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FLUTTER AND OSCILLATING AIR-FORCE CALCULATIONS FOR AN AIRFOIL IN A TWO-DIMENSIONAL SUPERSONIC FLOW

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SUMMARY

A connected account is given of the Possio theory of nonstationary flow for small disturbances in a two-dimensional supersonic flow and of its application to the determination of the aerodynamic forces on an oscillating airfoil. Further application is made to the problem of wing flutter in the degrees of freedom—torsion, bending, and aileron rotation. Numerical tables for flutter calculations are provided for various values of the Mach number greater than unity. Results for bending-torsion wing flutter are shown in figures and are discussed. The static instabilities of divergence and aileron reversal are examined as is a one-degree-of-freedom case of torsional oscillatory instability.

INTRODUCTION

The problem of flutter or aerodynamic instability for high-speed aircraft is of considerable importance and hence interest is directed to the aerodynamic problem of the oscillating airfoil moving forward at high speed. Although for conventional aircraft the subsonic and the near-sonic or transonic speed ranges are still of main interest, the supersonic speed range is becoming increasingly significant.

A theoretical treatment of the oscillating airfoil of infinite aspect ratio moving at supersonic speed has been given by Possio (reference 1). This treatment is based on the theory of small perturbations to the main stream, thus is essentially an acoustic theory, and leads to linearization of the equation satisfied by the velocity potential. The airfoil is therefore assumed to be very thin, at small angle of attack, and the flow is assumed nonviscous, unseparated, and free from strong shocks.

The small-disturbance linearized theory, being much less complicated than a more rigorous nonlinear theory, is to be regarded as an expedient which allows an initial theoretical solution. The theory permits the occurrence of weak (infinitesimally small) shocks and thus the basic trends and effects of the parameters of the simplified problem can be indicated. The theory reduces to that of Ackeret in the stationary (static) case and, like it, is not expected to be valid too near $M=1$. In view of the restrictions and assumptions in the analysis, important modifications may be required in certain cases for thick finite airfoils; but even here the simple theory for thin wing sections may serve as a basis.

In addition to Possio's brief work, an equivalent extended treatment has been given by Borbely (reference 2) which utilizes contour integrations to carry out the solution of the partial differential equation for the velocity potential according to the Heaviside operator method or Laplace transform method. Recently, another equivalent treatment has been given in England by Temple and Jahn employing the method of characteristics. In reference 1 a few curves are given for the aerodynamic coefficients but no numerical values are tabulated. Reference 2 contains no numerical results. Temple and Jahn recognize the lack of numerical results and supply some initial calculations for the functions necessary for flutter calculations.

A paper has recently appeared by Schwarz (reference 3) devoted to computing and tabulating the key mathematical functions that arise in the theory. The present paper makes use of reference 3 to supply more extensive numerical tables for application of the theory. The formulas of the theory are recast in more familiar form for application to the flutter problem and a series of calculations on bending-torsion flutter are carried out and discussed. The performance of similar calculations for wing-aileron flutter is indicated. Brief discussions also are given of the static instabilities, divergence and aileron reversal, and of a one-degree-of-freedom torsional oscillatory instability.

For completeness, a connected account of the Possio theory is presented since the original presentation in Italian is quite terse and also since it is believed that this treatment is the simplest and most suitable for general extensions. The extension of its application to include the aileron is given.

AIR FORCES AND MOMENTS ON AN OSCILLATING AIRFOIL MOVING AT SUPERSONIC SPEED IN TWO-DIMENSIONAL FLOW

DIFFERENTIAL EQUATION FOR THE VELOCITY POTENTIAL

The differential equation satisfied by the velocity potential in fixed coordinates in the case of infinitesimal disturbances is the wave equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\nabla^2\phi \tag{1}$$

where c is the velocity of sound in the undisturbed medium. $\left(\text{For the adiabatic equation of state } c^2=\frac{dp}{d\rho}=\gamma\frac{p}{\rho}.\right)$

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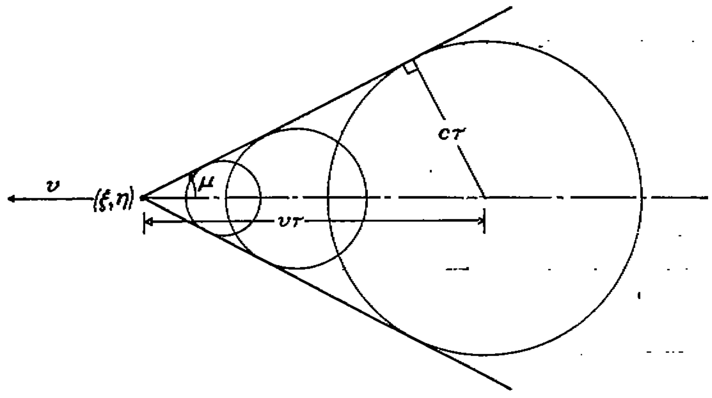


FIGURE 1.—Mach angle μ. The disturbance at point (ξ, η) moving forward with supersonic velocity v influences the angular region having half vertex angle $\mu=\sin^{-1}\frac{c}{v}$.

Referred to a system of rectangular coordinates moving forward at a constant supersonic speed v in the negative x-direction, the wave equation satisfied by the velocity potential in two-dimensional flow becomes

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}+\frac{2v}{c^2}\frac{\partial^2\phi}{\partial x\,\partial t}+\left[\left(\frac{v}{c}\right)^2-1\right]\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}=0 \tag{2}$$

It is proposed to treat the effect of a slightly cambered thin airfoil moving forward at a supersonic speed v at small (zero) angle of attack as that of a distribution of small disturbances placed along the x-axis and hence to utilize equation (2). The velocity components in the x- and y-directions relative to the moving airfoil are, respectively,

$$v_x=\frac{\partial\phi}{\partial x}$$

and

$$v_y=\frac{\partial\phi}{\partial y}$$

which may be considered the additional components to the main stream due to the disturbance created by the presence of the airfoil. Relative to coordinates fixed in space, the velocity components are $v+v_x$ and v_y.

EFFECT OF A SOURCE

Equation (2) is linear and solutions are therefore additive. An important particular solution of equation (2) having the property of a source pulse is

$$\phi_0=\frac{A(\xi, \eta, T)}{\sqrt{c^2(t-T)^2-[x-\xi-v(t-T)]^2-(y-\eta)^2}} \tag{3}$$

This solution may be considered to give the effect at a point (x, y) at time t of a disturbance of magnitude A originating at a point (ξ, η) at an earlier time T. The potential ϕ_0 is thus a retarded potential and the elapsed time at (x, y) since the creation of the disturbance is $\tau=t-T$.

Unlike the situation for a subsonic flow, for a supersonic flow the effect of the disturbance is propagated only downstream; that is, the point being influenced (x, y) is always considered to be aft of the point of disturbance (ξ, η). Equation (3) is thus valid in the angular region with vertex at (ξ, η) and bounded by two straight lines making the Mach angles $\pm\mu=\pm\sin^{-1}\frac{c}{v}=\pm\sin^{-1}\frac{1}{M}$ with respect to the x-axis. (See fig. 1.) Upstream from this angular region the value of ϕ_0 is zero. It follows also that disturbances in the wake need not be considered and the solution to the boundary problem may be attempted by a distribution of potentials of the type ϕ_0 taken along the projection of the airfoil on the x-axis.

A disturbance at (ξ, η) created at time T is first felt at a point (x, y) after a certain time τ_1 has elapsed. The point (x, y) penetrates the wave front of the disturbed region and because it is moving at a speed greater than that of the wave front it emerges from the disturbed region at a later time τ_2. Thus, the duration of this initial disturbance at (x, y) is $\tau_2-\tau_1$. (See fig. 2.) The transition at (x, y) from a region of quiescence to a region of disturbance and vice versa is associated with the vanishing of the denominator in equation (3). The values of τ_1 and τ_2 for a disturbance created on the axis $\eta=0$ are thus given by

$$\tau_{1,2}=\frac{M(x-\xi)\mp\sqrt{(x-\xi)^2-y^2(M^2-1)}}{c(M^2-1)} \tag{4}$$

where the minus sign is associated with τ_1 and the plus sign with τ_2 and where $M=\frac{v}{c}$. It may also be observed that a negative quantity under the radical sign in equation (3) is to be interpreted as associated with an undisturbed region (that is, with $\phi=0$).

POTENTIAL FOR A DISTRIBUTION OF SOURCES

The total effect at any point (x, y) is the sum of the effects of disturbances originating between the leading edge $\xi=0$ and the intersection of the Mach line through (x, y) with the ξ-axis

$$\xi=\xi_1=x-y\sqrt{M^2-1}$$

(since only disturbances created forward of the Mach angle region can affect (x, y); see fig. 3).

The total potential at (x, y) at any time t is thus given by

$$\begin{aligned}\phi(x, y, t)&=\int_0^{\xi_1}\int_{\tau_1}^{\tau_2}\frac{A(\xi, 0, t-\tau)}{\sqrt{c^2\tau^2-(x-\xi-v\tau)^2-y^2}}\,d\tau\,d\xi\\&=\frac{1}{\sqrt{v^2-c^2}}\int_0^{\xi_1}\int_{\tau_1}^{\tau_2}\frac{A(\xi, 0, t-\tau)}{\sqrt{(\tau-\tau_1)(\tau_2-\tau)}}\,d\tau\,d\xi\end{aligned} \tag{5}$$

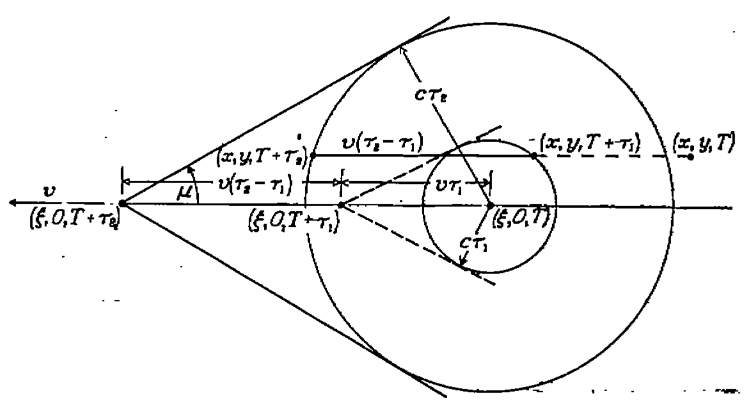


FIGURE 2.—Influence of impulse created at point $(\xi, 0)$ at time $t=T$ on a point (x, y) fixed relative to $(\xi, 0)$ and moving with supersonic velocity v. (Observe that the disturbance influences the point (x, y) only during the time interval $\tau_2-\tau_1$.)

BOUNDARY CONDITION AND STRENGTH OF DISTRIBUTION

The function $A(\xi, 0, t-\tau)$ giving the magnitude of the source distribution is now to be determined by the usual boundary condition of tangential flow along the airfoil. If the ordinate of any point of the mean line defining the airfoil is given as $y=y_m(x, t)$, the boundary condition may be written

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=0}=w(x,t)=\frac{dy}{dt}$$

$$=v\,\frac{\partial y_m}{\partial x}+\frac{\partial y_m}{\partial t} \tag{6}$$

where $w(x, t)$ thus represents the vertical velocity induced by the source distribution in order to realize tangential flow at the airfoil boundary. (In the nonstationary case as in the stationary case (corresponding to the Ackeret treatment), the two surfaces of the airfoil may be considered as acting independently of each other. For the purpose of obtaining the oscillating forces in the linear treatment it is sufficient, however, to consider separately the upper and lower sides of only the mean line.)

The evaluation of $\frac{\partial\phi}{\partial y}$ as y approaches zero may be readily obtained by use of the variable θ instead of τ where $2\tau=(\tau_2-\tau_1)\cos\theta+\tau_2+\tau_1$. This substitution in equation (5) yields

$$\phi=\frac{1}{\sqrt{v^2-c^2}}\int_0^{\xi_1}\int_0^{\pi}A\left(\xi,0,t-\frac{\tau_2+\tau_1}{2}-\frac{\tau_2-\tau_1}{2}\cos\theta\right)d\theta\,d\xi$$

By differentiation with regard to y and with the aid of an integration by parts

$$\frac{\partial\phi}{\partial y}=\frac{1}{\sqrt{v^2-c^2}}\frac{\partial\xi_1}{\partial y}\,\pi A\left(\xi_1,0,t-\frac{My}{c\sqrt{M^2-1}}\right)+$$

$$\frac{1}{\sqrt{v^2-c^2}}\,\frac{y}{c\sqrt{M^2-1}}\int_0^{\xi_1}\int_0^{\pi}\frac{\partial^2A}{\partial t^2}\sin^2\theta\,d\theta\,d\xi$$

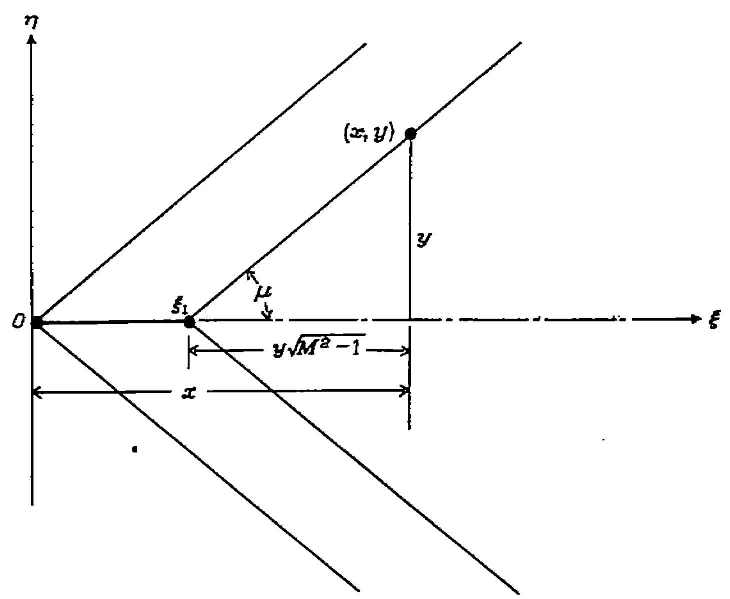


FIGURE 3.—Sketch showing that only disturbances created forward of the Mach angle region with vertex at ξ_1 can affect (x, y).

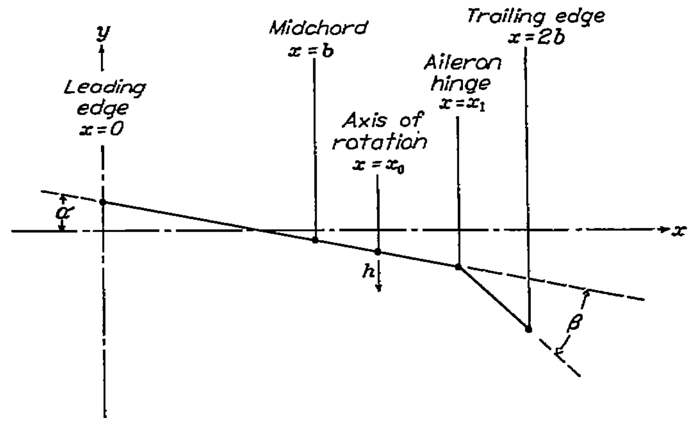


FIGURE 4.—Sketch illustrating the three degrees of freedom h, α, and β of the oscillating airfoil.

Since $\xi_1=x-y\sqrt{M^2-1}$, there results in the limit as y approaches zero on the positive side the important relation

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=+0}=-\frac{\pi}{c}\,A(x,0,t)$$

or, briefly,

$$A(x,t)=-\frac{c}{\pi}\,w(x,t) \tag{7}$$

For y approaching 0 on the negative side an equal and opposite result is obtained and hence the distribution of singularities to be utilized to replace the airfoil is of the source-sink type. Thus ϕ is to be understood in the subsequent analysis to be prefixed by a $\pm$ sign, $+$ for the upper side and $-$ for the lower side.

The total potential for $y=0$ may now be expressed by means of equations (5) and (7) as

$$\phi(x,t)=-\frac{1}{\pi}\,\frac{1}{\sqrt{M^2-1}}\int_0^x\int_{\tau_1}^{\tau_2}\frac{w(\xi,\,t-\tau)}{\sqrt{(\tau-\tau_1)(\tau_2-\tau)}}\,d\tau\,d\xi \tag{8}$$

where, from equation (4) with $y=0$,

$$\tau_1=\frac{x-\xi}{c}\,\frac{1}{M+1}$$

and

$$\tau_2=\frac{x-\xi}{c}\,\frac{1}{M-1}$$

APPLICATION TO OSCILLATING AIRFOIL

The general result given by equation (8) may now be applied for definiteness to the case of an airfoil performing small sinusoidal oscillations in several degrees of freedom. Let the wing undergo the following motions: a motion due to displacement h (velocity $\dot{h}$) in a vertical direction; a torsional motion consisting of a turning about $x=x_0$ with instantaneous angle of attack α; a rotation of an aileron about its hinge at $x=x_1$ with instantaneous aileron angle β measured with respect to α. (See fig. 4.)

In accordance with equation (6) the vertical velocity at any point x of the airfoil situated at $0\leqq x\leqq 2b$ (of chord $2b$ and leading edge at $x=0$) is easily recognized to be

$$w(x,t)=-[\dot{h}+v\alpha+(x-x_0)\dot{\alpha}+v\beta+(x-x_1)\dot{\beta}] \tag{9}$$

where the β-terms are to be interpreted as zero for $x<x_1$ (and where the minus sign is introduced because the vertical velocity w is positive upwards whereas the terms within the brackets are positive downwards).

It is convenient in treating sinusoidal motion to utilize the complex notation

$$\left.\begin{aligned} h&=h_0e^{i\omega t}\\ \alpha&=\alpha_0e^{i\omega t}\\ \beta&=\beta_0e^{i\omega t}\end{aligned}\right\} \tag{10}$$

where h_0, α_0, and β_0 are complex amplitudes and hence include phase angles.

Since the further analysis is concerned only with exponential time variations of the type given in equation (10), the function $w(\xi,\ t-\tau)$ occurring in equation (8) is of the form $w(\xi)e^{i\omega(t-\tau)}$, which may also be written for convenience as $w(\xi,t)e^{-i\omega\tau}$. The potential ϕ given by equation (8) may now be written as

$$\phi(x,t)=-\frac{1}{\sqrt{M^2-1}}\int_0^x w(\xi,t)\,I(\xi,x)\,d\xi \tag{11}$$

where

$$I(\xi,x)=\frac{1}{\pi}\int_{\tau_1}^{\tau_2}\frac{e^{-i\omega\tau}}{\sqrt{(\tau-\tau_1)(\tau_2-\tau)}}d\tau$$

The integration with regard to τ may be readily performed by substitution of the variable θ where $2\tau=(\tau_2-\tau_1)\cos\theta+\tau_2+\tau_1$. Then

$$I(\xi,x)=\frac{1}{\pi}e^{-i\omega(\tau_2+\tau_1)/2}\int_0^\pi e^{-i\omega\cos\theta(\tau_2-\tau_1)/2}\ d\theta$$

With τ_1 and τ_2 replaced by their values as given for equation (8) and with the aid of the Bessel function relation

$$\frac{1}{\pi}\int_0^\pi e^{-i\lambda\cos\theta}\ d\theta=J_0(\lambda)$$

it is recognized that

$$I(\xi,x)=e^{-i\omega\frac{x-\xi}{c}\frac{M}{M^2-1}}J_0\left(\frac{x-\xi}{c}\,\frac{\omega}{M^2-1}\right) \tag{12}$$

Throughout the subsequent analysis it is convenient to employ the variables x and ξ in a new sense to mean nondimensional quantities obtained by dividing the old variables by the chord $2b$. The retaining of the symbols x and ξ for the nondimensional variables should lead to no confusion.

The potential ϕ of equation (11) is then

$$\phi(x,t)=\frac{2b}{\sqrt{M^2-1}}\int_0^x[v\alpha+\dot{h}+2b(\xi-x_0)\,\dot{\alpha}+v\beta+2b(\xi-x_1)\dot{\beta}]I(\xi,x)d\xi \tag{13}$$

where with the introduction of the important frequency parameters

$$k=\frac{\omega b}{v}$$

$$\overline{\omega}=\frac{2kM^2}{M^2-1}$$

the function $I(\xi,\ x)$ becomes

$$I(\xi,\ x)=e^{-i\overline{\omega}(x-\xi)}J_0\left[\frac{\overline{\omega}}{M}(x-\xi)\right] \tag{12'}$$

Thus, $I(\xi,\ x)$ is a function of the variable $x-\xi$ and of two parameters M and $\overline{\omega}$ or, alternatively, M and k.

