0-1}}\left(\cos^{-1}c-c\sqrt{1-c^2}\right)\right]$$

Putting $\Delta\Gamma=U\mathrm{d}x_0$ and integrating

$$M_\beta=-\frac{\rho vb^2}{\pi}\left[\left\{\sqrt{1-c^2}\left(1+\frac{c}{2}\right)-\cos^{-1}c\left(c+\frac{1}{2}\right)\right\}\int_1^\infty\frac{x_0}{\sqrt{x_0^2-1}}U\mathrm{d}x_0+\left(\cos^{-1}c-c\sqrt{(1-c^2)}\right)\frac{1}{2}\int_1^\infty\sqrt{\frac{x_0+1}{x_0-1}}U\mathrm{d}x_0\right] \qquad \text{(V)}$$

Further, for the moment on the entire airfoil around a

$$\int_{-1}^{+1}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial x_0}\right)(x-a)\mathrm{d}x=-\frac{\Delta\Gamma}{2\pi}\frac{1}{\sqrt{x_0^2-1}}\left[\left(x_0+\frac{x}{2}-a\right)\sqrt{1-x^2}+\left(\frac{1}{2}-x_0a\right)\cos^{-1}x\right]_{-1}^{+1}=+\frac{\Delta\Gamma}{2\pi}\frac{1}{\sqrt{x_0^2-1}}\left(\frac{1}{2}-x_0a\right)\pi$$

and

$$\Delta M_\alpha=-\rho vb^2\Delta\Gamma\frac{\frac{1}{2}-x_0a}{\sqrt{x_0^2-1}}$$

Integrated, this becomes

$$M_\alpha=-\rho vb^2\int_1^\infty\frac{\frac{1}{2}-x_0a}{\sqrt{x_0^2-1}}U\mathrm{d}x_0$$

$$=-\rho vb^2\int_1^\infty\left\{\frac{\frac{1}{2}+\frac{1}{2}x_0}{\sqrt{x_0^2-1}}-\frac{x_0\left(a+\frac{1}{2}\right)}{\sqrt{x_0^2-1}}\right\}U\mathrm{d}x_0$$

$$=-\rho vb^2\int_1^\infty\left\{\frac{1}{2}\sqrt{\frac{x_0+1}{x_0-1}}-\left(a+\frac{1}{2}\right)\frac{x_0}{\sqrt{x_0^2-1}}\right\}U\mathrm{d}x_0 \qquad \text{(VI)}$$

THE MAGNITUDE OF THE CIRCULATION

The magnitude of the circulation is determined by the Kutta condition, which requires that no infinite velocities exist at the trailing edge, or, at $x=1$

$$\frac{\partial}{\partial x}(\varphi_\Gamma+\varphi_\alpha+\varphi_{\dot h}+\varphi_{\dot\alpha}+\varphi_\beta+\varphi_{\dot\beta})=\text{finite}$$

Introducing the values of φ_α, etc. from page 5 and φ_Γ from $\frac{\partial\varphi}{\partial x}$ page 6 gives the important relation:

$$\frac{1}{2\pi}\int_1^\infty\frac{\sqrt{x_0+1}}{\sqrt{x_0-1}}U\mathrm{d}x_0=v\alpha+\dot h+b\left(\frac{1}{2}-a\right)\dot\alpha+\frac{T_{10}}{\pi}v\beta+b\frac{T_{11}}{2\pi}\dot\beta \qquad \text{(VII)}$$

This relation must be [illegible]ed to comply with the Kutta condition, which [illegible] that the flow shall leave the airfoil at the trailing edge.

It is observed that the relation reduces to that of the Kutta condition for stationary flow on putting $x_0=\infty$, and in subsequence omitting the variable parameters $\dot{\alpha}$, $\dot{\beta}$, and $\dot{h}$.

Let us write

$$\frac{1}{2\pi}\int_1^\infty \sqrt{\frac{x_0+1}{x_0-1}}\,U\mathrm{d}x_0 = v\alpha+\dot{h}+b\left(\frac{1}{2}-a\right)\dot{\alpha} + \frac{T_{10}}{\pi}v\beta+b\,\frac{T_{11}}{2\pi}\dot{\beta}=Q$$

Introduced in (IV)

$$P=-2\pi\rho v b Q\,\frac{\int_1^\infty \frac{x_0}{\sqrt{x_0^2-1}}U\mathrm{d}x_0}{\int_1^\infty \sqrt{\frac{x_0+1}{x_0-1}}\,U\mathrm{d}x_0}$$

from (V)

$$M_\beta=-2\rho v b^2\left[\left(\sqrt{1-c^2}\left(1+\frac{c}{2}\right)-\cos^{-1}c\left(c+\frac{1}{2}\right)\right)\times \frac{\int_1^\infty \frac{x_0}{\sqrt{x_0^2-1}}\,U\mathrm{d}x_0}{\int_1^\infty \sqrt{\frac{x_0+1}{x_0-1}}\,U\mathrm{d}x_0}+\frac{1}{2}\left(\cos^{-1}c-c\sqrt{1-c^2}\right)\right]Q$$

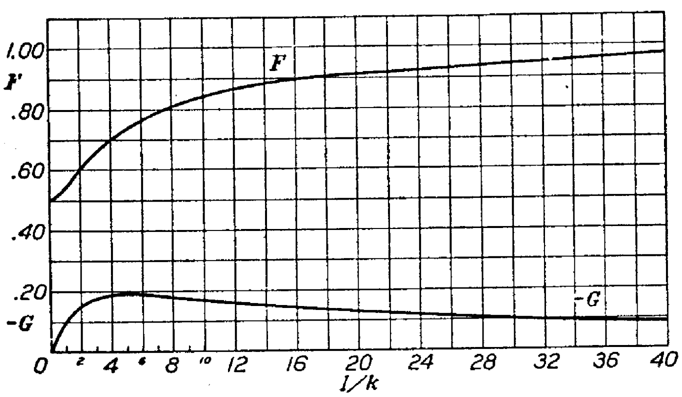

FIGURE 4.—The functions F and G against $\frac{1}{k}$.

and from (VI)

$$M_\alpha=-2\pi\rho v b^2\left[\frac{1}{2}-\left(a+\frac{1}{2}\right)\frac{\int_1^\infty \frac{x_0}{\sqrt{x_0^2-1}}\,U\mathrm{d}x_0}{\int_1^\infty \sqrt{\frac{x_0+1}{x_0-1}}\,U\mathrm{d}x_0}\right]Q$$

Introducing

$$C=\frac{\int_1^\infty \frac{x_0}{\sqrt{x_0^2-1}}\,U\mathrm{d}x_0}{\int_1^\infty \sqrt{\frac{x_0+1}{x_0-1}}\,U\mathrm{d}x_0}$$

we obtain finally

$$P=-2\rho v b\pi C Q \qquad \text{(VIII)}$$

$$M_\beta=-2\rho v b^2\left[\left(\sqrt{1-c^2}\left(1+\frac{c}{2}\right)-\cos^{-1}c\left(c+\frac{1}{2}\right)\right)C+\frac{1}{2}\left(\cos^{-1}c-c\sqrt{1-c^2}\right)\right]Q=-\rho v b^2(T_{12}C-T_4)Q \qquad \text{(IX)}$$

$$M_\alpha=2\pi\rho v b^2\left[\left(a+\frac{1}{2}\right)C-\frac{1}{2}\right]Q \qquad \text{(X)}$$

where Q is given above and $C=C(k)$ will be treated in the following section.

VALUE OF THE FUNCTION $C(k)$

Put $U=U_0e^{i\left[k\left(\frac{s}{b}-x_0\right)+\varphi\right]}$

where $s=vt$ $(s\rightarrow\infty)$, the distance from the *first* vortex element to the airfoil, and k a positive constant determining the wave length,

then

$$C(k)=\frac{\int_1^\infty \frac{x_0}{\sqrt{x_0^2-1}}e^{-ikx_0}\mathrm{d}x_0}{\int_1^\infty \frac{x_0+1}{\sqrt{x_0^2-1}}e^{-ikx_0}\mathrm{d}x_0} \qquad \text{(XI)}$$

These integrals are known, see next part, formulas (XIV)—(XVII) and we obtain [3]

$$C(k)=\frac{-\frac{\pi}{2}J_1+i\frac{\pi}{2}Y_1}{-\frac{\pi}{2}J_1-\frac{\pi}{2}Y_0+i\frac{\pi}{2}Y_1-i\frac{\pi}{2}J_0}=\frac{-J_1+iY_1}{-(J_1+Y_0)+i(Y_1-J_0)}$$

$$=\frac{(-J_1+iY_1)[-(J_1+Y_0)-i(Y_1-J_0)]}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

$$=\frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

$$-i\frac{Y_1(J_1+Y_0)-J_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}=F+iG$$

where

$$F=\frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2} \qquad \text{(XII)}$$

$$G=-\frac{Y_1Y_0+J_1J_0}{(J_1+Y_0)^2+(Y_1-J_0)^2} \qquad \text{(XIII)}$$

These functions, which are of fundamental importance in the theory of the oscillating airfoil are given graphically against the argument $\frac{1}{k}$ in figure 4.

SOLUTION OF THE DEFINITE INTEGRALS IN C BY MEANS OF BESSEL FUNCTIONS

We have

$$K_n(z)=\int_0^\infty e^{-z\cosh t}\cosh nt\,\mathrm{d}t$$

(Formula (34), p. 51—Gray, Mathews & MacRobert: Treatise on Bessel Functions. London, 1922)

where

$$K_n(t)=e^{\frac{in\pi}{2}}G_n(it)$$

(Eq. (28), sec. 3, p. 23, same reference)

and

$$G_n(x)=-\overline{Y}_n(x)+\left[\log 2-\gamma+\frac{i\pi}{2}\right]J_n(x)$$

but

$$\overline{Y}_n(x)=\frac{\pi}{2}Y_n(x)+(\log 2-\gamma)J_n(x)$$

(where $Y_n(x)$ is from N. Nielsen: Handbuch der Theorie der Cylinderfunktionen. Leipzig, 1904).

[3] This may also be expressed in Hänkel functions, $C=\frac{H_1^{(2)}}{H_1^{(2)}+i\,H_0^{(2)}}$

Thus,

$$G_n(x) = -\frac{\pi}{2}[Y_n(x) - iJ_n(x)]$$

We have

$$K_0(-ik) = \int_0^\infty e^{ik\cosh t}\,dt = \int_1^\infty \frac{e^{ikx}}{\sqrt{x^2-1}}\,dx$$

or

$$-\frac{\pi}{2}Y_0(k) + i\frac{\pi}{2}J_0(k) = \int_1^\infty \frac{\cos kx dx}{\sqrt{x^2-1}} + i\int_1^\infty \frac{\sin kx dx}{\sqrt{x^2-1}}$$

Thus,

$$\int_1^\infty \frac{\cos kx dx}{\sqrt{x^2-1}} = -\frac{\pi}{2}Y_0(k) \qquad \text{(XIV)}$$

$$\int_1^\infty \frac{\sin kx dx}{\sqrt{x^2-1}} = \frac{\pi}{2}J_0(k) \qquad \text{(XV)}$$

Further,

$$K_1(-ik) = \int_0^\infty e^{ik\cosh t}\cosh t dt = \int_1^\infty \frac{e^{ikx} x dx}{\sqrt{x^2-1}}$$

$$iG_1(k) = -i\frac{\pi}{2}Y_1(k) - \frac{\pi}{2}J_1(k)$$

$$= \int_1^\infty \frac{x}{\sqrt{x^2-1}}(\cos kx + i\sin kx)\,dx$$

Thus,

$$\int_1^\infty \frac{x\cos kx dx}{\sqrt{x^2-1}} = -\frac{\pi}{2}J_1(k) \qquad \text{(XVI)}$$

$$\int_1^\infty \frac{x\sin kx dx}{\sqrt{x^2-1}} = -\frac{\pi}{2}Y_1(k) \qquad \text{(XVII)}$$

TOTAL AERODYNAMIC FORCES AND MOMENTS

TOTAL FORCE

From equations (I) and (VIII) we obtain

$$P = -\rho b^2(v\pi\dot\alpha + \pi\ddot h - \pi ba\ddot\alpha - vT_4\dot\beta - T_1 b\ddot\beta) - 2\pi\rho vbC\left\{v\alpha + \dot h + b\left(\frac{1}{2}-a\right)\dot\alpha + \frac{1}{\pi}T_{10}v\beta + b\frac{1}{2\pi}T_{11}\dot\beta\right\} \qquad \text{(XVIII)}$$

TOTAL MOMENTS

From equations (II) and (IX) we obtain similarly

$$M_\beta = -\rho b^2\left[\left\{-2T_9 - T_1 + T_4\left(a-\frac{1}{2}\right)\right\}vb\dot\alpha + 2T_{13}b^2\ddot\alpha + \frac{1}{\pi}v^2\beta(T_5 - T_4T_{10}) - \frac{1}{2\pi}vb\dot\beta T_4T_{11} - \frac{1}{\pi}T_3b^2\ddot\beta - T_1b\ddot h\right] - \rho vb^2T_{12}C\left\{v\alpha + \dot h + b\left(\frac{1}{2}-a\right)\dot\alpha + \frac{1}{\pi}T_{10}v\beta + b\frac{1}{2\pi}T_{11}\dot\beta\right\} \qquad \text{(XIX)}$$

