REPORT No. 27.

THEORY OF AN AIRPLANE ENCOUNTERING GUSTS, III.

By Edwin Bidwell Wilson.

The following is an immediate continuation of my two previous papers on the same subject which appeared in these Reports, First Year, pages 52–75, and Third Year, pages 405–431, the latter being a reprint of my contribution to the Proceedings American Philosophical Society (Philadelphia), volume 56, pages 212–248. In my second paper I pointed out: (1) That the study of a gust of the type $Je^{-nt}\sin pt$, tuned both in damping and in period to the natural motion of the machine, might be important; (2) that the solution for this case by Brodetsky treated only the particular integral without taking account of the fact that the constants of integration in the complementary function might be such as largely to upset any conclusion that such a gust produces violent motions; (3) that a new method for solving linear equations had been developed by Bromwich, which was suited to determine the motion for any particular gust, when the machine started from equilibrium, without the trouble of determining the constants of integration in the complementary function.

I shall now apply Bromwich's method to the calculation of the effect of a head-on gust of the form $u_1 = Je^{-nt}\sin pt$ where $n = .0654$, $p = .187$, as in the case of the slow oscillation for the Curtiss JN2. Inasmuch as the method depends on the use of the theory of functions of a complex variable, which is a mathematical subject of prime importance to any aeronautical engineer who would have that knowledge of fluid motion which is regarded in high quarters as essential, and further inasmuch as neither the theory of functions nor Bromwich's special method is likely to become as familiar as they should be to engineers or physicists without detailed directions for and examples of the application of such ways of calculating, I may be pardoned for the somewhat lengthy presentation of matters elementary for the pure mathematician.

Suppose it be required to solve the equation

$$\frac{d^2x}{dt^2} + 2v\frac{dx}{dt} + (v^2+n^2)x = Fe^{-vt}\cos(nt+\omega) \tag{1}$$

with the supposition that the damped harmonic force on the right is applied at and from the time $t=0$, and that at $t=0$ the system is at rest in its position of equilibrium, i. e., $x=0$ and $dx/dt=0$.

Now trigonometric terms are generally treated by their exponential equivalents, through the formulas

$$\cos y = \frac{e^{iy}+e^{-iy}}{2},\quad \sin y = \frac{e^{iy}-e^{-iy}}{2i},$$

$$e^{iy} = \cos y + i\sin y,\ \text{where } i=\sqrt{-1}.$$

The solution of (1) may be obtained by solving the equation obtained by replacing $\cos(nt+\omega)$ by its value as the sum of two exponential expressions. A method generally better is to replace the equation by

$$\begin{aligned}\frac{d^2x}{dt^2} + 2v\frac{dx}{dt} + (v^2+n^2)x &= Fe^{-vt}e^{i(nt+\omega)} \\ &= Fe^{i\omega}e^{(-v+in)t}\end{aligned} \tag{2}$$

and take the real part of the solution since the real part of e^{iy} is $\cos y$.

Bromwich states that the solution of (2) subject to the stated conditions is

$$x=\frac{1}{2\pi i}\int_{0}\xi e^{\lambda t}dt,$$

where

$$\xi=\frac{Fe^{i\omega}}{(\lambda^2+2\upsilon\lambda+\upsilon^2+n^2)(\lambda+\upsilon-in)}$$

$$=\frac{Fe^{i\omega}}{(\lambda+\upsilon-in)^2(\lambda+\upsilon+in)},$$

and where the integral must be taken around a curve in the complex λ-plane sufficiently large to include all the points, $\lambda=-\upsilon-in$ and $\lambda=-\upsilon+in$ (double), where ξ becomes infinite. He further points out that the value obtained for x is none other than the sum of the residues of $\xi e^{\lambda t}$ by virtue of Cauchy's theorem.