It is desirable to express the potential ϕ as the sum of the separate effects due to position and motion of the airfoil associated with the individual terms in equation (13). Thus

$$\phi(x,\ t)=\phi_\alpha+\phi_{\dot{h}}+\phi_{\dot{\alpha}}+\phi_\beta+\phi_{\dot{\beta}} \tag{14}$$

where

$$\phi_\alpha=\frac{2b}{\sqrt{M^2-1}}\ v\alpha\int_0^x I(\xi,\ x)d\xi$$

$$\phi_{\dot{h}}=\frac{2b}{\sqrt{M^2-1}}\ \dot{h}\int_0^x I(\xi,\ x)d\xi$$

$$\phi_{\dot{\alpha}}=\frac{4b^2}{\sqrt{M^2-1}}\ \dot{\alpha}\int_0^x (\xi-x_0)I(\xi,\ x)d\xi$$

$$\phi_\beta=\frac{2b}{\sqrt{M^2-1}}\ v\beta\int_{x_1}^x I(\xi,\ x)d\xi$$

$$\phi_{\dot{\beta}}=\frac{4b^2}{\sqrt{M^2-1}}\ \dot{\beta}\int_{x_1}^x (\xi-x_1)I(\xi,\ x)d\xi$$

FORCES AND MOMENTS

The basic pressure formula in the theory of small disturbances is

$$p=-\rho\frac{d\phi}{dt}$$

which in the present case of the moving airfoil may be expressed as

$$p=-\rho\left(\frac{\partial\phi}{\partial t}+v\frac{\partial\phi}{\partial x}\right)$$

where ρ is the density in the undisturbed medium. The local pressure difference on the airfoil surface between the upper and lower surfaces at any point x (nondimensional) is

$$p'=-2\rho\left(\frac{\partial\phi}{\partial t}+\frac{v}{2b}\frac{\partial\phi}{\partial x}\right) \tag{15}$$

The total force (positive downward) on the airfoil is

$$\begin{aligned}P&=2b\int_0^1 p'dx\\ &=-2\rho v\int_0^1\frac{\partial\phi}{\partial x}\ dx-4\rho b\int_0^1\dot{\phi}\ dx\end{aligned} \tag{16}$$

The moment (positive clockwise; fig. 4) on the entire airfoil about any point x_0 is

$$\begin{aligned}M_\alpha&=4b^2\int_0^1 (x-x_0)p'dx\\ &=-4\rho bv\int_0^1\frac{\partial\phi}{\partial x}(x-x_0)dx-8\rho b^2\int_0^1\dot{\phi}(x-x_0)dx\end{aligned} \tag{17}$$

Similarly, the moment (positive clockwise; fig. 4) on the aileron about the hinge point x_1 is

$$\begin{aligned}M_\beta&=4b^2\int_{x_1}^1(x-x_1)\ p'dx\\ &=-4\rho bv\int_{x_1}^1\frac{\partial\phi}{\partial x}\ (x-x_1)\ dx-8\rho b^2\int_{x_1}^1\dot{\phi}\ (x-x_1)\ dx\end{aligned} \tag{18}$$

In the further reduction of equations (16) to (18), with the potential ϕ replaced by its separated form given in equation (14), the following sets of integral evaluations are required:

$$\int_0^1 \frac{\partial \phi_\alpha}{\partial x}\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\alpha r_1\,(M,k)$$

$$\int_0^1 \frac{\partial \phi_{\dot\alpha}}{\partial x}\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\alpha\,[r_2(M,k)-x_0r_1\,(M,k)]$$

$$\int_{x_1}^1 \frac{\partial \phi_\beta}{\partial x}\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\beta t_1\,(M,k,x_1)$$

$$\int_{x_1}^1 \frac{\partial \phi_{\dot\beta}}{\partial x}\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\beta t_2\,(M,k,x_1)$$

$$\int_0^1 \phi_\alpha\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\alpha r_2\,(M,k)$$

$$\int_0^1 \phi_{\dot\alpha}\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\alpha\left[\frac{1}{2}\,r_3(M,k)-x_0r_2(M,k)\right]$$

$$\int_{x_1}^1 \phi_\beta\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\beta\,t_2(M,k,x_1)$$

$$\int_{x_1}^1 \phi_{\dot\beta}\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\beta\,\frac{1}{2}\,t_3(M,k,x_1)$$

$$\int_0^1 \frac{\partial \phi_\alpha}{\partial x}\,x\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\alpha\,q_1\,(M,k)$$

$$\int_0^1 \frac{\partial \phi_{\dot\alpha}}{\partial x}\,x\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\alpha\left[\frac{1}{2}\,q_2(M,k)-x_0q_1(M,k)\right]$$

$$\int_{x_1}^1 \frac{\partial \phi_\beta}{\partial x}\,x\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\beta\,[s_1(M,k,x_1)+x_1t_1(M,k,x_1)]$$

$$\int_{x_1}^1 \frac{\partial \phi_{\dot\beta}}{\partial x}\,x\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\beta\left[\frac{1}{2}\,s_2(M,k,x_1)+x_1t_2(M,k,x_1)\right]$$

$$\int_0^1 \phi_\alpha x\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\alpha\,\frac{1}{2}\,q_2(M,k)$$

$$\int_0^1 \phi_{\dot\alpha} x\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\alpha\left[\frac{1}{6}\,q_3(M,k)-\frac{1}{2}\,x_0q_2(M,k)\right]$$

$$\int_{x_1}^1 \phi_\beta x\,dx=\frac{2b}{\sqrt{M^2-1}}\,v\beta\left[\frac{1}{2}\,s_2(M,k,x_1)+x_1t_2(M,k,x_1)\right]$$

$$\int_{x_1}^1 \phi_{\dot\beta} x\,dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\beta\left[\frac{1}{6}\,s_3(M,k,x_1)+\frac{1}{2}\,x_1t_3(M,k,x_1)\right]$$

$$\int_{x_1}^1 \frac{\partial \phi_\alpha}{\partial x}\,(x-x_1)dx=\frac{2b}{\sqrt{M^2-1}}\,v\alpha\,p_1(M,k,x_1)$$

$$\int_{x_1}^1 \frac{\partial \phi_{\dot\alpha}}{\partial x}\,(x-x_1)dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\alpha\left[\frac{1}{2}\,p_2(M,k,x_1)-x_0p_1(M,k,x_1)\right]$$

$$\int_{x_1}^1 \frac{\partial \phi_\beta}{\partial x}\,(x-x_1)dx=\frac{2b}{\sqrt{M^2-1}}\,v\beta\,s_1(M,k,x_1)$$

$$\int_{x_1}^1 \frac{\partial \phi_{\dot\beta}}{\partial x}\,(x-x_1)dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\beta\,\frac{1}{2}\,s_2(M,k,x_1)$$

$$\int_{x_1}^1 \phi_\alpha(x-x_1)dx=\frac{2b}{\sqrt{M^2-1}}\,v\alpha\,\frac{1}{2}\,p_2(M,k,x_1)$$

$$\int_{x_1}^1 \phi_{\dot\alpha}(x-x_1)dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\alpha\left[\frac{1}{6}\,p_3(M,k,x_1)-\frac{1}{2}\,x_0p_2(M,k,x_1)\right]$$

$$\int_{x_1}^1 \phi_\beta(x-x_1)dx=\frac{2b}{\sqrt{M^2-1}}\,v\beta\,\frac{1}{2}\,s_2(M,k,x_1)$$

$$\int_{x_1}^1 \phi_{\dot\beta}(x-x_1)dx=\frac{4b^2}{\sqrt{M^2-1}}\,\dot\beta\,\frac{1}{6}\,s_3(M,k,x_1)$$

The functions defined by the foregoing integral evaluations are further discussed in the following section; first, however, the force and moments (equations (16) to (18)) are given in their final forms as

$$P=-\frac{4\rho b}{\sqrt{M^2-1}}\left[v(v\alpha+\dot h-2bx_0\dot\alpha)r_1+2b(2v\dot\alpha+\ddot h-2bx_0\ddot\alpha)r_2+\right.$$
$$\left.4b^2\ddot\alpha\,\frac{r_3}{2}+v^2\beta t_1+4bv\dot\beta t_2+4b^2\ddot\beta\,\frac{t_3}{2}\right] \tag{16'}$$

$$M_\alpha=-\frac{8\rho b^2}{\sqrt{M^2-1}}\left[v(v\alpha+\dot h-2bx_0\dot\alpha)q_1+2b(2v\dot\alpha+\ddot h-2bx_0\ddot\alpha)\,\frac{q_2}{2}+\right.$$
$$4b^2\ddot\alpha\,\frac{q_3}{6}+v^2\beta(s_1+x_1t_1)+4bv\dot\beta\left(\frac{s_2}{2}+x_1t_2\right)+$$
$$\left.4b^2\ddot\beta\left(\frac{s_3}{6}+x_1\,\frac{t_3}{2}\right)\right]-2bx_0P \tag{17'}$$

$$M_\beta=-\frac{8\rho b^2}{\sqrt{M^2-1}}\left[v\,(v\alpha+\dot h-2bx_0\dot\alpha)\,p_1+2b\,(2v\dot\alpha+\ddot h-2bx_0\ddot\alpha)\,\frac{p_2}{2}+\right.$$
$$\left.4b^2\ddot\alpha\,\frac{p_3}{6}+v^2\beta s_1+4bv\dot\beta\,\frac{s_2}{2}+4b^2\ddot\beta\,\frac{s_3}{6}\right] \tag{18'}$$

REDUCTION AND EVALUATION OF FOREGOING INTEGRALS

It is convenient to introduce the substitution $u=x-\xi$ and to express the function $I(\xi,x)$ (equation (12′)) as

$$I\,(\xi,x)=I\,(u)=e^{-i\bar\omega u}J_0\left(\frac{\bar\omega}{M}u\right) \tag{19}$$

The various functions defined by the foregoing sets of integrals may now be expressed as follows:

$$r_1\,(M,k)=\int_0^1 I\,(u)\,du$$

$$r_2\,(M,k)=\int_0^1\int_0^x I\,(u)\,du\,dx$$

$$r_3\,(M,k)=2\int_0^1\int_0^x (x-u)\,I\,(u)\,du\,dx$$

$$q_1\,(M,k)=\int_0^1 uI\,(u)\,du$$

$$q_2\,(M,k)=2\int_0^1\int_0^x xI\,(u)\,du\,dx$$

$$q_3\,(M,k)=6\int_0^1\int_0^x x\,(x-u)\,I(u)\,du\,dx$$

$$p_1(M, k, x_1)=\int_{x_1}^{1}(u-x_1)I(u)du$$

$$p_2(M, k, x_1)=2\int_{x_1}^{1}\int_{0}^{x}(x-x_1)I(u)du\,dx$$

$$p_3(M, k, x_1)=6\int_{x_1}^{1}\int_{0}^{x}(x-x_1)(x-u)I(u)du\,dx$$

$$t_1(M, k, x_1)=\int_{0}^{1-x_1}I(u)du$$

$$t_2(M, k, x_1)=\int_{0}^{1-x_1}\int_{0}^{x}I(u)du\,dx$$

$$t_3(M, k, x_1)=2\int_{0}^{1-x_1}\int_{0}^{x}(x-u)I(u)du\,dx$$

$$s_1(M, k, x_1)=\int_{0}^{1-x_1}uI(u)du$$

$$s_2(M, k, x_1)=2\int_{0}^{1-x_1}\int_{0}^{x}xI(u)du\,dx$$

$$s_3(M, k, x_1)=6\int_{0}^{1-x_1}\int_{0}^{x}x(x-u)I(u)du\,dx$$

Borbely (reference 2) has shown by means of reduction formulas that the six r- and q-functions may be obtained from a single integral. In a similar manner it may be indicated how the foregoing 15 functions may be obtained from the evaluation of the same integral. The reduction is accomplished in two stages. First, consider integrals of the following type:

$$\left.\begin{aligned} f_\lambda&=f_\lambda(M, \overline{\omega})=\int_0^1 I(u)u^\lambda du \\ g_\lambda&=f_\lambda(M, \overline{\omega}x_1)=\frac{1}{x_1^{\lambda+1}}\int_0^{x_1} I(u)u^\lambda du \\ h_\lambda&=f_\lambda[M, \overline{\omega}(1-x_1)]=\frac{1}{(1-x_1)^{\lambda+1}}\int_0^{1-x_1} I(u)u^\lambda du \end{aligned}\right\} \qquad (20)$$

By integration by parts it can be readily verified that the following relations hold:

$$r_1=f_0$$
$$r_2=f_0-f_1$$
$$r_3=f_0-2f_1+f_2$$
$$q_1=f_1$$
$$q_2=f_0-f_2$$
$$q_3=2f_0-3f_1+f_3$$
$$p_1=q_1-x_1r_1+x_1^2(g_0-g_1)$$
$$p_2=q_2-2x_1r_2+x_1^3(g_0-2g_1+g_2)$$
$$p_3=q_3-3x_1r_3+x_1^4(g_0-3g_1+3g_2-g_3)$$

$$t_1=(1-x_1)h_0$$
$$t_2=(1-x_1)^2(h_0-h_1)$$
$$t_3=(1-x_1)^3(h_0-2h_1+h_2)$$
$$s_1=(1-x_1)^2h_1$$
$$s_2=(1-x_1)^3(h_0-h_2)$$
$$s_3=(1-x_1)^4(2h_0-3h_1+h_3)$$

The final stage in the reduction of these functions is to utilize the following recursion formula (reference 2) obtained by integration by parts:

$$\begin{aligned}\frac{M^2-1}{M^2}\overline{\omega}f_\lambda(M,\overline{\omega})=&\left[i+(1-\lambda)\frac{1}{\overline{\omega}}\right]e^{-i\overline{\omega}}J_0\left(\frac{\overline{\omega}}{M}\right)-\frac{1}{M}e^{-i\overline{\omega}}J_1\left(\frac{\overline{\omega}}{M}\right)+\\ &i(1-2\lambda)f_{\lambda-1}(M,\overline{\omega})+\\ &(1-\lambda)^2\frac{1}{\overline{\omega}}f_{\lambda-2}(M,\overline{\omega}) \end{aligned}\qquad (21)$$

where $\lambda\geqq 1$ and f with a negative subscript is to be interpreted as zero. $\left(\text{Observe that } \frac{M^2-1}{M^2}\overline{\omega}=2k.\right)$

The function $f_\lambda(M, \overline{\omega})$ may clearly refer also to the foregoing g- and h-functions, if $\overline{\omega}$ is replaced by the appropriate parameter; namely, $\overline{\omega}x_1$ for g_λ and $\overline{\omega}(1-x_1)$ for h_λ. (See equations (20).) The recursion relation (equation (21)) thus reduces the various functions to the single function

$$f_0\ (M,\overline{\omega})=\frac{1}{\overline{\omega}}\int_0^{\overline{\omega}}e^{-iu}\ J_0\left(\frac{u}{M}\right)du \qquad (22)$$

which is therefore the only integral needed in the evaluation of the forces and moments.

The important integral in equation (22) has been recently made the subject of a mathematical investigation by Schwarz (reference 3). Schwarz gives tables of the values of its real and imaginary parts to eight decimal places for $0\leqq\overline{\omega}\leqq 5$ and for $1\leqq M\leqq 10$ for conveniently small intervals. For values of $\overline{\omega}>5$ not given in Schwarz' tables, the function f_0 may be evaluated by means of the following series development (reference 2):

$$f_0\ (M,\overline{\omega})=e^{-i\overline{\omega}}\sum_{n=0}^{\infty}\left(\frac{M^2-1}{M^2}\overline{\omega}\right)^n\frac{1}{2^n n!\ (2n+1)}[J_n(\overline{\omega})+iJ_{n+1}(\overline{\omega})] \qquad (23)$$

Table I gives values of the functions $f_0(M,\overline{\omega})$ based on the tables of Schwarz and on equation (23) for selected values of the Mach number $M=\frac{10}{9}, \frac{5}{4}, \frac{10}{7}, \frac{5}{3}, 2, \frac{5}{2}, \frac{10}{3}$, and 5 and for various appropriate values of $\overline{\omega}$ $\left(\text{or } \frac{1}{k}\right)$. Later use is made of the values given in table I for obtaining tables for flutter calculations.

EQUATIONS OF MOTION AND DETERMINANTAL EQUATION FOR FLUTTER CONDITION

The equations of motion and the border-line condition of unstable equilibrium yielding the flutter speed and frequency may be obtained exactly as in the incompressible case treated, for example, in reference 4. The two-dimensional treatment (infinite aspect ratio) is retained herein. Modifications due to assumed vibration modes of the finite wing may of course be introduced as in current practice (for example, reference 5). The modification of the forces and moments due to the three-dimensional nature of the flow is a more difficult problem which remains to be studied.

The equilibrium of the vertical forces, of the moments about the torsional axis $x=x_0$, and of the moments on the aileron about its hinge $x=x_1$ yields the three equations

$$\left.\begin{aligned}&\ddot{h}M+\ddot{\alpha}S_\alpha+\ddot{\beta}S_\beta+hC_h=P\\&\ddot{\alpha}I_\alpha+\ddot{\beta}[I_\beta+2b(x_1-x_0)S_\beta]+\ddot{h}S_\alpha+\alpha C_\alpha=M_\alpha\\&\ddot{\beta}I_\beta+\ddot{\alpha}[I_\beta+2b(x_1-x_0)S_\beta]+\ddot{h}S_\beta+\beta C_\beta=M_\beta\end{aligned}\right\}\tag{24}$$

where the various parameters are defined in the list of notation. (See appendix.)

In order to define the borderline condition of unstable equilibrium separating damped and undamped oscillations, the variables h, α, and β are used in the sinusoidal exponential form given in equation (10). For the desired condition, it is necessary that the equations (24) have a (nontrivial) solution for the complex amplitudes h_0, α_0, and β_0, or that the following determinantal equation hold:

$$\begin{vmatrix}\overline{A}_{ch}&A_{c\alpha}&A_{c\beta}\\A_{ah}&\overline{A}_{a\alpha}&A_{a\beta}\\A_{bh}&A_{b\alpha}&\overline{A}_{b\beta}\end{vmatrix}=0\tag{25}$$

where the complex elements of the determinant in separated form are

$$\overline{A}_{ch}=\Omega_hX-\mu+L_1+iL_2$$
$$A_{c\alpha}=-\mu x_\alpha+L_3+iL_4$$
$$A_{c\beta}=-\mu x_\beta+L_5+iL_6$$
$$A_{ah}=-\mu x_\alpha+M_1+iM_2$$
$$\overline{A}_{a\alpha}=\Omega_\alpha X-\mu r_\alpha^2+M_3+iM_4$$
$$A_{a\beta}=-\mu[r_\beta^2+2(x_1-x_0)x_\beta]+M_5+iM_6$$
$$A_{bh}=-\mu x_\beta+N_1+iN_2$$
$$A_{b\alpha}=-\mu[r_\beta^2+2(x_1-x_0)x_\beta]+N_3+iN_4$$
$$\overline{A}_{b\beta}=\Omega_\beta X-\mu r_\beta^2+N_5+iN_6$$

and where the L's, M's, and N's are defined by the force and moment equations (16′), (17′), and (18′) expressed in the following forms:

$$\left.\begin{aligned}P&=-4\rho bv^2k^2e^{i\omega t}\left[\left(\frac{h_0}{b}\right)(L_1+iL_2)+\alpha_0(L_3+iL_4)+\beta_0(L_5+iL_6)\right]\\M_\alpha&=-4\rho b^2v^2k^2e^{i\omega t}\left[\left(\frac{h_0}{b}\right)(M_1+iM_2)+\alpha_0(M_3+iM_4)+\beta_0(M_5+iM_6)\right]\\M_\beta&=-4\rho b^2v^2k^2e^{i\omega t}\left[\left(\frac{h_0}{b}\right)(N_1+iN_2)+\alpha_0(N_3+iN_4)+\beta_0(N_5+iN_6)\right]\end{aligned}\right\}\tag{26}$$

Hence,

$$L_1+iL_2=\frac{1}{\sqrt{M^2-1}}\left(-2r_2+\frac{i}{k}r_1\right)$$

$$L_3+iL_4=\frac{1}{\sqrt{M^2-1}}\left[-2r_3+\frac{2i}{k}r_2-\frac{i}{k}\left(-2r_2+\frac{i}{k}r_1\right)-2x_0\left(-2r_2+\frac{i}{k}r_1\right)\right]$$

$$L_5+iL_6=\frac{1}{\sqrt{M^2-1}}\left[-2t_3+\frac{2i}{k}t_2-\frac{i}{k}\left(-2t_2+\frac{i}{k}t_1\right)\right]$$

$$M_1+iM_2=\frac{1}{\sqrt{M^2-1}}\left[-2q_2+\frac{2i}{k}q_1-2x_0\left(-2r_2+\frac{i}{k}r_1\right)\right]$$

$$M_3+iM_4=\frac{1}{\sqrt{M^2-1}}\left\{-\frac{4}{3}q_3+\frac{2i}{k}q_2-\frac{i}{k}\left(-2q_2+\frac{2i}{k}q_1\right)-2x_0\left[-2r_3+\frac{2i}{k}r_2-\frac{i}{k}\left(-2r_2+\frac{i}{k}r_1\right)-2q_2+\frac{2i}{k}q_1-2x_0\left(-2r_2+\frac{i}{k}r_1\right)\right]\right\}$$

$$M_5+iM_6=\frac{1}{\sqrt{M^2-1}}\left\{-\frac{4}{3}s_3+\frac{2i}{k}s_2-\frac{i}{k}\left(-2s_2+\frac{2i}{k}s_1\right)+2(x_1-x_0)\left[-2t_3+\frac{2i}{k}t_2-\frac{i}{k}\left(-2t_2+\frac{i}{k}t_1\right)\right]\right\}$$

$$N_1+iN_2=\frac{1}{\sqrt{M^2-1}}\left(-2p_2+\frac{2i}{k}p_1\right)$$

$$N_3+iN_4=\frac{1}{\sqrt{M^2-1}}\left[-\frac{4}{3}p_3+\frac{2i}{k}p_2-\frac{i}{k}\left(-2p_2+\frac{2i}{k}p_1\right)-2x_0\left(-2p_2+\frac{2i}{k}p_1\right)\right]$$

$$N_5+iN_6=\frac{1}{\sqrt{M^2-1}}\left[-\frac{4}{3}s_3+\frac{2i}{k}s_2-\frac{i}{k}\left(-2s_2+\frac{2i}{k}s_1\right)\right]$$

The determinantal equation (25) with the foregoing complex elements is equivalent to two real simultaneous equations and hence may be solved for two unknowns. In a given case the usual unknowns are the flutter speed v and the flutter frequency ω or, more conveniently, the related nondimensional parameters X and $1/k$. The parameter X appears linearly and only in the major diagonal elements (with bars), while the parameter $1/k$ appears transcendentally in every element of the determinant. Hence an obvious procedure, though not the simplest for obtaining the simultaneous solutions of the two equations, is to fix values of $1/k$, to solve for the roots of the two polynomials in X, to plot graphically these roots against $1/k$, and to note the points of intersection.

In a systematic numerical study of flutter any two parameters may be utilized as unknowns instead of X and $1/k$, a procedure which is often more convenient. A discussion of such procedure and the use of a method of elimination for simplifying the calculations is given in the appendix of reference 6.

The application to the two-degree-of-freedom subcase of bending-torsion flutter is treated more fully in the following section.

APPLICATION TO BENDING-TORSION FLUTTER

The determinantal equation in the two degrees of freedom h and α is

$$\begin{vmatrix} \overline{A}_{ch} & A_{c\alpha} \\ A_{\alpha h} & \overline{A}_{\alpha\alpha} \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \Omega_h X - \mu + L_1 + iL_2 & -\mu x_\alpha + L_3 + iL_4 \\ -\mu x_\alpha + M_1 + iM_2 & \Omega_\alpha X - \mu r_\alpha^2 + M_3 + iM_4 \end{vmatrix} = 0 \quad (27)$$

The two equations in X obtained by equating the real and imaginary parts separately to zero are

$$\left.\begin{aligned} \Omega_h\Omega_\alpha X^2 + [\Omega_\alpha(L_1-\mu) + \Omega_h(M_3-\mu r_\alpha^2)]X + C_R = 0 \\ (\Omega_\alpha L_2 + \Omega_h M_4)X + C_I = 0 \end{aligned}\right\} \quad (27')$$

where

$$C_R = \mu[x_\alpha(M_1+L_3) - (M_3-\mu r_\alpha^2) - L_1 r_\alpha^2 - \mu x_\alpha^2] + D_R$$

$$C_I = \mu[x_\alpha(M_2+L_4) - M_4 - L_2 r_\alpha^2] + D_I$$

and where

$$D_R = L_1M_3 - L_3M_1 - L_2M_4 + L_4M_2$$

$$D_I = L_1M_4 - L_4M_1 + L_2M_3 - L_3M_2$$

For convenience in numerical tabulation, it is desirable to introduce primed quantities, independent of the parameter x_0, defined by the following relations:

$$\left.\begin{aligned} L_3 &= L_3' - 2x_0L_1 \\ L_4 &= L_4' - 2x_0L_2 \\ M_1 &= M_1' - 2x_0L_1 \\ M_2 &= M_2' - 2x_0L_2 \\ M_3 &= M_3' - 2x_0[(M_1'+L_3') - 2x_0L_1] \\ M_4 &= M_4' - 2x_0[(M_2'+L_4') - 2x_0L_2] \end{aligned}\right\} \quad (28)$$

In table II convenient expressions for the quantities L_1, L_2, L_3', L_4', M_1', M_2', M_3', and M_4' are given and tabulated together with the combinations $M_1'+L_3'$ and $M_2'+L_4'$. Clearly these quantities depend on the function f_0 given in table I and hence the tabulation is made for the same values of M and $1/k$ (or $\overline{\omega}$). In addition, table II contains values for the quantities D_R and D_I which, in fact, are independent of x_0 and may be expressed as

$$D_R = L_1M_3' - L_3'M_1' - L_2M_4' + L_4'M_2'$$

$$D_I = L_1M_4' - L_4'M_1' + L_2M_3' - L_3'M_2'$$

The numerical application in the case of bending-torsion flutter has been performed for various selected examples. In most of the calculations the numerical procedure was to fix values of $1/k$, eliminate X, and solve for the parameter x_α. Interpolation was also used to obtain additional points in order to improve the fairing of some of the curves. Values of $1/k$ less than 1 did not yield any flutter points in this procedure. Results are shown plotted in a number of figures (figs. 5 to 20); however, before these figures are discussed, it is desirable to explain the significance of the parameters and the numerical values assigned to them.

The parameter μ may be considered to signify the wing density and three selected values 3.927, 7.854, and 15.708 in the order of increasing wing density have been mainly used in the calculations. (These values correspond to values of $\frac{1}{\kappa}=5$, 10, and 20 in the notation of reference 4.) Alternatively, an increase in μ may be interpreted as an increase in altitude for a fixed wing density. The parameter μ may be expected to range up to high values for actual supersonic wings at high altitude. Only a few calculations, however, have been made for high values of μ ($\mu=78.54$, $\frac{1}{\kappa}=100$; see fig. 18).