From equations (III) and (X)

$$M_\alpha = -\rho b^2\left[\pi\left(\frac{1}{2}-a\right)vb\dot\alpha + \pi b^2\left(\frac{1}{8}+a^2\right)\ddot\alpha + (T_4+T_{10})v^2\beta + \left(T_1 - T_8 - (c-a)T_4 + \frac{1}{2}T_{11}\right)vb\dot\beta - \left(T_7 + (c-a)T_1\right)b^2\ddot\beta - a\pi b\ddot h\right] + 2\rho vb^2\pi\left(a+\frac{1}{2}\right)C\left\{v\alpha + \dot h + b\left(\frac{1}{2}-a\right)\dot\alpha + \frac{1}{\pi}T_{10}v\beta + b\frac{1}{2\pi}T_{11}\dot\beta\right\} \qquad \text{(XX)}$$

DIFFERENTIAL EQUATIONS OF MOTION

Expressing the equilibrium of the moments about a of the entire airfoil, of the moments on the aileron about c, and of the vertical forces, we obtain, respectively, the following three equations:

α: $\quad -I_\alpha\ddot\alpha - I_\beta\ddot\beta - b(c-a)S_\beta\ddot\beta - S_\alpha\ddot h - \alpha C_\alpha + M_\alpha = 0$

β: $\quad -I_\beta\ddot\beta - I_\beta\ddot\alpha - b(c-a)\ddot\alpha S_\beta - \ddot h S_\beta - \beta C_\beta + M_\beta = 0$

h: $\quad -\ddot h M - \ddot\alpha S_\alpha - \ddot\beta S_\beta - hC_h + P = 0$

Rearranged:

α: $\quad \ddot\alpha I_\alpha + \ddot\beta(I_\beta + b(c-a)S_\beta) + \ddot h S_\alpha + \alpha C_\alpha - M_\alpha = 0$

β: $\quad \ddot\alpha(I_\beta + b(c-a)S_\beta) + \ddot\beta I_\beta + \ddot h S_\beta + \beta C_\beta - M_\beta = 0$

h: $\quad \ddot\alpha S_\alpha + \ddot\beta S_\beta + \ddot h M + hC_h - P = 0$

The constants are defined as follows:

- ρ, mass of air per unit of volume.
- b, half chord of wing.
- M, mass of wing per unit of length.
- S_α, S_β, static moments of wing (in slugs-feet) per unit length of wing-aileron and aileron, respectively. The former is referred to the axis a; the latter, to the hinge c.
- I_α, I_β, moments of inertia per unit length of wing-aileron and aileron about a and c, respectively.
- C_α, torsional stiffness of wing around a, corresponding to unit length.
- C_β, torsional stiffness of aileron around c, corresponding to unit length.
- C_h, stiffness of wing in deflection, corresponding to unit length.

DEFINITION OF PARAMETERS USED IN EQUATIONS

- $\kappa = \frac{\pi\rho b^2}{M}$, the ratio of the mass of a cylinder of air of a diameter equal to the chord of the wing to the mass of the wing, both taken for equal length along span.

$r_\alpha=\sqrt{\frac{I_\alpha}{Mb^2}}$, the radius of gyration divided by b.

$x_\alpha=\frac{S_\alpha}{Mb}$, the center of gravity distance of the wing from a, divided by b.

$\omega_\alpha=\sqrt{\frac{C_\alpha}{I_\alpha}}$, the frequency of torsional vibration around a.

$r_\beta=\sqrt{\frac{I_\beta}{Mb^2}}$, *reduced* radius of gyration of aileron divided by b, that is, the radius at which the entire mass of the airfoil would have to be concentrated to give the moment of inertia of the aileron I_β.

$x_\beta=\frac{S_\beta}{Mb}$, *reduced* center of gravity distance from c.

$\omega_\beta=\sqrt{\frac{C_\beta}{I_\beta}}$, frequency of torsional vibration of aileron around c.

$\omega_h=\sqrt{\frac{C_h}{M}}$, frequency of wing in deflection.

FINAL EQUATIONS IN NONDIMENSIONAL FORM

On introducing the quantities M_α, M_β, and P, replacing T_9 and T_{13} from page 5, and reducing to nondimensional form, we obtain the following system of equations:

$$\text{(A)}\quad \ddot{\alpha}\left[r_\alpha^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$

$$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot{h}\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

$$\text{(B)}\quad \ddot{\alpha}\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$$

$$+\beta\left[\frac{C_\beta}{Mb^2}+\frac{1}{\pi^2}\frac{v^2}{b^2}\kappa(T_5-T_4T_{10})\right]+\ddot{h}\left(x_\beta-\frac{1}{\pi}\kappa T_1\right)\frac{1}{b}+\frac{T_{12}}{\pi}\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

$$\text{(C)}\quad \ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$$

$$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

SOLUTION OF EQUATIONS

As mentioned in the introduction, we shall only have to specify the conditions under which an unstable equilibrium may exist, no general solution being needed. We shall therefore introduce the variables at once as sine functions of the distance s or, in complex form with $\frac{1}{k}$ as an auxiliary parameter, giving the ratio of the wave length to 2π times the half chord b:

$$\alpha=\alpha_0 e^{ik\frac{s}{b}} \quad = \alpha_0 e^{i\omega t}$$

$$\beta=\beta_0 e^{i\left(k\frac{s}{b}+\varphi_1\right)}$$

and

$$h=h_0 e^{i\left(k\frac{s}{b}+\varphi_2\right)}$$

where s is the distance from the airfoil to the *first* vortex element, $\frac{ds}{dt}=v$, and φ_1 and φ_2 are phase angles of β and h with respect to α.

Having introduced these quantities in our system of equations, we shall divide through by $\left(\frac{v}{b}k\right)^2\kappa$.

We observe that the velocity v is then contained in only *one* term of each equation. We shall consider this term containing v as the unknown parameter ΩX. To distinguish terms containing X we shall employ a bar; terms without bars do not contain X.

We shall resort to the following notation, taking care to retain a perfectly cyclic arrangement. Let the letter A refer to the coefficients in the first equation not containing $C(k)$ or X, B to similar coefficients of the second equation, and C to those in the third equation. Let the first subscript α refer to the first variable α, the subscript β to the second, and h to the third. Let the second subscripts 1, 2, 3 refer to the second derivative, the first derivative, and the argument of each variable, respectively. $A_{\alpha 1}$ thus refers to the coefficient in the first equation associated with the second derivative of α and not containing $C(k)$ or

X; C_{h3} to the constant in the third equation attached to h, etc. These coefficients[4] are as follows:

$$A_{\alpha 1}=\frac{r_\alpha^2}{\kappa}+\left(\frac{1}{8}+a^2\right)$$

$$A_{\alpha 2}=\left(\frac{1}{2}-a\right)$$

$$A_{\alpha 3}=0$$

$$A_{\beta 1}=\frac{r_\beta^2}{\kappa}-\frac{T_7}{\pi}+(c-a)\left(\frac{x_\beta}{\kappa}-\frac{T_1}{\pi}\right)$$

$$A_{\beta 2}=\frac{1}{\pi}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$

$$A_{\beta 3}=\frac{1}{\pi}(T_4+T_{10})$$

$$A_{h1}=\frac{x_\alpha}{\kappa}-a$$

$$A_{h2}=0$$

$$A_{h3}=0$$

$$B_{\alpha 1}=\frac{r_\beta^2}{\kappa}-\frac{T_7}{\pi}+(c-a)\left(\frac{x_\beta}{\kappa}-\frac{T_1}{\pi}\right) \qquad (=A_{\beta 1})$$

$$B_{\alpha 2}=\frac{1}{\pi}\left(p-T_1-\frac{1}{2}T_4\right)$$

$$B_{\alpha 3}=0$$

$$B_{\beta 1}=\frac{r_\beta^2}{\kappa}-\frac{1}{\pi^2}T_3$$

$$B_{\beta 2}=-\frac{1}{2\pi^2}T_4T_{11}$$

$$B_{\beta 3}=\frac{1}{\pi^2}(T_5-T_4T_{10})$$

$$B_{h1}=\frac{x_\beta}{\kappa}-\frac{1}{\pi}T_1$$

$$B_{h2}=0$$

$$B_{h3}=0$$

$$C_{\alpha 1}=\frac{x_\alpha}{\kappa}-a \qquad (=A_{h1})$$

$$C_{\alpha 2}=1$$

$$C_{\alpha 3}=0$$

$$C_{\beta 1}=\frac{x_\beta}{\kappa}-\frac{1}{\pi}T_1 \qquad (=B_{h1})$$

$$C_{\beta 2}=-\frac{1}{\pi}T_4$$

$$C_{\beta 3}=0$$

$$C_{h1}=\frac{1}{\kappa}+1$$

$$C_{h2}=0$$

$$C_{h3}=0$$

[4] The factor $\frac{1}{k}$ or $\frac{1}{k^2}$ is not included in these constants. See the expressions for the R's and I's on next page.

The solution of the instability problem as contained in the system of three equations A, B, and C is given by the vanishing of a third-order determinant of complex numbers representing the coefficients. The solution of particular subcases of two degrees of freedom is given by the minors involving the particular coefficients. We shall denote the case *torsion-aileron* (α, β) as case 3, *aileron-deflection* (β, h) as case 2, and *deflection-torsion* (h, α) as case 1. The determinant form of the solution is given in the major case and in the three possible subcases, respectively, by:

$$\overline{D}=\begin{vmatrix}\bar{R}_{a\alpha}+iI_{a\alpha}, & R_{a\beta}+iI_{a\beta}, & R_{ah}+iI_{ah}\\ R_{b\alpha}+iI_{b\alpha}, & \bar{R}_{b\beta}+iI_{b\beta}, & R_{bh}+iI_{bh}\\ R_{c\alpha}+iI_{c\alpha}, & R_{c\beta}+iI_{c\beta}, & \bar{R}_{ch}+iI_{ch}\end{vmatrix}=0$$

and

$$\overline{M}_{ch}=\begin{vmatrix}\bar{R}_{a\alpha}+iI_{a\alpha}, & R_{a\beta}+iI_{a\beta}\\ R_{b\alpha}+iI_{b\alpha}, & \bar{R}_{b\beta}+iI_{b\beta}\end{vmatrix}=0 \qquad \text{Case 3}$$

$$\overline{M}_{a\alpha}=\begin{vmatrix}\bar{R}_{b\beta}+iI_{b\beta}, & R_{bh}+iI_{bh}\\ R_{c\beta}+iI_{c\beta}, & \bar{R}_{ch}+iI_{ch}\end{vmatrix}=0 \qquad \text{Case 2}$$

$$\overline{M}_{b\beta}=\begin{vmatrix}\bar{R}_{ch}+iI_{ch}, & R_{c\alpha}+iI_{c\alpha}\\ R_{ah}+iI_{ah}, & \bar{R}_{a\alpha}+iI_{a\alpha}\end{vmatrix}=0 \qquad \text{Case 1}$$

REAL EQUATIONS **IMAGINARY EQUATIONS**

$$\begin{vmatrix}\bar{R}_{a\alpha}R_{a\beta}\\ R_{b\alpha}\bar{R}_{b\beta}\end{vmatrix}-\begin{vmatrix}I_{a\alpha}I_{a\beta}\\ I_{b\alpha}I_{b\beta}\end{vmatrix}=0 \quad \begin{vmatrix}\bar{R}_{a\alpha}R_{a\beta}\\ I_{b\alpha}I_{b\beta}\end{vmatrix}+\begin{vmatrix}I_{a\alpha}I_{a\beta}\\ R_{b\alpha}\bar{R}_{b\beta}\end{vmatrix}=0 \quad \text{Case 3}$$

$$\begin{vmatrix}\bar{R}_{b\beta}R_{bh}\\ R_{c\beta}\bar{R}_{ch}\end{vmatrix}-\begin{vmatrix}I_{b\beta}I_{bh}\\ I_{c\beta}I_{ch}\end{vmatrix}=0 \quad \begin{vmatrix}\bar{R}_{b\beta}R_{bh}\\ I_{c\beta}I_{ch}\end{vmatrix}+\begin{vmatrix}I_{b\beta}I_{bh}\\ R_{c\beta}\bar{R}_{ch}\end{vmatrix}=0 \quad \text{Case 2}$$

$$\begin{vmatrix}\bar{R}_{ch}R_{c\alpha}\\ R_{ah}\bar{R}_{a\alpha}\end{vmatrix}-\begin{vmatrix}I_{ch}I_{c\alpha}\\ I_{ah}I_{a\alpha}\end{vmatrix}=0 \quad \begin{vmatrix}\bar{R}_{ch}R_{c\alpha}\\ I_{ah}I_{a\alpha}\end{vmatrix}+\begin{vmatrix}I_{ch}I_{c\alpha}\\ R_{ah}\bar{R}_{a\alpha}\end{vmatrix}=0 \quad \text{Case 1}$$

NOTE.—Terms with bars contain X; terms without bars do not contain X.

