REPORT No. 839

THEORETICAL LIFT AND DRAG OF THIN TRIANGULAR WINGS AT SUPERSONIC SPEEDS

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SUMMARY

A method is derived for calculating the lift and the drag due to lift of point-forward triangular wings and a restricted series of sweptback wings at supersonic speeds. The elementary or "supersonic source" solution of the linearized equation of motion is used to find the potential function of a line of doublets. The flow about the triangular flat plate is then obtained by a surface distribution of these doublet lines. The lift-curve slope of triangular wings is found to be a function of the ratio of the tangent of the apex angle to the tangent of the Mach angle. As the apex angle approaches and becomes greater than the Mach angle, the lift coefficient of the triangular wing becomes equal to that of a two-dimensional supersonic airfoil at the same Mach number.

The drag coefficient due to lift of triangular wings with leading edges well behind the Mach cone is shown to be close to that of elliptically loaded wings of the same aspect ratio in subsonic flight. The resultant force on wings with leading edges outside the Mach cone, however, is shown to act normal to the surface and thus an induced drag equal to the lift times the angle of attack is obtained.

INTRODUCTION

In reference 1, Jones calculated the lift of thin point-forward triangular wings for the cases in which the apex angle of the wing was very small. It was pointed out that the results obtained should be applicable in both supersonic and subsonic flight, the criterion for the case of supersonic flight being that the apex angle be small as compared with the Mach angle of the flow. The present paper, making use of less restricted theory, extends Jones' work to the case of triangular wings having large apex angles and traveling at supersonic speeds. A recent paper was published by H. J. Stewart (reference 2) in which the lift of triangular wings has been computed, but the method used appears to be entirely different.

In the present theory, the linearized equation of motion was used and the results must therefore be restricted to small angles of attack and moderate supersonic Mach numbers. The solution which has been found should hold good for large values of the apex angle up to and coincident with the Mach angle. Jones (reference 3) and Puckett (reference 4) have found solutions for the drag of triangular wings of small thickness at zero angle of attack. The solutions are applicable to wings having the leading edges either in or out of the Mach cone springing from the apex of the wing. Puckett has pointed out that, for the case where the leading edge is ahead of the Mach cone, these solutions can also be used to calculate the lift; thus, with the present solution, the lift for the whole range of apex angles at supersonic speeds may be obtained. The pressure distributions and lift-curve slopes obtained in the present paper can be used to obtain the lift and drag characteristics of a limited series of sweptback wings. The drag due to lift of the triangular wing has been calculated and a suction force has been found to exist on the leading edge. In order to use the suction force, however, it appears necessary to provide an airfoil section with a rounded leading edge. The author is indebted to Mr. Arthur Kantrowitz of the Langley Memorial Aeronautical Laboratory for suggesting the method used to calculate the induced drag.

SYMBOLS

α	angle of attack
A	aspect ratio $\left(\frac{b^2}{S}\right)$
b	maximum span of wing
$\beta=\sqrt{M^2-1}$	
C	tangent of apex angle
C_L	lift coefficient $\left(\frac{L}{qS}\right)$
C_{D_i}	drag coefficient due to lift $\left(\frac{D_i}{qS}\right)$
D_i	drag force due to lift
E	source strength
ϵ	apex angle of wing measured from flight direction
$f(\sigma)$	doublet-distribution function
F	suction force on wing leading edge
$\tau=\frac{z_1}{x_1}$	
$i=\sqrt{-1}$	
I	strength of line doublet
$\sigma=\frac{y_1}{x_1}$	
c_r	length of wing or root chord
L	lift force
$\lambda=\int_{-C}^{C}\frac{\beta^2\sigma^2}{\sqrt{C^2-\sigma^2}\sqrt{1-\beta^2\sigma^2}}\tanh^{-1}\sqrt{1-\beta^2\sigma^2}\,d\sigma$	
M	Mach number
μ	Mach angle $\left(\sin^{-1}\frac{1}{M}\right)$

Δp	lifting pressure
q	dynamic pressure $\left(\frac{1}{2}\rho V^2\right)$
R	distance along leading edge from wing apex
ρ	density of the fluid
S	wing area
s_n	distance normal to leading edge
u	velocity increment in x-direction $\left(\frac{\partial\phi}{\partial x}\right)$
U	velocity increment normal to leading edge $\left(\frac{\partial\phi}{\partial s_n}\right)$
v	velocity increment in y-direction $\left(\frac{\partial\phi}{\partial y}\right)$
V	flight velocity
w	velocity in z-direction $\left(\frac{\partial\phi}{\partial z}\right)$
W	resultant velocity in z-direction created by the doublet distribution
x, y, z	coordinates of an arbitrary field point
x_1, y_1, z_1	coordinates of a source or doublet
ϕ	disturbance-potential function
ϕ_0	potential of a supersonic source
ϕ_1	potential of a line of supersonic sources
ϕ_2	potential of a line of supersonic doublets

Subscripts:

n	normal to leading edge
Δ	triangular-wing condition
∞	infinite-span or two-dimensional wing condition