The parameter ω_h/ω_α is the ratio of the wing bending frequency to the wing torsional frequency and may be expected normally to be less than unity. The three values 0, 0.707, and 1 have been largely used in the calculations although other values up to 2 have also been studied.

The parameter x_0 represents the position of the elastic axis measured from the leading edge and the three values 0.4, 0.5, and 0.6 represent, respectively, positions at 40, 50, and 60 percent chord. (These values correspond to values of $a=-0.2$, 0, and 0.2 in the notation of reference 4.)

The parameter x_α represents the distance of the center of gravity from the elastic axis. For example, $x_\alpha=0.2$ represents a position of the center of gravity 10 percent of the chord behind the elastic axis. In many of the calculations x_α has been regarded as variable.

The parameter r_α^2 represents the radius of gyration of the wing about the elastic axis and has been kept fixed at the value $r_\alpha^2=0.25$.

The ordinate in figures 5 to 20 is the nondimensional flutter coefficient $v/b\omega_\alpha$ where $b\omega_\alpha$ is a convenient reference speed. This coefficient is also a function of the Mach number $M=\frac{v}{c}$ and several values of M have been employed in the calculations.

In a plot of the flutter coefficient $v/b\omega_\alpha$ against M, straight lines drawn from the origin at angle δ and intersecting the curves may be given an interesting interpretation (fig. 17). The slope of the line is given by $\frac{v/b\omega_\alpha}{v/c}=\frac{c}{b\omega_\alpha}$ or $\cot\delta=\frac{b\omega_\alpha}{c}$. Thus, $\cot\delta$ is directly proportional to the product of the chord and the torsional frequency divided by the velocity of sound. The question of whether at a given value of M the value of $b\omega_\alpha$ which will just prevent flutter is also sufficient to prevent flutter at neighboring higher values of M is answered by the simple criterion of whether $\cot\delta$ increases or decreases. In figure 17 two typical flutter curves are shown. In curve B the value of $b\omega_\alpha$ just necessary to prevent flutter at a speed corresponding to the value of M at P_2 is insufficient to prevent flutter at any higher value of M for which the flutter curve is below the straight line $0P_2$. For the type of curve A a maximum value of δ occurs at the "design critical points" P_1. The value of $b\omega_\alpha$ just necessary to prevent flutter at a speed corresponding to the value of M at P_1 is also sufficient to prevent flutter at all higher speeds.

The following salient facts may be extracted by inspection of the figures. Flutter exists or is possible for various ranges of the parameters but, in general, compared with subsonic cases the ranges of the parameters yielding flutter are more restricted.

The chordwise position of the aerodynamic center, the center of the oscillating pressure, is an important factor in the consideration of flutter. In the static case the midchord is the aerodynamic center for $M>>1$. For subsonic speeds, $M<<1$, the linearized theory indicates the quarter-chord position as the aerodynamic center. It should be expected that in the transonic region near $M=1$ the aerodynamic center may shift considerably. From this point of view alone conclusions drawn from the simple theory for the range near $M=1$ may require large modifications. The nature of the modifications may be roughly inferred by further experimental and theoretical study of the behavior of center-of-pressure locations.

For low values of the ratio of bending frequency to torsional frequency $\frac{\omega_h}{\omega_\alpha}\approx 0$ the position of the center of gravity relative to the aerodynamic center is important. For center-of-gravity positions forward of the midchord no flutter exists, whereas for positions behind the midchord there is a sharp decrease in the flutter coefficient from infinity; the position of the elastic axis influences the value of the flutter coefficient in this region, forward positions being more favorable (figs. 5 (a) to 16 (a)).

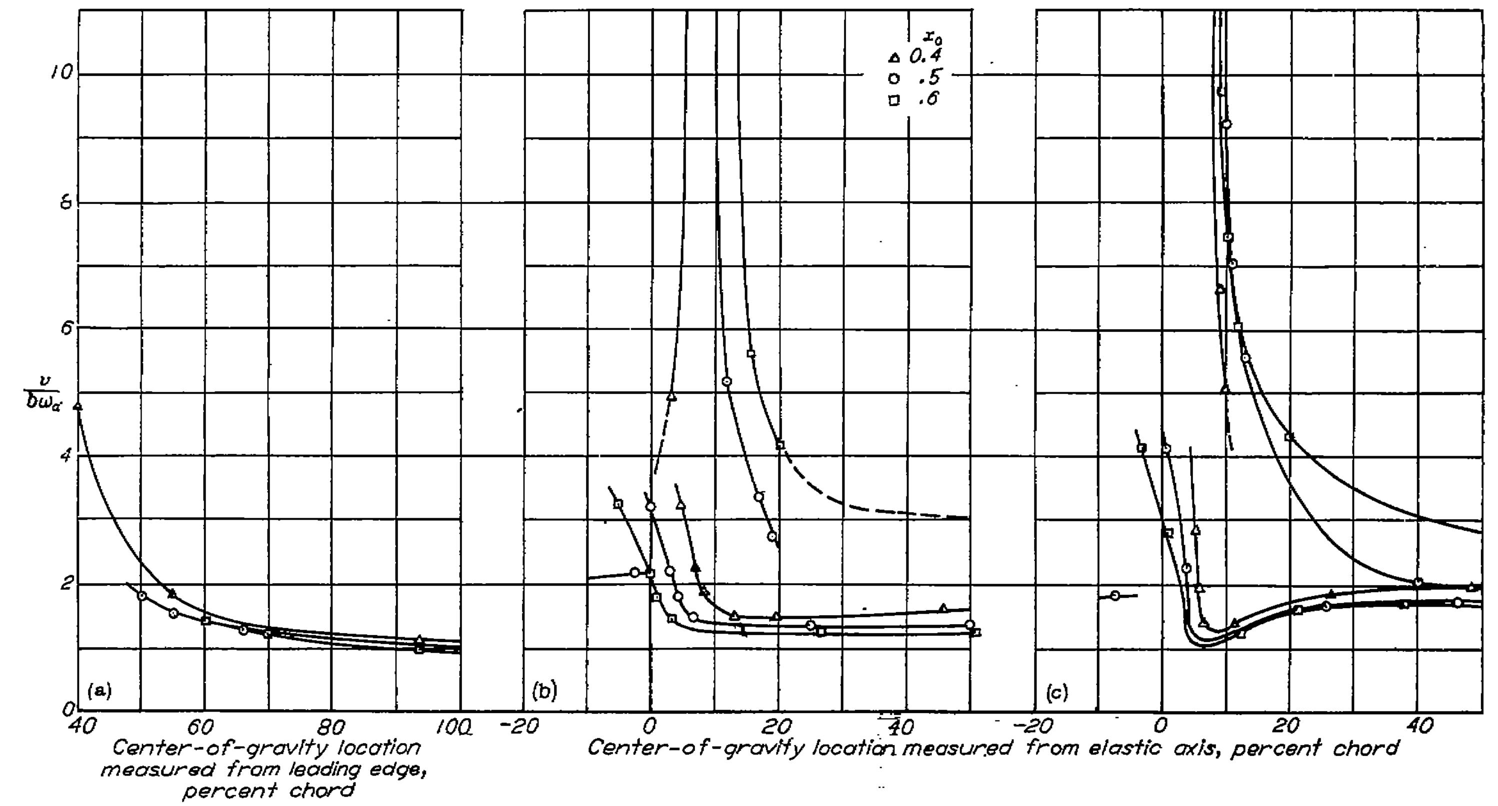
For values of $\frac{\omega_h}{\omega_\alpha}\approx 1$ the position of the center of gravity relative to the elastic axis becomes of more importance. For center-of-gravity positions forward of the elastic axis no flutter exists, whereas for positions behind the elastic axis flutter does occur, and a relative minimum coefficient appears for center-of-gravity positions only slightly (a few percent of the chord) behind the elastic axis.

The intermediate case, for which $\frac{\omega_h}{\omega_\alpha}=0.707$, shows a blending of the effects in which the center-of-gravity position relative both to the aerodynamic center and to the elastic axis is significant.

In figures 12 and 14 there are shown, for reference, some numerical values of ω/ω_α, the ratio of the flutter frequency to the torsional frequency.

The effect of the wing density parameter μ is rather complicated but, in general, an increase in μ yields a corresponding increase in the flutter coefficient. For low values of ω_h/ω_α and for high wing densities this increase is expected to be proportional to $\sqrt{\mu}$. In the resonance-like region near $\frac{\omega_h}{\omega_\alpha}=1$ and for small values of x_α the flutter coefficient is relatively unaffected by the value of μ, and in this region the structural damping may be expected to be particularly effective in increasing the flutter coefficient.

For values of the Mach number near unity (for example $M=\frac{10}{9}$, a value for which the validity of the theory is in question), the flutter calculations become difficult to plot because of the appearance of other branches. In some cases (for instance, $x_0=0.6$) the flutter instability appears limited to a definite range of flutter speed coefficients. Calculations to include damping were performed to verify the existence of the range. (The appearance of these other branches seems to involve values of $1/k$ for which the quantity M_4 is negative. The condition of negative M_4 is significant for the one-degree-of-freedom instability discussed in the next section.)

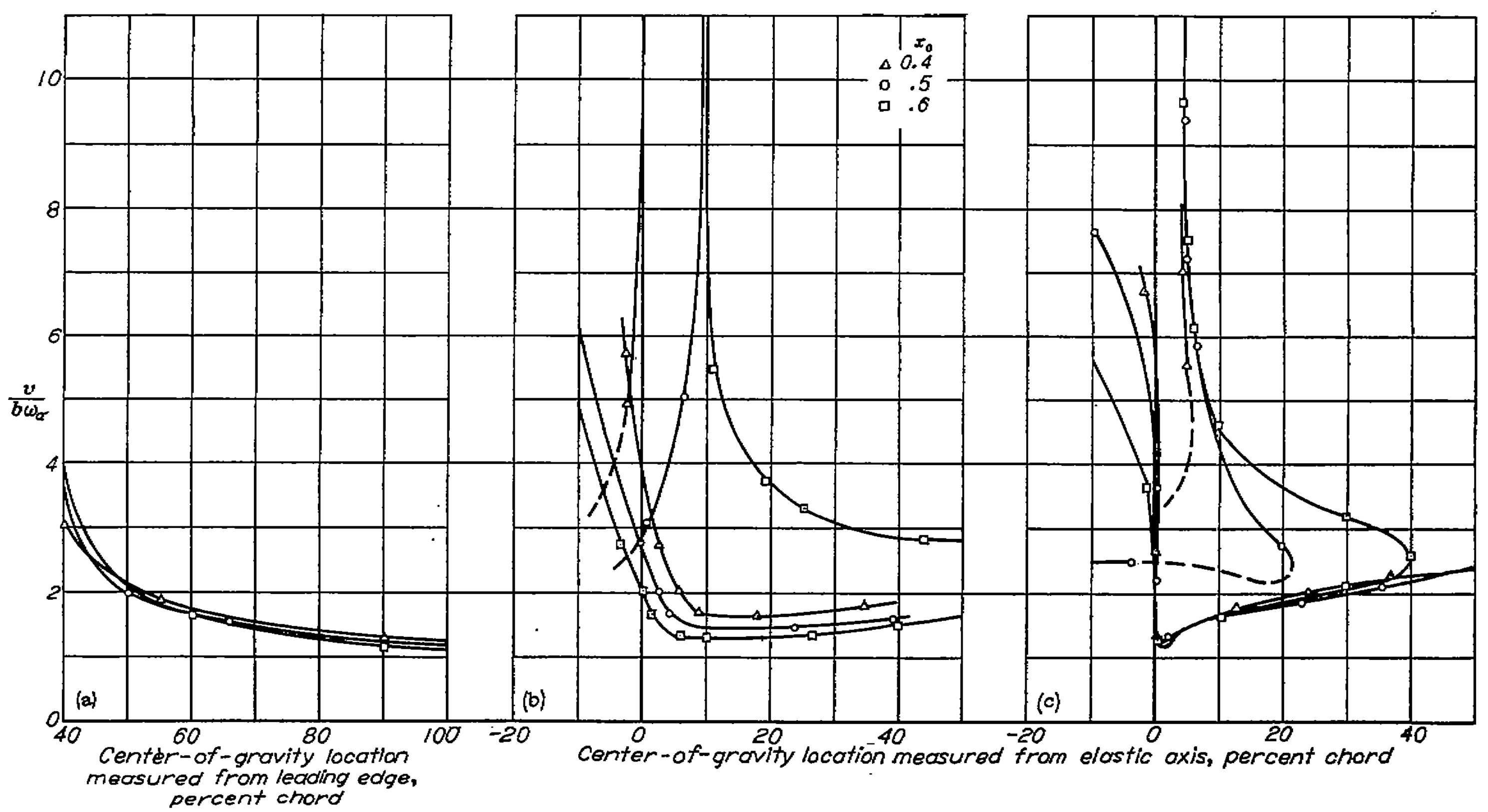


(a) $\frac{\omega_h}{\omega_\alpha}=0$.

(b) $\frac{\omega_h}{\omega_\alpha}=0.707$.

(c) $\frac{\omega_h}{\omega_\alpha}=1.0$.

FIGURE 5.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=\frac{10}{9}$; $\mu=3.927$.

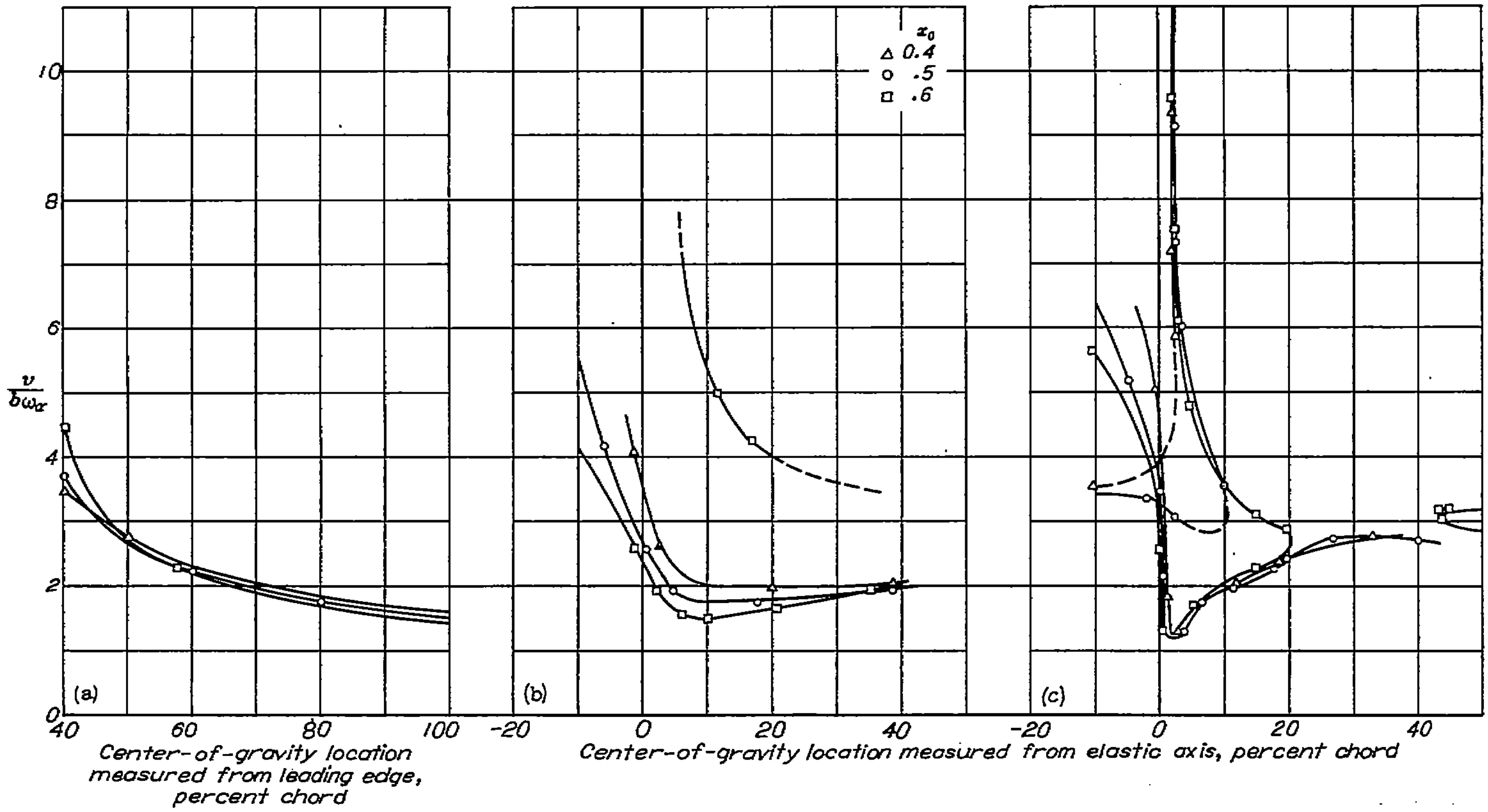


(a) $\frac{\omega_h}{\omega_\alpha}=0$.

(b) $\frac{\omega_h}{\omega_\alpha}=0.707$.

(c) $\frac{\omega_h}{\omega_\alpha}=1.0$.

FIGURE 6.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=\frac{10}{9}$; $\mu=7.854$.

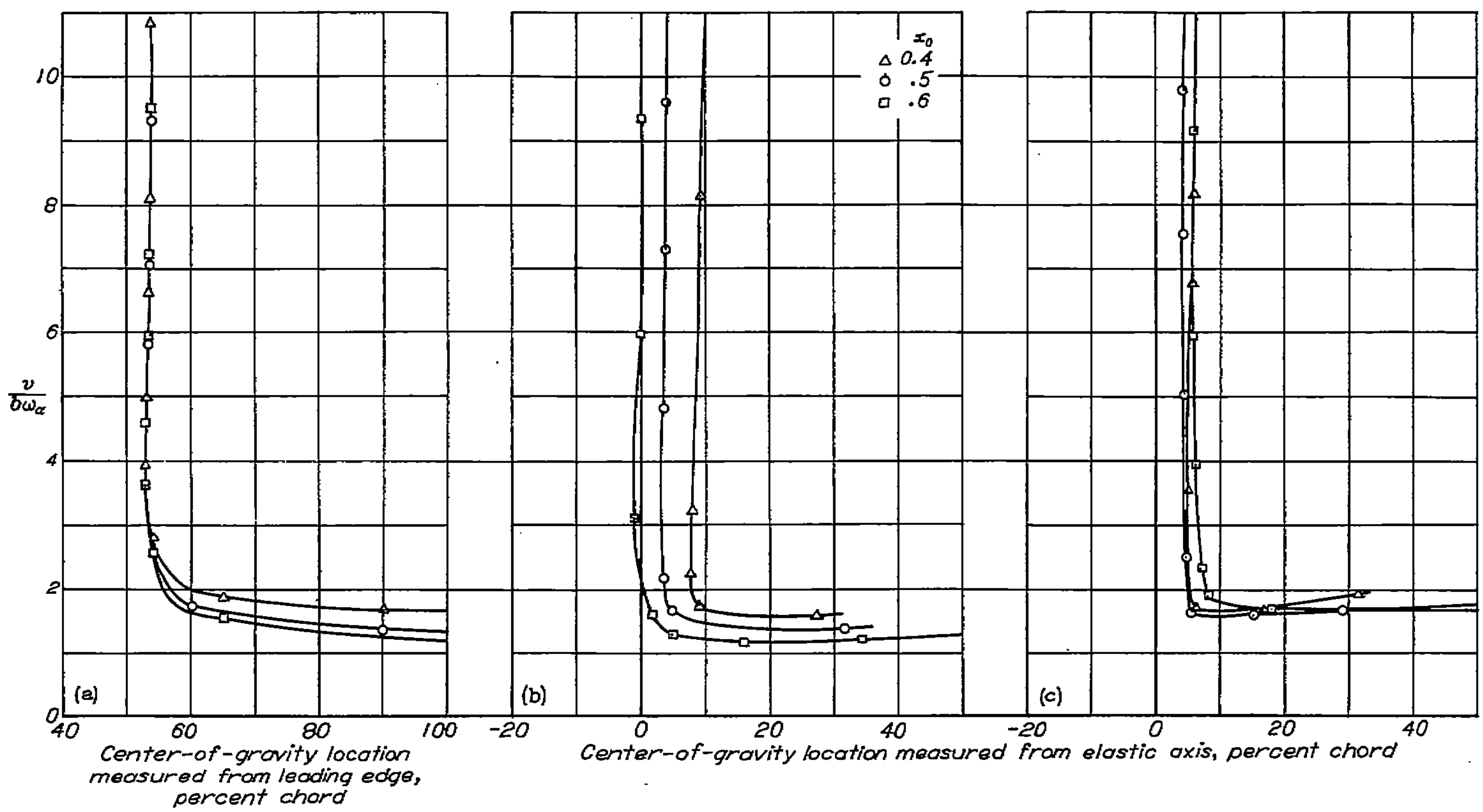


(a) $\frac{\omega_h}{\omega_\alpha}=0$.

(b) $\frac{\omega_h}{\omega_\alpha}=0.707$.

(c) $\frac{\omega_h}{\omega_\alpha}=1.0$.

FIGURE 7.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=\frac{10}{9}$; $\mu=15.708$.

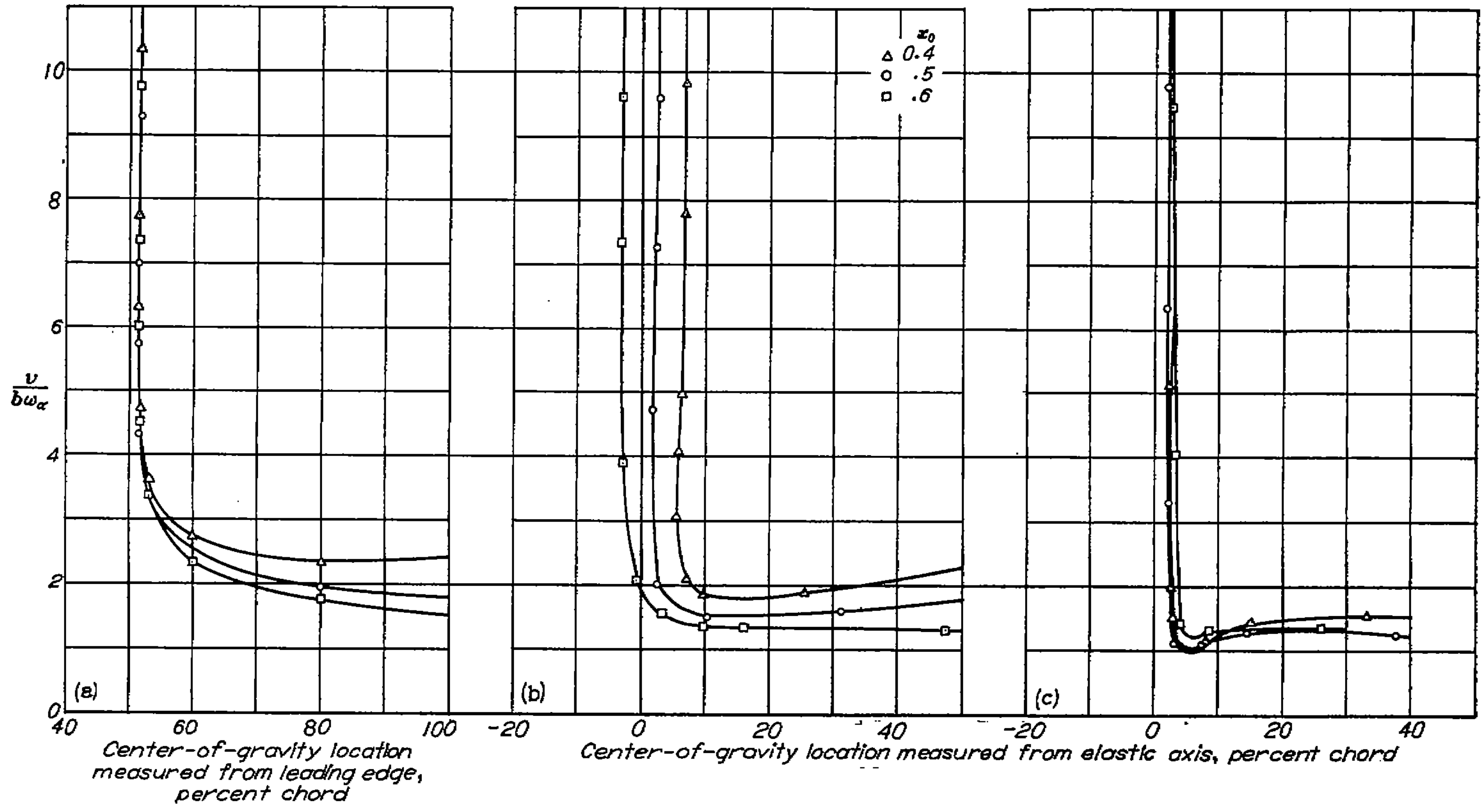


(a) $\frac{\omega_h}{\omega_\alpha}=0$.