Now a residue may be given a simple definition. Suppose

$$f(x)=\frac{\phi(x)}{(x-a)^n},$$

where $\phi(x)$ does not vanish when $x=a$, i. e., no factor $x-a$ may be canceled out, the fraction is in its lowest terms. The function $\phi(x)$ may be expanded by Taylor's theorem about $x=a$ as:

$$\phi(x)=\phi(a)+\phi'(a)(x-a)+\tfrac{1}{2}\phi''(a)(x-a)^2+\cdot\cdot\cdot\cdot\cdot$$

and

$$f(x)=\frac{A_0}{(x-a)^n}+\frac{A_1}{(n-a)^{n-1}}+\cdot\cdot\cdot\cdot\cdot\cdot+\frac{A_{n-1}}{x-a}+A_n+A_{n+1}(x-a)+\cdot\cdot\cdot\cdot\cdot,$$

where $A_i=\phi^{(i)}(a)/i!$ with $\phi^{(i)}(a)$ denoting the ith derivative of $\phi(x)$ taken for the value $x=a$. The coefficient A_{n-1} which occurs over the factor $x-a$ is called the "residue of $f(x)$ at $x=a$."

In case $n=1$, that is, if $f(x)=\frac{\phi(x)}{(x-a)}$, the residue of f at a is $\phi(a)$.

In case $n=2$, that is, if $f(x)=\frac{\phi(x)}{(x-a)^2}$, the residue of f at a is $\phi'(a)$.

As applied to the case in hand where

$$f(\lambda)=\xi e^{\lambda t}=\frac{Fe^{i\omega}e^{\lambda t}}{(\lambda+\upsilon-in)^2(\lambda+\upsilon+in)},$$

the function for the consideration of the value $\lambda=-\upsilon-in$ may be written

$$f(\lambda)=\frac{Fe^{i\omega}e^{\lambda t}(\lambda-\upsilon-in)^{-2}}{\lambda+\upsilon+in},$$

and the residue of f at $\lambda=-\upsilon-in$ is obtained by substituting $\lambda=-\upsilon-in$ in the numerator; hence residue of f at $-\upsilon-in=Fe^{i\omega}e^{(-\upsilon-in)t}(-2in)^{-2}$; and for the consideration of the value $\lambda=-\upsilon+in$,

$$f(\lambda)=\frac{Fe^{i\omega}e^{\lambda t}(\lambda+\upsilon+in)^{-1}}{(\lambda+\upsilon-in)^2},$$

and the residue of f at $\lambda=-\upsilon+in$ is the value for this value of λ of the derivative:

$$\frac{d}{d\lambda}\frac{Fe^{i\omega}e^{\lambda t}}{\lambda+\upsilon+in}=\frac{Fe^{i\omega}[(\lambda+\upsilon+in)te^{\lambda t}-e^{\lambda t}]}{(\lambda+\upsilon+in)^2}.$$

which value is

$$\text{residue of } f(\lambda) \text{ at } -\upsilon+in=Fe^{i\omega}e^{(-\upsilon+in)t}(2int-1)(2in)^{-2}.$$

Hence the solution for x in (2) is the sum of the residues, or

$$x=\frac{-F}{4n}e^{-\nu}\ e^{i\omega}\left[e^{-int}+e^{int}(2int-1)\right].$$

And the solution for x in (1) is the real part of the above obtained by substituting for $e^{i\omega}$, e^{-in}, e^{int} their expressions in terms of trigonometric functions. This gives:

$$x=\frac{Fe^{-\nu t}}{2n^2}\left[nt\sin\ (nt+\omega)-\sin\ \omega\ \sin\ nt\right],$$

the result stated by Bromwich, without giving the steps in detail.

If it had been required to solve the equation:

$$\frac{d^2x}{dt^2}+2\nu\frac{dx}{dt}+(\nu^2+n^2)\ x=Fe^{-\nu t}\sin\ (nt+\omega), \tag{1a}$$

the only change would have been to take the imaginary part, instead of the real part, and throw out the factor $i=\sqrt{-1}$. Thus,

$$\begin{aligned}x&=\frac{-F}{4n^2}\ e^{-\nu t}\ e^{i\omega}\ [2int\ e^{int}-(e^{int}-e^{-int})]\\&=\frac{-F}{4n^2}\ e^{-\nu t}\ e^{i\omega}\ [2int\ (\cos\ nt+i\sin\ nt)-2i\sin\ nt]\\&=-\frac{F}{4n^2}\ e^{-\nu t}\ (\cos\ \omega+i\sin\ \omega)\ [2int\cos\ nt-2nt\sin\ nt-2i\sin\ nt]\\&=-\frac{F}{4n^2}\ e^{-\nu t}\ (-2nt\cos\ \omega\sin\ nt-2nt\sin\ \omega\cos\ nt+2\sin\ \omega\sin\ nt)+\\&\qquad i\ (2nt\cos\ \omega\cos\ nt-2nt\sin\ \omega\sin\ nt-2\sin\ \omega\sin\ nt)\end{aligned}$$