The 9 quantities $R_{a\alpha}$, $R_{a\beta}$, etc., refer to the real parts and the 9 quantities $I_{a\alpha}$, $I_{a\beta}$, etc., to the imaginary parts of the coefficients of the 3 variables α, β, and h in the 3 equations A, B, C on page 10. Denoting the coefficients of $\ddot{\alpha}$, $\dot{\alpha}$, and α in the first equation by p, q, and r,

$$R_{a\alpha}+iI_{a\alpha}=\frac{1}{\kappa}\left[-p+iq\frac{b}{kv}+r\left(\frac{b}{kv}\right)^2\right]$$

which, separated in real and imaginary parts, gives the quantities $R_{a\alpha}$ and $I_{a\alpha}$. Similarly, the remaining quantities R and I are obtained. They are all functions of k or $C(k)$. The terms with bars $\bar{R}_{a\alpha}$, $\bar{R}_{b\beta}$, and $\bar{R}_{ch}$ are seen to be the only ones containing the unknown X. The quantities Ω and X will be defined shortly. The quantities R and I are given in the following list:

$$\begin{cases} \bar{R}_{a\alpha} = -A_{\alpha 1} + \Omega_\alpha X + \frac{1}{k}2\left(\frac{1}{2}+a\right)\left[\left(\frac{1}{2}-a\right)G - \frac{1}{k}F\right] & (1) \\ R_{a\beta} = -A_{\beta 1} + \frac{1}{k^2}A_{\beta 3} + \frac{1}{k}\frac{1}{\pi}\left(a+\frac{1}{2}\right)\left[T_{11}G - 2\frac{1}{k}T_{10}F\right] & (2) \\ R_{ah} = -A_{h1} + \frac{1}{k}2\left(a+\frac{1}{2}\right)G & (3) \end{cases}$$

$$\begin{cases} R_{b\alpha} = -B_{\alpha 1} - \frac{1}{k}\frac{T_{12}}{\pi}\left[\left(\frac{1}{2}-a\right)G - \frac{1}{k}F\right] & (4) \\ \bar{R}_{b\beta} = -B_{\beta 1} + \frac{1}{k^2}B_{\beta 3} + \Omega_\beta X - \frac{1}{k}\frac{T_{12}}{2\pi^2}\left[T_{11}G - 2T_{10}\frac{1}{k}F\right] & (5) \\ R_{bh} = -B_{h1} - \frac{1}{k}\frac{T_{12}}{\pi}G & (6) \end{cases}$$

$$\begin{cases} R_{c\alpha} = -C_{\alpha 1} - \frac{1}{k}2\left[\left(\frac{1}{2}-a\right)G - \frac{1}{k}F\right] & (7) \\ R_{c\beta} = -C_{\beta 1} - \frac{1}{k}\frac{1}{\pi}\left[T_{11}G - 2T_{10}\frac{1}{k}F\right] & (8) \\ \bar{R}_{ch} = -C_{h1} + \Omega_h X - \frac{1}{k}2G & (9) \end{cases}$$

$$\begin{cases} I_{a\alpha} = -\frac{1}{k}\left[2\left(a+\frac{1}{2}\right)\left\{\left(\frac{1}{2}-a\right)F + \frac{1}{k}G\right\} - A_{\alpha 2}\right] & (11) \\ I_{a\beta} = -\frac{1}{k}\left[\frac{1}{\pi}\left(a+\frac{1}{2}\right)\left(T_{11}F + 2\frac{1}{k}T_{10}G\right) - A_{\beta 2}\right] & (12) \\ I_{ah} = -\frac{1}{k}2\left(a+\frac{1}{2}\right)F & (13) \end{cases}$$

$$\begin{cases} I_{b\alpha} = \frac{1}{k}\left[\frac{T_{12}}{\pi}\left\{\left(\frac{1}{2}-a\right)F + \frac{1}{k}G\right\} + B_{\alpha 2}\right] & (14) \\ I_{b\beta} = \frac{1}{k}\left[\frac{T_{12}}{2\pi^2}\left(T_{11}F + 2\frac{1}{k}T_{10}G\right) + B_{\beta 2}\right] & (15) \\ I_{bh} = \frac{1}{k}\frac{T_{12}}{\pi}F & (16) \end{cases}$$

$$\begin{cases} I_{c\alpha} = \frac{1}{k}\left[2\left\{\left(\frac{1}{2}-a\right)F + \frac{1}{k}G\right\} + C_{\alpha 2}\right] & (17) \\ I_{c\beta} = \frac{1}{k}\left[\frac{1}{\pi}\left(T_{11}F + 2\frac{1}{k}T_{10}G\right) + C_{\beta 2}\right] & (18) \\ I_{ch} = \frac{1}{k}2F & (19) \end{cases}$$

The solution as given by the three-row determinant shall be written explicitly in X. We are immediately able to put down for the general case a cubic equation in X with complex coefficients and can easily segregate the three subcases. The quantity D is as before the value of the determinant, but with the term containing X missing. The quantities $M_{a\alpha}$, $M_{b\beta}$, and M_{ch} are the minors of the elements in the diagonal squares $a\alpha$, $b\beta$, and ch, respectively. They are expressed explicitly in terms of R and I under the subcases treated in the following paragraphs.

$$\bar{D} = \begin{vmatrix} A_{a\alpha}+\Omega_\alpha X & A_{a\beta} & A_{ah} \\ A_{b\alpha} & A_{b\beta}+\Omega_\beta X & A_{bh} \\ A_{c\alpha} & A_{c\beta} & A_{ch}+\Omega_h X \end{vmatrix} = 0$$

where $A_{a\alpha} = R_{a\alpha} + iI_{a\alpha}$ etc.

Complex cubic equation in X:

$$\Omega_\alpha\Omega_\beta\Omega_h X^3 + (\Omega_\alpha\Omega_\beta A_{ch} + \Omega_\beta\Omega_h A_{a\alpha} + \Omega_h\Omega_\alpha A_{b\beta})X^2 + (\Omega_\alpha M_{a\alpha} + \Omega_\beta M_{b\beta} + \Omega_h M_{ch})X + D = 0 \qquad \text{(XXI)}$$

Case 3, (α, β):

$$\Omega_\alpha\Omega_\beta X^2 + (\Omega_\alpha A_{b\beta} + \Omega_\beta A_{a\alpha})X + M_{ch} = 0 \qquad \text{(XXII)}$$

Case 2, (β, h):

$$\Omega_\beta\Omega_h X^2 + (\Omega_\beta A_{ch} + \Omega_h A_{b\beta})X + M_{a\alpha} = 0 \qquad \text{(XXIII)}$$

Case 1, (h, α):

$$\Omega_h\Omega_\alpha X^2 + (\Omega_h A_{a\alpha} + \Omega_\alpha A_{ch})X + M_{b\beta} = 0 \qquad \text{(XXIV)}$$

$$\Omega_\alpha X = \frac{C_\alpha}{k^2 M v^2 \kappa} = \left(\frac{\omega_\alpha r_\alpha}{\omega_r r_r}\right)^2 \frac{1}{\kappa}\left(\frac{b r_r \omega_r}{vk}\right)^2$$

$$\Omega_\beta X = \frac{C_\beta}{k^2 M v^2 \kappa} = \left(\frac{\omega_\beta r_\beta}{\omega_r r_r}\right)^2 \frac{1}{\kappa}\left(\frac{b r_r \omega_r}{vk}\right)^2$$

$$\Omega_h X = \frac{C_h b^2}{k^2 M v^2 \kappa} = \left(\frac{\omega_h}{\omega_r r_r}\right)^2 \frac{1}{\kappa}\left(\frac{b r_r \omega_r}{vk}\right)^2$$

and finally

$$X = \frac{1}{\kappa}\left(\frac{b r_r \omega_r}{vk}\right)^2$$

We are at liberty to introduce the reference parameters ω_r and r_r, and the convention adopted is: ω_r is the last ω in cyclic order in each of the subcases 3, 2, and 1.

Then $\Omega_n = \left(\frac{\omega_n r_n}{\omega_{n+1} r_{n+1}}\right)^2$ and $\Omega_{n+1} = 1$, thus for

Case 3, $\Omega_\alpha = \left(\frac{\omega_\alpha r_\alpha}{\omega_\beta r_\beta}\right)^2$ and $\Omega_\beta = 1$

Case 2, $\Omega_\beta = \left(\frac{\omega_\beta r_\beta}{\omega_h}\right)^2$ and $\Omega_h = 1$

Case 1, $\Omega_h = \left(\frac{\omega_h}{\omega_\alpha r_\alpha}\right)^2$ and $\Omega_\alpha = 1$

To treat the general case of three degrees of freedom (equation (XXI)), it is observed that the real part of the equation is of third degree while the imaginary part furnishes an equation of second degree. The problem is to find values of X satisfying both equations. We shall adopt the following procedure: Plot graphically X against $\frac{1}{k}$ for both equations. The points of intersection are the solutions. We are only concerned with positive values of $\frac{1}{k}$ and positive values of X. Observe that we do not have to solve for k, but may reverse the process by choosing a number of values of k and solve for X. The plotting of X against $\frac{1}{k}$ for the second-degree equation is simple enough, whereas the [illegible] is somewhat more laborious [illegible] equation. However, the general case is of less practical importance than are the three subcases. The equation simplifies considerably, becoming of second degree in X.

We shall now proceed to consider these three subcases. By virtue of the cyclic arrangement, we need only consider the first case (α, β). The complex quadratic equations (XXII)-(XXIV) all resolve themselves into two independent statements, which we shall for convenience denote "Imaginary equation" and "Real equation", the former being of first and the latter of second degree in X. All constants are to be resolved into their real and imaginary parts, denoted by an upper index R or I, respectively.

Let $M_{a\alpha}=M^R_{a\alpha}+iM^I_{a\alpha}$ and let similar expressions denote $M_{b\beta}$ and M_{ch}

Case 3, (α,β). Separating equation (XXII) we obtain.

(1) Imaginary equation:

$$(\Omega_\alpha I_{b\beta}+\Omega_\beta I_{a\alpha})X+M^I_{ch}=0$$

$$X=-\frac{M^I_{ch}}{\Omega_\alpha I_{b\beta}+\Omega_\beta I_{a\alpha}}$$

(2) Real equation:

$$\Omega_\alpha\Omega_\beta X^2+(\Omega_\alpha R_{b\beta}+\Omega_\beta R_{a\alpha})X+M^R_{ch}=0$$

Eliminating X we get

$$\Omega_\alpha\Omega_\beta(M^I_{ch})^2-(\Omega_\alpha R_{b\beta}+\Omega_\beta R_{a\alpha})(\Omega_\alpha I_{b\beta}+\Omega_\beta I_{a\alpha})M^I_{ch}$$
$$+M^R_{ch}(\Omega_\alpha I_{b\beta}+\Omega_\beta I_{a\alpha})^2=0$$

By the convention adopted we have in this case:

$$\omega_r=\omega_\beta, \qquad \Omega_\alpha=\left(\frac{\omega_\alpha}{\omega_\beta}\right)^2\left(\frac{r_\alpha}{r_\beta}\right)^2, \qquad \text{and } \Omega_\beta=1$$

Arranging the equation in powers of Ω_α we have:

$$\Omega_\alpha^2[-M^I_{ch}(R_{b\beta}I_{b\beta})+M^R_{ch}I_{b\beta}^2]+\Omega_\alpha[(M^I_{ch})^2$$
$$-M^I_{ch}(R_{a\alpha}I_{b\beta}+I_{a\alpha}R_{b\beta})+2M^R_{ch}I_{a\alpha}I_{b\beta}]$$
$$+[-M^I_{ch}R_{a\alpha}I_{a\alpha}+M^R_{ch}I_{a\alpha}^2]=0$$

But we have

$$(M^I_{ch})^2-M^I_{ch}(R_{a\alpha}I_{b\beta}+I_{a\alpha}R_{b\beta})$$
$$=M^I_{ch}[R_{a\alpha}I_{b\beta}-R_{a\beta}I_{b\alpha}+R_{b\beta}I_{a\alpha}-R_{b\alpha}I_{a\beta}-R_{a\alpha}I_{b\beta}-R_{b\beta}I_{a\alpha}]$$
$$=-M^I_{ch}(R_{a\beta}I_{b\alpha}+I_{a\beta}R_{b\alpha})$$

Finally, the equation for Case 3 (α, β) becomes:

$$\Omega_\alpha^2(M^R_{ch}I_{b\beta}^2-M^I_{ch}R_{b\beta}I_{b\beta})+\Omega_\alpha[-M^I_{ch}(R_{a\beta}I_{b\alpha}+I_{a\beta}R_{b\alpha})$$
$$+2M^R_{ch}I_{a\alpha}I_{b\beta}]+M^R_{ch}I_{a\alpha}^2-M^I_{ch}R_{a\alpha}I_{a\alpha}=0 \quad \text{(XXV)}$$

where

$$M^R_{ch}=R_{a\alpha}R_{b\beta}-R_{a\beta}R_{b\alpha}-I_{a\alpha}I_{b\beta}+I_{a\beta}I_{b\alpha}$$
$$M^I_{ch}=R_{a\alpha}I_{b\beta}-R_{a\beta}I_{b\alpha}+I_{a\alpha}R_{b\beta}-I_{a\beta}R_{b\alpha}$$