(b) $\frac{\omega_h}{\omega_\alpha}=0.707$.

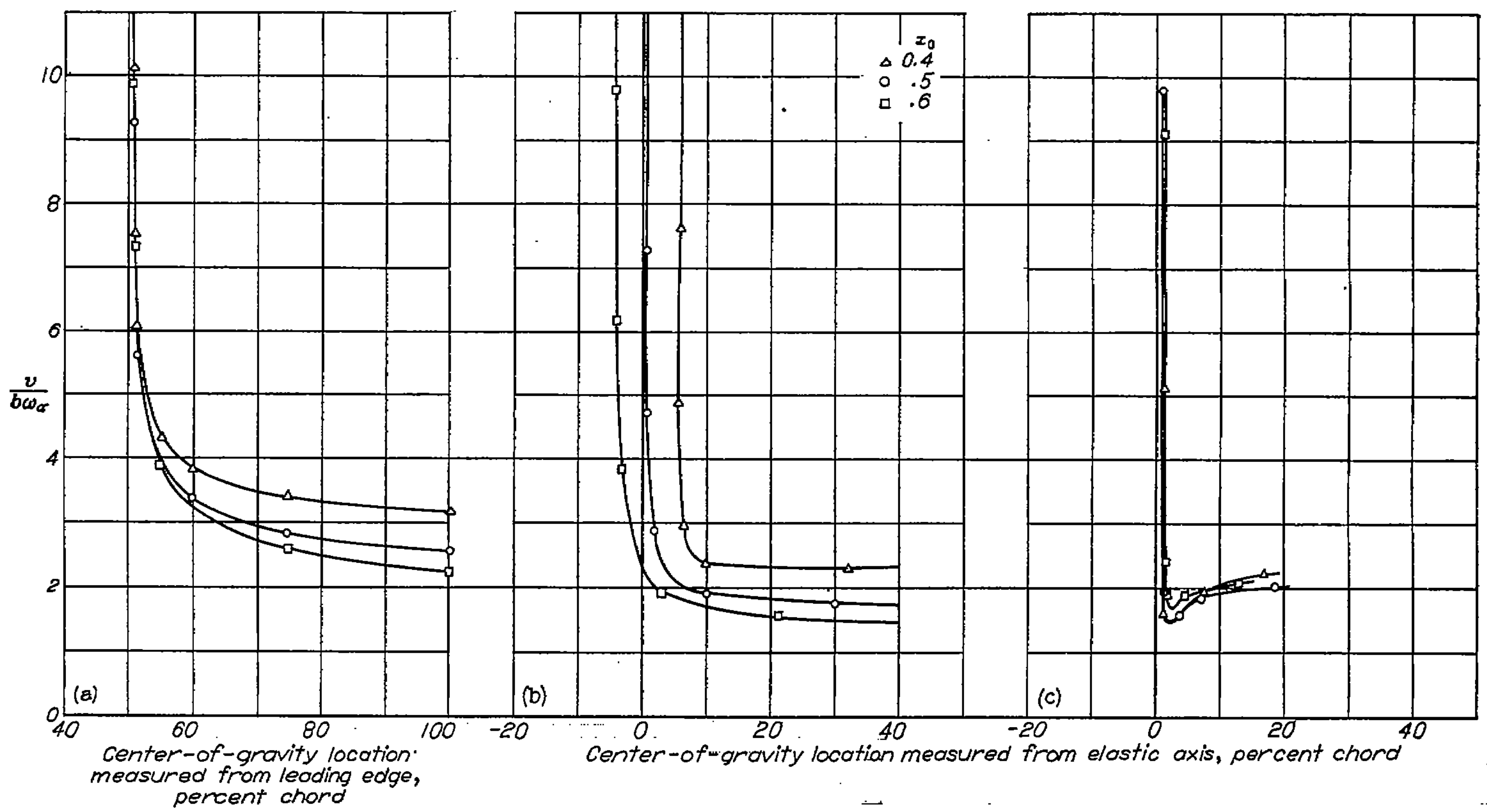
(c) $\frac{\omega_h}{\omega_\alpha}=1.0$.

FIGURE 8.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=\frac{10}{7}$; $\mu=3.927$.



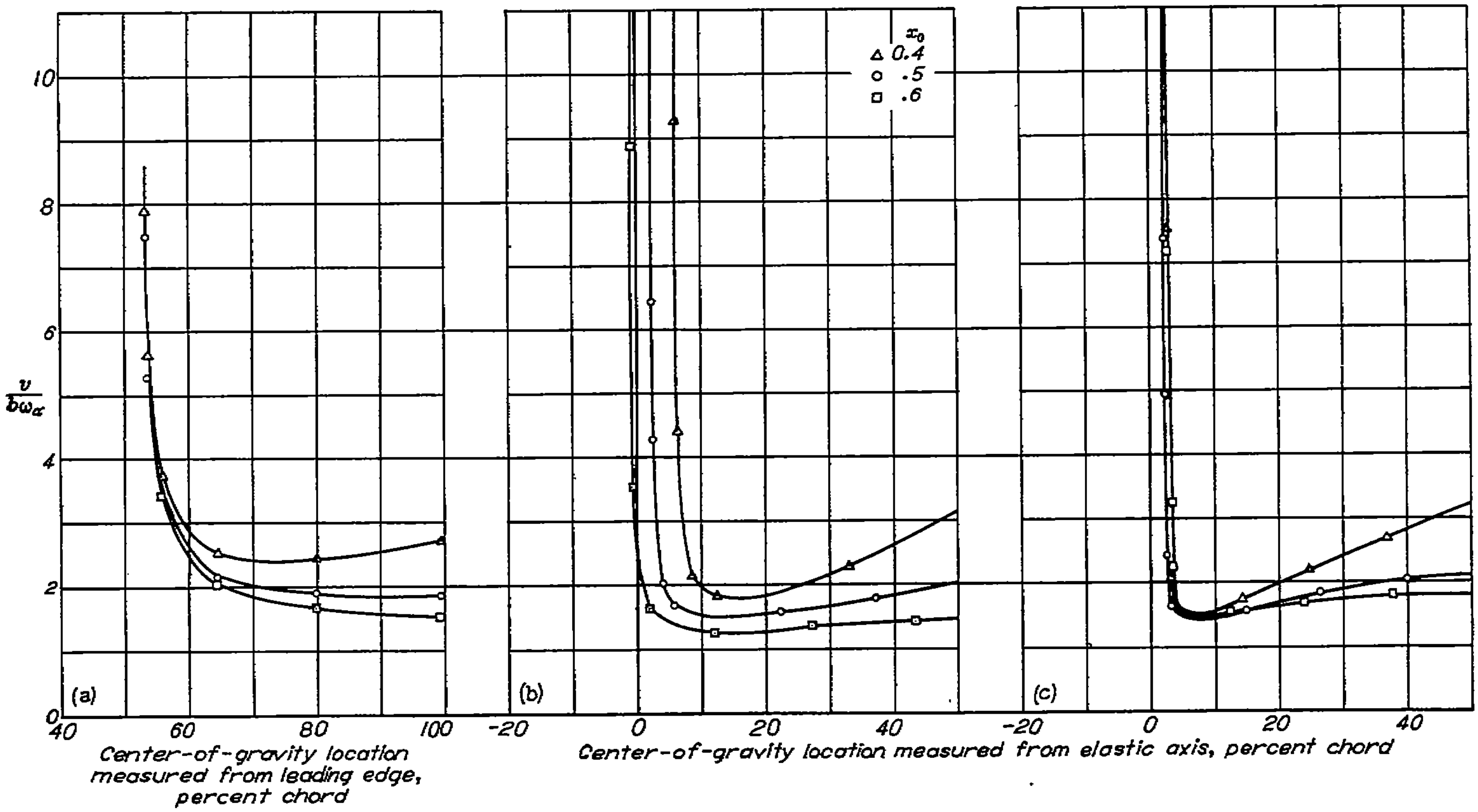
(a) $\frac{\omega_h}{\omega_\alpha}=0.$ (b) $\frac{\omega_h}{\omega_\alpha}=0.707.$ (c) $\frac{\omega_h}{\omega_\alpha}=1.0.$

FIGURE 9.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=\frac{10}{7}$; $\mu=7.854$.



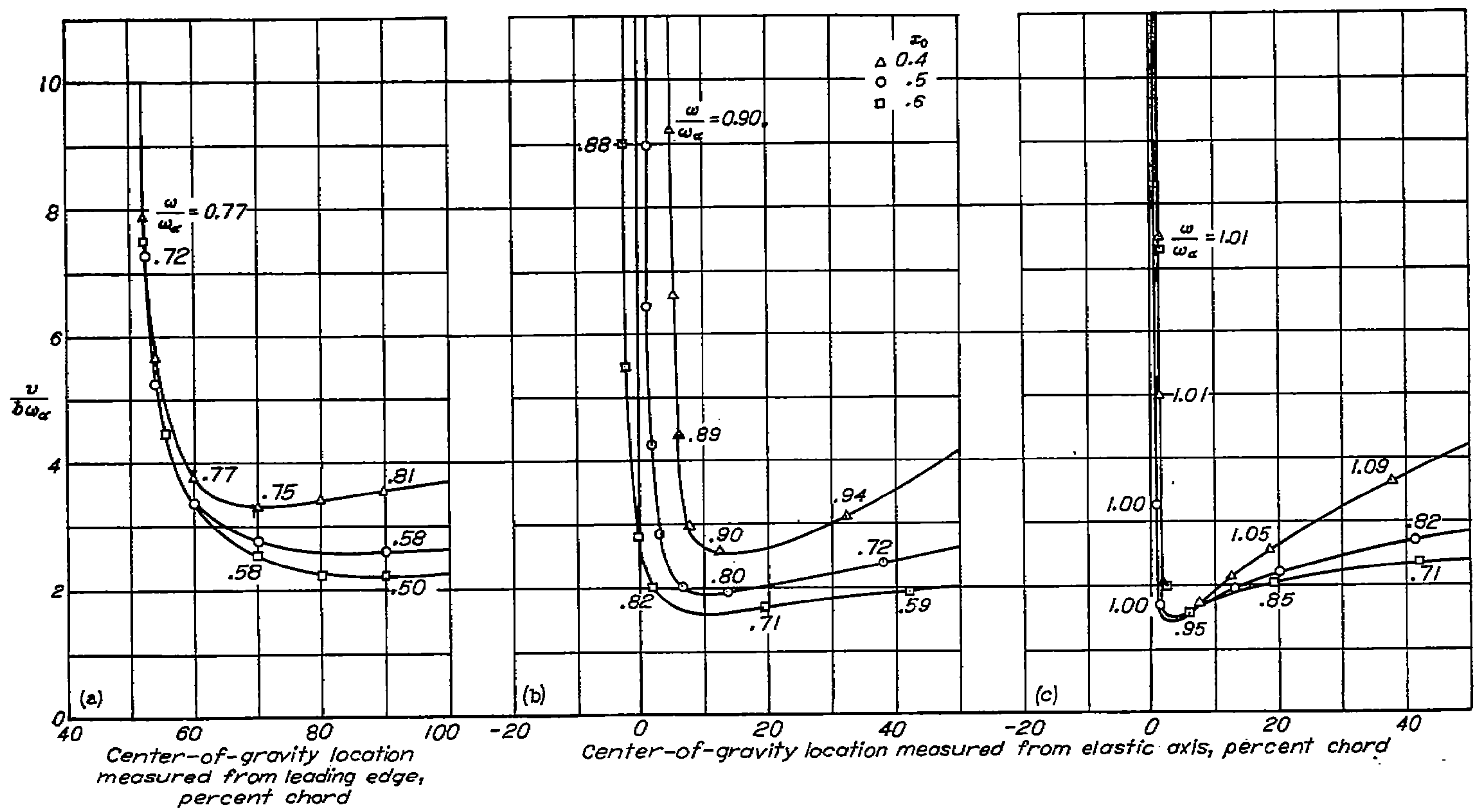
(a) $\frac{\omega_h}{\omega_\alpha}=0.$ (b) $\frac{\omega_h}{\omega_\alpha}=0.707.$ (c) $\frac{\omega_h}{\omega_\alpha}=1.0.$

FIGURE 10.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=\frac{10}{7}$; $\mu=15.708$.



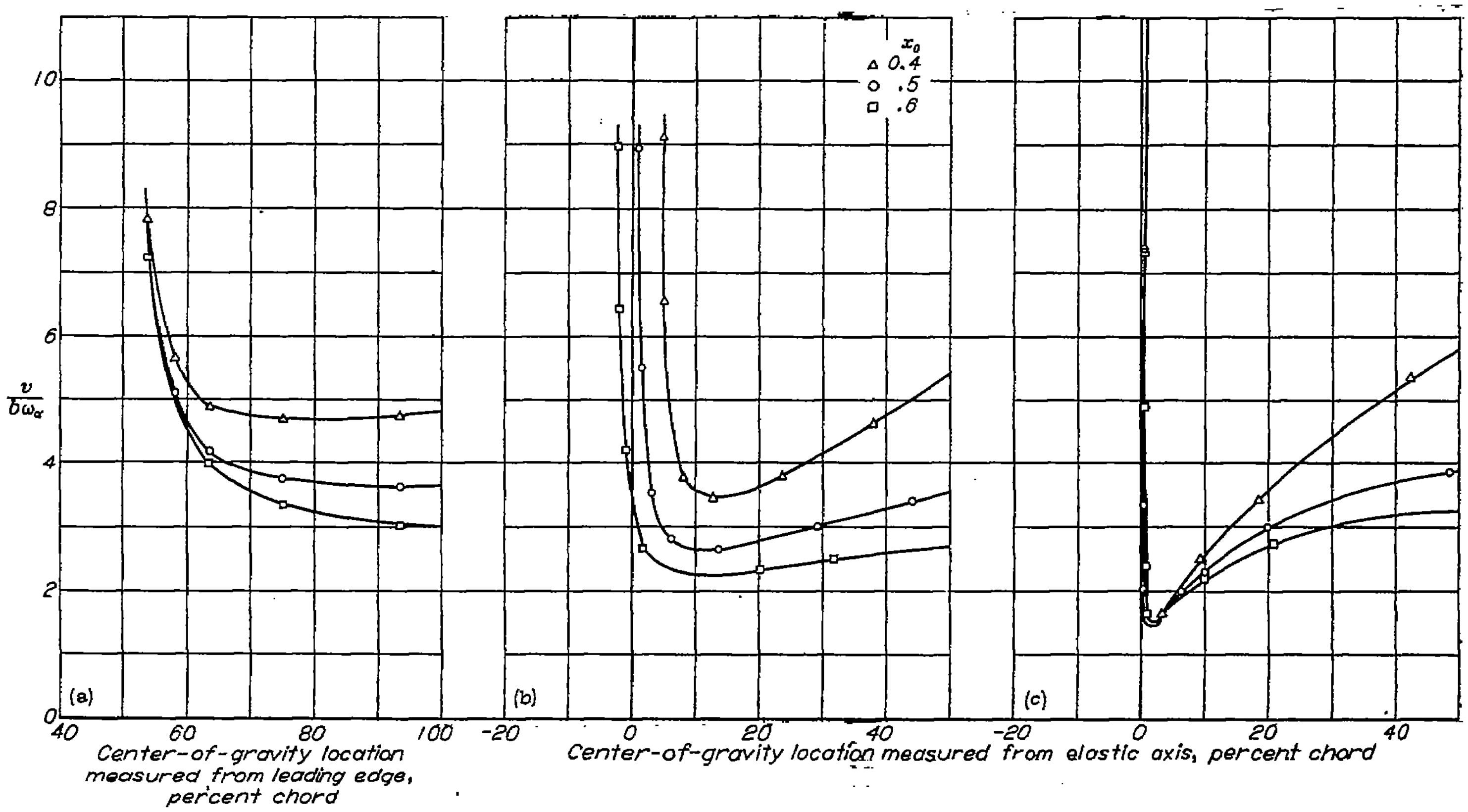
(a) $\frac{\omega_h}{\omega_\alpha}=0.$ (b) $\frac{\omega_h}{\omega_\alpha}=0.707.$ (c) $\frac{\omega_h}{\omega_\alpha}=1.0.$

FIGURE 11.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=2$; $\mu=3.927$.



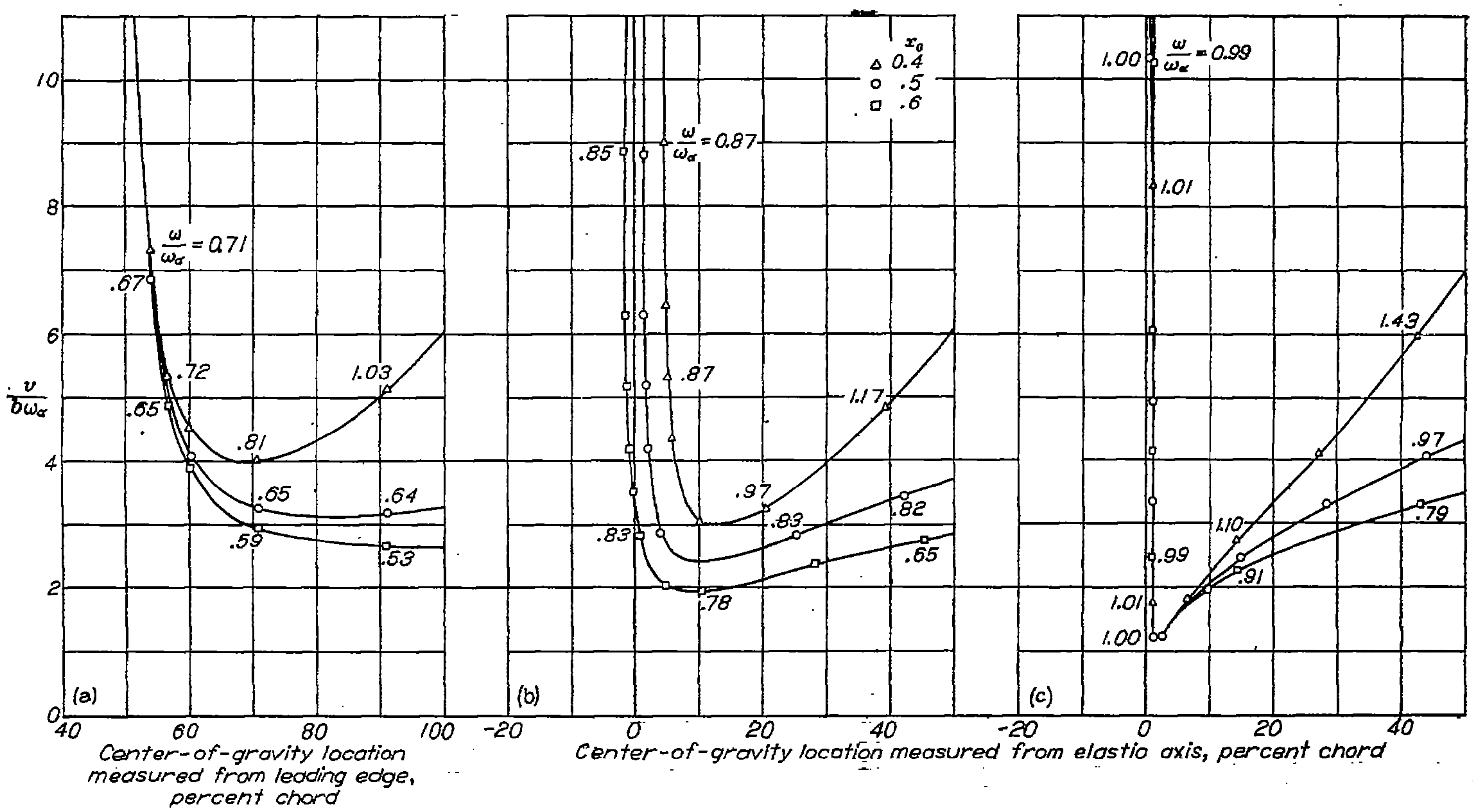
(a) $\frac{\omega_h}{\omega_\alpha}=0.$ (b) $\frac{\omega_h}{\omega_\alpha}=0.707.$ (c) $\frac{\omega_h}{\omega_\alpha}=1.0.$

FIGURE 12.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=2$; $\mu=7.854$.



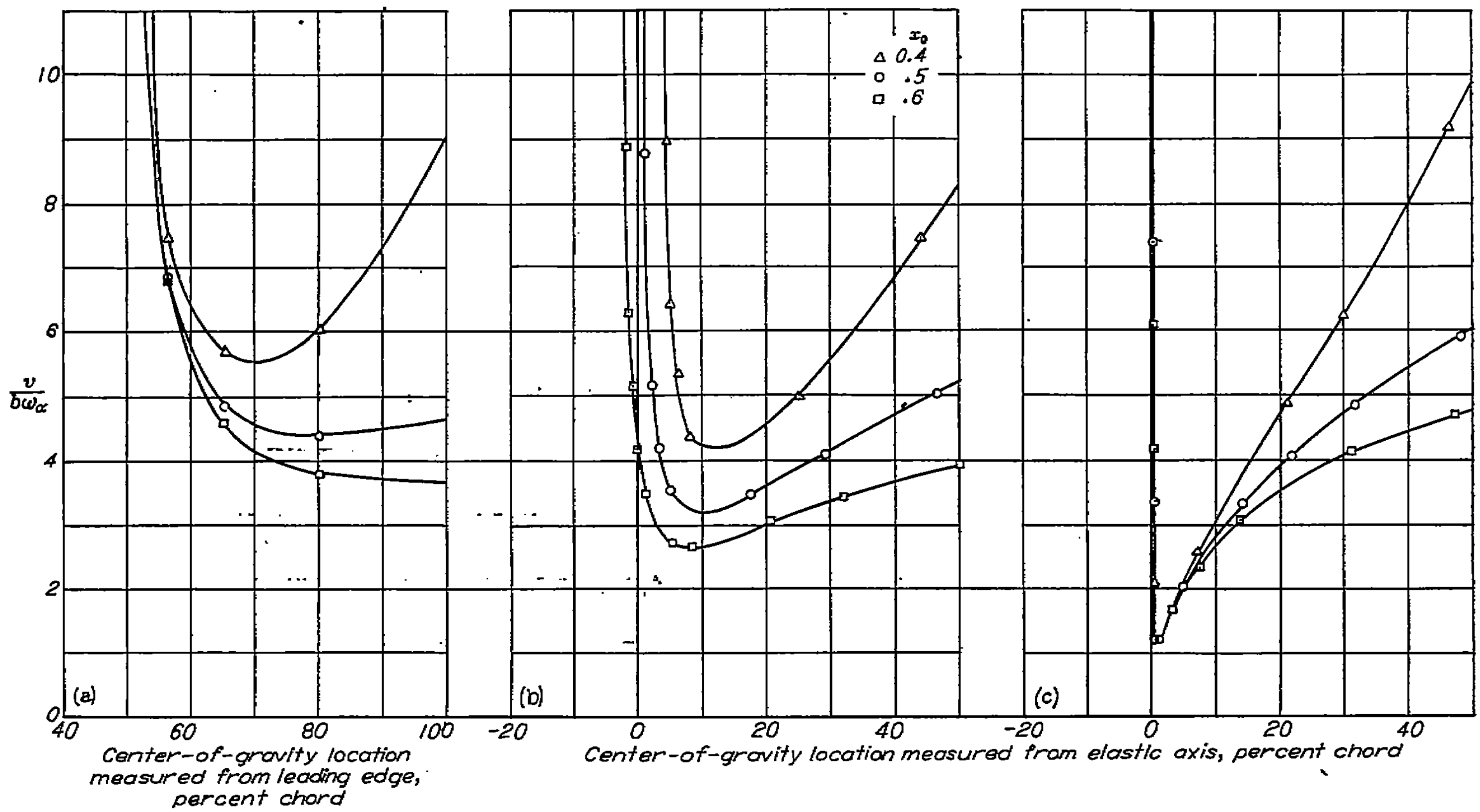
(a) $\frac{\omega_h}{\omega_\alpha}=0$. (b) $\frac{\omega_h}{\omega_\alpha}=0.707$. (c) $\frac{\omega_h}{\omega_\alpha}=1.0$.