and the solution of (3) is

$$x=\frac{F}{2n^2}\ e^{-\nu t}\ [-nt\cos\ (nt+\omega)+\cos\ \omega\sin\ nt].$$

For the head gust the equations to be solved are

$$\begin{aligned}(D-X_u)u-X_ww-(X_qD+g)\theta&=X_uu_1=X_uJ\ e^{-nt}\sin\ pt,\\-Z_uu+(D-Z_w)w-(Z_q+U)\ D\theta&=Z_uu_1=Z_uJ\ e^{-nt}\sin\ pt,\\-M_uu-M_ww+(k^2_BD^2-M_qD)\ \theta&=M_uJe^{-nt}\sin\ pt.\end{aligned}$$

These are replaced by equations with $Je^{-nt+pit}$ instead of $Je^{-nt}\sin pt$ and the equations for ξ, η, ζ become

$$\begin{aligned}(\lambda-X_u)\ \xi-X_w\eta-(X_q\lambda+g)\ \zeta&=X_uJ/(\lambda+n-pi)\\-Z_u\xi+(\lambda-Z_w)\eta-(Z_q+U)\ \lambda\zeta&=Z_uJ/(\lambda+n-pi)\\-M_u\xi-M_w\eta+(k^2_B\lambda^2-M_q\lambda)\ \zeta&=M_uJ/(\lambda+n-pi).\end{aligned}$$

Next ξ, η, ζ are obtained by solution and multiplied by $e^{\lambda t}$. The results are as follows:

$$\begin{aligned}\xi e^{\lambda t}&=\frac{-(.128\lambda^3-1.16\lambda^2-3.385\lambda-.917)\ Je^{\lambda t}}{(\lambda^4-8.49\lambda^3-24.5\lambda^2-3.385\lambda-.917)\ (\lambda+n-pi)},\\\eta e^{\lambda t}&=\frac{-(\lambda^2.557\lambda-2.458)Je^{\lambda t}}{(\lambda^4-8.49\lambda^3-24.5\lambda^2-3.385\lambda-.917)\ (\lambda+n-pi)},\\\zeta e^{\lambda t}&=\frac{-.02851\lambda Je^{\lambda t}}{(\lambda^4-8.49\lambda^3-24.5\lambda^2-3.385\lambda-.917)\ (\lambda+n-pi)}.\end{aligned}$$

The solutions for u, w, θ, respectively, are the sums of the residues of these three expressions

The denominator factors (since $n=.0654$, $p=.187$) into $(\lambda+.0654+.187i)(\lambda+.0654-.187i)^2\ (\lambda-4.18+2.43i)(\lambda+4.18-2.43i)$. It is necessary to calculate the residues for each of the following values of λ: $-4.18\pm2.43\ i, -.0654-.187i, -.0654+.187i$. The first three correspond to single factors of the denominator and are obtained merely by discarding that factor and substituting the value of λ in the rest of the expression; the fourth requires that the double factor be discarded and that the value of λ be substituted in the derivative of what remains.

The whole of the calculation need not be made. The interest lies almost entirely in the up and down motion which is given by

$$z = \int (w - 115.5\theta)\, dt$$

and for which w and θ alone need to be determined. One of the advantages of the Bromwich method lies precisely in this ability to calculate just those elements needed.

We shall begin with θ and figure the residues of:

$$\frac{-.02851\ \lambda J\ e^{\lambda t}}{(\lambda+.0654+.187i)\ (\lambda+.0654-.187i)^2\ (\lambda+4.18\pm 2.43i)}.$$

For that at $\lambda = -4.18+2.43\ i$ calculate

$$R_1 = \frac{.02851\ (-4.18+2.43i)\ e^{-4.18t}\ (\cos 2.43t + i \sin 2.43t)}{(-4.115+2.617i)\ (-4.115+2.243i)^2\ (4.86i)},$$

discarding for the moment the factor $-J$ common to all residues. The rules for dealing with imaginaries by trigonometry are helpful. We need the magnitudes and angles of the quantities:

$$\begin{aligned} &\text{mag } (-4.18+2.43i) = 4.834, && \log \text{mag} = 0.6843, && \text{ang} = 149^\circ.83, \\ &\text{mag } (-4.115+2.617i) = 4.875, && \log \text{mag} = 0.6879, && \text{ang} = 147^\circ.54, \\ &\text{mag } (-4.115+2.243i) = 4.792, && \log \text{mag} = 0.6805, && \text{ang} = 151^\circ.40. \end{aligned}$$