The remaining cases may be obtained by cyclic rearrangement:

Case 2, (β,h) $\qquad \omega_r=\omega_h \qquad \Omega_\beta=\left(\frac{\omega_\beta}{\omega_h}\right)^2 r_\beta^2 \qquad \Omega_h=1$

$$\Omega_\beta^2(M^R_{a\alpha}I_{ch}^2-M^I_{a\alpha}R_{ch}I_{ch})+\Omega_\beta[-M^I_{a\alpha}(R_{bh}I_{c\beta}+I_{bh}R_{c\beta})$$
$$+2M^R_{a\alpha}I_{b\beta}I_{ch}]+M^R_{a\alpha}I_{b\beta}^2-M^I_{a\alpha}R_{b\beta}I_{b\beta}=0 \quad (XXVI)$$

where
$$M^R_{a\alpha}=R_{b\beta}R_{ch}-R_{bh}R_{c\beta}-I_{b\beta}I_{ch}+I_{bh}I_{c\beta}$$
$$M^I_{a\alpha}=R_{b\beta}I_{ch}-R_{bh}I_{c\beta}+I_{b\beta}R_{ch}-I_{bh}R_{c\beta}$$

Case 1, (h,α) $\qquad \omega_r=\omega_\alpha \qquad \Omega_h=\left(\frac{\omega_h}{\omega_\alpha}\right)^2\frac{1}{r_\alpha^2} \qquad \Omega_\alpha=1$

$$\Omega_h^2(M^R_{b\beta}I_{a\alpha}^2-M^I_{b\beta}R_{a\alpha}I_{a\alpha})+\Omega_h[-M^I_{b\beta}(R_{c\alpha}I_{ah}+I_{c\alpha}R_{ah})$$
$$+2M^R_{b\beta}I_{ch}I_{a\alpha}]+M^R_{b\beta}I_{ch}^2-M^I_{b\beta}R_{ch}I_{ch}=0 \quad (XXVII)$$

where
$$M^R_{b\beta}=R_{ch}R_{a\alpha}-R_{c\alpha}R_{ah}-I_{ch}I_{a\alpha}+I_{c\alpha}I_{ah}$$
$$M^I_{b\beta}=R_{ch}I_{a\alpha}-R_{c\alpha}I_{ah}+I_{ch}R_{a\alpha}-I_{c\alpha}R_{ah}$$

Equations (XXV), (XXVI), and (XXVII) thus give the solutions of the cases: *torsion-aileron*, *aileron-deflection*, and *deflection-torsion*, respectively. The quantity Ω may immediately be plotted against $\frac{1}{k}$ for any value of the independent parameters. The coefficients in the equations are all given in terms of R and I, which quantities have been defined above. Routine calculations and graphs giving Ω against $\frac{1}{k}$ are contained in Appendix I and Appendix II. Knowing related values of Ω and $\frac{1}{k}$, X is immediately expressed as a function of Ω by means of the first-degree equation. The definition of X and Ω for each subcase is given above. The cyclic arrangement of all quantities is very convenient as it permits identical treatment of the three subcases.

It shall finally be repeated that the above solutions represent the *border case* of unstable equilibrium. The plot of X against Ω gives a boundary curve between the stable and the unstable regions in the $X\Omega$ plane. It is preferable, however, to plot the quantity $\frac{1}{k^2}\frac{1}{X}$ instead of X, since this quantity is proportional to the square of the flutter speed. The stable area can easily be identified by inspection as it will contain the axis $\frac{1}{k^2}\frac{1}{X}=0$, if the combination is stable for zero velocity.

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LANGLEY FIELD, VA., *May 2, 1934.*

APPENDIX I

PROCEDURE IN SOLVING NUMERICAL EXAMPLES

(1) Determine the R's and I's, nine of each for a major case of three degrees of freedom, or those pertaining to a particular subcase, 4 R's and 4 I's. Refer to the following for the R's and I's involved in each case:

The numerals 1 to 9 and 11 to 19 are used for convenience.

(Major case) Three degrees of freedom

1	$R_{a\alpha}$	$I_{a\alpha}$	11
2	$R_{a\beta}$	$I_{a\beta}$	12
3	R_{ah}	I_{ah}	13
4	$R_{b\alpha}$	$I_{b\alpha}$	14
5	$R_{b\beta}$	$I_{b\beta}$	15
6	R_{bh}	I_{bh}	16
7	$R_{c\alpha}$	$I_{c\alpha}$	17
8	$R_{c\beta}$	$I_{c\beta}$	18
9	R_{ch}	I_{ch}	19

(Case 3) Torsional-aileron (α, β)

1	$R_{a\alpha}$	$I_{a\alpha}$	11
2	$R_{a\beta}$	$I_{a\beta}$	12
4	$R_{b\alpha}$	$I_{b\alpha}$	14
5	$R_{b\beta}$	$I_{b\beta}$	15

(Case 2) Aileron-deflection (β, h)

5	$R_{b\beta}$	$I_{b\beta}$	15
6	R_{bh}	I_{bh}	16
8	$R_{c\beta}$	$I_{c\beta}$	18
9	R_{ch}	I_{ch}	19

(Case 1) Deflection-torsion (h, α)

7	$R_{c\alpha}$	$I_{c\alpha}$	17
9	R_{ch}	I_{ch}	19
1	$R_{a\alpha}$	$I_{a\alpha}$	11
3	R_{ah}	I_{ah}	13

It has been found convenient to split the R's in two parts $R=R'+R''$, the former being independent of the argument $\frac{1}{k}$. The quantities I and R'' are functions of the two independent parameters a and c only.[5] The formulas are given in the following list.

$$R''_{a\alpha}=\frac{1}{k}2\left(a+\frac{1}{2}\right)\left\{\left(\frac{1}{2}-a\right)G-\frac{F}{k}\right\} \tag{1}$$

$$R''_{a\beta}=\frac{1}{k}\frac{1}{\pi}\left\{(T_4+T_{10})\frac{1}{k}+\left(a+\frac{1}{2}\right)\left(T_{11}G-\frac{2}{k}T_{10}F\right)\right\} \tag{2}$$

$$R''_{ah}=\frac{1}{k}2\left(a+\frac{1}{2}\right)G \tag{3}$$

$$R''_{b\alpha}=-\frac{1}{k}\frac{T_{12}}{\pi}\left\{\left(\frac{1}{2}-a\right)G-\frac{F}{k}\right\} \tag{4}$$

$$R''_{b\beta}=-\frac{1}{k}\frac{1}{\pi^2}\left\{\frac{T_{12}}{2}\left(T_{11}G-\frac{2}{k}T_{10}F\right)-\frac{1}{k}(T_5-T_4T_{10})\right\} \tag{5}$$

$$R''_{bh}=-\frac{1}{k}\frac{T_{12}}{\pi}G \tag{6}$$

$$R''_{c\alpha}=-\frac{1}{k}2\left\{\left(\frac{1}{2}-a\right)G-\frac{F}{k}\right\} \tag{7}$$

$$R''_{c\beta}=-\frac{1}{k}\frac{1}{\pi}\left(T_{11}G-2T_{10}\frac{F}{k}\right) \tag{8}$$

$$R''_{ch}=-\frac{1}{k}2G \tag{9}$$

$$I_{a\alpha}=-2\left(a+\frac{1}{2}\right)\left\{\left(\frac{1}{2}-a\right)F+\frac{1}{k}G\right\}+\frac{1}{2}-a \tag{11}$$

$$I_{a\beta}=-\frac{1}{\pi}\left\{\left(a+\frac{1}{2}\right)\left(T_{11}F+\frac{2}{k}T_{10}G\right)+2p+\left(\frac{1}{2}-a\right)T_4\right\} \tag{12}$$

$$I_{ah}=-2\left(a+\frac{1}{2}\right)F \tag{13}$$

$$I_{b\alpha}=\frac{T_{12}}{\pi}\left\{\left(\frac{1}{2}-a\right)F+\frac{1}{k}G\right\}+\frac{1}{\pi}\left(p-T_1-\frac{1}{2}T_4\right) \tag{14}$$

Where $p=-\frac{1}{3}(1-c^2)^{3/2}$

$$I_{b\beta}=\frac{1}{2\pi^2}\left\{T_{12}\left(T_{11}F+\frac{2}{k}T_{10}G\right)-T_4T_{11}\right\} \tag{15}$$

$$I_{bh}=\frac{T_{12}}{\pi}F \tag{16}$$

$$I_{c\alpha}=2\left\{\left(\frac{1}{2}-a\right)F+\frac{1}{k}G\right\}+1 \tag{17}$$

$$I_{c\beta}=\frac{1}{\pi}\left\{\left(T_{11}F+\frac{2}{k}T_{10}G\right)-T_4\right\} \tag{18}$$

$$I_{ch}=2F \tag{19}$$

[5] The quantities I given in the appendix and used in the following calculations are seen to differ from the I's given in the body of the paper by the factor $\frac{1}{k}$. It may be noticed that this factor drops out in the first-degree equations.

Choosing certain values of a and c and employing the values of the T's given by the formulas of the report (p. 5) or in table I and also using the values of F and G (formulas (XII) and (XIII)) or table II, we evaluate the quantities I and R'' for a certain number of $\frac{1}{k}$ values. The results of this evaluation are given in tables III and IV, which have been worked out for $a=0, -0.2$, and -0.4, and for $c=0.5$ and $c=0$. The range of $\frac{1}{k}$ is from 0 to 40. These tables save the work of calculating the I's and R'''s for almost all cases of practical importance. Interpolation may be used for intermediate values. This leaves the quantities R' to be determined. These, being independent of $\frac{1}{k}$, are as a result easy to obtain. Their values, using the same system of numbers for identification, and referring to the definition of the original independent variables on pages 9 and 10, are given as follows:

$$R'_{a\alpha} = -\frac{r_\alpha^2}{\kappa} - \left(\frac{1}{8} + a^2\right) \tag{1}$$

$$R'_{a\beta} = -\frac{r_\beta^2}{\kappa} - (c-a)\frac{x_\beta}{\kappa} + \frac{T_7}{\pi} + (c-a)\frac{T_1}{\pi} \tag{2}$$

$$R'_{ah} = -\frac{x_\alpha}{\kappa} + a \tag{3}$$

$$R'_{b\alpha} = \text{same as } R'_{a\beta} \tag{4}$$

$$R'_{b\beta} = -\frac{r_\beta^2}{\kappa} + \frac{1}{\pi^2}T_3 \tag{5}$$

$$R'_{bh} = -\frac{x_\beta}{\kappa} + \frac{1}{\pi}T_1 \tag{6}$$

$$R'_{c\alpha} = \text{same as } R'_{ah} \tag{7}$$

$$R'_{c\beta} = \text{same as } R'_{bh} \tag{8}$$

$$R'_{ch} = -\frac{1}{\kappa} - 1 \tag{9}$$

Because of the symmetrical arrangement in the determinant, the 9 quantities are seen to reduce to 6 quantities to be calculated. It is very fortunate, indeed, that all the remaining variables segregate themselves in the 6 values of R' which are independent of $\frac{1}{k}$, while the more complicated I and R'' are functions solely of c and a. In order to solve any problem it is therefore only necessary to refer to tables III and IV and then to calculate the 6 values of R'.

The quantities (1) to (9) and (11) to (19) thus having been determined, the plot of Ω against $\frac{1}{k}$, which constitutes our method of solution, is obtained by solving the equation $a\Omega^2+b\Omega+c=0$. The constants a, b, and c are obtained automatically by computation according to the following scheme:

Case 3

Find products 1.5 2.4 11.15 12.14

Then $M^R_{ch} = 1.5 - 2.4 - \frac{1}{k^2}(11.15 - 12.14)$

Find products 1.15 2.14 11.5 12.4

Then $M^I_{ch} = 1.15 - 2.14 + 11.5 - 12.4$

and $a = M^R_{ch}(15)^2 - M^I_{ch}(5.15)$

$b = -M^I_{ch}(2.14 + 12.4) + 2M^R_{ch}(11.15)$

$c = M^R_{ch}(11)^2 - M^I_{ch}(1.11)$ Find Ω_α

Solution: $\frac{1}{X} = -\frac{\Omega_\alpha(15)+11}{M^I_{ch}}$

Similarly

Case 2

5.9 6.8 15.19 16.18

$M^R_{a\alpha} = 5.9 - 6.8 - \frac{1}{k^2}(15.19 - 16.18)$

5.19 6.18 15.9 16.8

$M^I_{a\alpha} = 5.19 - 6.18 + 15.9 - 16.8$

$a = M^R_{a\alpha}(19)^2 - M^I_{a\alpha}(9.19)$

$b = -M^I_{a\alpha}(6.18 + 16.8)$

$+ 2M^R_{a\alpha}(6.18 + 16.8)$

$c = M^R_{a\alpha}(15)^2 - M^I_{a\alpha}(5.15)$ Find Ω_β

$\frac{1}{X} = \frac{\Omega_\beta(19)+15}{M^I_{a\alpha}}$

and

Case 1

9.1 7.3 19.11 17.13

$M^R_{b\beta} = 9.1 - 7.3 - \frac{1}{k^2}(19.11 - 17.13)$

9.11 7.13 19.1 17.3

$M^I_{b\beta} = 9.11 - 7.13 + 19.1 - 17.3$

$a = M^R_{b\beta}(11)^2 - M^I_{b\beta}(1.11)$

$b = -M^I_{b\beta}(7.13 + 17.3) + 2M^R_{b\beta}(19.11)$

$c = M^R_{b\beta}(19)^2 - M^I_{b\beta}(9.19)$ Find Ω_h

$\frac{1}{X} = \frac{\Omega_h(11)+19}{M^I_{b\beta}}$

Ω_α is defined as $\left(\frac{\omega_\alpha r_\alpha}{\omega_\beta r_\beta}\right)^2$ for case 3;

Ω_β is defined as $\left(\frac{\omega_\beta r_\beta}{\omega_h}\right)^2$ for case 2; and

Ω_h is defined as $\left(\frac{\omega_h}{\omega_\alpha r_\alpha}\right)^2$ for case 1.