FIGURE 13.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=2$; $\mu=15.708$.



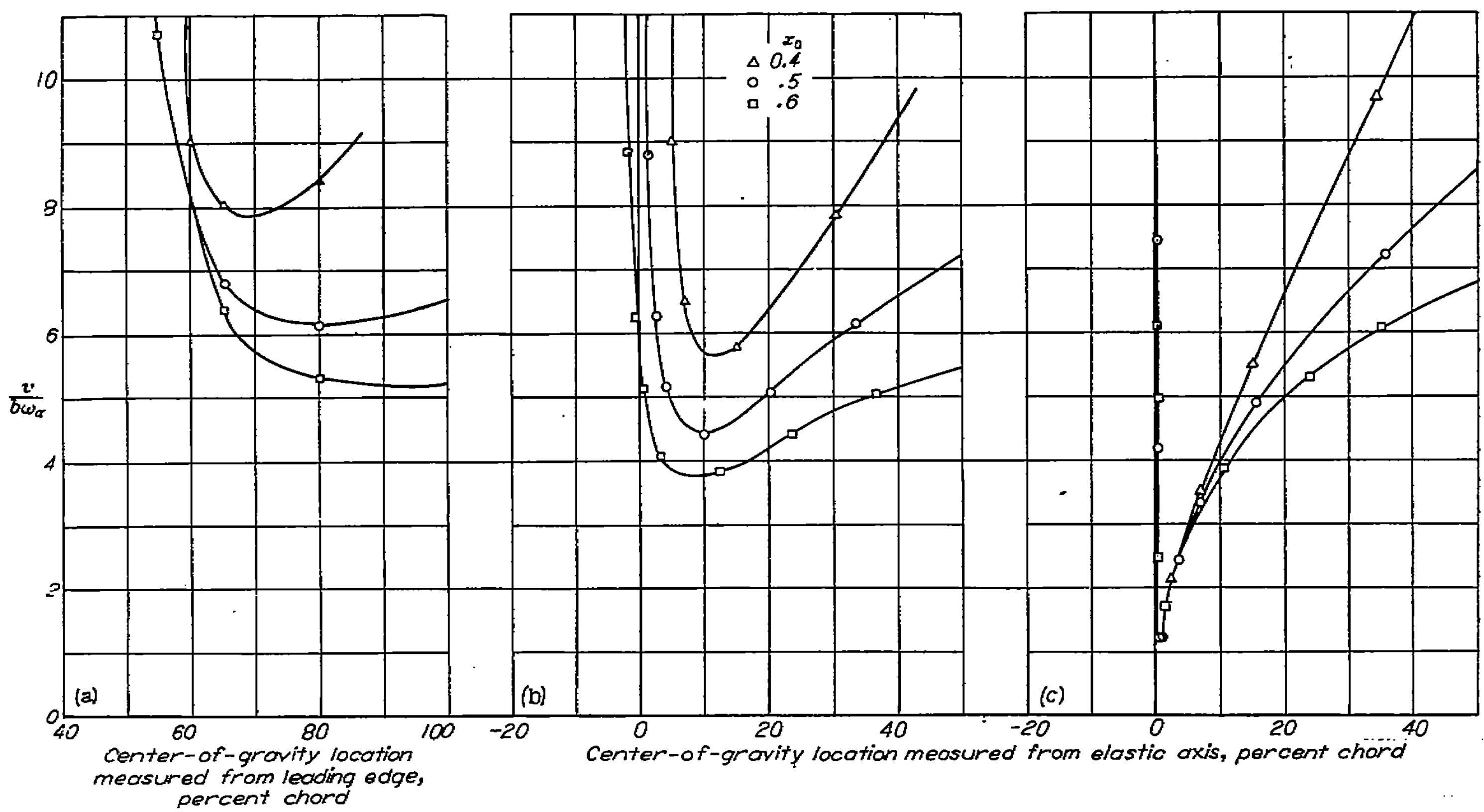
(a) $\frac{\omega_h}{\omega_\alpha}=0$. (b) $\frac{\omega_h}{\omega_\alpha}=0.707$. (c) $\frac{\omega_h}{\omega_\alpha}=1.0$.

FIGURE 14.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=5$; $\mu=3.927$.



(a) $\frac{\omega_h}{\omega_\alpha}=0.$ (b) $\frac{\omega_h}{\omega_\alpha}=0.707.$ (c) $\frac{\omega_h}{\omega_\alpha}=1.0.$

FIGURE 15.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=5$; $\mu=7.854$.



(a) $\frac{\omega_h}{\omega_\alpha}=0.$ (b) $\frac{\omega_h}{\omega_\alpha}=0.707.$ (c) $\frac{\omega_h}{\omega_\alpha}=1.0.$

FIGURE 16.—The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M=5$; $\mu=15.708$.

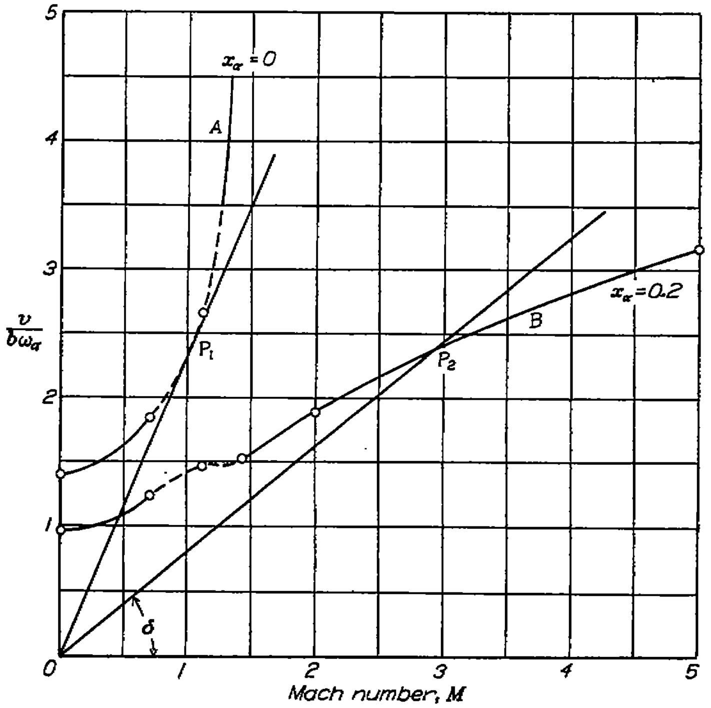


FIGURE 17.—The flutter coefficient against Mach number for two locations of the center of gravity. Other parameters are $\frac{\omega_h}{\omega_\alpha}=0.707$; $a=0$; $\mu=7.854$.

A plot of the flutter coefficient against Mach number for two values of x_α is shown in figure 17. The significance of the straight lines drawn from the origin has already been discussed. The type of curve A is representative of the effect of forward location of the center of gravity and the type of curve B is representative of rearward locations of the center of gravity. Figure 18 gives a plot of the flutter coefficient against M for various values of the wing density parameter μ and for a rearward location of the center of gravity. The subsonic values for $M=0$ and $M=0.7$ shown on these curves and on some of the other figures have been either taken from reference 7 or calculated in the manner outlined therein. The subsonic and supersonic parts of the curves (figs. 17 and 18) have been arbitrarily joined by a dashed smooth curve in the transonic range. In figure 19 there is given a cross plot of flutter coefficient against frequency ratio ω_h/ω_α, for various values of M, and in figure 20 is given a similar cross plot for three values of the elastic-axis parameter x_0.

An indication of the effect of structural damping in increasing the flutter speed in a few examples may be obtained from the following table, where g_α and g_h are the torsional and flexural damping coefficients, respectively, and where $M=\frac{10}{7}$, $\mu=7.854$, $a=0$, and $x_\alpha=0.2$:

ω_h/ω_α	g_α	g_h	ω/ω_α	$v/b\omega_\alpha$
0	0	0	0.673	2.438
0	.05	0	.648	2.551
0	.10	0	.628	2.669
.707	0	0	.777	1.535
.707	.05	0	.771	1.553
.707	.10	0	.766	1.569
.707	0	.05	.788	1.592
.707	0	.10	.797	1.642
.707	.05	.05	.782	1.628
.707	.10	.10	.784	1.725

STATIC CASES—WING DIVERGENCE AND AILERON REVERSAL

It is of some interest to examine the expressions for the forces and moments in the limit case in which the frequency approaches zero. There follow then for the mean-line wing section the well-known static-case results which may of course be obtained directly without the use of a limiting process, as originally treated by Ackeret. Thus, with the use of the following relation easily verified from equations (20),

$$\lim_{k\to 0} f_\lambda(m,k)=\frac{1}{\lambda+1}$$

there are obtained from equations (16′) to (18′) for the lift

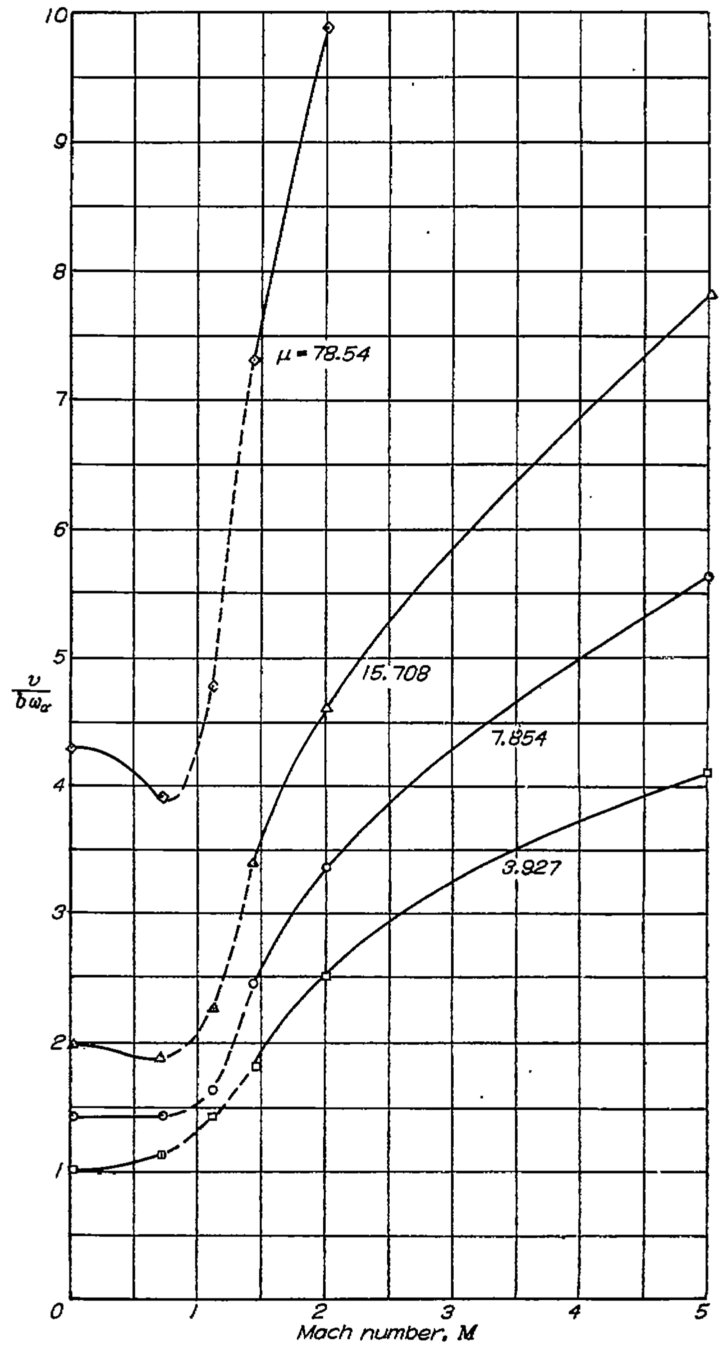


FIGURE 18.—The flutter coefficient against Mach number for several values of μ. Other parameters are $\frac{\omega_h}{\omega_\alpha}=0$; $x_\alpha=0.2$; $a=0$.

and moments in the static case,

$$L=-\frac{4\rho b v^2}{\sqrt{M^2-1}}[\alpha+(1-x_1)\beta]$$

$$M_\alpha=-\frac{4\rho b^2 v^2}{\sqrt{M^2-1}}[(1-2x_0)\alpha+(1-x_1)(1+x_1-2x_0)\beta]$$

$$M_\beta=-\frac{4\rho b^2 v^2}{\sqrt{M^2-1}}(1-x_1)^2(\alpha+\beta)$$

These relations for the mean-line wing section are now used to obtain the critical speeds for the static instabilities—wing divergence and wing-aileron reversal (for wing of infinite span). At the wing divergence speed the effective torsional stiffness of the wing vanishes, hence the total moment about the elastic axis is zero. The sum of the structural restoring moment and the aerodynamic twisting moment is

$$\alpha C_\alpha+\frac{4\rho b^2 v^2}{\sqrt{M^2-1}}\alpha(1-2x_0)$$

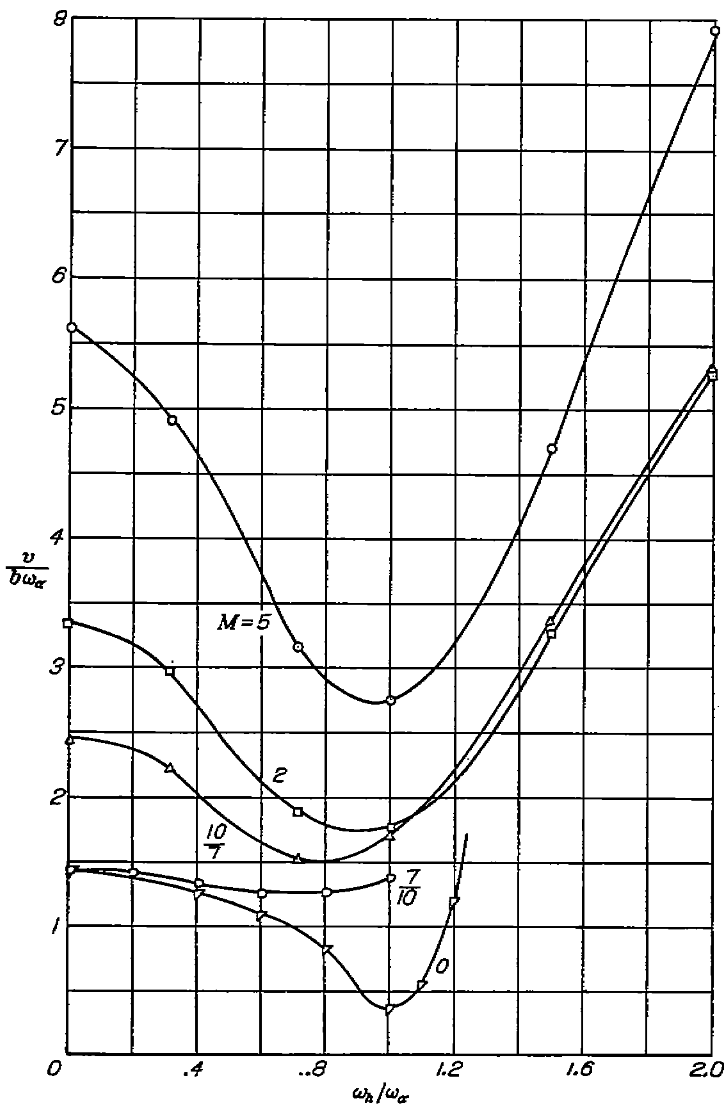


FIGURE 19.—The flutter coefficient against frequency ratio for several values of M. Other parameters are $a=0$; $x_\alpha=0.2$; $\mu=7.854$.

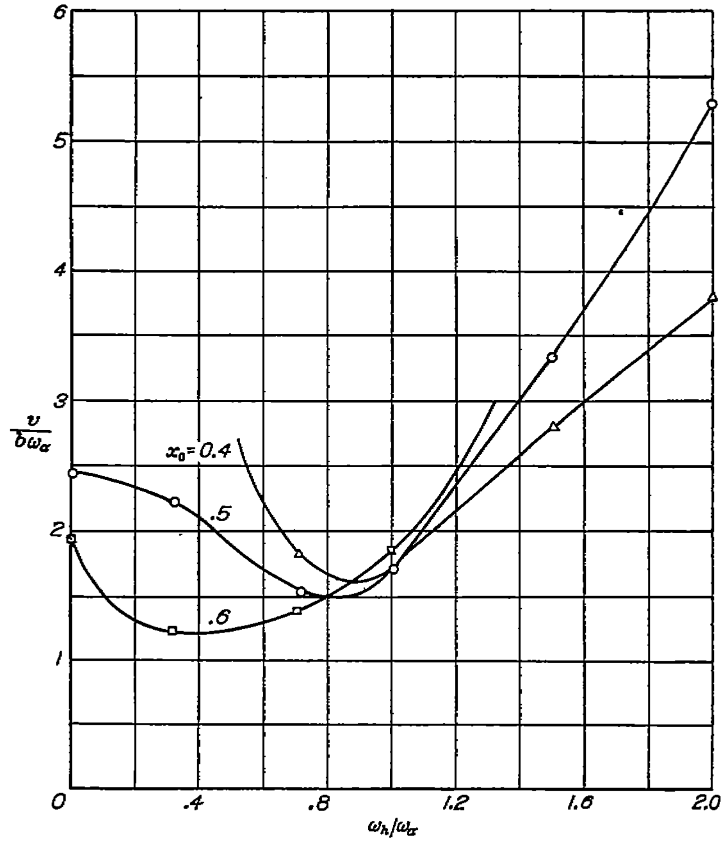


FIGURE 20.—The flutter coefficient against frequency ratio for three values of x_0. Other parameters are $M=\frac{10}{7}$; $x_\alpha=0.2$; $\mu=7.854$.

which when equated to zero yields the divergence speed

$$v_D=b\omega_\alpha(M^2-1)^{1/4}\sqrt{\mu r_\alpha^2}\,\frac{1}{\sqrt{2x_0-1}}$$

Thus, the divergence speed is real only for positions of the elastic axis behind the aerodynamic center (midchord, in the simple theory). This formula obviously should not be used for values of M too near unity.

For comparison it is of interest to note the corresponding result for the divergence speed in the subsonic case, where the aerodynamic center is (approx.) at the quarter-chord point. Thus,

$$v_D=b\omega_\alpha(1-M^2)^{1/4}\sqrt{\frac{r_\alpha^2}{\kappa}}\,\frac{1}{\sqrt{4x_0-1}}$$

where M is less than about 0.7.

The aileron reversal speed is determined by the condition that the change in angle of attack due to wing torsion nullifies the effect of movement of the aileron so as to yield no change in lift (in rolling moment, in the case of finite wing span). There are two equations to be satisfied for this condition; namely,

$$\alpha+(1-x_1)\beta=0$$

(that is, $L=0$) and

$$\alpha C_\alpha+\frac{4\rho b^2 v^2}{\sqrt{M^2-1}}[(1-2x_0)\alpha+(1-x_1)(1+x_1-2x_0)\beta]=0$$

The aileron reversal speed, obtained by elimination of α and β, is

$$v_R = b\omega_\alpha (M^2-1)^{1/4} \sqrt{\mu r_\alpha^2} \frac{1}{\sqrt{x_1}}$$

For hinge positions aft of the midchord, the factor $1/\sqrt{x_1}$ in this expression varies from 1.4 to 1.0. The aileron reversal speed is thus relatively unaffected by the position of the hinge. In general v_R may be expected to be lower than v_D.

ONE-DEGREE-OF-FREEDOM OSCILLATORY INSTABILITY

As was pointed out by Possio, the theory indicates the existence of a torsional instability which may arise for a wing having only one degree of freedom. This instability is due to the wing being negatively damped in torsion and is associated with the vanishing (and change in sign) of the torsional damping coefficient M_4 (equation (26)).

Certain considerations for the case of slow oscillations made by Possio (reference 1) and further discussed by Temple and Jahn serve to bring out the main results. Thus from equation (20), for slow oscillations,

$$f_\lambda(M,k) \approx \frac{1}{\lambda+1} - i\frac{2kM^2}{M^2-1}\frac{1}{\lambda+2}$$

and

$$M_4 \approx \frac{1}{\sqrt{M^2-1}} \frac{1}{k} \frac{2}{3}\left[4-9x_0+6x_0^2-\frac{M^2}{M^2-1}(2-3x_0)\right]$$

The condition $M_4(M, x_0)=0$ is shown plotted in figure 21, where the shaded area is the region in which the instability is possible (negative M_4). The maximum ranges for the parameters x_0 and M in this region are x_0 less than 2/3 and M less than $\sqrt{2.5}$ (and greater than unity).

(It may be appropriate to mention that a similar torsional instability is theoretically indicated even in the subsonic (incompressible) case for positions of the axis of rotation between the leading edge and the quarter-chord point. The combination of parameters required for this indicated instability, however, is not very likely.)

The torsional instability may be studied more fully in the general case. It is found that the range of instability for the parameters x_0 and M remains essentially as in the simple case (large $1/k$) but more information may be obtained regarding the critical speed and frequency. The moment equation is equivalent to $\bar{A}_{\alpha\alpha}=0$, or to the two equations

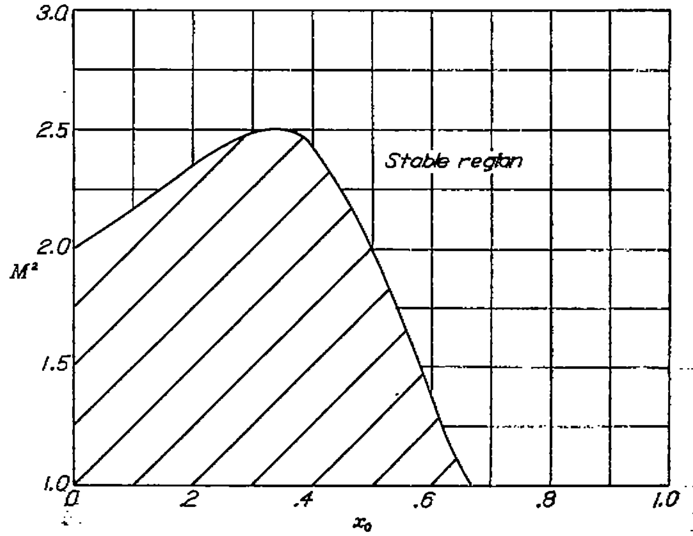


FIGURE 21.—Plot of $M_4(M,x_0)=0$.

$$\Omega_\alpha X - \mu r_\alpha^2 + M_3(M, x_0) = 0$$

$$M_4(M, x_0) + g_\alpha \Omega_\alpha X = 0$$

where the structural damping coefficient in torsion g_α has been introduced as in reference 6. The critical speed and frequency may be studied as functions of the parameters x_0, M, g_α and the product combination μr_α^2. Results of a few selected calculations are shown plotted in figure 22. Since instabilities are indicated for the range of near-sonic values ($1<M<1.58$), it would seem that a more comprehensive investigation of this problem is very desirable.