The angle of the coefficient in R_1 is $-30^\circ.51$; the log mag is 6.4038. Hence

$$R_1 = (.0002183 - .0001283\ i)\ e^{-4.18t}\ (\cos 2.43t + i \sin 2.43t).$$

Of this the imaginary part, rejecting i, is:

$$A = e^{-4.18t}\ (.0002183 \sin 2.43t - .0001283 \cos 2.436).$$

Turn next to the residue, omitting the factor $-J$, at $\lambda = -4.18-2.43i$, or

$$R_2 = \frac{.02851\ (-4.18-2.43i)\ e^{-4.18t}\ (\cos 2.43t - i \sin 2.43t)}{(-4.115-2.243i)\ (-4.115-2.617i)^2\ (-4.86i)}.$$

The angle of the coefficient is $26^\circ.65$; the log mag is 6.3964. Hence

$$R_2 = (.0002226 + .0001117i)\ e^{-4.18t}\ (\cos 2.43t - i \sin 2.45t).$$

Of this the imaginary part, neglecting i, is:

$$B = e^{-4.18t}(-.0002226 \sin 2.43t + .0001117 \cos 2.43t).$$

Adding A and B, the dependence of θ on the short oscillation is

$$-\theta/J = e^{-4.18t}(-.0000043 \sin 2.43t - .0000166 \cos 2.43t). \tag{3}$$

The effect is very small, indeed quite negligible (compared with (4) below) as might be imagined from the small results found for other types of gust in the two previous papers.

The residue at $\lambda = -.0654-.187i$ is

$$R_3 = \frac{.02851(-.0654-.187i)e^{-.0654t}(\cos .187t - i \sin 187t)}{(-.374i)^2(4.115+2.243i)\ (4.115-2.617i)}.$$

$$\text{mag } (-.0654-.187i) = .1981, \quad \log \text{mag} = 9.2969, \quad \text{ang} = 250^\circ.72.$$

The angle of the coefficient is $74^\circ.58$. The log mag is 7.2377. Hence

$$R_3 = e^{-.0654t}(\cos .187t - i \sin .187t)(.0004596 + .001666i).$$

Of this the imaginary part is:

$$C=e^{-.0654t}(-.0004596 \sin .187t+.001666 \cos .187t).$$

For $-\theta/J$ there remains only to calculate the residue at $\lambda=-.0654+187i$, which is the hardest of all, since this factor is squared. We have to find

$$\frac{d}{d\lambda}\left[\frac{.02851\lambda e^{\lambda t}}{(\lambda+.0654+.187i)\,(\lambda+4.18+2.43i)\,(\lambda+4.18-2.43i)}\right].$$

The derivative of a product is often calculated most easily by taking the logarithm before differentiation. For:

$$\frac{d \log f(x)}{dx}=\frac{1}{f(x)}\frac{df(x)}{dx} \text{ or } \frac{df}{dx}=f(x)\frac{d\log x}{dx}.$$

The derivative of the logarithm of the bracket with respect to λ is

$$D=\frac{1}{\lambda}+t-\frac{1}{\lambda+.0654+.187i}-\frac{1}{\lambda+418+2.43i}-\frac{1}{\lambda+4.18-2.43i}.$$

As $\lambda=-.0654+.187i$ we have

$$\begin{aligned}D&=\frac{1}{-.0654+.187i}+t-\frac{1}{.374i}-\frac{1}{4.115+2.617i}-\frac{1}{4.115-2.243i}\\&=-1.1629-4.766i+t+2.674i-.1731+.1101i-.1832-.0999i\\&=-1.985-1.992i+t \quad , \quad \text{ang}=225.°10 \quad , \quad \log \text{mag}=.4491.\end{aligned}$$

Then D must be multiplied by

$$E=\frac{.02851(-.0654+.187i)e^{-.0654t}(\cos .187t+i \sin .187t)}{(.374i)\,(4.115+2.617i)\,(4.115-2.243i)}.$$