The quantity $\frac{1}{X}$ is $\kappa\left(\frac{vk}{b\omega_r r_r}\right)^2$ by definition.

Since both Ω and $\frac{1}{X}$ are calculated for each value of $\frac{1}{k}$, we may plot $\frac{1}{k^2}\frac{1}{X}$ directly as a function of Ω. This quantity, which is proportional to the square of the flutter speed, represent[illegible] [illegible] Ω.

We shall [illegible] root of the above quantity, viz, $\frac{1}{k}\sqrt{\frac{1}{X}} = \frac{v\sqrt{\kappa}}{b\omega_r r_r}$, and will denote this quantity by F, which we shall term the "flutter factor." The flutter velocity is consequently obtained as

$$v=F\frac{b\omega_r r_r}{\sqrt{\kappa}}$$

Since F is nondimensional, the quantity $\frac{b\omega_r r_r}{\sqrt{\kappa}}$ must obviously be a velocity. It is useful to establish the significance of this velocity, with reference to which the flutter speed, so to speak, is measured. Observing that $\kappa=\frac{\pi\rho b^2}{M}$ and that the stiffness in case 1 is given by $\omega_\alpha=\sqrt{\frac{C_\alpha}{Mb^2r_\alpha^2}}$ this reference velocity may be written:

$$v_R=\frac{b\omega_\alpha r_\alpha}{\sqrt{\kappa}}=\frac{1}{b}\sqrt{\frac{C_\alpha}{\pi\rho}}\text{ or}$$

$$\pi\rho v_R^2b^2=C_\alpha$$

The velocity v_R is thus the velocity at which the total force on the airfoil $\pi\rho v_R^2 2b$ attacking with an arm $\frac{b}{2}$ equals the torsional stiffness C_α of the wing. This statement means, in case 1, that the reference velocity used is equal to the "divergence" velocity obtained with the torsional axis in the middle of the chord. This velocity is considerably smaller than the usual divergence velocity, which may be expressed as

$$v_D=v_R\frac{1}{\frac{1}{2}+a}$$

where a ranges from 0 to $-\frac{1}{2}$. We may thus express the flutter velocity as

$$v_F=v_RF$$

In case 3 the reference velocity has a similar significance, that is, it is the velocity at which the entire lift of the airfoil attacking with a leverage $\frac{1}{2}b$ equals numerically the torsional stiffness C_β of the aileron or movable tail surface.

In case 2, no suitable or useful significance of the reference velocity is available.

TABLE I.—VALUES OF T

	$c=1$	$c=\frac{1}{2}$	$c=0$	$c=-\frac{1}{2}$	$c=-1$
T_1	0	−0.1259	−0.6967	−1.6967	−3.1416
T_2	0	−0.2103	−1.5707	−4.8356	−9.8697
T_3	0	−.05313	−.8084	−3.8375	−11.1034
T_4	0	−.6142	−1.5708	−1.6614	−3.1416
T_5	0	−.9398	−3.4674	−6.9503	−9.8697
T_6	0	−0.2103	−1.5707	−4.8356	−9.8697
T_7	0	.0132	−.1964	−1.1913	−3.5343
T_8	0	.0903	−.3333	−1.4805	−3.1416
T_{10}	0	1.9132	2.5708	2.9604	3.1416
T_{11}	0	1.2990	3.5708	6.3538	9.4248
T_{12}	0	.07066	.4292	1.2990	3.1416

TABLE II.—TABLE OF THE BESSEL FUNCTIONS J_0, J_1 Y_0, Y_1 AND THE FUNCTIONS F AND G

$$F(k)=\frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

$$-G(k)=\frac{Y_1(J_1+Y_0)-J_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

k	$\frac{1}{k}$	J_0	J_1	Y_0	Y_1	F	$-G$
∞	0	--------	--------	--------	--------	0.5000	0
10	1/10	−0.2459	0.0435	0.0557	0.2490	.5006	0.0126
6	1/6	.1506	−.2767	−.2882	−.1750	.5018	.0207
4	1/4	−.3972	−.0660	−.0170	.3979	.5037	.0305
2	1/2	.2239	.5767	.5104	−.1071	.5129	.0577
1	1	.7652	.4401	.0882	−.7813	.5395	.1003
.8	1 1/4	.8463	.3688	−.0868	−.9780	.5541	.1165
.6	1 2/3	.9120	.2867	−.3085	−1.2604	.5788	.1378
.5	2	.9385	.2423	−.4444	−1.4714	.6030	.151
.4	2 1/2	.9604	.1960	−.6060	−1.7808	.6245	.166
.3	3 1/3	.9776	.1483	−.8072	−2.2929	.6650	.180
.2	5	.9900	.0995	−1.0810	−3.3235	.7276	.1886
.1	10	.9975	.0499	−1.5342	−7.0317	.8457	.1626
.05	20	--------	--------	--------	--------	.911	.132
.025	40	--------	--------	--------	--------	.965	.090
0	∞	--------	--------	--------	--------	1.000	0

TABLE III.—VALUES OF R

$\frac{1}{k}$	c	a	0	1/10	1/6	1/4	1/2	1	1 1/4	1 2/3	2	2 1/2	3 1/3	5	10	20	40
$R''_{a\alpha}$	(1)	0	0	−0.00564	−0.01566	−0.03529	−0.14265	−0.58965	−0.93656	−1.72330	−2.56300	−4.11000	−7.68720	−18.66150	−85.38300	−365.72000	−1,528.2000
		−.2	0	−.00353	−.00981	−.02208	−.08905	−.36586	−.58061	−1.06158	−1.57400	−2.51580	−4.68430	−11.31010	−51.42490	−219.74900	−917.3520
		−.4	0	−.00123	−.00341	−.00767	−.03084	−.12595	−.19936	−.36305	−.53676	−.85520	−1.58540	−3.80774	−17.20670	−73.35520	−305.9280
$R''_{a\beta}$	0	0	0	−.00163	−.00452	−.01020	−.04175	−.18016	−.29384	−.56223	−.87212	−1.43983	−2.84988	−7.46300	−38.29650	−172.36360	−741.7972
		−.2	0	.00030	.00083	.00184	.00679	.01922	.02266	.01629	−.01400	−.06803	−.29517	−1.29480	−10.24590	−52.49020	−241.3664
		−.4	0	.00222	.00617	.01388	.05531	.21861	.33914	.59499	.84414	1.30365	2.25914	4.87340	17.80470	67.38320	259.0648
	0.5	0	0	.00083	.00229	.00510	.01932	.06419	.08876	.12176	.12260	.12205	−.02900	−.93535	−10.48970	−59.16180	−268.7236
		−.2	0	.00214	.00595	.01336	.05278	.20325	.31065	.53062	.73222	1.10233	1.81135	3.55230	10.14740	31.49620	101.6340
		−.4	0	.00347	.00965	.02170	.08656	.34361	.53463	.94336	1.34762	2.09190	3.66913	8.08235	30.97980	120.89760	475.2592
R''_{ah}	(1)	0	0	−.00125	−.00345	−.00763	−.02890	−.10030	−.14560	−.22470	−.30200	−.41500	−.60000	−.94300	−1.62600	−2.64000	−3.6000
		−.2	0	−.00075	−.00207	−.00426	−.01734	−.06018	−.08736	−.13482	−.18120	−.24900	−.36000	−.56580	−.97560	−1.58400	−2.1600
		−.4	0	−.00201	−.00334	−.00502	−.01003	−.02006	−.02508	−.03236	−.04012	−.05015	−.06683	−.10030	−.20060	−.40120	−.8024
$R''_{b\alpha}$	0	0	0	.00077	.00214	.00482	.01949	.08055	.12821	.23541	.35010	.56143	1.05008	2.54920	11.66330	49.95700	208.7520
		−.2	0	.00080	.00223	.00503	.02027	.08329	.13219	.24169	.35836	.57276	1.06650	2.57490	11.70770	50.03000	208.8500
		−.4	0	.00084	.00233	.00523	.02106	.08603	.13616	.24796	.36661	.58410	1.08286	2.60069	11.75220	50.10160	208.9490
	0.5	0	0	.00013	.00035	.00079	.00321	.01327	.02112	.03878	.05767	.09248	.17296	.41988	1.92110	8.22870	34.3850
		−.2	0	.00013	.00037	.00083	.00334	.01372	.02177	.03981	.05903	.09434	.17566	.42413	1.92840	8.24060	34.4007
		−.4	0	.00014	.00038	.00086	.00347	.01417	.02243	.04084	.06039	.09621	.17836	.42837	1.93575	8.25246	34.4169
$R''_{b\beta}$	0	(2)	0	.00124	.00343	.00772	.03101	.12642	.19830	.35807	.52400	.82930	1.51630	3.54970	15.35120	64.02240	263.2340
	.5		0	.00031	.00087	.00196	.00785	.03170	.04980	.08935	.13000	.20440	.36940	.84970	3.55050	14.56740	59.3188
R''_{bh}	0	(2)	0	.00017	.00047	.00104	.00394	.01370	.01989	.03177	.04125	.05669	.08196	.12881	.22211	.36062	.4918
	.5		0	.00003	.00008	.00016	.00065	.00226	.00328	.00506	.00680	.00934	.01350	.02122	.03659	.05940	.0810
$R''_{c\alpha}$	(1)	0	0	.01128	.03132	.07058	.28530	1.17930	1.87710	3.44670	5.12600	8.22000	15.37450	37.32300	170.76600	731.44000	3,056.4000
		−.2	0	.01178	.03270	.07362	.29684	1.21954	1.93540	3.53860	5.24680	8.38600	15.61440	37.70020	171.41640	732.49600	3,057.8400
		−.4	0	.01228	.03408	.07668	.30838	1.25950	1.99360	[illegible]	5.36760	8.55200	15.85440	38.07740	172.06680	733.55200	3,059.2800
$R''_{c\beta}$	0	(2)	0	−.00963	−.02673	−.06018	−.24266	−1.00561	−1.58246	−2.89371	−4.29100	−6.85898	−15.49965	−30.84330	−140.26370	−599.41300	−2,502.3470
	.5		0	.00660	.01840	.04150	.16810	.69850	1.11453	2.05320	3.06224	4.92530	9.24438	22.54400	103.67300	444.86400	1,881.9900
R''_{ch}	(1)	(2)	0	.00250	.00690	.01420	.05780	.20060	.29120	.44940	.60400	.83000	1.20000	1.88600	3.25200	5.28000	7.2000

[1] Independent of c. [2] Independent of a.