It may be remarked that a similar analysis for pure bending exhibits no instability while the case of the aileron alone does exhibit a range where such instability may occur. This range for an aileron hinged at its leading edge is $1<M\leqq\sqrt{2}$.

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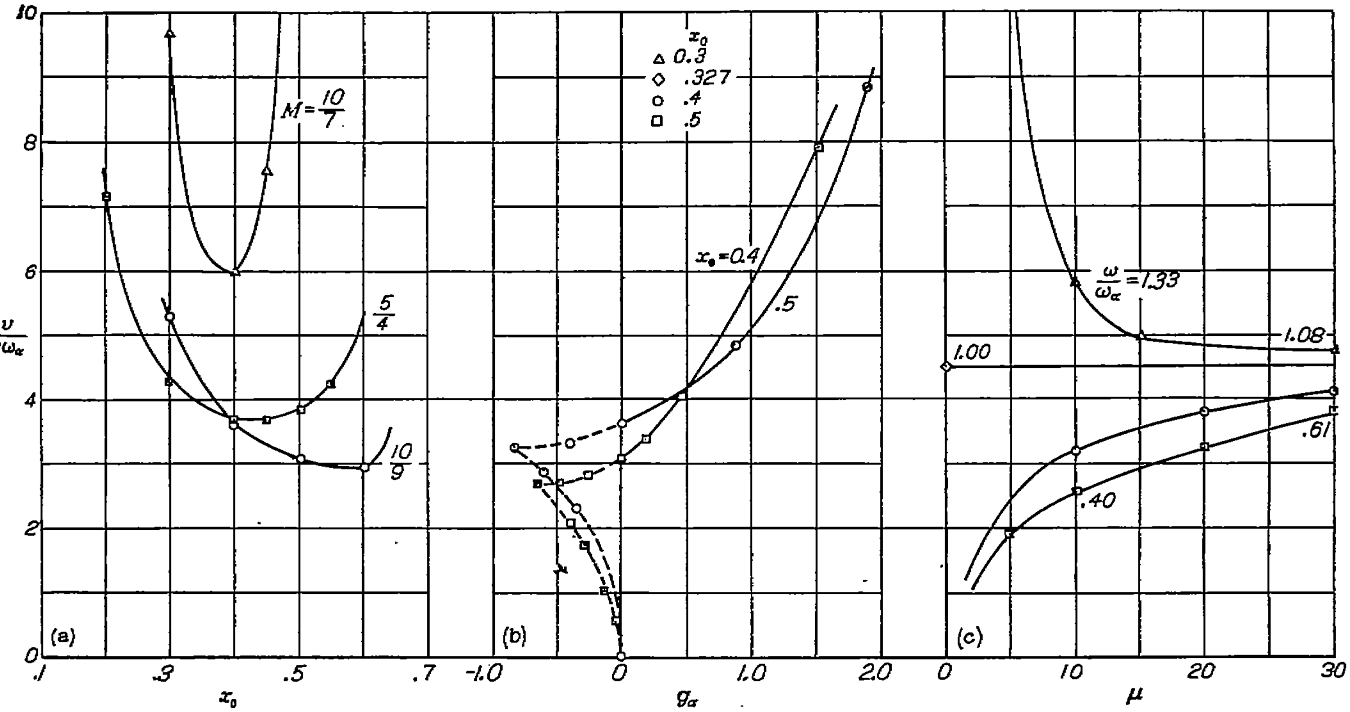

(a) Flutter coefficient against axis-of-rotation position for several values of M (μ=15.708). Note that the range of x_0 narrows with increase in M and disappears at M=1.58 and x_0=0.33.

(b) Flutter coefficient against torsional damping coefficient for two values of x_0 $\left(M=\frac{10}{9}; \mu=15.708\right)$. Negative damping values are shown dashed and have no physical existence.

(c) Flutter coefficient against wing density parameter μ for several values of x_0 $\left(M=\frac{10}{9}\right)$. The straight-line curve shown corresponds to $M_1=0$ (x_0=0.327).

FIGURE 22.—Curves for one-degree-of-freedom torsional instability.

APPENDIX

SYMBOLS

ϕ	disturbance velocity potential
t	time at which disturbance influence is felt
T	time at which disturbance is created
$\tau=t-T$	
p	pressure
p'	pressure difference
ρ	density
γ	adiabatic index (for air, $\gamma\approx1.4$)
v	velocity of main stream (supersonic)
c	velocity of sound in undisturbed medium
M	Mach number (v/c)
x	coordinate measured in direction of main stream
y	ordinate
x_0	abscissa of axis of rotation of wing section (elastic axis)
x_1	abscissa of aileron hinge
ξ, η	abscissa and ordinate of point of disturbance
b	one-half chord

After equation (12) the quantities x, y, x_0, x_1, and ξ are employed nondimensionally and are referred to the chord $2b$ as reference length.

$w(x,t)$	vertical velocity at position x on chord and at time t
h	vertical displacement of axis of rotation
α	angular displacement about axis of rotation
β	angular displacement of aileron; measured with respect to α
ω	angular frequency of oscillation
k	reduced frequency ($\omega b/v$)
$\overline{\omega}$	frequency parameter $\left(\frac{2kM^2}{M^2-1}\right)$
$I(\xi,x)$	function given in equations (12) and (12')
$J_n(\lambda)$	Bessel function of order n

The following additional symbols, employed in the flutter equations, conform to the notation used in references 4 and 6, in which the half-chord b is the unit reference length.

M — mass of wing per unit span

S_α — static moment of wing-aileron combination per unit span referred to elastic axis

S_β — static moment of aileron per unit span referred to aileron hinge

I_α — moment of inertia of wing-aileron combination about elastic axis per unit span

I_β — moment of inertia of aileron about its hinge per unit span

a — coordinate of elastic axis measured from midchord $(2x_0-1)$

c — coordinate of aileron hinge axis measured from midchord $(2x_1-1)$

x_α — location of center of gravity of wing-aileron system measured from elastic axis S_α/M_b; location of center of gravity in percent total chord measured from leading edge is $100\frac{1+a+x_\alpha}{2}=100\left(x_0+\frac{x_\alpha}{2}\right)$

x_β — reduced location of center of gravity of aileron referred to c (S_β/M_b)

r_α — radius of gyration of wing-aileron combination referred to a $\left(\sqrt{\frac{I_\alpha}{Mb^2}}\right)$

r_β — reduced radius of gyration of aileron referred to c $\left(\sqrt{\frac{I_\beta}{Mb^2}}\right)$

C_α — torsional stiffness of wing around a per unit span

C_β — torsional stiffness of aileron system around c per unit span

C_h — stiffness of wing in deflection

ω_α — natural angular frequency of torsional vibrations about elastic axis $\left(\sqrt{\frac{C_\alpha}{I_\alpha}}\right)$; $(\omega_\alpha=2\pi f_\alpha$, where f_α is in cycles per second)

ω_β — natural angular frequency of torsional vibrations of aileron around c $\left(\sqrt{\frac{C_\beta}{I_\beta}}\right)$

ω_h — natural angular frequency of wing in deflection $\left(\sqrt{\frac{C_h}{M}}\right)$

μ — wing density parameter $\left(\frac{\pi}{4}\frac{1}{\kappa}\text{ or }\frac{M}{4\rho b^2}\right)$ (Note that in the incompressible case (references 4 and 6) μ is replaced by $1/\kappa$.)

κ — ratio of mass of cylinder of air of diameter equal to chord of wing to mass of wing, both taken for equal length along span $\left(\frac{\pi\rho b^2}{M}\right)$ (This ratio may be expressed as $\kappa=0.24\left(\frac{b^2}{W}\right)\left(\frac{\rho}{\rho_0}\right)$ where W is weight in pounds per foot span, b is in feet, and ρ/ρ_0 is ratio of air density at altitude to that for normal standard air.)

g_α, g_β, g_h — structural damping coefficients (see reference 6)

$L_1, L_2, L_3, L_4, M_1, M_2, M_3, M_4$ — quantities defined in table II and by equations (26) and (28)

$v/b\omega_\alpha$ — flutter coefficient; that is, flutter speed divided by reference speed $b\omega_\alpha$

$$\Omega_\alpha X=\mu r_\alpha^2\left(\frac{\omega_\alpha}{\omega}\right)^2$$

$$\Omega_\beta X=\mu r_\beta^2\left(\frac{\omega_\beta}{\omega}\right)^2$$

$$\Omega_h X=\mu\left(\frac{\omega_h}{\omega}\right)^2$$

$X=\mu r_\alpha^2\left(\frac{\omega_\alpha}{\omega}\right)^2$ for case of bending torsion

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TABLE I.—VALUES OF $f_0(M, \bar{\omega}) = (f_0)_R + i(f_0)_I$

$$(f_0)_R = \frac{1}{\bar{\omega}} \int_0^{\bar{\omega}} J_0\left(\frac{u}{M}\right) \cos u \, du$$

$$(f_0)_I = -\frac{1}{\bar{\omega}} \int_0^{\bar{\omega}} J_0\left(\frac{u}{M}\right) \sin u \, du$$

$M = \frac{10}{9}$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.52632	0.02108	−0.14999
12.00	.87719	.08682	−.21130
10.00	1.05263	.10786	−.21774
9.00	1.16959	.12303	−.23999
8.00	1.31579	.15423	−.24647
7.00	1.50376	.16377	−.24545
6.00	1.75439	.17593	−.27930
5.00	2.10526	.23364	−.30624
4.20	2.50627	.26928	−.29526
3.60	2.92398	.26970	−.30153
3.20	3.28947	.27020	−.32996
2.80	3.75940	.28891	−.37917
2.50	4.21053	.32454	−.42368
2.20	4.78469	.38355	−.46577
2.10	5.01253	.40861	−.47752
1.90	5.54017	.46632	−.49523
1.68	6.26566	.53988	−.50275
1.50	7.01754	.60577	−.49720
1.40	7.51880	.64370	−.48896
1.30	8.09717	.68202	−.47679
1.10	9.56938	.75786	−.44031
1.00	10.5263	.79439	−.41597
.90	11.6959	.82931	−.38765
.84	12.5313	.84928	−.36881
.80	13.1579	.86213	−.35551
.70	15.0376	.89236	−.31977
.60	17.5439	.91957	−.28073
.56	18.7970	.92951	−.26426
.52	20.2429	.93888	−.24735
.36	29.2398	.97016	−.17575
.26	40.4858	.98431	−.12839
.16	65.7895	.99402	−.07962
.10	105.263	.99766	−.04991

$M = \frac{5}{4}$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.27778	−0.02589	−0.08630
10.00	.55556	.02530	−.22400
5.00	1.11111	.19004	−.33200
4.40	1.26263	.21539	−.32283
3.30	1.68350	.22644	−.36922
2.80	1.98413	.26381	−.43392
2.60	2.13675	.29327	−.46284
2.40	2.31481	.33256	−.49015
2.20	2.52525	.38184	−.51318
2.00	2.77778	.44053	−.52914
1.96	2.83447	.45329	−.53124
1.84	3.01932	.49337	−.53500
1.66	3.34672	.55768	−.53260
1.56	3.56125	.59498	−.52665
1.48	3.75375	.62531	−.51933
1.34	4.14594	.67878	−.50083
1.24	4.48029	.71672	−.48306
1.10	5.05051	.76858	−.45178
1.06	5.24109	.78294	−.44148
.98	5.66893	.81095	−.41909
.94	5.91017	.82452	−.40702
.88	6.31313	.84426	−.38785
.82	6.77507	.86317	−.36745
.78	7.12251	.87528	−.35318
.74	7.50751	.88695	−.33840
.70	7.93651	.89816	−.32312
.66	8.41751	.90890	−.30737
.60	9.25926	.92404	−.28289
.56	9.92063	.93348	−.26603
.52	10.6838	.94235	−.24877
.48	11.5741	.95065	−.23114
.44	12.6263	.95836	−.21316
.40	13.8889	.96545	−.19484
.36	15.4321	.97191	−.17623
.34	16.3399	.97491	−.16682
.32	17.3611	.97774	−.15735
.30	18.5185	.98041	−.14781
.28	19.8413	.98291	−.13822
.14	39.6825	.99570	−.06978
.06	92.5926	.99920	−.02998

$M = \frac{10}{7}$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.19608	0.01042	−0.05474
10.00	.39216	−.02790	−.18977
5.00	.78431	.13530	−.33799
3.90	1.00553	.15895	−.35747
3.10	1.26502	.19688	−.44344
2.40	1.63399	.32318	−.53919
2.00	1.96078	.44414	−.56786
1.60	2.45098	.59013	−.55477
1.34	2.92654	.68965	−.51771
1.18	3.32336	.74934	−.46289
1.06	3.69959	.79215	−.45080
.94	4.17188	.83250	−.41378
.88	4.45633	.85156	−.39350
.82	4.78240	.86976	−.37209
.78	5.02765	.88139	−.35722
.74	5.29942	.89258	−.34188
.70	5.60224	.90331	−.32610
.66	5.94177	.91356	−.30988
.62	6.32511	.92332	−.29326
.60	6.53595	.92801	−.28480
.56	7.00280	.93699	−.26760
.52	7.54148	.94543	−.25004
.48	8.16993	.95331	−.23214
.44	8.91266	.96062	−.21393
.40	9.80392	.96734	−.19543
.36	10.8932	.97346	−.17666
.34	11.5340	.97630	−.16718
.32	12.2549	.97897	−.15765
.30	13.0719	.98150	−.14806
.28	14.0056	.98386	−.13842
.26	15.0830	.98607	−.12873
.24	16.3399	.98812	−.11900
.22	17.8253	.99001	−.10923
.20	19.6078	.99173	−.09942
.16	24.5098	.99470	−.07971
.10	39.2157	.99793	−.04993
.06	65.3595	.99925	−.02998
.04	98.0392	.99967	−.02000

$M = \frac{5}{3}$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.15625	0.00827	−0.07002
10.00	.31250	−.03441	−.13439
5.00	.62500	.07303	−.32466
3.10	1.00806	.15700	−.49133
2.50	1.25000	.28500	−.57452
1.90	1.64474	.48401	−.60118
1.60	1.95312	.59910	−.57738
1.30	2.40385	.71425	−.52374
1.10	2.84091	.78650	−.47109
.94	3.32447	.83048	−.41970
.84	3.72024	.86976	−.38369
.76	4.11184	.89216	−.35268
.68	4.59559	.91277	−.32043
.62	5.04032	.92697	−.29509
.58	5.38793	.93582	−.27775
.54	5.78704	.94414	−.26009
.50	6.25000	.95193	−.24211
.48	6.51042	.95562	−.23301
.44	7.10227	.96258	−.21460
.42	7.44048	.96585	−.20530
.40	7.81250	.96898	−.19594
.38	8.22368	.97197	−.18651
.36	8.68056	.97481	−.17703
.34	9.19118	.97750	−.16750
.32	9.76562	.98005	−.15791
.30	10.4167	.98244	−.14828
.28	11.1607	.98469	−.13860
.26	12.0192	.98678	−.12888
.24	13.0208	.98873	−.11912
.22	14.2045	.99052	−.10932
.20	15.6250	.99216	−.09949
.18	17.3611	.99364	−.08963
.16	19.5312	.99497	−.07974
.08	39.0625	.99874	−.03996
.06	52.0833	.99930	−.02998
.04	78.1250	.99967	−.02000

$M = 2$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.13333	−0.01549	−0.06337
10.00	.26667	−.00697	−.09613
5.00	.53333	.00782	−.29542
2.70	.98765	.21306	−.59459
2.10	1.26984	.41160	−.63550
1.60	1.66667	.60681	−.59699
1.30	2.05128	.72238	−.53567
1.10	2.42424	.79366	−.47883
.90	2.96296	.85749	−.41012
.80	3.33333	.88589	−.37172
.74	3.60360	.90165	−.34750
.70	3.80952	.91160	−.33089
.64	4.16667	.92564	−.30533
.58	4.59770	.93857	−.27903
.54	4.93827	.94657	−.26113
.50	5.33333	.95404	−.24294
.46	5.79710	.96098	−.22449
.42	6.34921	.96738	−.20580
.40	6.66667	.97037	−.19637
.38	7.01754	.97323	−.18688
.36	7.40741	.97595	−.17735
.34	7.84314	.97852	−.16776
.30	8.88889	.98324	−.14846
.28	9.52381	.98539	−.13875
.26	10.2564	.98739	−.12900
.24	11.1111	.98925	−.11921
.22	12.1212	.99096	−.10939
.20	13.3333	.99252	−.09954
.18	14.8148	.99394	−.08967
.16	16.6667	.99521	−.07977
.14	19.0476	.99633	−.06984
.10	26.6667	.99813	−.04994
.06	44.4444	.99933	−.02998
.04	66.6667	.99970	−.02000
.02	133.333	.99995	−.01000

$M = \frac{5}{2}$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.11905	0.00672	−0.04538
10.00	.23810	.02216	−.06785
5.00	.47619	−.05520	−.25614
4.80	.49603	−.05939	−.27691
2.40	.99206	.29707	−.65923
1.90	1.25313	.49183	−.65291
1.40	1.70068	.60154	−.57127
1.20	1.98413	.76520	−.51685
1.04	2.28938	.81926	−.46491
.96	2.48016	.84430	−.43633
.86	2.76855	.87343	−.39832
.78	3.05250	.89498	−.36621
.72	3.30688	.90936	−.34120
.68	3.50140	.91938	−.32412
.62	3.84025	.93262	−.29792
.58	4.10509	.94083	−.28009
.52	4.57875	.95223	−.25283
.48	4.96032	.95918	−.23435
.46	5.17598	.96246	−.22502
.42	5.66893	.96863	−.20620
.38	6.26566	.97427	−.18718
.36	6.61376	.97688	−.17761
.34	7.00280	.97936	−.16798
.32	7.44048	.98170	−.15832
.30	7.93651	.98390	−.14861
.28	8.50340	.98596	−.13887
.26	9.15751	.98789	−.12910
.24	9.92063	.98967	−.11929
.22	10.8225	.99132	−.10945
.20	11.9048	.99282	−.09958
.18	13.2275	.99418	−.08970
.16	14.8810	.99540	−.07979
.14	17.0068	.99648	−.06986
.12	19.8413	.99741	−.05991
.06	39.6825	.99935	−.02998
.04	59.5238	.99970	−.02000
.02	119.048	.99995	−.01000

$M = \frac{10}{3}$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.10989	0.00961	−0.05304
10.00	.21978	.02509	−.08576
5.00	.43956	−.11067	−.21422
4.40	.49950	−.12439	−.31881
2.20	.99900	.37035	−.68957
1.80	1.22100	.53617	−.65969
1.30	1.69062	.73436	−.55335
1.10	1.99800	.80420	−.49023
.88	2.49750	.87133	−.40882
.80	2.74725	.89279	−.37644
.72	3.05250	.91252	−.34274
.66	3.33000	.92612	−.31666
.62	3.54484	.93460	−.29891
.58	3.78931	.94260	−.28091
.52	4.22654	.95367	−.25348
.46	4.77783	.96362	−.22544
.44	4.99500	.96668	−.21600
.42	5.23286	.96960	−.20652
.38	5.78369	.97507	−.18742
.36	6.10501	.97761	−.17781
.34	6.46412	.98001	−.16815
.32	6.86813	.98227	−.15846
.30	7.32601	.98441	−.14873
.28	7.84929	.98641	−.13896
.26	8.45309	.98827	−.12917
.24	9.15751	.99000	−.11935
.22	9.99001	.99160	−.10950
.20	10.9890	.99305	−.09962
.18	12.2100	.99437	−.08972
.16	13.7363	.99555	−.07981
.14	15.6986	.99659	−.06987
.12	18.3150	.99749	−.05992
.10	21.9780	.99826	−.04995
.06	36.6300	.99937	−.02998
.04	54.9451	.99972	−.02000
.02	109.890	.99995	−.01000

$M = 5$

$\bar{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
20.00	0.10417	−0.01855	−0.06012
10.00	.20833	−.00208	−.12707
5.00	.41667	−.15447	−.17735
4.20	.49603	−.16645	−.35826
2.10	.99206	.41123	−.70315
1.70	1.22549	.58072	−.65532
1.20	1.73611	.77361	−.52771
1.00	2.08333	.83908	−.45747
.84	2.48016	.88461	−.39451
.80	2.60417	.89496	−.37793
.72	2.89352	.91435	−.34383
.68	3.06373	.92336	−.32635
.66	3.15657	.92770	−.31751
.62	3.36022	.93602	−.29963
.56	3.72024	.94760	−.27234
.50	4.16667	.95808	−.24453
.46	4.52899	.96444	−.22573
.42	4.96032	.97030	−.20675
.38	5.48246	.97564	−.18759
.36	5.78704	.97812	−.17795
.34	6.12745	.98047	−.16827
.32	6.51042	.98269	−.15856
.30	6.94444	.98478	−.14881
.28	7.44048	.98673	−.13903
.26	8.01282	.98855	−.12923
.24	8.68056	.99024	−.11939
.22	9.46970	.99180	−.10953
.20	10.4167	.99321	−.09964
.18	11.5741	.99450	−.08974
.16	13.0208	.99566	−.07982
.14	14.8810	.99667	−.06988
.12	17.3611	.99755	−.05992
.10	20.8333	.99830	−.04996
.06	34.7222	.99938	−.02998
.04	52.0833	.99972	−.02000
.02	104.167	.99995	−.01000

TABLE II.—VALUES OF FUNCTIONS USED IN THE FLUTTER CALCULATIONS

The expressions employed in the calculations of this table are:

$$L_1=\frac{1}{\sqrt{M^2-1}}\left\{-2(f_0)_R+\frac{1}{k}\left[J_0\left(\frac{\bar\omega}{M}\right)\sin\bar\omega-\frac{1}{M}J_1\left(\frac{\bar\omega}{M}\right)\cos\bar\omega\right]\right\}$$

$$L_2=\frac{1}{\sqrt{M^2-1}}\left\{-2(f_0)_I+\frac{1}{k}\left[J_0\left(\frac{\bar\omega}{M}\right)\cos\bar\omega+\frac{1}{M}J_1\left(\frac{\bar\omega}{M}\right)\sin\bar\omega\right]\right\}$$

$$L_3'=L_1+\frac{1}{k}L_2+A_1$$

$$L_4'=L_2-\frac{1}{k}L_1+A_2$$

$$M_1'=L_1-A_1$$

$$M_2'=L_2-A_2$$

$$M_3'=\frac{4}{3}(L_1-B_1)+\frac{1}{k}(L_2+A_2)$$

$$M_4'=\frac{4}{3}(L_2-B_2)-\frac{1}{k}(L_1+A_1)$$

where

$$A_1=\frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{1}{2k^2}\left[\frac{1}{M}(f_0)_R-\frac{1}{M}J_0\left(\frac{\bar\omega}{M}\right)\cos\bar\omega-J_1\left(\frac{\bar\omega}{M}\right)\sin\bar\omega\right]$$

$$A_2=\frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{1}{2k^2}\left[\frac{1}{M}(f_0)_I+\frac{1}{M}J_0\left(\frac{\bar\omega}{M}\right)\sin\bar\omega-J_1\left(\frac{\bar\omega}{M}\right)\cos\bar\omega\right]$$

$$B_1=\frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{1}{2k^2}\left[-\frac{2}{\bar\omega}J_1\left(\frac{\bar\omega}{M}\right)\cos\bar\omega+\frac{1}{M}J_0\left(\frac{\bar\omega}{M}\right)\cos\bar\omega+J_1\left(\frac{\bar\omega}{M}\right)\sin\bar\omega\right]$$

$$B_2=\frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{1}{2k^2}\left[\frac{2}{\bar\omega}J_1\left(\frac{\bar\omega}{M}\right)\sin\bar\omega-\frac{1}{M}J_0\left(\frac{\bar\omega}{M}\right)\sin\bar\omega+J_1\left(\frac{\bar\omega}{M}\right)\cos\bar\omega\right]$$