THEORY FOR LIFTING TRIANGLE

The linearized equation of motion of a nonviscous compressible fluid may be written

$$\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=(M^2-1)\frac{\partial^2\phi}{\partial x^2} \tag{1}$$

where ϕ is the potential function assumed to represent the effect of a small disturbance set up by the body being considered. The body in this case is a triangular flat plate having its vertex at the center of the coordinate system and lying in the xy-plane (fig. 1). The problem is to find a solution of equation (1) that will satisfy the known boundary conditions which are: (1) that the flow be quiescent ahead of the Mach cone and (2) that the flow at the surface of the plate be tangent to that surface. Because of the linear character of the differential equation (1), more general potentials can be built up from simple well-known solutions such as the one for a single source.

$$\phi_0=\frac{E}{\sqrt{x^2-\beta^2(y^2+z^2)}} \tag{2}$$

where $\beta=\sqrt{M^2-1}$. The potential of a line of sources with strength proportional to x can be found as follows:

$$\phi_1=\int_0^{x'}\frac{Ex_1\,dx_1}{\sqrt{(x-x_1)^2-\beta^2(y-\sigma x_1)^2-\beta^2(z-\tau x_1)^2}} \tag{3}$$

where $\sigma=\frac{y_1}{x_1}$, $\tau=\frac{z_1}{x_1}$, and x' is the value of x_1 for which the

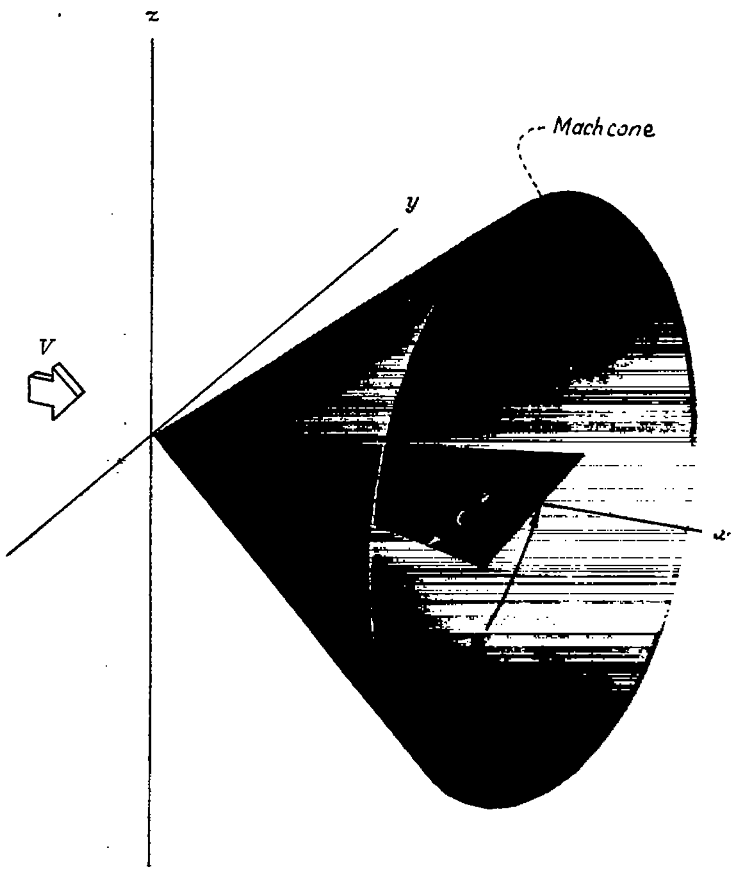

FIGURE 1.—Coordinate system.

denominator of the integrand is zero. Physically interpreted, the range of integration is from the origin to the last source point which can influence the field point. Performing the integration yields

$$\phi_1=E\left\{-\frac{\sqrt{x^2-\beta^2(y^2+z^2)}}{1-\beta^2\sigma^2-\beta^2\tau^2}-\frac{x-\beta^2\sigma y-\beta^2\tau z}{(1-\beta^2\sigma^2-\beta^2\tau^2)^{3/2}}\text{ctnh}^{-1}\frac{x-\beta^2\sigma y-\beta^2\tau z}{\sqrt{(1-\beta^2\sigma^2-\beta^2\tau^2)[x^2-\beta^2(y^2+z^2)]}}\right\} \tag{4}$$

If two such source lines of opposite strength are brought together from the z-direction at the xy-plane while the product of source strength and the angle between them is kept constant, the potential of a line of doublets in the xy-plane at an angle $\tan^{-1}\sigma$ from the x-axis is obtained. Thus differentiating with respect to τ and setting $\tau=0$ gives

$$\phi_2=\frac{-Iz\beta^2}{(1-\beta^2\sigma^2)^{3/2}}\left(\frac{\zeta}{\zeta^2-1}-\text{ctnh}^{-1}\zeta\right) \tag{5}$$

where

$$\zeta=\frac{x-\beta^2\sigma y}{\sqrt{(1-\beta^2\sigma^2)[x^2-\beta^2(y^2+z^2)]}}$$

and I is the doublet strength. Differentiating the potential function with respect to z gives the vertical velocity w:

$$w=\frac{-I\beta^2}{(1-\beta^2\sigma^2)^{3/2}}\left(\frac{\zeta}{\zeta^2-1}-\text{ctnh}^{-1}\zeta\right)+2Iz^2\frac{\beta^2(x-\beta^2\sigma y)\sqrt{x^2-\beta^2(y^2+z^2)}}{\{(x-\beta^2\sigma y)^2-(1-\beta^2\sigma^2)[x^2-\beta^2(y^2+z^2)]\}^2} \tag{6}$$