TABLE IV.—VALUES OF I

$\frac{1}{k}$	c	a	0	1/10	1/6	1/4	1/2	1	1 1/4	1 2/3	2	2 1/2	3 1/3	5	10	20	40
$I_{a\alpha}$	(1)	0	0.25000	0.25096	0.25255	0.25578	0.27240	0.33055	0.36855	0.44030	0.50050	0.60275	0.76750	1.07920	1.70320	2.68450	3.61750
		−.2	.49000	.49050	.49131	.49302	.50189	.53359	.55464	.59472	.62794	.68671	.78070	.96021	1.32040	1.90140	2.45470
		−.4	.81000	.81014	.81037	.81086	.81145	.82395	.82938	.84176	.85186	.87059	.90030	.95763	1.07300	1.26400	1.44630
$I_{a\beta}$	0	0	.17805	.17874	.17985	.18219	.19433	.23768	.26645	.32132	.36664	.44690	.57526	.82035	1.31213	2.10476	2.85963
		−.2	.39170	.39212	.39278	.39418	.40147	.42748	.44474	.47761	.50485	.55300	.63002	.77708	1.07215	1.54773	2.00065
		−.4	.60531	.60545	.60567	.60614	.60857	.61724	.62299	.63395	.64303	.65908	.68475	.73377	.82313	.99065	1.14163
	0.5	0	.13252	.13317	.13425	.13640	.14742	.18544	.20914	.25611	.29514	.35951	.46379	.65973	1.05124	1.65524	2.22869
		−.2	.21297	.21336	.21401	.21530	.22191	.24472	.25894	.28712	.31054	.34916	.41173	.52929	.76420	1.12651	1.47067
		−.4	.29342	.29354	.29376	.29419	.29640	.30400	.30891	.31813	.32594	.33881	.35966	.39884	.47714	.59792	.71260
I_{ah}	(1)	0	−.50000	−.50060	−.50180	−.50370	−.51290	−.53950	−.55410	−.57880	−.60300	−.62450	−.66500	−.72760	−.84570	−.94100	−.96500
		−.2	−.30000	−.30036	−.30108	−.30222	−.30774	−.32370	−.33246	−.34728	−.36180	−.37470	−.39900	−.43656	−.50752	−.56460	−.57900
		−.4	−.10000	−.10012	−.10036	−.10074	−.10258	−.10790	−.11082	−.11576	−.12060	−.12490	−.13300	−.14552	−.16914	−.18220	−.19300
$I_{b\alpha}$	0	0	.39023	.39010	.38988	.38944	.38717	.37923	.37404	.36424	.35601	.34204	.31954	.27696	.19172	.05766	−.06980
		−.2	.40389	.40378	.40359	.40320	.40119	.39397	.38918	.38005	.37249	.35911	.33771	.29683	.21469	.08255	−.04344
		−.4	.41755	.41746	.41730	.41697	.41520	.40871	.40432	.39586	.38896	.37617	.35598	.31671	.23793	.10744	−.01707
	0.5	0	.07438	.07435	.07433	.07424	.07387	.07256	.07171	.07009	.06874	.06644	.06273	.05572	.04168	.01960	−.00327
		−.2	.07663	.07661	.07658	.07651	.07618	.07499	.07420	.07270	.07145	.06925	.06572	.05899	.04548	.02370	.00295
		−.4	.07887	.07885	.07882	.07867	.07848	.07741	.07668	.07529	.07416	.07205	.06871	.06226	.04928	.02779	.00728
$I_{b\beta}$	0	(2)	.32297	.32288	.32273	.32241	.32075	.31483	.31090	.30342	.29721	.28625	.26872	.23524	.16806	.05979	−.04333
	.5		.04270	.04270	.04270	.04270	.04240	.04150	.04095	.03930	.03904	.03760	.03386	.03080	.02200	.00845	−.00470
I_{bh}	0	(2)	.06830	.06840	.06850	.06880	.07010	.07370	.07570	.07910	.08240	.08530	.09080	.09940	.11550	.12440	.13180
	.5		.01125	.01126	.01129	.01133	.01154	.01214	.01247	.01302	.01357	.01405	.01496	.01637	.01903	.02117	.02171
$I_{c\alpha}$	(1)	0	1.50000	1.49808	1.49490	1.48844	1.45520	1.33890	1.26290	1.11940	.99900	.74950	.46500	−.15840	−1.40630	−3.36900	−5.23500
		−.2	1.70000	1.69832	1.69562	1.68992	1.66036	1.55470	1.48454	1.35092	1.24020	1.04430	.73100	.13264	−1.06802	−3.00460	−4.84900
		−.4	1.90000	1.89856	1.89634	1.89140	1.88552	1.77050	1.70618	1.58244	1.48410	1.31410	.99700	.42370	−.72974	−2.64020	−4.46300
$I_{c\beta}$	0	(2)	1.06830	1.06690	1.06470	1.06000	1.03580	.94900	.89150	.78190	.69110	.52840	.27380	−.21640	−1.20010	−2.78550	−4.29530
	.5		.40220	.40100	.39880	.39450	.37240	.29640	.24640	.15510	.07610	−.05180	−.26030	−.65220	−1.43520	−2.64380	−3.79010
I_{ch}	(1)	(2)	1.00000	1.00120	1.00360	1.00740	1.02580	1.07900	1.10820	1.15760	1.20600	1.24900	1.33000	1.45520	1.69140	1.82200	1.93000

[1] Independent of c. [2] Independent of a.

APPENDIX II

NUMERICAL CALCULATIONS

A number of routine examples have been worked out to illustrate typical results. A "standard" case has been chosen, represented by the following constants:

$$\kappa=0.1,\ c=0.5,\ a=-0.4,\ x_\alpha=0.2,$$

$$r_\alpha^2=0.25,\ x_\beta=\frac{1}{80},\ r_\beta^2=\frac{1}{160}$$

$$\omega_\alpha,\ \omega_\beta,\ \omega_h \text{ variable.}$$

We will show the results of a numerical computation of the three possible subcases in succession.

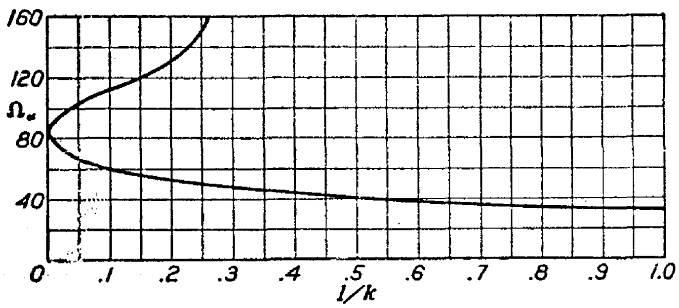

FIGURE 5.—Case 3, Torsion-aileron (α,β): Standard case. Showing Ω_α against $\frac{1}{k}$.

Case 3, Torsion-aileron (α,β): Figure 5 shows the Ω_α against $\frac{1}{k}$ relation and figure 6 the final curve

$$F=\kappa\left(\frac{v}{\omega_\beta r_\beta b}\right)^2 \text{against } \Omega_\alpha=\left(\frac{\omega_\alpha r_\alpha}{\omega_\beta r_\beta}\right)^2=40\left(\frac{\omega_\alpha}{\omega_\beta}\right)^2$$

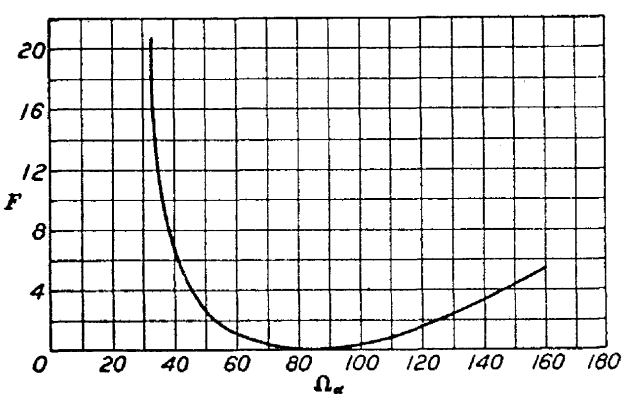

FIGURE 6.—Case 3, Torsion-aileron (α,β): Standard case. Showing flutter factor F against Ω_α.

Case 2, Aileron-flexure (β, h): Figure 7 shows the Ω_β against $\frac{1}{k}$ relation[6] and figure 8 the final curve $\kappa\left(\frac{v}{\omega_h b}\right)^2$ against $\Omega_\beta=\left(\frac{\omega_\beta r_\beta}{\omega_h}\right)^2=\frac{1}{160}\left(\frac{\omega_\beta}{\omega_h}\right)^2$

[6] It is realized that considerable care must be exercised to get these curves reasonably accurate.

The heavy line shows the standard case, while the remaining curves show the effect of a change in the value of x_β to $\frac{1}{40}$ and $\frac{1}{160}$.

Case 1, Flexure-torsion (h, α): Figure 9 shows again

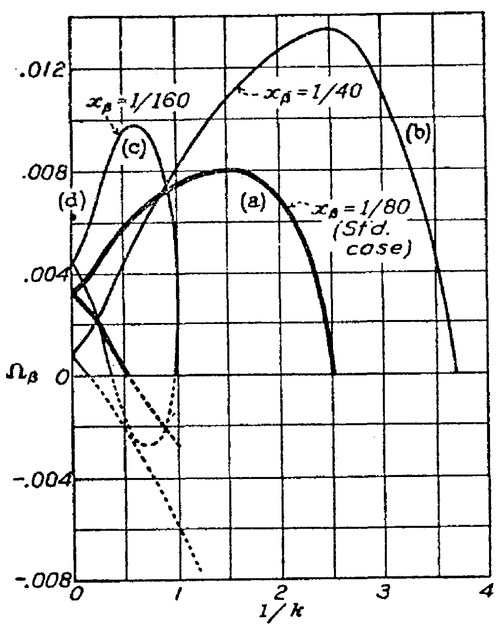

FIGURE 7.—Case 2, Aileron-deflection (β, h): (a) Standard case. (b), (c), (d) indicate dependency on x_β. Case (d), $x_\beta=-0.004$, reduces to a point.

the Ω_h against $\frac{1}{k}$ relation and figure 10 the final result

$$\kappa\left(\frac{v}{\omega_\alpha r_\alpha b}\right)^2 \text{ against } \Omega_h=\left(\frac{\omega_h}{\omega_\alpha r_\alpha}\right)^2=4\left(\frac{\omega_h}{\omega_\alpha}\right)^2$$

Case 1, which is of importance in the propeller theory, has been treated in more detail. The quantity F shown in the figures is $\sqrt{\kappa}\ \frac{v}{\omega_\alpha r_\alpha b}$.

Figure 11 shows the dependency on $\frac{\omega_h}{\omega_\alpha}=\frac{\omega_1}{\omega_2}$; figure 12 shows the dependency on the location of the axis a; figure 13 shows the dependency on the radius of gyration $r_\alpha=r$; and figure 14 shows the dependency on the location of the center of gravity x, for three different combinations of constants.

EXPERIMENTAL RESULTS

Detailed discussion of the experimental work will not be given in this paper, but shall be reserved for a later report. The experiments given in the following are restricted to wings of a large aspect ratio, arranged with two or three degrees of freedom in accordance with the

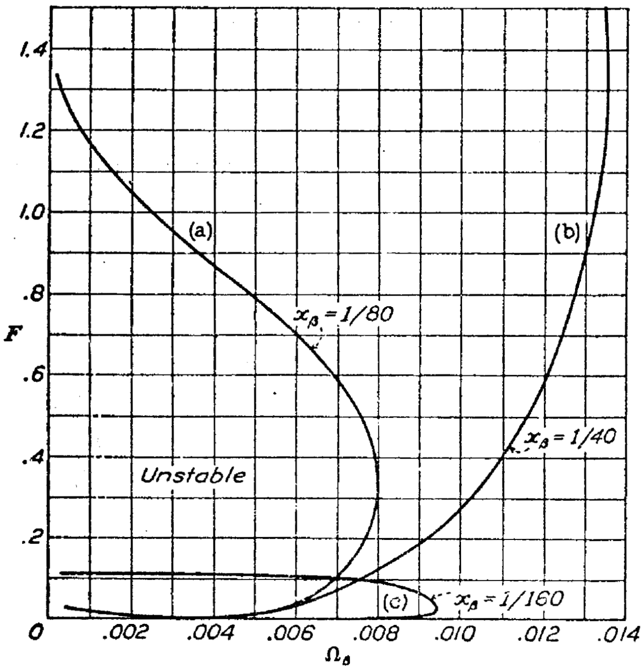

FIGURE 8.—Case 2, Aileron-deflection (β, h): Final curves giving flutter factor F against Ω_β corresponding to cases shown in figure 7.

theoretical cases. The wing is free to move parallel to itself in a vertical direction (h); is equipped with an

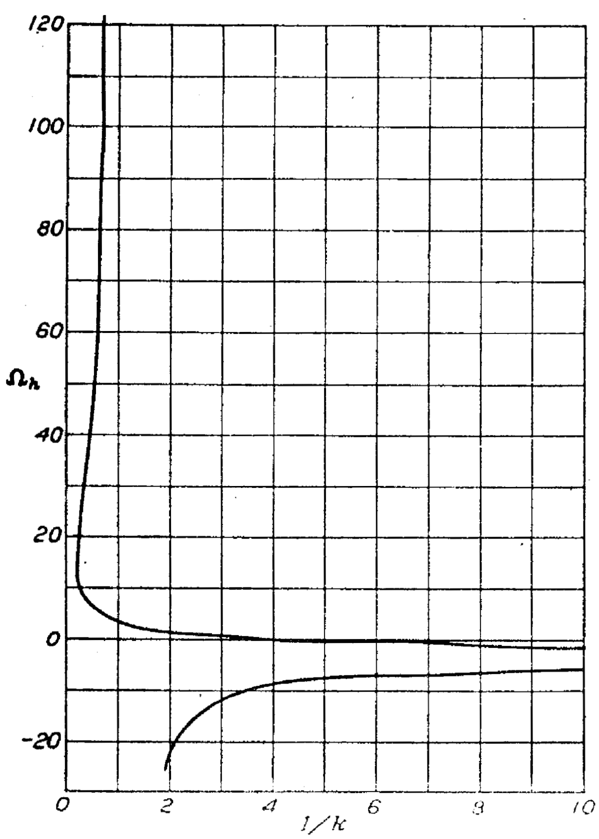

FIGURE 9.—Case 1, Flexure-torsion (h, α): Standard case. Showing Ω_h against $\frac{1}{k}$.

axis in roller bearings at (a) (fig. 2) for torsion, and with an aileron hinged at (c). Variable or exchangeable springs restrain the wing to its equilibrium position.