The angle of the coefficient is 15.°42; the log mag is 6.8106. When multiplied by D we have

$$[t(.0006233+.0001716i)+(-.0008935-.001583)]\, e^{-.0654t}(\cos .187t+i \sin 187t).$$

Taking the imaginary part, we have:

$$F=e^{-.0654t}\,[t\,(.0006233 \sin .187t+.0001716 \cos .187t)+(-.0008935 \sin .187t-.001583 \cos .187t)].$$

On adding C and F the effect on the long oscillation is

$$\begin{aligned}-\theta/J=e^{-.0654t}[t(&.000623 \sin .187t+.000172 \cos .187t)\\&+(-.001353 \sin .187t+.000083 \cos .187t)]. \qquad (4)\end{aligned}$$

The accuracy, of course, at this point is not great. The total result for θ should give $\theta=0$ and $d\theta/dt=0$ when $t=0$, and it does within the estimated remaining accuracy.

Although the short oscillation is of importance if the values of dw/dt or $d\theta/dt$ are desired, it is (as seen above and in previous papers) of very little use in considering the variable w or θ at least in forced motions where the applied force operates relatively slowly (mild or moderate as distinguished from sharp gusts); it is quite insignificant for the path. The above calculations could therefore be abridged somewhat by the device mentioned in my second paper of a partial resolution with partial fractions:

$$\frac{1}{(\lambda+.0654\pm.187i)\,(\lambda+4.18\pm2.43i)}=\frac{.016\lambda+.089}{(\lambda+4.18\pm2.43i)}+\frac{-.01601\lambda+.04263}{(\lambda+.0654\pm.187i)}.$$

The residues to be calculated for $-\theta/J$ are those of

$$\frac{.02851\lambda(.016\lambda+.089)}{(\lambda+4.18\pm2.43i)\,(\lambda+.0654-.187i)} \text{ and } \frac{.02851\lambda(-.01601\lambda+.04263)}{(\lambda+.0654+.187i)\,(\lambda+.0654-187i)^2}.$$

In the first there is no residue for $\lambda = -.0654 - .187i$ and the residue for $\lambda = -.0654 + .187i$ is negligible compared with that of the second expression for $\lambda = -.0654 + .187i$ (which is the easy one to calculate). As the residues at $\lambda = -4.18 \pm 2.43i$ refer to the short oscillation, they need not be calculated, and therefore the only calculations really necessary for discussing the path are those for the residues of the second expression. The work was carried through on this basis and checked with that obtained by the complete analysis above to within 2 or 3%

On the abbreviated plan just outlined let us calculate the result for $-w/J$. The residues are required for

$$\frac{\lambda^2(.557\lambda+2.458)e^{\lambda t}(-.01601\lambda+.04263)}{(\lambda+.0654+.187i)\,(\lambda+.0654-.187i)^2}$$

at $\lambda = -.0654 - .187i$ and $\lambda = -.0654 + .187i$. The first is $\frac{-e^{\lambda t}}{.1399}(-.0654-.187i)^2\,(2.422-.1042i)$ $(.04368+.002994i) = e^{-.0654t}\,(\cos .187t - i\sin .187t)\,(.02375-.01797i)$.
Of this the imaginary part is

$$G = e^{-.0654t}(-.02375\sin .187t - .01797\cos .187t).$$

The second is

$$\frac{d}{d\lambda}\left[\frac{\lambda^2(.557\lambda+2.458)\;e^{\lambda t}\,(-.01601\lambda+.04263)}{\lambda+.0654+.187i}\right]_{\lambda=-.0654+.187i.}$$

Apply logarithmic differentiation as before, and the residue is seen to be the value of the bracket multiplied by

$$\frac{2}{\lambda}+\frac{.557}{.557\lambda+2.458}+t-\frac{.01601}{-.01601\lambda+.04263}-\frac{1}{\lambda+.0654+.187i},$$

or

$$\frac{2}{-.0654+.187i}+\frac{.557}{2.422-.1042i}+t-\frac{.01601}{.04368-.002994i}-\frac{1}{.374i} = -3.258-9.532i+t+2.674i+.2295$$
$$+.00987i-.3648-.02506i = t-3.391-6.873i$$

The value of the bracket itself, apart from $e^{\lambda t}$, is $-.006718+.008884i$.