It will be noticed that the line doublet creates a conical field as the velocity is only a function of z/x and y/x. Since the triangular flat plate is a conical body which creates a conical field, an attempt will be made to build up the flow about the lifting triangle by a suitable distribution of line doublets inasmuch as the addition of two or more conical fields having the same vertex always creates another conical field. The distribution of line doublets must satisfy the boundary conditions at the body surface which may be written:

$$\underset{z=0}{W}=V\alpha \tag{7}$$

where W is the resultant vertical velocity of the line-doublet distribution. If the distribution of line doublets is $f(\sigma)$

$$\underset{z\to 0}{W}=\int_{-C}^{C} f(\sigma)w\left(\sigma;\frac{y}{x};\frac{z}{x}\right)d\sigma \qquad \left(-C<\frac{y}{x}<C\right) \tag{8}$$

in which $\tan^{-1}C=\epsilon$, the angle of the leading edge. The distribution function $f(\sigma)$ can be found in a rather simple way by analogy with the solution for incompressible two-dimensional flow about a flat plate. Differentiating equation (8) with respect to y/x and setting $z=0$ gives

$$\underset{z=0}{\frac{dW}{d\left(\frac{y}{x}\right)}}=2I\int_{-C}^{C}\frac{f(\sigma)\sqrt{1-\beta^2\left(\frac{y}{x}\right)^2}}{\left(\sigma-\frac{y}{x}\right)^3}d\sigma=0 \tag{9}$$

which is of the same form as the integral equation obtained when the incompressible flow normal to a two-dimensional flat plate is constructed by a doublet distribution. The expression $f(\sigma)$ for the incompressible case would be

$$f(\sigma)=\sqrt{C^2-\sigma^2} \tag{10}$$

That this expression is a solution of equation (8) must be verified by substitution in equation (8), inasmuch as equation (9) is a divergent integral. This proof is carried out in appendix A. The value of the velocity in the x-direction u can now be obtained

$$u=\int_{-C}^{C} f(\sigma)\frac{\partial\phi_2}{\partial x}d\sigma \tag{11}$$

The integration indicated in equation (11) is presented in appendix B. The expression obtained from equation (11) on the lifting surface ($z=0$) gives

$$u=\pm\frac{\pi C^2 I}{\sqrt{C^2-\left(\frac{y}{x}\right)^2}} \tag{12}$$

the sign of the expression being opposite for the two sides of the plate. The result in equation (12) shows that the shape of the pressure distribution is independent of aspect ratio.

The lift for the isosceles triangle with root chord c_r is

$$L=\int\Delta p\, dS=\int_{-C}^{C}\rho Vuc_r^2\, d\sigma \tag{13}$$

and substituting equation (12) in equation (13) and integrating gives

$$L=\rho VIc_r^2\pi^2C^2 \tag{14}$$

and

$$C_L=\frac{L}{qS}=\frac{2I\pi^2C}{V} \tag{15}$$

The value of the constant I must be obtained by solving equation (8).

The value of the normal velocity at the plate and hence the angle of attack may be found by integration of equation (8) and letting $z=0$. The integration is involved and the method of integration is given in appendix A. The resulting expression for W is obtained as

$$\underset{z=0}{W}=I\left(\pi+\int_{-C}^{C}\frac{\beta^2\sigma^2}{\sqrt{C^2-\sigma^2}\sqrt{1-\beta^2\sigma^2}}\tanh^{-1}\sqrt{1-\beta^2\sigma^2}\,d\sigma\right) \tag{16}$$

From this equation the value of I may be calculated. If the numerator and denominator of the integrand are multiplied by β the resulting integral can be seen to be dependent upon only the quantity βC or $\tan\epsilon/\tan\mu$. The value of the integral may be obtained easily by making the substitution $\beta^2C^2-\beta^2\sigma^2=n^2$ and plotting the resultant expression. This procedure has been followed for values of βC between 0 and 1 and the result is given in figure 2. The value of I is found to be

$$I=\frac{V\alpha}{\pi+\lambda} \tag{17}$$

where λ is the integral term of equation (16). The lift-curve slope is now

$$\frac{C_L}{\alpha}=\frac{2\pi^2C}{\pi+\lambda} \tag{18}$$

As βC approaches zero, λ also approaches zero and the lift-curve slope from equation (18) approaches Jones' value. (See reference 1.) Equation (18) shows that the lift-curve slope is a function of only the apex angle and the parameter $\tan\epsilon/\tan\mu$. It is interesting to note that mathematically there is a finite lift-curve slope at the Mach number 1.0.