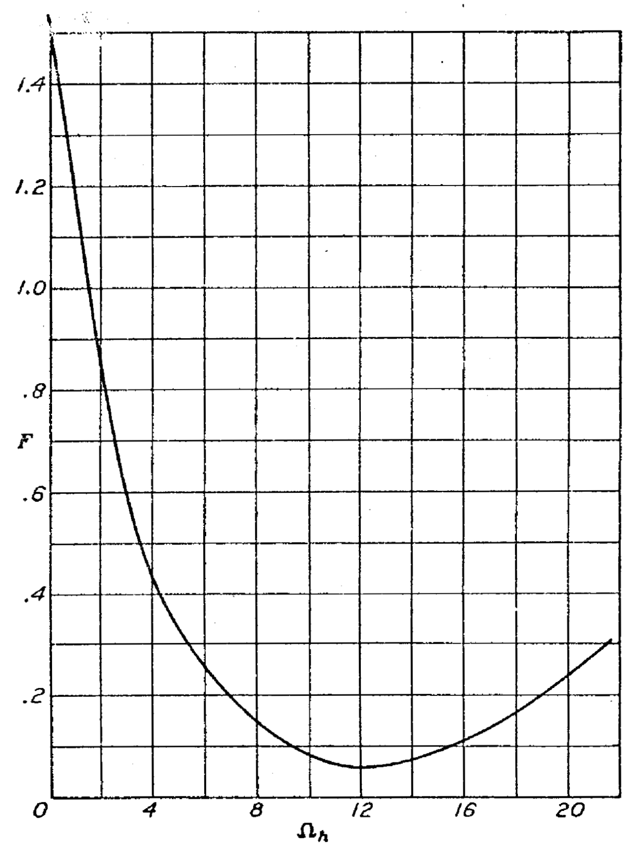

FIGURE 10.—Case 1, Flexure-torsion (h, α): Standard case. Showing flutter factor F against Ω_h.

We shall present results obtained on two wings, both of symmetrical cross section 12 percent thick, and with chord $2b = 12.7$ cm, tested at 0°.

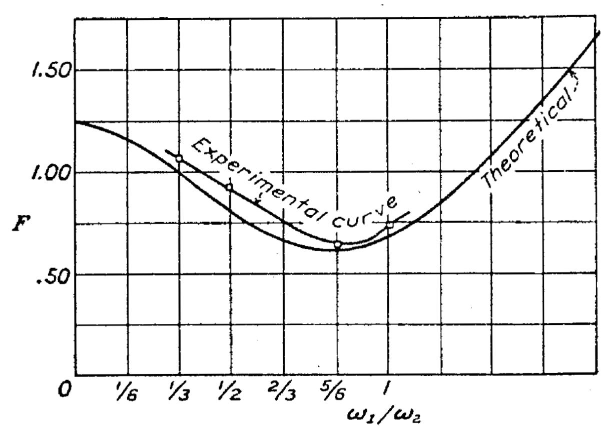

FIGURE 11.—Case 1, Flexure-torsion (h, α): Showing dependency of F on $\frac{\omega_h}{\omega_\alpha}$. The upper curve is experimental. Airfoil with $r = \frac{1}{2}$; $a = -0.4$; $x = 0.2$; $4\kappa = .01$; $\frac{\omega_1}{\omega_2}$ variable.

Wing A, aluminum, with the following constants:

$$\kappa = \frac{1}{416},\ a = -0.4,\ x_\alpha = \text{[illegible]},\ 0.173, \text{ and } 0.038,$$

respectively;

$$r_\alpha^2 = 0.33 \text{ and } \omega_\alpha = 7 \times 2\pi$$

Wing B, wood, with flap, and the constants:

$\kappa=\frac{1}{100}$, $c=0.5$, $a=-0.4$, $x_\alpha=0.192$, $r_\alpha^2=0.178$, $x_\beta=0.019$, $r_\beta^2=0.0079$, and ω_α kept constant $=17.6\times 2\pi$

The results for wing A, case 1, are given in figure 15; and those for wing B, cases 2 and 3, are given in figures 16 and 17, respectively. The abscissas are the frequency ratios and the ordinates are the velocities in cm/sec. Compared with the theoretical results calculated for the three test cases, there is an almost perfect

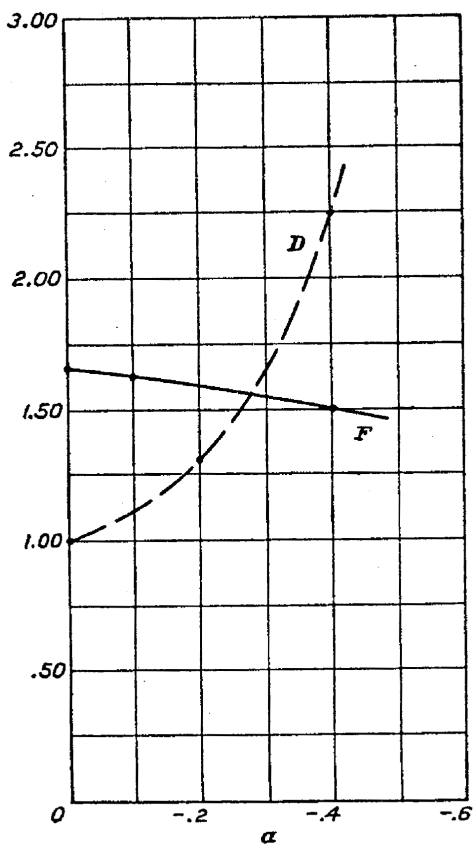

FIGURE 12.—Case 1, Flexure-torsion (h, α): Showing dependency of F on location of axis of rotation a. Airfoil with $r=\frac{1}{2}$; $x=0.2$; $\kappa=\frac{1}{4}$; $\frac{\omega_1}{\omega_2}=\frac{1}{6}$; a variable.

agreement in case 1 (fig. 15). Not only is the minimum velocity found near the same frequency ratio, but the experimental and theoretical values are, furthermore, very nearly alike. Very important is also the fact that the peculiar shape of the response curve in case 2, predicted by the theory, repeats itself experimentally. The theory predicts a *range* of instabilities extending from a small value of the velocity to a definite upper limit. It was very gratifying to observe that the upper branch of the curve not only existed but that it was remarkably definite. A small increase in speed near this upper limit would suffice to change the condition from violent flutter to complete rest, no range of transition being observed. The experimental cases 2 and 3 are compared with theoretical results given by the dotted lines in both figures (figs. 16 and 17).

The conclusion from the experiments is briefly that the general shapes of the predicted response curves re-

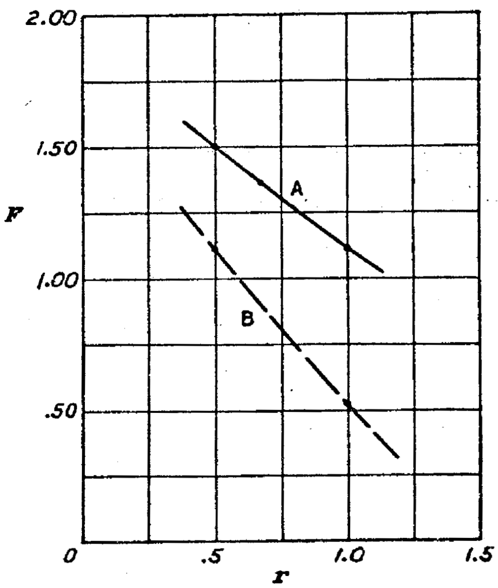

FIGURE 13.—Case 1, Flexure-torsion (h, α): Showing dependency of F on the radius of gyration $r_\alpha=r$.

A, airfoil with $a=-0.4$; $\kappa=\frac{1}{4}$; $x=0.2$; $\frac{\omega_1}{\omega_2}=\frac{1}{6}$; r variable.

B, airfoil with $a=-0.4$; $\kappa=\frac{1}{4}$; $x=0.2$; $\frac{\omega_1}{\omega_2}=1.00$; r variable.

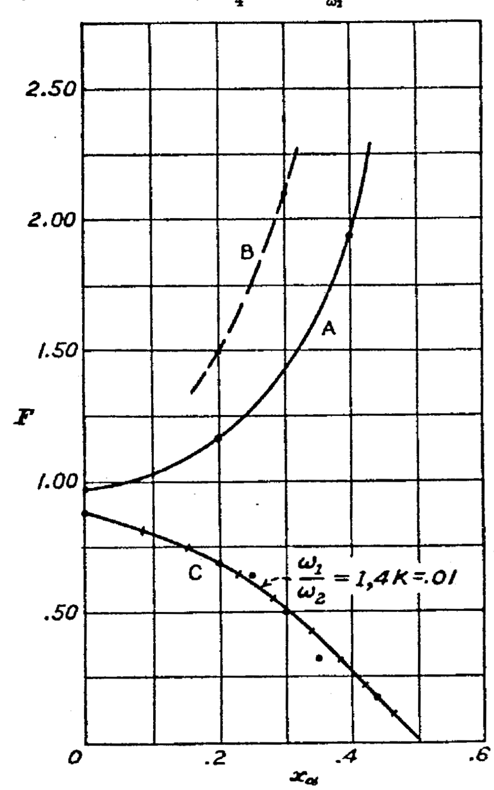

FIGURE 14.—Case 1, Flexure-torsion (h, α): Showing dependency of F on x_α, the location of the center of gravity.

A, airfoil with $r=\frac{1}{2}$; $a=-0.4$; $\kappa=\frac{1}{400}$; $\frac{\omega_1}{\omega_2}=\frac{1}{6}$; x variable.

B, airfoil with $r=\frac{1}{2}$; $a=-0.4$; $\kappa=\frac{1}{4}$; $\frac{\omega_1}{\omega_2}=\frac{1}{6}$; x variable.

C, airfoil with $r=\frac{1}{2}$; $a=-0.4$; $\kappa=\frac{1}{100}$; $\frac{\omega_1}{\omega_2}=1$; x variable.

peat themselves satisfactorily. Next, that the influence of the internal friction[7] obviously is quite appreci-

[7] This matter is the subject of a paper now in preparation.

able in case 3. This could have been expected since the predicted velocities and thus also the air forces on the aileron are very low, and no steps were taken to eliminate the friction in the hinge. The outline of the stable region is rather vague, and the wing is subject

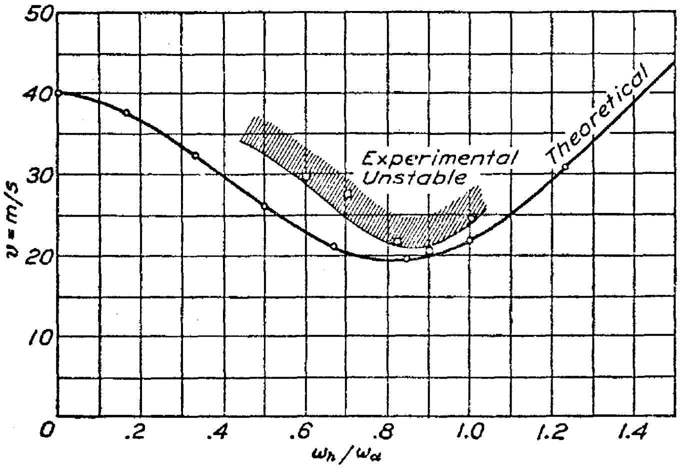

FIGURE 15.—Case 1. Wing A. Theoretical and experimental curves giving flutter velocity v against frequency ratio $\frac{\omega_h}{\omega_\alpha}$. Deflection-torsion.

to temporary vibrations at much lower speeds than that at which the violent flutter starts. The above experiments are seen to refer to cases of exaggerated unbalance, and therefore of low flutter speeds. It is evident that the internal friction is less important at larger velocities. The friction does in all cases *increase* the speed at which flutter starts.

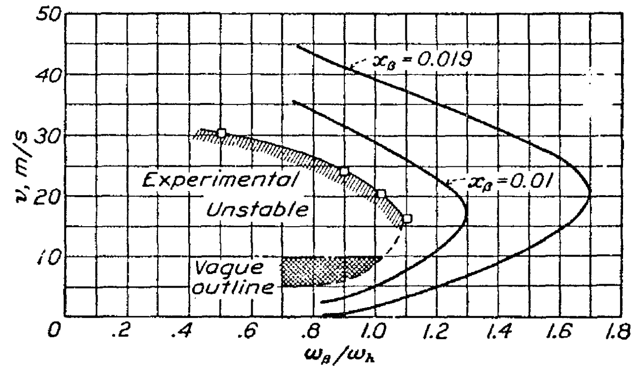

FIGURE 16.—Case 2. Wing B. Theoretical and experimental curves giving flutter velocity v against frequency ratio $\frac{\omega_\beta}{\omega_h}$. Aileron-deflection ($\beta$, h).