$\bar\omega$	$\frac{1}{k}$	L_1	L_2	L_3'	L_4'	M_1'	M_2'	M_3'	M_4'	$M_1'+L_3'$	$M_2'+L_4'$	D_R	D_I
							$M=\frac{10}{9}$						
20.00	0.52632	−0.02525	0.44559	0.25959	0.44106	−0.07557	0.46341	0.24942	0.60938	0.18402	0.90447	−0.05382	0.00879
12.00	.87719	.03452	.68634	.74823	.67602	−.07713	.66637	.72093	.98267	.67110	1.34239	−.14137	.08227
10.00	1.05263	.06401	.80285	1.07622	.79116	−.10309	.74716	1.02212	1.20708	.97313	1.53832	−.20161	.17583
9.00	1.16959	.12910	.88824	1.33276	.81901	−.03568	.80648	1.25375	1.27722	1.29708	1.62549	−.20455	.23290
8.00	1.31579	.17553	1.03257	1.73315	.88014	−.02346	.95403	1.65937	1.43563	1.70969	1.83417	−.31078	.33258
7.00	1.50376	.21865	1.15544	2.23400	.97017	−.05919	1.01191	2.07687	1.71360	2.17481	1.98208	−.41190	.57119
6.00	1.75439	.39623	1.32462	2.99883	.88853	.11753	1.06558	2.65307	1.76331	3.11636	1.95411	−.69014	.91306
5.00	2.10526	.62771	1.73018	4.57331	.08852	.33722	1.46206	4.15796	1.77406	4.91053	2.15058	−1.00521	1.41382
4.20	2.50627	.74997	2.15463	6.63529	.62547	.26471	1.80417	6.00140	2.37986	6.90060	2.42964	−1.25482	2.57886
3.60	2.92398	.96344	2.44397	8.78186	.26279	.29113	1.80808	7.32174	2.83636	9.07299	2.07087	−1.95944	4.07196
3.20	3.28947	1.30408	2.69560	10.8827	−.62644	.59242	1.72808	8.27851	2.42162	11.4751	1.10104	−3.26196	7.03836
2.80	3.75940	1.89448	3.14356	14.3133	−2.60192	1.29358	1.70692	9.96924	.55905	15.6069	−.83500	−5.95366	10.4733
2.50	4.21053	2.52780	3.74107	18.6678	−5.21239	2.13970	2.05114	12.6210	−2.42793	20.8075	−3.16125	−9.64841	13.0414
2.20	4.78469	3.32068	4.70908	25.9083	−9.21741	3.26461	2.74715	17.8619	−7.42043	29.1729	−6.47026	−15.6451	18.3895
2.10	5.01253	3.61621	5.14361	29.3190	−10.9472	3.69590	3.10809	20.5534	−9.65451	33.0149	−7.83911	−18.4005	20.1390
1.90	5.54017	4.24469	6.22882	38.3694	−15.1403	4.62868	4.08162	28.1102	−15.1781	42.9981	−11.0587	−25.5359	24.1373
1.68	6.26566	4.97649	7.84048	53.3432	−21.1367	5.73565	5.63664	41.3948	−23.2324	59.0788	−15.5001	−30.9439	29.4956
1.50	7.01754	5.58839	9.58798	71.7863	−27.4388	6.67459	7.39792	58.4583	−31.7953	78.4609	−20.0409	−50.5037	34.8859
1.40	7.51880	5.92672	10.7730	85.6551	−31.6318	7.19803	8.61585	71.5756	−37.5200	92.8531	−23.0159	−60.0713	38.4026
1.30	8.09717	6.25999	12.1474	103.163	−36.4354	7.71630	10.0420	88.3498	−44.0911	110.879	−26.3934	−73.2599	42.3941
1.10	9.56938	6.80867	15.6376	154.723	−48.4330	8.71601	13.6921	138.521	−60.5158	163.439	−34.7409	−109.784	52.3172
1.00	10.5263	7.19773	17.8823	193.444	−56.0456	9.18676	16.0447	176.599	−70.9259	202.631	−40.0009	−130.928	58.0173
.90	11.6959	7.47913	20.5938	246.191	−65.1695	9.63112	18.8819	228.744	−83.3799	255.822	−46.2876	−173.704	66.2006
.84	12.5313	7.63826	22.5094	287.466	−71.5798	9.88297	20.8811	269.682	−92.1133	297.349	−50.6987	−202.306	71.0094
.80	13.1579	7.73991	23.9352	320.873	−76.3365	10.0440	22.3660	302.372	−98.5851	330.417	−53.9705	−225.182	75.5558
.70	15.0376	7.97728	28.1632	429.041	−90.3851	10.4208	26.7525	410.539	−117.658	439.462	−63.6326	−300.368	87.4609
.60	17.5439	8.18864	33.7070	596.971	−108.716	10.7570	32.4691	578.023	−142.462	607.728	−76.2469	−416.339	103.198
.56	18.7970	8.26537	36.4475	690.756	−117.752	10.8790	35.2825	671.647	−154.664	701.635	−82.4695	−480.792	110.924
.52	20.2429	8.33744	39.5891	807.080	−128.095	10.9940	38.4990	787.819	−168.607	818.074	−89.5960	−561.174	119.708
.36	29.2398	8.57650	58.8154	1725.58	−191.186	11.3760	58.0419	1705.75	−253.355	1730.91	−133.144	−1195.89	173.362
.26	40.4858	8.68382	82.4588	3344.24	−268.548	11.5444	81.8939	3324.24	−357.043	3355.78	−186.654	−2291.25	239.714
.16	65.7895	8.75727	135.137	8896.41	−440.657	11.6657	134.794	8875.77	−586.749	8908.08	−305.863	−6161.76	264.494
.10	105.263	8.78470	216.902	22887.6	−707.584	11.7082	216.682	22817.4	−943.012	22849.3	−490.902	−15722.7	643.309
							$M=\frac{5}{4}$						
20.00	0.27778	−0.00103	0.22815	0.06045	0.21882	0.00087	0.23777	0.05814	0.29553	0.06132	0.45659	−0.01551	−0.00160
10.00	.55556	−.01996	.41500	.25390	.41670	−.06327	.42439	.24339	.57278	.19063	.84109	−.04965	.00819
5.00	1.11111	.07962	.79349	1.07903	.74308	−.03812	.75543	1.04705	1.10037	1.04091	1.49851	−.18729	.13163
4.40	1.26263	.09225	.89629	1.39479	.84102	−.07860	.83508	1.34400	1.30141	1.31619	1.67610	−.23051	.22001
3.30	1.68350	.25179	1.11026	2.36647	.94772	.00623	.84890	2.05296	1.67536	2.37270	1.79662	−.55339	.63036
2.80	1.98413	.47142	1.32708	3.30046	.80052	.27547	.91828	2.70703	1.59376	3.57593	1.71880	−1.01298	1.09251
2.60	2.13675	.59314	1.46975	3.88620	.66943	.44059	1.00268	3.15058	1.45701	4.32679	1.67211	−1.31370	1.30322
2.40	2.31481	.73357	1.66044	4.67228	.48252	.63847	1.14028	3.77937	1.23812	5.31075	1.62280	−1.71629	1.54788
2.20	2.52525	.89075	1.91297	5.74578	.22823	.86646	1.34833	4.68307	.91962	6.61224	1.57656	−2.25852	1.83276
2.00	2.77778	1.06163	2.24455	7.23819	−.10719	1.11998	1.64732	5.99657	.48138	8.35812	1.54013	−2.99707	2.16703
1.96	2.83447	1.09710	2.32220	7.60326	−.18545	1.17314	1.72014	6.32512	.37724	8.77640	1.53409	−3.17543	2.24095
1.84	3.01932	1.20545	2.58151	8.56557	−.44540	1.33674	1.96875	7.48002	.02815	10.2053	1.52335	−3.78773	2.47007
1.66	3.34672	1.37142	3.05697	11.3836	−.91551	1.59007	2.43968	9.82591	−.61190	12.9737	1.52417	−4.98415	2.89089
1.56	3.56125	1.46411	3.37532	13.2158	−1.22532	1.73277	2.76191	11.5697	−1.03745	14.9486	1.53659	−5.84313	3.15485
1.48	3.75375	1.53786	3.66846	14.9806	−1.50242	1.84688	3.05660	13.2624	−1.41954	16.8275	1.55418	−6.66353	3.38831
1.34	4.14594	1.66481	4.25359	18.9204	−2.06073	2.04435	3.66571	17.0793	−2.19177	20.9647	1.60498	−8.47728	3.85561
1.24	4.48029	1.75284	4.75682	22.6356	−2.52797	2.18197	4.18841	20.7101	−2.83902	24.8176	1.66044	−10.1722	4.24665
1.10	5.05051	1.87067	5.61041	29.7097	−3.30445	2.36702	5.07747	27.6722	−3.91428	32.0767	1.77302	−13.3754	4.90165
1.06	5.24109	1.90291	5.89388	32.2783	−3.55838	2.41782	5.37279	30.2103	−4.26548	34.6961	1.81441	−14.5338	5.11805
.98	5.66893	1.96520	6.52649	38.4125	−4.11894	2.51610	6.03135	36.2859	−5.03977	40.9256	1.91241	−17.2914	5.59984
.94	5.91017	1.99514	6.88089	42.0940	−4.42961	2.56342	6.39980	39.9393	−5.46825	44.6574	1.97019	−18.9423	5.86978
.88	6.31313	2.03842	7.46934	48.5998	−4.94077	2.63193	7.01066	46.4046	−6.17199	51.2317	2.06989	−21.8566	6.31771
.82	6.77507	2.07961	8.13877	56.6027	−5.51597	2.69723	7.70405	54.3690	−6.96218	59.2999	2.18808	−25.4363	6.82606
.78	7.12251	2.10583	8.63895	63.0038	−5.94185	2.73883	8.22097	60.7458	−7.54621	65.7426	2.27912	−28.2928	7.21026
.74	7.50751	2.13102	9.19010	70.4784	−6.40791	2.77884	8.78960	68.1969	−8.18410	73.2572	2.38169	−31.0290	7.62955
.70	7.93651	2.15513	9.80097	79.2786	−6.92064	2.81714	9.41840	76.9748	−8.88488	82.0957	2.49776	−35.5491	8.09848
.66	8.41751	2.17813	10.4820	89.7347	−7.48844	2.85375	10.1180	87.4097	−9.65901	92.5884	2.62958	−40.2130	8.62434
.60	9.25926	2.21045	11.6654	109.529	−8.46664	2.90518	11.3304	107.174	−10.9903	112.434	2.86371	−49.0229	9.52380
.56	9.92063	2.23051	12.5890	126.414	−9.22393	2.93717	12.2738	124.041	−12.0179	129.351	3.04991	−56.5440	10.2582
.52	10.6838	2.24932	13.6489	147.353	−10.0875	2.96709	13.3542	144.963	−13.1887	150.320	3.20066	−65.8407	11.0690
.48	11.5741	2.26687	14.8791	173.751	−11.0839	2.99512	14.6053	171.344	−14.5358	176.746	3.52136	−77.5946	12.0059
.44	12.6263	2.28312	16.3258	207.680	−12.2490	3.02103	16.0733	205.257	−16.1087	210.701	3.82437	−92.6756	13.1083
.40	13.8889	2.29805	18.0537	252.297	−13.6328	3.04483	17.8228	249.862	−17.9740	255.342	4.18999	−112.484	14.4990
.36	15.4321	2.31164	20.1562	312.609	−15.3084	3.06669	19.9473	310.162	−20.2255	315.676	4.63894	−139.384	16.1738
.34	16.3399	2.31792	21.3892	351.057	−16.2875	3.07652	21.1915	348.604	−21.5442	354.134	4.90396	−156.341	17.1073
.32	17.3611	2.32386	22.7736	396.937	−17.3847	3.08630	22.5872	394.478	−23.0120	400.023	5.20248	−176.961	18.1605
.30	18.5185	2.32945	24.3397	452.300	−18.6232	3.09500	24.1648	449.831	−24.6772	455.395	5.54158	−201.400	19.1671
.28	19.8413	2.33469	26.1264	519.946	−20.0328	3.10368	25.9624	517.481	−26.5612	523.050	5.92953	−231.739	21.0328
.14	39.6825	2.36140	52.7445	2094.62	−40.8806	3.14468	52.6634	2092.06	−54.4906	2097.76	11.7828	−925.551	34.7301
.06	92.5926	2.36874	123.386	11426.2	−95.9104	3.18836	123.354	11428.4	−125.022	11429.4	27.4433	−5776.74	29.8099

TABLE II.—VALUES OF FUNCTIONS USED IN THE FLUTTER CALCULATIONS—Continued

$\bar{\omega}$	$\frac{1}{k}$	L_1	L_2	L_3'	L_4'	M_1'	M_2'	M_3'	M_4'	$M_1'+L_3'$	$M_2'+L_4'$	D_R	D_I
							$M=\frac{10}{7}$						
20.00	0.19608	0.00227	0.13710	0.02700	0.13688	0.00443	0.13688	0.02705	0.18353	0.03143	0.27376	−0.00648	−.00018
10.00	.39216	−.00911	.27591	.10722	.26624	−.01725	.28916	.10313	.35992	.08997	.55540	−.02141	−.00124
5.00	.78431	−.00598	.50830	.45680	.50919	−.06970	.51311	.45040	.70878	.38710	1.02230	−.07021	.02603
3.90	1.00553	.01036	.60695	.73498	.64457	−.10395	.55892	.69894	.93421	.63103	1.20349	−.12311	.09011
3.10	1.26502	.10629	.73206	1.14404	.73643	−.00539	.59322	1.02716	1.10167	1.13865	1.32965	−.25428	.19434
2.40	1.63399	.28611	1.00934	1.98049	.76349	.24197	.78890	1.74282	1.16631	2.22246	1.55239	−.55617	.34069
2.00	1.96078	.42338	1.32284	2.99507	.74175	.44550	1.07377	2.67381	1.14150	3.44057	1.81552	−.91582	.47385
1.60	2.45098	.57188	1.83902	4.97860	.68975	.67257	1.58663	4.57135	1.06421	5.65117	2.27638	−1.59692	.65230
1.34	2.92654	.66667	2.35411	7.40271	.64192	.82004	2.11529	6.94229	.98523	8.22275	2.75721	−2.41036	.81646
1.18	3.32336	.72169	2.78331	9.78696	.60897	.90636	2.55920	9.29616	.93408	10.6933	3.16817	−3.20292	.94948
1.06	3.69959	.76043	3.18711	12.3445	.58376	.96741	2.97716	11.8325	.89151	13.3119	3.56092	−4.04780	1.07312
.94	4.17188	.79648	3.68899	15.9586	.55944	1.02439	3.49574	15.4270	.84908	16.9830	4.05518	−5.23712	1.22612
.88	4.45633	.81335	3.96867	18.3504	.54814	1.05112	3.80467	17.8097	.82867	19.4015	4.35281	−6.02275	1.31764
.82	4.78240	.82937	4.33003	21.2901	.53783	1.07654	4.15586	20.7408	.80934	22.3666	4.69369	−6.96717	1.42136
.78	5.02765	.83956	4.58539	23.6401	.53172	1.09272	4.41806	23.0853	.79732	24.7328	4.94978	−7.75737	1.50010
.74	5.29942	.84933	4.86709	26.3831	.52637	1.10825	4.70685	25.8230	.78622	27.4913	5.23322	−8.65588	1.58598
.70	5.60224	.85968	5.17957	29.6115	.52197	1.12312	5.02666	29.0463	.77632	30.7346	5.54863	−9.71303	1.69078
.66	5.94177	.86759	5.52836	33.4462	.51876	1.13729	5.38297	32.8763	.76791	34.5835	5.90173	−10.9677	1.78838
.62	6.32511	.87604	5.92041	38.0486	.51702	1.15075	5.78275	37.4742	.76137	39.1994	6.29977	−12.4734	1.90912
.60	6.53595	.88010	6.13534	40.7032	.51679	1.15721	6.00162	40.1266	.75895	41.8604	6.51841	−13.3416	1.97511
.56	7.00280	.88785	6.60968	46.8924	.51795	1.16958	6.48398	46.3117	.75622	48.0620	7.00193	−15.3666	2.12176
.52	7.54148	.89512	7.15470	54.5661	.52169	1.18116	7.03721	53.9815	.75676	55.7473	7.55890	−17.8745	2.28953
.48	8.16993	.90189	7.78797	64.2390	.52870	1.19196	7.67886	63.6508	.76165	65.4310	8.20756	−21.0362	2.48497
.44	8.91266	.90816	8.53348	76.6703	.53994	1.20200	8.43258	76.0790	.77230	77.8723	8.97252	−25.1030	2.71955
.40	9.80392	.91392	9.42482	93.0168	.55673	1.21121	9.33288	92.4224	.79027	94.2280	9.88961	−30.4485	2.99778
.36	10.8932	.91915	10.5105	115.112	.58098	1.21953	10.4274	114.514	.81772	116.332	11.0084	−37.6635	3.32361
.34	11.5340	.92157	11.1476	129.196	.59684	1.22388	11.0689	128.598	.83620	130.419	11.6657	−42.2590	3.54192
.32	12.2549	.92386	11.8633	146.004	.61562	1.22703	11.7891	145.404	.85902	147.231	12.4047	−47.7516	3.75375
.30	13.0719	.92601	12.6732	166.284	.63818	1.23051	12.6035	165.684	.88705	167.515	13.2417	−54.3875	4.02221
.28	14.0056	.92803	13.5976	191.064	.66519	1.23364	13.5324	190.463	.91941	192.298	14.1976	−62.4490	4.31785
.26	15.0830	.92991	14.6628	221.781	.69760	1.23678	14.6021	221.179	.96218	223.018	15.2997	−72.5388	4.66702
.24	16.3399	.93166	15.9040	260.493	.73689	1.23964	15.8479	259.890	1.01340	261.733	16.5848	−85.2274	5.05421
.22	17.8253	.93326	17.3693	310.237	.78518	1.24191	17.3178	309.634	1.07027	311.479	18.1030	−101.310	5.52723
.20	19.6078	.93473	19.1258	375.641	.84441	1.24435	19.0791	375.032	1.14853	376.885	19.9235	−122.731	5.91765
.16	24.5098	.93726	23.9493	587.618	1.01451	1.24876	23.9120	587.005	1.37985	588.867	24.9265	−192.393	7.26316
.10	39.2157	.94000	38.3920	1506.20	1.55291	1.25288	38.3685	1505.59	2.08335	1507.45	39.9214	−492.234	11.9893
.06	65.3595	.94113	64.0369	4186.04	2.54033	1.25802	64.0217	4185.51	3.61239	4187.30	66.5620	−1395.70	29.8922
.04	98.0392	.94146	96.0788	9420.14	3.77746	1.23919	96.0801	9418.46	3.48453	9421.38	99.8576	−2778.09	−175.059
							$M=\frac{5}{3}$						
20.00	0.15625	−0.00090	0.09297	0.01471	0.09352	−0.00198	0.09255	0.01474	0.12493	0.01273	0.18607	−0.00294	0.00008
10.00	.31250	−.00024	.19313	.05802	.18525	.00185	.20108	.05672	.24911	.05987	.38633	−.01098	−.00111
5.00	.62500	−.01971	.36096	.24221	.36086	−.05602	.37339	.23936	.49241	.18619	.73425	−.03415	.00647
3.10	1.00806	.03778	.51751	.61729	.54668	−.02005	.45027	.57390	.76842	.59724	.99695	−.11745	.05904
2.50	1.25000	.11109	.66519	.97181	.62689	.08187	.56462	.89651	.88199	1.05368	1.19151	−.21271	.09430
1.90	1.64474	.20716	.96017	1.76808	.73691	.22547	.84272	1.65534	1.02910	1.99355	1.57963	−.42253	.14645
1.60	1.95312	.25811	1.20754	2.57072	.81960	.30398	1.09136	2.43919	1.13661	2.87470	1.91096	−.62989	.18407
1.30	2.40385	.30697	1.57362	4.01643	.94318	.38027	1.46616	3.86735	1.29656	4.39670	2.40934	−.99761	.23635
1.10	2.84091	.33078	1.92656	5.71955	1.06737	.42719	1.82901	5.55996	1.45784	6.14674	2.89638	−1.42724	.28548
.94	3.32447	.35828	2.31278	7.94412	1.20910	.46121	2.22538	7.77701	1.64279	8.40533	3.43448	−1.98627	.33876
.84	3.72024	.37045	2.62575	10.0288	1.32773	.48051	2.54562	9.35747	1.79822	10.5093	3.87335	−2.50302	.38190
.76	4.11184	.37939	2.93308	12.3244	1.44693	.49472	2.85924	12.1500	1.95482	12.8191	4.30617	−3.08406	.42432
.68	4.59559	.38759	3.31011	15.4793	1.59609	.50777	3.24294	15.3021	2.15130	15.9871	4.83903	−3.87400	.47657
.62	5.04032	.39321	3.65468	18.6904	1.73469	.51673	3.59275	18.5113	2.33422	19.2071	5.32744	−4.67758	.52442
.58	5.38793	.39670	3.92288	21.4073	1.84380	.52230	3.86456	21.2270	2.47842	21.9296	5.70836	−5.35735	.56135
.54	5.78704	.39999	4.22983	24.7506	1.96975	.52753	4.17518	24.5692	2.64505	25.2781	6.14493	−6.19330	.60422
.50	6.25000	.40305	4.58479	28.9286	2.11660	.53243	4.53388	28.7461	2.83356	29.4610	6.65048	−7.23871	.65357
.48	6.51042	.40451	4.78400	31.4201	2.19951	.53474	4.73500	31.2371	2.94944	31.9548	6.93451	−7.86162	.68100
.44	7.10227	.40724	5.23577	37.4613	2.38860	.53911	5.19061	37.2778	3.20019	38.0004	7.57921	−9.37212	.74390
.42	7.44048	.40852	5.49340	41.1494	2.49701	.54116	5.45019	40.9650	3.34409	41.6906	7.94720	−10.2946	.77993
.40	7.81250	.40975	5.77642	45.4047	2.61648	.54311	5.73518	45.2197	3.50264	45.9478	8.35166	−11.3577	.81802
.38	8.22368	.41092	6.08882	50.3494	2.74879	.54499	6.04956	50.1641	3.67846	50.8944	8.79835	−12.5950	.86195
.36	8.68056	.41203	6.43551	56.1411	2.89612	.54676	6.39822	55.9556	3.87415	56.6879	9.29434	−14.0424	.91250
.34	9.19118	.41309	6.82254	62.9848	3.06104	.54846	6.78726	62.7989	4.09350	63.5333	9.84830	−15.7550	.96592
.32	9.76562	.41408	7.25745	71.1517	3.24691	.55008	7.22420	70.9651	4.34048	71.7017	10.4711	−17.7948	1.02296
.30	10.4167	.41502	7.74982	81.0058	3.45784	.55153	7.71856	80.8193	4.62114	81.5573	11.1765	−20.2582	1.09761
.28	11.1607	.41590	8.31196	93.0463	3.69940	.55296	8.28278	92.8591	4.94261	93.5993	11.9822	−23.2722	1.16910
.26	12.0192	.41672	8.95997	107.971	3.97847	.55428	8.93280	107.784	5.31420	108.525	12.9113	−27.0066	1.26740
.24	13.0208	.41748	9.71531	126.781	4.30446	.55550	9.69021	126.593	5.74828	127.336	13.9947	−31.7120	1.36439
.22	14.2045	.41818	10.6072	150.951	4.69027	.55659	10.5842	150.763	6.26151	151.508	15.2745	−37.7461	1.48560
.20	15.6250	.41882	11.6767	182.729	5.15358	.55767	11.6558	182.540	6.87952	183.287	16.8094	−45.7121	1.61942
.18	17.3611	.41940	12.9830	225.679	5.72058	.55863	12.9641	225.491	7.63546	226.238	18.6847	−56.4691	1.83115
.16	19.5312	.41992	14.6148	285.725	6.43004	.55947	14.5980	285.537	8.58106	286.284	21.0280	−71.4066	2.05854
.08	39.0625	.42139	29.2800	1144.03	12.8288	.56266	29.2706	1143.88	17.1434	1144.59	42.0994	−268.132	6.36769
.06	52.0833	.42159	39.0498	2034.13	17.0995	.56920	39.0425	2033.99	22.6518	2034.69	56.1420	−496.917	9.36991
.04	78.1250	.42177	58.5853	4577.25	25.6355	.57088	58.5845	4576.81	34.8559	4577.82	84.2200	−1222.90	−22.0494