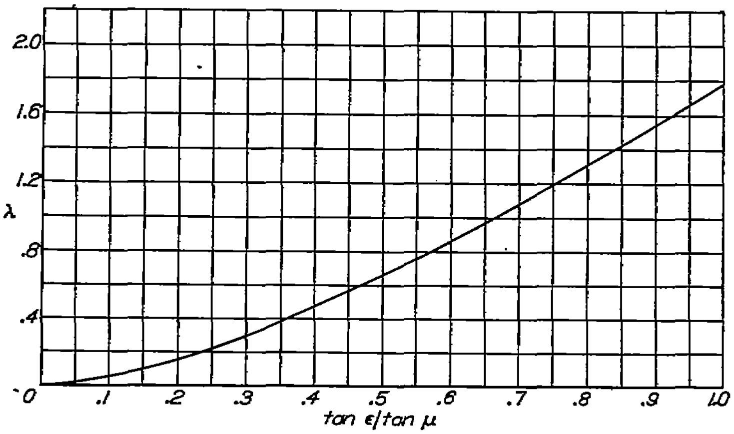

FIGURE 2.—Variation of λ with $\frac{\tan\epsilon}{\tan\mu}$.

The theory is not valid, however, near $M=1$ because of the original assumptions used in obtaining equation (1). The lift-curve slopes of two triangular wings are plotted in figure 3 against Mach number.

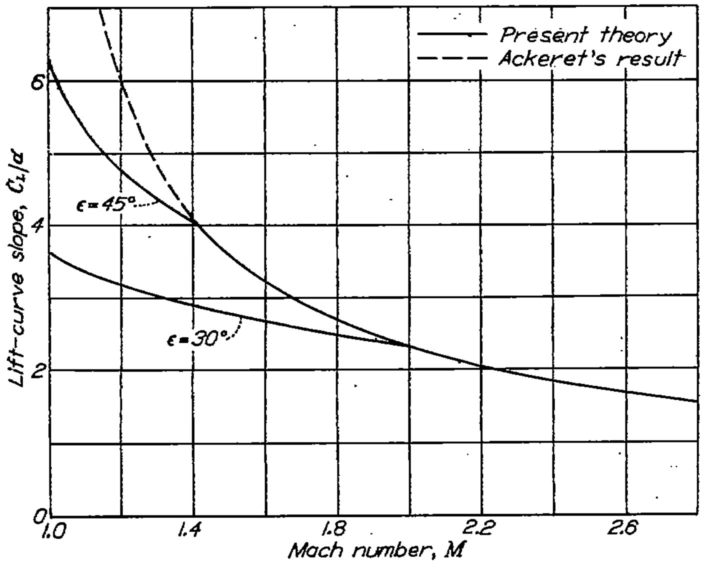

FIGURE 3.—Variation with Mach number of C_L/α for two triangular wings.

Rewriting equation (18) in a convenient form gives

$$\left(\frac{C_L}{\alpha}\right)_\Delta=\frac{2\pi^2\dfrac{\tan \epsilon}{\tan \mu}}{(\pi+\lambda)\sqrt{M^2-1}} \tag{19}$$

Now, according to Ackeret's result (reference 5) the lift-curve slope of a two-dimensional flat plate is

$$\left(\frac{C_L}{\alpha}\right)_\infty=\frac{4}{\sqrt{M^2-1}} \tag{20}$$

A single curve for all Mach numbers can therefore be plotted if the ratio tan ϵ/tan μ is used for the abscissa and $\dfrac{(C_L/\alpha)_\Delta}{(C_L/\alpha)_\infty}$ is used as the ordinate. This curve is shown in figure 4.

It can be seen that as the apex angle approaches the Mach angle the triangular wing provides the same lift coefficient as a two-dimensional wing at the same Mach number.

The case of the triangular wing having the leading edge ahead of the Mach cone from the apex has been treated in reference 4. It was found that the lift coefficient obtained is the same as that of a two-dimensional airfoil flying at the same Mach number. The curve shown in figure 4 therefore becomes flat at values of $\dfrac{\tan \epsilon}{\tan \mu}>1$. A typical pressure distribution over a wing having $\dfrac{\tan \epsilon}{\tan \mu}>1$ is shown in figure 5.