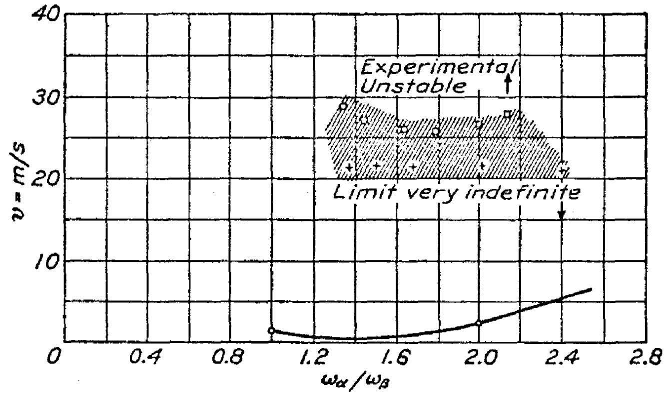

FIGURE 17.—Case 3. Theoretical curve giving flutter velocity against the frequency ratio $\frac{\omega_\alpha}{\omega_\beta}$. The experimental unstable area is indefinite due to the importance of internal friction at very small velocities. Torsion-aileron (α, β).

APPENDIX III

EVALUATION OF φ_β

$$\int_c^1 \log\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}\,dx_1$$

$$=\left[x_1\log\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_1)^2+(y+y_1)^2}\right]_c^1-2y\int_c^1\frac{x_1dx_1}{y_1(x-x_1)}$$

$$=-2c\log\frac{1-xc-y\sqrt{1-c^2}}{(x-c)}-2y\int_c^1\frac{x_1dx_1}{\sqrt{1-x_1^2}(x-x_1)}$$

$$+\int_c^1\frac{x_1dx_1}{\sqrt{1-x_1^2}(x-x_1)}=\int\frac{dx_1}{\sqrt{1-x_1^2}}$$

$$+x\int\frac{dx_1}{(x_1-x)\sqrt{1-x_1^2}}\ \text{[Putting } x_1=\cos\theta\text{]}$$

$$=-\theta-\frac{x}{\sqrt{1-x^2}}\log\frac{1-x\cos\theta+\sqrt{1-x^2}\sin\theta}{\cos\theta-x}\Bigg|_{\cos\theta=c}^{\cos\theta=1}$$

$$=\cos^{-1}c+\frac{x}{\sqrt{1-x^2}}\log\frac{1-cx+\sqrt{1-x^2}\sqrt{1-c^2}}{c-x}$$

$$=\cos^{-1}c+\frac{x}{\sqrt{1-x^2}}\log\frac{c-x}{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}$$

$$\varphi\frac{2\pi}{\epsilon}=-2c\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})+2c\log(x-c)$$

$$-2\sqrt{1-x^2}\cos^{-1}c-2x\log(c-x)$$

$$+2x\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$=2(x-c)\log\left(\frac{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}{x-c}\right)$$

$$-2\sqrt{1-x^2}\cos^{-1}c$$

EVALUATION OF φ_β

$$\varphi_x=\int_c^1\{\log[(x-x_1)^2+(y-y_1)^2]$$

$$-\log[(x-x_1)^2+(y+y_1)^2]\}(x_1-c)dx_1$$

$$=\frac{(x_1-c)^2}{2}\{\log[(x-x_1)^2+(y-y_1)^2]$$

$$-\log[(x-x_1)^2+(y+y_1)^2]\}]_c^1$$

$$+y\int_c^1(x_1-c)^2\frac{dx_1}{y_1(x-x_1)}$$

$$\int_c^1\frac{(x_1-c)^2dx_1}{y_1(x-x_1)}=\int_c^1\frac{(x_1-c)^2dx_1}{1-x_1^2(x-x_1)}=-\int\frac{(\cos\theta-c)^2d\theta}{x-\cos\theta}$$

$x_1=\cos\theta,\ y_1=\sin\theta,\ dx_1=-\sin\theta d\theta$

$$\int_c^1\frac{(x_1-c)^2dx_1}{y_1(x-x_1)}=\sin\theta+(x-2c)\theta-(x-c)^2\int_c^1\frac{d\theta}{x-\cos\theta}$$

$$\int_c^1\frac{d\theta}{x-\cos\theta}=\int_c^1\frac{d(\pi+\theta)}{x+\cos(\pi+\theta)}$$

$$=\frac{1}{\sqrt{1-x^2}}\log\frac{1-x\cos\theta-\sqrt{1-x^2}\sin\theta}{x-\cos\theta}\Bigg|_{\cos\theta=c}^{\cos\theta=1}$$

$$=\frac{1}{\sqrt{1-x^2}}\left[\log\frac{1-x}{x-1}-\log\frac{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}{x-c}\right]$$

$$=-\frac{1}{\sqrt{1-x^2}}\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$+\frac{1}{\sqrt{1-x^2}}\log(x-c)$$

$$\frac{2\pi}{\epsilon}\varphi_x=\sqrt{1-x^2}\Big[-\sqrt{1-c^2}-(x-2c)\cos^{-1}c$$

$$+\frac{(x-c)^2}{\sqrt{1-x^2}}\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$-\frac{(x-c)^2}{\sqrt{1-x^2}}\log(x-c)\Big]$$

$$\frac{2\pi}{\epsilon}\varphi_x=-\sqrt{1-c^2}\sqrt{1-x^2}-\cos^{-1}c(x-2c)\sqrt{1-x^2}$$

$$+(x-c)^2\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$-(x-c)^2\log(x-c)$$

EVALUATION OF T_3

$$\int_c^1\frac{2\pi}{\epsilon}\varphi_x(x-c)dx=-\sqrt{1-c^2}\int(x-c)\sqrt{1-x^2}\,dx$$

$$-\cos^{-1}c\int(x-c)(x-2c)\sqrt{1-x^2}\,dx$$

$$+\frac{(x-c)^4}{4}\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$-\frac14\int(x-c)^3dx-\sqrt{\frac{1-c^2}{4}}\frac{(x-c)^3}{\sqrt{1-x^2}}dx$$

$$-\int(x-c)^3\log(x-c)dx;\ x=\cos\theta,\ dx=-\sin\theta d\theta$$

$$\frac{2\pi}{\epsilon}\int_c^1\varphi_x(x-c)dx=\sqrt{1-c^2}\int(\cos\theta-c)\sin^2\theta d\theta$$

$$+\cos^{-1}c\int(\cos\theta-c)(\cos\theta-2c)\sin^2\theta d\theta$$

$$+\frac{(x-c)^4}{4}\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$-\frac14\int(x-c)^3dx+\frac{\sqrt{1-c^2}}{4}\int(\cos\theta-c)^3d\theta$$

$$-\frac{(x-c)^4}{4}\log(x-c)+\frac14\int(x-c)^3dx$$

$$\frac{2\pi}{\epsilon}\int_c^1\varphi_x(x-c)dx=-\cos^{-1}c\int\cos^4\theta d\theta$$

$$+\left(3c\cos^{-1}c-\sqrt{1-c^2}+\frac{\sqrt{1-c^2}}{4}\right)\int\cos^3\theta d\theta$$

$$+\left(\cos^{-1}c-2c^2\cos^{-1}c+c\sqrt{1-c^2}-\frac34c\sqrt{1-c^2}\right)\int\cos^2\theta d\theta$$

$$+\left(-3c\cos^{-1}c+\sqrt{1-c^2}+\frac{3c^2\sqrt{1-c^2}}{4}\right)\int\cos\theta d\theta$$

$$+\left(2c^2\cos^{-1}c-c\sqrt{1-c^2}-\frac{c^3\sqrt{1-c^2}}{4}\right)\int d\theta$$

$$=-\cos^{-1}c\left[\frac{\cos^3\theta\sin\theta}{4}+\frac{3}{4}\left(\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}\right)\right]$$

$$+\frac{1}{3}\left(3c\cos^{-1}c-\frac{3}{4}\sqrt{1-c^2}\right)\sin\theta(\cos^2\theta+2)$$

$$+\left(\cos^{-1}c-2c^2\cos^{-1}c+\frac{c\sqrt{1-c^2}}{4}\right)\left(\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}\right)$$

$$+\left(-3c\cos^{-1}c+\sqrt{1-c^2}+\frac{3c^2\sqrt{1-c^2}}{4}\right)\sin\theta$$

$$+\left(2c^2\cos^{-1}c-c\sqrt{1-c^2}-\frac{c^3\sqrt{1-c^2}}{4}\right)\theta$$

$$=\cos^{-1}c\left(\frac{3}{8}\pi+\frac{\pi}{2}-3\pi\right)=-\frac{9}{8}\pi\cos^{-1}c$$

$$\frac{2\pi}{e}\int_c^1\varphi_x(x-c)\mathrm{d}x$$

$$=\cos^{-1}c\left[\frac{c^3\sqrt{1-c^2}}{4}+\frac{3\cos^{-1}c}{8}+\frac{3c\sqrt{1-c^2}}{8}\right]$$

$$-\left[c\cos^{-1}c-\frac{\sqrt{1-c^2}}{4}\right](c^2\sqrt{1-c^2}+2\sqrt{1-c^2})$$

$$-\left(\cos^{-1}c-2c^2\cos^{-1}c+\frac{c\sqrt{1-c^2}}{4}\right)\left(\frac{\cos^{-1}c+c\sqrt{1-c^2}}{2}\right)$$

$$-\left(-3c\cos^{-1}c+\sqrt{1-c^2}+\frac{3c^2\sqrt{1-c^2}}{4}\right)\sqrt{1-c^2}$$

$$-\left(2c^2\cos^{-1}c-c\sqrt{1-c^2}-\frac{c^3\sqrt{1-c^2}}{4}\right)\cos^{-1}c$$

$$=\cos^{-1}c\left[\frac{3}{8}-\frac{1}{2}+c^2-2c^2\right]$$

$$+\sqrt{1-c^2}\cos^{-1}c\left[\frac{c^3}{4}+\frac{3c}{8}-c^3-2c-\frac{c}{2}+c^3+3c+c\right.$$

$$\left.+\frac{c^3}{4}-\frac{c}{8}\right]+\frac{c^2(1-c^2)}{4}+\frac{(1-c^2)}{2}-\frac{c^2(1-c^2)}{8}$$

$$-(1-c^2)-\frac{3c^2(1-c^2)}{4}=-\left(\frac{1}{8}+c^2\right)(\cos^{-1}c)^2$$

$$+\frac{c\sqrt{1-c^2}\cos^{-1}c}{4}(7+2c^2)-\frac{(1-c^2)}{8}(5c^2+4)\quad(=T_3)$$

EVALUATION OF T_5

$$\int_c^1\left\{2(x-c)\log\frac{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}{x-c}\right.$$

$$\left.-2\sqrt{1-x^2}\cos^{-1}c\right\}\mathrm{d}x=T_5=-2\int(x-c)\log(x-c)\mathrm{d}x$$

$$+2\int(x-c)\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})\mathrm{d}x$$

$$-2\cos^{-1}c\int\sqrt{1-x^2}\,\mathrm{d}x=-\frac{2(x-c)^2}{2}\log(x-c)$$

$$+\int(x-c)\mathrm{d}x+2\cos^{-1}c\int\sin^2\theta\mathrm{d}\theta$$

$$+(x-c)^2\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$-\int(x-c)^2\frac{-c+x\dfrac{\sqrt{1-c^2}}{\sqrt{1-x^2}}}{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}\mathrm{d}x$$

Now

$$\int(x-c)^2\frac{-c+x\dfrac{\sqrt{1-c^2}}{\sqrt{1-x^2}}}{1-cx-\sqrt{1-x^2}\sqrt{1-c^2}}\,\mathrm{d}x$$

$$=\int\left\{-c+c^2x-c\sqrt{1-c^2}\sqrt{1-x^2}+x\frac{\sqrt{1-c^2}}{\sqrt{1-x^2}}-cx^2\frac{\sqrt{1-c^2}}{\sqrt{1-x^2}}\right.$$

$$\left.+(1-c^2)x\right\}\mathrm{d}x=\int\frac{(x-c)\sqrt{1-x^2}+(x-c)\sqrt{1-c^2}}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$=\int(x-c)\mathrm{d}x+\sqrt{1-c^2}\int\frac{(x-c)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$T_5=-(x-c)^2\log(x-c)+2\cos^{-1}c\int\sin^2\theta\mathrm{d}\theta$$

$$+(x-c)^2\log(1-cx-\sqrt{1-x^2}\sqrt{1-c^2})$$

$$+\sqrt{1-c^2}\int(\cos\theta-c)\mathrm{d}\theta$$

$$=\frac{2\cos^{-1}c}{2}(\theta-\sin\theta\cos\theta)+\sqrt{1-c^2}\sin\theta$$

$$-c\sqrt{1-c^2}\,\theta\Big|_{\cos\theta=c}^{\cos\theta=1}$$

$$=-(1-c^2)-(\cos^{-1}c)^2+2c\sqrt{1-c^2}\cos^{-1}c$$