TABLE II.—VALUES OF FUNCTIONS USED IN THE FLUTTER CALCULATIONS—Continued

$\bar{\omega}$	$\frac{1}{k}$	L_1	L_2	L_3'	L_4'	M_1'	M_2'	M_3'	M_4'	$M_1'+L_3'$	$M_2'+L_4'$	D_R	D_I
							$M=2$						
20.00	0.13333	−0.00008	0.06698	0.00885	0.06657	−0.00009	0.06739	0.00881	0.08885	0.00876	0.13396	−0.00140	−0.00001
10.00	.26667	.00177	.13612	.03543	.13288	.00439	.13889	.03521	.17813	.03982	.27177	−.00588	−.00010
5.00	.53333	−.01646	.26350	.14409	.26138	−.03647	.27440	.14234	.35343	.10762	.53578	−.01849	.00168
2.70	.98765	.03592	.44538	.49654	.45368	.01517	.40159	.47173	.62041	.51171	.85527	−.08471	.02610
2.10	1.26984	.07918	.60315	.84503	.55683	.07924	.54892	.80838	.75833	.92427	1.10575	−.15468	.03964
1.60	1.66667	.11850	.84295	1.50219	.69968	.13972	.78872	1.45559	.94730	1.64191	1.48840	−.28407	.05608
1.30	2.05128	.14082	1.07856	2.31942	.83947	.17466	1.02878	2.26740	1.13198	2.49408	1.86825	−.44309	.07214
1.10	2.42424	.15433	1.30577	3.27821	.97664	.19595	1.26078	3.22296	1.31341	3.47416	2.23742	−.62865	.08667
.90	2.96296	.16626	1.63075	4.94952	1.17699	.21485	1.59188	4.89145	1.57885	5.16437	2.76887	−.95123	.10731
.80	3.33333	.17153	1.85225	6.29401	1.31584	.22822	1.81690	6.23471	1.76306	6.51723	3.13274	−1.21039	.12135
.74	3.60360	.17444	2.01310	7.37542	1.41759	.22786	1.97999	7.31545	1.89814	7.60328	3.39758	−1.41879	.13157
.70	3.80952	.17627	2.13526	8.25611	1.49530	.23078	2.10371	8.19571	2.00137	8.48689	3.59901	−1.58846	.13921
.64	4.16667	.17886	2.34649	9.89987	1.63041	.23489	2.31733	9.83886	2.18092	10.1348	3.94774	−1.90491	.15263
.58	4.59770	.18123	2.60050	12.0801	1.79398	.23868	2.57381	12.0186	2.39832	12.3188	4.36774	−2.32474	.16896
.54	4.93827	.18270	2.80063	13.9546	1.92342	.24102	2.77564	13.8928	2.57053	14.1956	4.69906	−2.68551	.18170
.50	5.33333	.18407	3.03226	16.2970	2.07384	.24321	3.00899	16.2348	2.77070	16.5402	5.08283	−3.13657	.19625
.46	5.79710	.18534	3.30361	19.2768	2.25071	.24523	3.28209	19.2143	3.00610	19.5220	5.53280	−3.71002	.21357
.42	6.34921	.18651	3.62600	23.1481	2.46157	.24710	3.60626	23.0854	3.28681	23.3952	6.06483	−4.40253	.30381
.40	6.66667	.18705	3.81112	25.5836	2.58293	.24798	3.79228	25.4707	3.44842	25.7816	6.37521	−4.91468	.24585
.38	7.01754	.18757	4.01558	28.3055	2.71716	.24880	3.99759	28.2425	3.62714	28.5543	6.71475	−5.44770	.25914
.36	7.40741	.18807	4.24247	31.5522	2.86638	.24960	4.22544	31.4891	3.82589	31.8018	7.09182	−6.07278	.27372
.34	7.84314	.18854	4.49589	35.3856	3.03325	.25085	4.47978	35.3254	4.04820	35.6389	7.51303	−6.81126	.28984
.30	8.88889	.18940	5.10339	45.4905	3.43409	.25173	5.08912	45.4272	4.58222	45.7422	8.52321	−8.75577	.32951
.28	9.52381	.18979	5.47184	52.2400	3.67763	.25236	5.45849	52.1765	4.90674	52.4924	9.13612	−10.0553	.35311
.26	10.2564	.19016	5.89669	60.6062	3.95877	.25293	5.88428	60.5427	5.28130	60.8591	9.84305	−11.6640	.39068
.24	11.1111	.19049	6.39204	71.1502	4.28689	.25346	6.38060	71.0803	5.71849	71.4037	10.6675	−13.6924	.40827
.22	12.1212	.19081	6.97712	84.6987	4.67484	.25398	6.93659	84.6351	6.23567	84.9527	11.6414	−16.3019	.45064
.20	13.3333	.19109	7.67884	102.512	5.14053	.25441	7.66929	102.448	6.85610	102.766	12.8098	−19.7250	.48987
.18	14.8148	.19135	8.53607	126.588	5.70988	.25486	8.52745	126.524	7.61560	126.843	14.2373	−24.3684	.54691
.16	16.6667	.19158	9.60715	160.247	6.42183	.25515	9.59954	160.182	8.56338	160.502	16.0214	−30.8224	.59706
.14	19.0476	.19179	10.9837	209.341	7.33737	.25559	10.9770	209.277	9.78581	209.597	18.3144	−40.3103	.70109
.10	26.6667	.19211	15.3864	410.432	10.2685	.25587	15.3814	410.372	13.6875	410.688	25.6499	−78.8381	1.13108
.06	44.4444	.19232	25.6542	1140.82	17.1103	.25623	25.6507	1140.28	22.7618	1140.58	42.7610	−216.790	2.97544
.04	66.6667	.19240	38.4862	2565.87	25.6606	.25653	38.4854	2565.73	34.2170	2566.13	64.1460	−493.900	−3.33473
.02	133.333	.19241	76.9781	10263.9	51.3241	.22448	76.9777	10263.7	64.1563	10264.1	128.302	−1317.02	−10.4670
							$M=\frac{5}{2}$						
20.00	0.11905	0.00029	0.04770	0.00567	0.04760	0.00059	0.04776	0.00567	0.06351	0.00626	0.09536	−0.00076	−0.00001
10.00	.23810	.00080	.09534	.02271	.09517	.00160	.09532	.02273	.12728	.02431	.19049	−.00308	−.00005
5.00	.47619	−.01004	.19080	.09083	.18862	−.02004	.19777	.08981	.25367	.07079	.38639	−.01018	.00041
4.80	.49603	−.01083	.19674	.09847	.19634	−.02254	.20253	.09718	.26426	.07593	.39887	−.01106	.00074
2.40	.99206	.02406	.37551	.40186	.37419	.01879	.35297	.39036	.50433	.42065	.72716	−.05546	.00584
1.90	1.25313	.04114	.49084	.65264	.46379	.04474	.46633	.63795	.62350	.69738	.93012	−.09271	.01369
1.40	1.70068	.05800	.69385	1.22505	.61784	.07098	.67123	1.20742	.82810	1.29603	1.28907	−.17679	.01905
1.20	1.98413	.06409	.82238	1.67931	.71598	.08056	.80161	1.66066	.95850	1.75987	1.51759	−.24317	.02329
1.04	2.28938	.06852	.96011	2.24752	.82209	.08757	.94126	2.22813	1.09956	2.33509	1.76335	−.32604	.02710
.96	2.48016	.07056	1.04583	2.64413	.88859	.09081	1.02807	2.62441	1.18800	2.73494	1.91666	−.38355	.02947
.86	2.76855	.07298	1.17494	3.30417	.98930	.09458	1.15865	3.28405	1.32199	3.39875	2.14795	−.48001	.03303
.78	3.05250	.07468	1.30159	4.02513	1.08866	.09735	1.28658	4.00472	1.45423	4.12248	2.37524	−.58494	.03647
.72	3.30688	.07589	1.41473	4.73083	1.17780	.09928	1.40071	4.71024	1.57289	4.83011	2.57851	−.68768	.03903
.68	3.50140	.07665	1.50107	5.30865	1.24602	.10049	1.48778	5.28792	1.66373	5.40914	2.73375	−.77180	.04201
.62	3.84025	.07772	1.65116	6.39410	1.36499	.10219	1.63888	6.37320	1.82215	6.49629	3.00387	−.93547	.04639
.58	4.10509	.07838	1.76825	7.31232	1.45804	.10325	1.75069	7.29132	1.94610	7.41557	3.21473	−1.00337	.04939
.52	4.57875	.07930	1.97728	9.10725	1.62463	.10471	1.95581	9.08610	2.16800	9.21197	3.59144	−1.32315	.05509
.48	4.96032	.07986	2.14533	10.6056	1.75892	.10561	2.13564	10.6744	2.34692	10.8012	3.89456	−1.55580	.05962
.46	5.17598	.08012	2.24024	11.6497	1.83485	.10603	2.23093	11.6284	2.44810	11.7557	4.06578	−1.69410	.06233
.42	5.66893	.08062	2.45695	13.9827	2.00848	.10682	2.44841	13.9614	2.67948	14.0895	4.45689	−2.03383	.06855
.38	6.26566	.08107	2.71898	17.0908	2.21879	.10754	2.71122	17.0694	2.95974	17.1983	4.93001	−2.48597	.07578
.36	6.61376	.08128	2.87170	19.0474	2.34151	.10788	2.86434	19.0259	3.12329	19.1553	5.20585	−2.77008	.07971
.34	7.00280	.08148	3.04229	21.3598	2.47869	.10820	3.03532	21.3378	3.30615	21.4675	5.51401	−3.10712	.08466
.32	7.44048	.08166	3.23412	24.1182	2.63306	.10850	3.22755	24.0967	3.51190	24.2267	5.86061	−3.50866	.09002
.30	7.93651	.08184	3.45141	27.4470	2.80805	.10878	3.44524	27.4255	3.74514	27.5558	6.25329	−3.99279	.09619
.28	8.50340	.08201	3.69964	31.5145	3.00807	.10904	3.69388	31.4928	4.01176	31.6235	6.70195	−4.58423	.10225
.26	9.15751	.08216	3.98593	36.5562	3.23890	.10929	3.98057	36.5345	4.31947	36.6655	7.21947	−5.31799	.11035
.24	9.92063	.08230	4.31980	42.9102	3.50826	.10951	4.31485	42.8886	4.67848	43.0197	7.82311	−6.24185	.11995
.22	10.8225	.08243	4.71428	51.0749	3.82663	.10972	4.70968	51.0533	5.10290	51.1846	8.53631	−7.42973	.12992
.20	11.9048	.08256	5.18738	61.8097	4.20872	.10992	5.18323	61.7882	5.61232	61.9196	9.39135	−8.90447	.14578
.18	13.2275	.08266	5.76550	76.3185	4.67580	.11008	5.76178	76.2967	6.23456	76.4286	10.4376	−11.1006	.15888
.16	14.8810	.08276	6.48794	96.6021	5.25968	.11025	6.48462	96.5805	7.01349	96.7124	11.7443	−14.0535	.18114
.14	17.0068	.08285	7.41657	126.187	6.01052	.11037	7.41365	126.166	8.01420	126.297	13.4242	−18.3524	.21331
.12	19.8413	.08293	8.65446	171.771	7.01164	.11054	8.65193	171.750	9.34986	171.882	15.6636	−24.9084	.24310
.06	39.6825	.08308	17.3164	687.213	14.0211	.11088	17.3148	687.204	18.6993	687.324	31.3359	−100.132	.94255
.04	59.5238	.08312	25.9767	1546.28	21.0206	.11230	25.9762	1546.24	28.1290	1546.39	47.0058	−229.553	−.28947
.02	119.048	.08311	51.9558	6185.28	42.0625	.09993	51.9556	6185.23	54.7873	6185.38	94.0181	−765.177	−1.53023

TABLE II.—VALUES OF FUNCTIONS USED IN THE FLUTTER CALCULATIONS—Concluded

$\bar{\omega}$	$\frac{1}{k}$	L_1	L_2	L_3'	L_4'	M_1'	M_2'	M_3'	M_4'	$M_1'+L_3'$	$M_2'+L_4'$	D_R	D_I
$M=\frac{10}{3}$													
20.00	0.10989	−0.00012	0.03287	0.00363	0.03296	−0.00026	0.03279	0.00363	0.04396	0.00337	0.06575	−0.00036	0
10.00	.21978	−.00010	.06520	.01451	.06591	−.00039	.06451	.01453	.08800	.01412	.13042	−.00148	.00003
5.00	.43956	−.00468	.13262	.05779	.13131	−.00885	.13599	.05732	.17594	.04894	.26730	−.00523	.00008
4.40	.49950	−.00527	.14749	.07452	.14885	−.01138	.14876	.07360	.19960	.06314	.29761	−.00684	.00041
2.20	.99900	.01148	.29222	.30418	.28968	.01070	.28310	.29991	.38813	.31488	.57298	−.03117	.00288
1.80	1.22100	.01675	.36313	.45807	.35199	.01882	.35381	.45309	.47080	.47689	.70580	−.04766	.00372
1.30	1.69062	.02295	.51431	.88688	.48373	.02851	.50608	.88113	.64618	.91539	.98981	−.09259	.00538
1.10	1.99800	.02511	.61299	1.24305	.57022	.03192	.60561	1.23704	.76184	1.27497	1.17583	−.12998	.00641
.88	2.49750	.02717	.77266	1.94886	.71108	.03520	.76642	1.94260	.94891	1.98406	1.47745	−.20406	.00808
.80	2.74725	.02783	.85222	2.36068	.78153	.03625	.84646	2.35434	1.04283	2.39693	1.62799	−.24724	.00889
.72	3.05250	.02843	.94928	2.91733	.86775	.03721	.94402	2.91092	1.15771	2.95454	1.81177	−.30561	.00988
.66	3.33000	.02885	1.03736	3.47426	.94617	.03787	1.03249	3.46780	1.26222	3.51213	1.97866	−.36399	.01050
.62	3.54484	.02911	1.10548	3.93870	1.00690	.03828	1.10088	3.93221	1.34316	3.97698	2.10778	−.41267	.01150
.58	3.78931	.02935	1.18293	4.50253	1.07604	.03867	1.17860	4.49601	1.43530	4.54120	2.25464	−.47179	.01230
.52	4.22654	.02969	1.32129	5.60466	1.19972	.03921	1.31738	5.59811	1.60015	5.64387	2.51710	−.58733	.01373
.46	4.77783	.02909	1.49554	7.16572	1.35572	.03970	1.49205	7.15913	1.80809	7.20542	2.84777	−.75105	.01556
.44	4.99500	.03009	1.56413	7.88317	1.41719	.03985	1.56079	7.82656	1.89003	7.87302	2.97798	−.82097	.01622
.42	5.23256	.03018	1.63923	8.59823	1.48451	.03999	1.63603	8.59161	1.97978	8.63822	3.12054	−.90116	.01705
.38	5.78369	.03034	1.81305	10.5066	1.64046	.04026	1.81015	10.4999	2.18767	10.5469	3.45061	−1.10131	.01865
.36	6,10501	.03042	1.91440	11.7079	1.73144	.04038	1.91164	11.7012	2.30895	11.7483	3.64308	−1.22718	.01981
.34	6.46412	.03049	2.02764	13.1274	1.83313	.04050	2.02503	13.1208	2.44452	13.1679	3.85816	−1.37607	.02117
.32	6.86813	.03056	2.15500	14.8218	1.94754	.04061	2.15254	14.8147	2.59705	14.8619	4.10008	−1.55364	.02251
.30	7.32601	.03063	2.29929	16.8652	2.07722	.04071	2.29699	16.8585	2.76993	16.9059	4.37421	−1.76773	.02366
.28	7.84929	.03069	2.46416	19.3625	2.22543	.04081	2.46201	19.3558	2.96753	19.4033	4.68744	−2.02059	.02537
.26	8.45309	.03075	2.65435	22.4581	2.39645	.04090	2.65235	22.4514	3.19555	22.4990	5.04880	−2.35404	.02738
.24	9.15751	.03080	2.87619	26.3593	2.59601	.04098	2.87433	26.3527	3.46158	26.4003	5.47034	−2.76291	.03028
.22	9.99001	.03085	3.13830	31.3723	2.83185	.04106	3.13660	31.3657	3.77602	31.4134	5.96845	−3.28842	.03283
.20	10.9890	.03089	3.45278	37.9638	3.11487	.04113	3.45123	37.9567	4.15336	38.0044	6.56610	−3.97945	.03624
.18	12.2100	.03093	3.83708	46.8714	3.46079	.04120	3.83568	46.8648	4.61461	46.9126	7.29647	−4.91372	.04044
.16	13.7363	.03097	4.31737	59.3253	3.89325	.04125	4.31613	59.3185	5.19110	59.3665	8.20938	−6.21820	.04438
.14	15.6986	.03100	4.93480	77.4002	4.44927	.04129	4.93372	77.4834	5.93236	77.5315	9.38299	−8.12114	.05033
.12	18.3150	.03103	5.75794	105.478	5.19060	.04136	5.75700	105.471	6.92114	105.519	10.9476	−11.0590	.05892
.10	21.9780	.03105	6.91021	151.894	6.22800	.04138	6.90942	151.887	8.30478	151.935	13.1380	−15.9210	.07175
.06	36.6300	.03109	11.5187	421.950	10.3805	.04157	11.5181	421.948	13.8451	421.992	21.8986	−44.3360	.22906
.04	54.9451	.03109	17.2788	949.406	15.5706	.04130	17.2786	949.391	20.7524	949.447	32.8492	−99.2323	−.06717
.02	109.890	.03109	34.5585	3797.66	31.1423	.03810	34.5584	3797.65	41.1535	3797.70	65.7007	−372.597	.12712
$M=5$													
20.00	0.10417	−0.00002	0.02084	0.00217	0.02083	−0.00004	0.02085	0.00216	0.02778	0.00213	0.04168	−0.00014	0
10.00	.20833	−.00022	.04122	.00868	.04165	−.00053	.04084	.00868	.05556	.00815	.08249	−.00059	.00001
5.00	.41667	−.00147	.08369	.03463	.08321	−.00270	.08477	.03448	.11119	.03193	.16798	−.00221	.00001
4.20	.49603	−.00159	.09822	.04905	.09686	−.00351	.09836	.04872	.13215	.04554	.19722	−.00316	.00010
2.10	.99206	.00350	.19647	.19843	.19557	.00347	.19389	.19732	.26113	.20190	.38946	−.01338	.00053
1.70	1.22549	.00495	.24454	.30387	.24102	.00572	.24199	.30260	.32169	.30959	.48301	−.02058	.00068
1.20	1.73611	.00659	.34988	.61232	.34059	.00830	.34772	.61089	.45437	.62062	.68831	−.04160	.00099
1.00	2.08333	.00715	.42136	.88295	.40636	.00917	.41947	.88148	.54470	.89212	.82783	−.06001	.00120
.84	2.48016	.00753	.50290	1.25255	.48586	.00979	.50125	1.25104	.64800	1.26234	.98711	−.08518	.00143
.80	2.60417	.00762	.52835	1.38123	.51009	.00993	.52677	1.37972	.68030	1.39116	1.03686	−.09394	.00150
.72	2.89352	.00778	.58771	1.70593	.56663	.01019	.58627	1.70439	.75567	1.71612	1.15290	−.11604	.00166
.68	3.06373	.00786	.62260	1.91289	.59989	.01031	.62123	1.91135	.80001	1.92320	1.22112	−.13012	.00177
.66	3.15657	.00790	.64163	2.03076	.61804	.01037	.64029	2.02922	.82420	2.04113	1.25833	−.13813	.00184
.62	3.36022	.00797	.68335	2.30166	.65785	.01048	.68209	2.30011	.87727	2.31214	1.33994	−.15656	.00194
.56	3.72024	.00806	.75707	2,82198	.72622	.01063	.75592	2.82043	.97109	2.83261	1.48414	−.19197	.00216
.50	4.16667	.00815	.84844	3.54068	.81550	.01078	.84740	3.53912	1.08745	3.55146	1.66290	−.24091	.00243
.46	4.52899	.00821	.92255	4.18378	.88635	.01086	.92160	4.18222	1.18190	4.19464	1.80795	−.28460	.00261
.42	4.96032	.00826	1.01076	5.01927	.97069	.01093	1.00989	5.01770	1.29435	5.03020	1.98058	−.34140	.00286
.38	5.48246	.00830	1.11751	6.13226	1.07279	.01101	1.11671	6.13069	1.43048	6.14327	2.18950	−.41721	.00321
.36	5.78704	.00832	1.17976	6.83291	1.13236	.01105	1.17901	6.83133	1.50300	6.84396	2.31137	−.46492	.00331
.34	6.12745	.00834	1.24933	7.66080	1.19894	.01108	1.24862	7.65922	1.59806	7.67138	2.44756	−.52124	.00351
.32	6.51042	.00836	1.32758	8.64874	1.27384	.01111	1.32691	8.64716	1.69852	8.65985	2.60075	−.58845	.00374
.30	6.94444	.00838	1.41626	9.84077	1.35872	.01113	1.41563	9.83919	1.81170	9.85190	2.77435	−.66947	.00402
.28	7.44048	.00839	1.51760	11.2973	1.45574	.01116	1.51701	11.2957	1.94105	11.3085	2.97275	−.76867	.00428
.26	8.01282	.00841	1.63451	13.1027	1.56768	.01119	1.63396	13.1011	2.09030	13.1139	3.20164	−.89153	.00463
.24	8.68056	.00842	1.77090	15.3780	1.69828	.01121	1.77039	15.3764	2.26444	15.3892	3.46867	−1.04640	.00504
.22	9.46970	.00844	1.93206	18.3017	1.85265	.01123	1.93160	18.3001	2.47024	18.3129	3.78425	−1.24515	.00537
.20	10.4167	.00845	2.12544	22.1457	2.03787	.01125	2.12502	22.1441	2.71720	22.1569	4.16289	−1.50675	.00593
.18	11.5741	.00846	2.36179	27.3411	2.26425	.01127	2.36140	27.3395	3.01906	27.3524	4.62565	−1.86013	.00691
.16	13.0208	.00847	2.65719	34.6044	2.54727	.01128	2.65685	34.6029	3.39637	34.6157	5.20412	−2.35434	.00781
.14	14.8810	.00848	3.03697	45.1986	2.91112	.01130	3.03667	45.1970	3.88152	45.2099	5 94779	−3.07542	.00872
.12	17.3611	.00849	3.54331	61.5215	3.39624	.01131	3.54306	61.5199	4.52844	61.5328	6.93930	−4.18609	.00975
.10	20.8333	.00849	4.25216	88.5924	4.07547	.01132	4.25196	88.5906	5.43403	88.6037	8.32743	−6.02636	.01007
.06	34.7222	.00850	7.08739	246.095	6.79234	.01136	7.08722	246.095	9.05726	246.106	13 8796	−16.7574	.04166
.04	52.0833	.00850	10.6313	553.719	10.1884	.01137	10.6312	553.716	13.5864	553.730	20.8196	−37.7154	.02312
.02	104.167	.00850	21.2628	2214.89	20.3777	.01054	21.2628	2214.88	27.0871	2214.90	41.6405	−147.179	−.19717

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