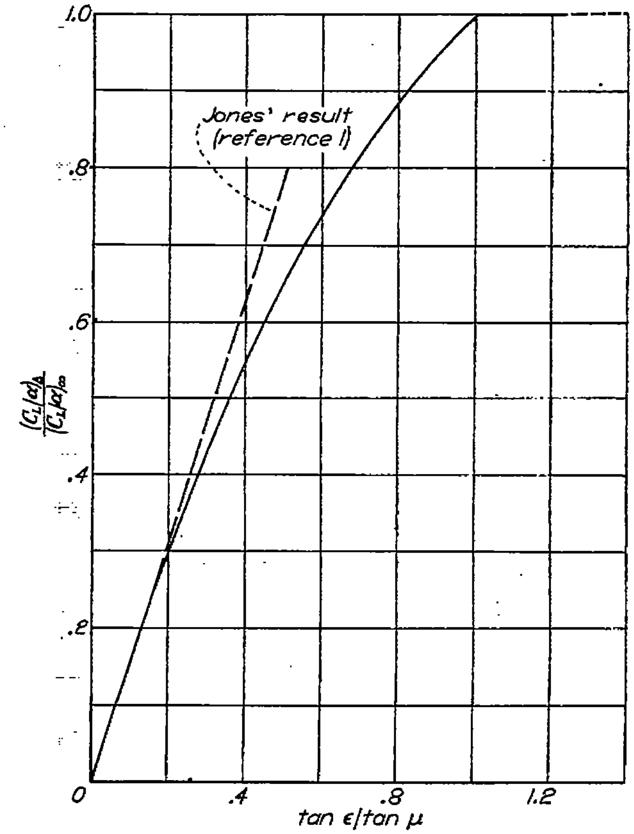

FIGURE 4.—Lift-curve slope of triangular wings.

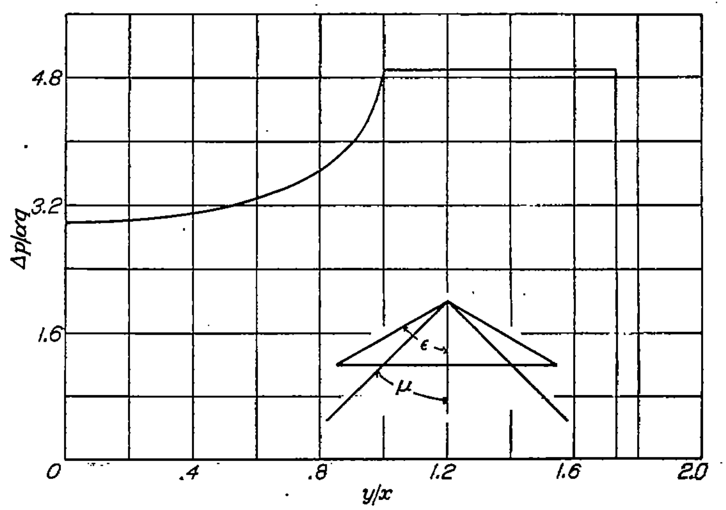

FIGURE 5.—Typical pressure distribution for triangular wings with $\frac{\tan \epsilon}{\tan \mu}>1$. $M=1.414$; $\epsilon=60°$.

DRAG DUE TO LIFT

The thin-airfoil theory used herein gives the result that the resultant force is directed normal to the plate, a result quite like that obtained from the thin-airfoil theory at subsonic speeds. In the solution for subsonic speeds, however, a simple extension described in reference 6 permits calculating the force due to the suction on the leading edge. It is reasonable to suppose that the same method is feasible for the triangular wing, as the pressure distribution in the neighborhood of the leading edge is identical in the limit with that for a two-dimensional flat plate in subsonic flow; that is, the velocities normal to the leading edge are in the form

$$\underset{s_n\to 0}{U}=\frac{G}{\sqrt{s_n}} \tag{21}$$

where s_n is the distance measured normal to the edge and G is a constant. According to reference 6 the force normal to the edge in the direction of the velocity is

$$F=\rho\pi G^2 \tag{22}$$

Equations (21) and (22) are, however, based on incompressible-flow relations and must be corrected for compressibility. A simple extension is found if the well-known concepts of the Prandtl-Glauert rule are applied. For the two-dimensional case it is found that the effect of compressibility on a flow having a given vorticity distribution in a plane is to reduce the velocities normal to the surface by the factor $\sqrt{1-M'^2}$ where M' is the Mach number of the flow. Therefore, if the strength of each vortex element is increased by a factor $\frac{1}{\sqrt{1-M'^2}}$, the resultant velocities normal to the surface will again be equal to those of the incompressible flow. In this case, however, the tangential velocities at the surface and therefore the forces on the surface are increased by the ratio $\frac{1}{\sqrt{1-M'^2}}$. This concept is well known in thin-airfoil theory where it gives the result that the lift-curve slopes of thin airfoils are increased by $\frac{1}{\sqrt{1-M'^2}}$. As the total resistance for $M<1$ is still zero, this result indicates that the leading-edge suction force has been increased over that of the incompressible flow by the ratio $\frac{1}{\sqrt{1-M'^2}}$.