Hence the residue is

$$e^{-.0654t}(\cos .187t + i\sin .187t)\,[t(-.006718+.008884i)+(.08385+.01607i)].$$

Of this the imaginary part is

$$H = e^{-.0654t}[t(-.006718\sin .187t + .0160\cos .187t) + (.08385\sin .187t + .01607\cos .187t)].$$

Adding G and H, the result for $-w/J$ is

$$-w/J = e^{-.0654t}[t(-.00672\sin .187t + .00888\cos .187t) + (.0610\sin .187t - .0019\cos .187t).$$

The last term should, of course, check out so as to give zero when $t=0$, but owing to the omission of the terms corresponding to the short oscillation the check can not be expected to be exact. Moreover the derivative should also vanish, but has the value $+.02$. This would correspond to a term $-w/J = .005e^{-4.18t}\cos 2.43t$. Now when treating the gust $J(1-e^{-.2t})$, which in its initial effects should not be far different from $Je^{-.0654t}\sin .187t$, the term $-w/J = e^{-4.18t}$ $(.004\cos 2.43t - .003\sin 2.43t)$, for which the derivative has the value $-.024$, was actually found. Such a failure to check as occurs in the value of $-w/J$ here determined can not be regarded as an indication of error in the calculation; and an independent calculation checks well with the value above given for $-w/J$.

The path in space is given by

$$z = \int_0^t (w+115.5\theta)dt = J\int_0^t e^{-.0654t}[t(-.062\sin .187t - .029\cos .187t) + (.095\sin .187t -$$
$$.008\cos .187t)]dt.$$

To integrate an expression like $e^{-at}t \cos bt$ or $e^{-at}t \sin bt$, the simplest thing is to integrate $te^{ct} = te^{(-a+bi)t}$, which may be found in integral tables, and take real and imaginary parts.

$$\int te^{ct}dt = \frac{te^{ct}}{e} - \frac{e^{ct}}{c^2} = e^{-at}\left[\frac{t}{-a+bi} - \frac{1}{(-a+be)^2}\right](\cos bt + i \sin bt),$$

$$\int te^{-at} \cos bt\, dt = e^{-at}\left[\frac{t(b \sin bt - a \cos bt)}{a^2+b^2} + \frac{(b^2-a^2) \cos bt + 2ab \sin bt}{(a^2+b^2)^2}\right],$$

$$\int te^{-at} \sin bt\, dt = e^{-at}\left[\frac{-t(a \sin bt + b \cos bt)}{a^2+b^2} + \frac{(b^2-a^2) \sin bt - 2ab \cos bt}{(a^2+b^2)^2}\right].$$

Then

$$z = Je^{-.0654t}[t(.34 \cos .187t - .03 \sin .187t) - 1.8 \sin .187t].$$

The values for $-z/J$ are, for $t=10$, 12, 14, respectively, about 1.4, 1.9, 2.0; and from then on the values decrease. Compare this for $-z/J$ in the case of the periodic gust $u_1 = J \sin .2t$, where the corresponding values are 1.9, 2.6, 3.0. It is seen that the damped periodic gust is decidedly less effective than the undamped gust. This is precisely what I foresaw, and indeed what must be admitted a priori unless astonishing powers of discrimination are given to the machine. A gust $Je^{-.0654t} \sin .187t$ or $J \sin .2t$ does not differ in general character from the gust of the form $J(1-e^{-.2t})$ or $J(1-e^{-.187t})$ during the first rise from zero to a maximum—the plot of the intensity is nearly a straight line until the maximum is approached and the slopes of the lines are nearly equal. The machine is by its inertia an integrating, rather than a differentiating device, and should give similar displacements in all four cases. The damping in the first case tends merely to diminish the effect. The maximum rise in the third and fourth types is about $-3.5J$, in the second case about $-3J$ (as is natural since the third and fourth gusts persist where the second begins to decrease after about 8 seconds), and the damped gust gives $-2J$ (as is again natural, since the maximum of that gust is only about $.6J$ owing to the damping factor). The successive forced oscillations in the case of the damped gust drop off very rapidly, whereas the straight periodic gust brings a decided resonance into play after the natural motion subsides. The conclusion is that the constants of integration are such as to mask the effect of the damped gust in the first quarter period, whereas the damping makes the effect small at the subsequent times of maxima.

DEPARTMENT OF PHYSICS,
MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
October 15, 1918.