It appears that the incompressible equations governing the leading-edge suction force (equations (21) and (22)) must be corrected as follows:

The leading-edge suction force on a two-dimensional plate will be

$$F_n=\frac{\rho\pi G^2}{\sqrt{1-M'^2}} \tag{23}$$

when the vorticity distribution γ at the leading edge or the tangential velocities are given by the following relations:

$$\underset{s_n\to 0}{\gamma}=\frac{2G}{\sqrt{1-M'^2}\sqrt{s_n}}$$

or

$$\underset{s_n\to 0}{U}=\frac{G}{\sqrt{1-M'^2}\sqrt{s_n}} \tag{24}$$

The value of the velocity in the y-direction on the triangular wing has been calculated to be

$$v=\frac{I\pi\frac{y}{x}}{\sqrt{C^2-\left(\frac{y}{x}\right)^2}}$$

Combining this expression with equation (12) gives

$$\underset{\frac{y}{x}\to C}{U}=\pm\frac{CI\pi\sqrt{1+C^2}}{\sqrt{C^2-\left(\frac{y}{x}\right)^2}}$$

$$=\pm\frac{I\pi\sqrt{\frac{CR}{2}}\sqrt{1-M'^2}}{\sqrt{s_n}\sqrt{1-M'^2}} \tag{25}$$

where $R=x\sqrt{1+C^2}$ and M' is the component of the flight Mach number normal to the leading edge. The normal force on a small element dR of leading edge from equation (23) becomes

$$\frac{dF_n}{dR}=\frac{\rho\pi CRI^2\pi^2\sqrt{1-M'^2}}{2} \tag{26}$$

and the force on one edge of the isosceles triangle with root chord c_r is:

$$F_n=\frac{\rho\pi^3I^2C\sqrt{1-M'^2}}{2}\int_0^{c_r\sqrt{1+C^2}}R\,dR$$

$$=\frac{\rho\pi^3I^2Cc_r^2(1+C^2)\sqrt{1-M'^2}}{4} \tag{27}$$

and the force in the flight direction from two edges becomes

$$F=2F_n\sin\epsilon$$

$$=\frac{\rho\pi^3C^2\sqrt{1+C^2}\,c_r^2I^2\sqrt{1-M'^2}}{2} \tag{28}$$

Substituting from equation (14)

$$F=\frac{L^2\sqrt{1+C^2}\sqrt{1-M'^2}}{\pi b^2 q}$$

$$=\frac{L\alpha}{2}\left(\frac{\sqrt{1+C^2}\sqrt{1-M'^2}}{1+\frac{\lambda}{\pi}}\right) \tag{29}$$

where b is the maximum span of the triangular wing. The induced drag or, more exactly, the drag resulting from the lift may be written

$$D_i=L\alpha-F$$

$$=\frac{L\alpha}{2}\left(2-\frac{\sqrt{1+C^2}\sqrt{1-M'^2}}{1+\frac{\lambda}{\pi}}\right) \tag{30}$$

Writing the identity

$$\sqrt{1-M'^2}=\frac{\sqrt{1-\beta^2C^2}}{\sqrt{1+C^2}} \tag{31}$$

equation (30) becomes

$$D_i=\frac{L\alpha}{2}\left(2-\frac{\sqrt{1-\beta^2C^2}}{1+\frac{\lambda}{\pi}}\right) \tag{32}$$

It will be noticed that this result is identical with Jones' result (reference 1) in the limiting case of $C=0$. The induced drag coefficient is found to be

$$C_{D_i}=\frac{C_L^2}{\pi A}\left[2\left(1+\frac{\lambda}{\pi}\right)-\sqrt{1-\beta^2C^2}\right] \tag{33}$$

where A is the aspect ratio. Equation (33) indicates that the triangular wings can obtain a considerable suction force at the leading edge and that the drag coefficient due to lift of slender wings is very close to that obtained from elliptical wings of the same aspect ratio at subsonic speeds. It should be pointed out, however, that as soon as the wing leading edge passes through the Mach cone, the possibility of obtaining a leading-edge suction is gone and the resultant force must become normal to the plate surface. This transition corresponds quite similarly to the case of a two-dimensional airfoil passing through the speed of sound.

DISCUSSION AND CONCLUSIONS

The lift at supersonic speeds of triangular wings having straight trailing edges has been shown to approach the lift of a two-dimensional airfoil as the leading edge approaches the Mach cone springing from the apex of the triangle. For the case where the triangular wing lies behind the Mach cone, a suction has been found to exist on the leading edge. In order to utilize this suction force in practice it would appear necessary, as in subsonic flow, to provide an airfoil section with a rounded leading edge. Triangular wings should be capable of higher lift-drag ratios than unswept wings at supersonic speeds when operating with their leading edges not too far behind the Mach cone; the improvement should be due to both reduced wave drag and reduced induced drag.

The lift and drag of a series of limited sweptback wings may also be calculated with the method developed. It will be noted that the pressure distribution over the triangular wing cannot be changed if the trailing edge is cut off from the tip to the center line along an angle always greater than the Mach angle. This fact arises from the nature of the supersonic flow in which disturbances cannot propagate any farther forward than the Mach cone from the origin of the disturbance. The aforementioned procedure produces therefore a series of tapered sweptback wings having pointed tips. A new series can also be constructed by cutting off the tips along lines having angles greater than the Mach angle. In each case the pressures over the remaining portions of the wing will be the same as though the cutbacks had not been made.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *November 29, 1946.*

APPENDIX A

CALCULATION OF VERTICAL VELOCITY INCREMENT

The value of W is, from equations (6) and (8),

$$W=I\int_{-C}^{C}\sqrt{C^2-\sigma^2}\left\{\frac{\beta^2\,d\sigma}{(1-\beta^2\sigma^2)^{3/2}}\left[\mathrm{ctnh}^{-1}\zeta-\frac{\zeta}{(\zeta^2-1)}\right]+2z^2\,d\sigma\,\frac{(x-\beta^2\sigma y)\sqrt{x^2-\beta^2(y^2+z^2)}}{[\sigma^2(x^2-\beta^2z^2)-2xy\sigma+y^2+z^2]^2}\right\} \tag{A1}$$

Integrating by parts gives

$$W=I\int_{-C}^{C}\left\{\frac{-\beta^2\sigma}{\sqrt{C^2-\sigma^2}}\int\frac{1}{(1-\beta^2\sigma^2)^{3/2}}\left(\frac{\zeta}{\zeta^2-1}-\mathrm{ctnh}^{-1}\zeta\right)d\sigma\right\}d\sigma$$
$$+I\sqrt{x^2-\beta^2(y^2+z^2)}\int_{-C}^{C}\frac{\sigma}{\sqrt{C^2-\sigma^2}}\left[\frac{\sigma x-y}{\sigma^2(x^2-\beta^2z^2)-2xy\sigma+y^2+z^2}+\frac{x}{z\sqrt{x^2-\beta^2(y^2+z^2)}}\tan^{-1}\frac{(x^2-\beta^2z^2)\sigma-xy}{z\sqrt{x^2-\beta^2(y^2+z^2)}}\right]d\sigma \tag{A2}$$

The integration of the term under the indefinite integral can be performed by parts to obtain the result

$$\frac{\sigma}{\sqrt{1-\beta^2\sigma^2}}\left(\frac{\zeta}{\zeta^2-1}-\mathrm{ctnh}^{-1}\zeta\right)-\frac{\sqrt{x^2-\beta^2(y^2+z^2)}}{\beta^2(x^2-\beta^2z^2)}\left[\frac{\sigma x(x^2-2\beta^2y^2-\beta^2z^2)+\beta^2y(y^2+z^2)}{\sigma^2(x^2-\beta^2z^2)-2xy\sigma+y^2+z^2}\right]+\frac{x}{\beta^2z}\tan^{-1}\frac{(x^2-\beta^2z^2)\sigma-xy}{z\sqrt{x^2-\beta^2(y^2+z^2)}}$$

Substituting in equation (A2) and rearranging terms gives:

$$W=I\int_{-C}^{C}\frac{\sigma^2\beta^2}{\sqrt{1-\beta^2\sigma^2}\sqrt{C^2-\sigma^2}}\left(\mathrm{ctnh}^{-1}\zeta-\frac{\zeta}{\zeta^2-1}d\sigma\right)+\frac{I\sqrt{x^2-\beta^2(y^2+z^2)}}{(x^2-\beta^2z^2)}\int_{-C}^{C}\frac{\sigma}{\sqrt{C^2-\sigma^2}}\left\{\frac{2[x^2-\beta^2(y^2+z^2)]\sigma x-y(x^2-\beta^2y^2-2\beta^2z^2)}{[\sigma^2(x^2-\beta^2z^2)-2xy\sigma+y^2+z^2]}\right\}d\sigma \tag{A3}$$

It will be noticed in the preceding operation that all the terms containing the singularity of the form $1/z$ cancel. If z is made to approach zero and terms are collected, equation (A3) becomes

$$W=I\int_{-C}^{C}\frac{\sigma^2\beta^2}{\sqrt{1-\beta^2\sigma^2}\sqrt{C^2-\sigma^2}}\,\mathrm{ctnh}^{-1}\zeta\,d\sigma+I\sqrt{1-\beta^2\left(\frac{y}{x}\right)^2}\int_{-C}^{C}\frac{\sigma}{\sqrt{C^2-\sigma^2}}\left[\frac{d\sigma}{\sigma-\frac{y}{x}}+\beta^2\left(\frac{y}{x}\right)\right]d\sigma \tag{A4}$$

Completing the integration of the second terms gives

$$W=I\int_{-C}^{C}\frac{\beta^2\sigma^2}{\sqrt{1-\beta^2\sigma^2}\sqrt{C^2-\sigma^2}}\,\mathrm{ctnh}^{-1}\left[\frac{1-\beta^2\sigma\left(\frac{y}{x}\right)}{\sqrt{1-\beta^2\sigma^2}\sqrt{1-\beta^2\left(\frac{y}{x}\right)^2}}\right]d\sigma+I\pi\sqrt{1-\beta^2\left(\frac{y}{x}\right)^2} \tag{A5}$$

Differentiating expression (A5) with respect to y/x and performing the integration gives

$$\frac{\partial W}{\partial\left(\frac{y}{x}\right)}=\frac{\pi\beta^2\left(\frac{y}{x}\right)}{\sqrt{1-\beta^2\left(\frac{y}{x}\right)^2}}-\frac{\pi\beta^2\left(\frac{y}{x}\right)}{\sqrt{1-\beta^2\left(\frac{y}{x}\right)^2}}=0$$

the vertical velocity is therefore constant over the plate surface and the expression $f(\sigma)=\sqrt{C^2-\sigma^2}$ is truly a solution of equation (8). It is possible that this solution is not unique; however, other solutions would undoubtedly lead to physically impossible conditions.

APPENDIX B

CALCULATION OF AXIAL VELOCITY INCREMENT

The value of the x-component of velocity may be written from equation (11)

$$u=2zI\sqrt{x^2-\beta^2(y^2+z^2)}\int_{-C}^{C}\sqrt{C^2-\sigma^2}\,\frac{\sigma yx-y^2-z^2}{[\sigma^2(x^2-\beta^2z^2)-2\sigma yx+y^2+z^2]^2}\,d\sigma \tag{B1}$$

This integral can be broken up into two separate integrals, as follows:

$$u=\frac{zI\sqrt{x^2-\beta^2(y^2+z^2)}}{(x^2-\beta^2z^2)^2}\left[xy\int_{-C}^{C}\frac{\sqrt{C^2-\sigma^2}\left(2\sigma-\frac{2xy}{x^2-\beta^2z^2}\right)}{\left(\sigma^2-\frac{2xy\sigma}{x^2-\beta^2z^2}+\frac{y^2+z^2}{x^2-\beta^2z^2}\right)^2}d\sigma-\left(y^2+z^2-\frac{x^2y^2}{x^2-\beta^2z^2}\right)\int_{-C}^{C}\frac{\sqrt{C^2-\sigma^2}\,d\sigma}{\left(\sigma^2-\frac{2xy\sigma}{x^2-\beta^2z^2}+\frac{y^2+z^2}{x^2-\beta^2z^2}\right)^2}\right] \tag{B2}$$

The first integral appearing in equation (B2) can be integrated by parts to give, for the first complete term, the integral

$$-\frac{Izxy\sqrt{x^2-\beta^2(y^2+z^2)}}{(x^2-\beta^2z^2)^2}\int_{-C}^{C}\frac{\sigma}{\sqrt{C^2-\sigma^2}}\,\frac{d\sigma}{\left(\sigma^2-\frac{2xy\sigma}{x^2-\beta^2z^2}+\frac{y^2+z^2}{x^2-\beta^2z^2}\right)} \tag{B3}$$

Evaluation of this integral, which may be found in reference 7, equation (228), gives finally

$$\pm\frac{Ixy\pi}{2(x^2-\beta^2z^2)}\left\{\frac{xy-iz\sqrt{x^2-\beta^2(y^2+z^2)}}{\sqrt{C^2-\left[\frac{xy-iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}}+\frac{xy+iz\sqrt{x^2-\beta^2(y^2+z^2)}}{\sqrt{C^2-\left[\frac{xy+iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}}\right\} \tag{B4}$$

the sign of the expression being opposite for the two sides of the plate. The second integral term of equation (B2) can be integrated by breaking the integrand up into four partial fractions, as follows:

$$\begin{aligned}\sqrt{C^2-\sigma^2}\Bigg\{&\frac{-(x^2-\beta^2z^2)^3d\sigma}{4iz^3\sqrt{[x^2-\beta^2(y^2+z^2)]^3}\left[\sigma-\frac{xy-iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]}-\frac{(x^2-\beta^2z^2)^2\,d\sigma}{4z^2\sqrt{[x^2-\beta^2(y^2+z^2)]^2}\left[\sigma-\frac{xy-iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}\\&+\frac{(x^2-\beta^2z^2)^3d\sigma}{4iz^3\sqrt{[x^2-\beta^2(y^2+z^2)]^3}\left[\sigma-\frac{xy+iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]}-\frac{(x^2-\beta^2z^2)^2\,d\sigma}{4z^2\sqrt{[x^2-\beta^2(y^2+z^2)]^2}\left[\sigma-\frac{xy+iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}\Bigg\}\end{aligned} \tag{B5}$$

The expressions (B5) may now be integrated (reference 7, equation (207)) giving the expression for the complete second term of equation (B2)

$$\begin{aligned}&\pm\frac{\pi I}{2}\left\{\sqrt{C^2-\left[\frac{xy-iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}-\sqrt{C^2-\left[\frac{xy+iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}\right\}\\&\qquad+\frac{i\pi Iz\sqrt{x^2-\beta^2(y^2+z^2)}}{2(x^2-\beta^2z^2)^2}\left\{\frac{xy-iz\sqrt{x^2-\beta^2(y^2+z^2)}}{\sqrt{C^2-\left[\frac{xy-iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}}-\frac{xy+iz\sqrt{x^2-\beta^2(y^2+z^2)}}{\sqrt{C^2-\left[\frac{xy+iz\sqrt{x^2-\beta^2(y^2+z^2)}}{x^2-\beta^2z^2}\right]^2}}\right\}\end{aligned} \tag{B6}$$

Combining equations (B4) and (B6) and setting $z=0$ yields for u on the surface

$$u=\pm\frac{\pi IC^2}{\sqrt{C^2-\left(\frac{y}{x}\right)^2}} \tag{B7}$$

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