NACA TN 1981

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1981

CRITICAL STRESS OF RING-STIFFENED CYLINDERS IN TORSION

By Manuel Stein, J. Lyell Sanders, Jr., and Harold Crate

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Washington
November 1949

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SUMMARY

A chart in terms of nondimensional parameters is presented for the theoretical critical stress in torsion of simply supported cylinders stiffened by identical equally spaced rings of zero torsional stiffness. The results are obtained by solving the equation of equilibrium by means of the Galerkin method. Comparison of the theoretical results with experimental results indicates that ring-stiffened cylinders buckle, on the average, at a buckling stress about 15 percent below the theoretical buckling stress.

INTRODUCTION

One of the problems that confronts the aircraft designer is that of predicting the strength of stiffened shell structures. The stress conditions to which such structures are subjected are shear arising from transverse loads or torsion and compression arising from longitudinal loads or bending. As an approach to the general problem, the case of shear alone was chosen for investigation.

For unstiffened cylinders reference 1 shows empirically that the buckling load in transverse shear is related to the buckling load in torsion; therefore the present investigation was restricted to torsion alone. Because the diagonal type of buckles that form in the walls of an unstiffened cylinder in shear or torsion would be more effectively resisted by circumferential stiffeners than by longitudinal stiffeners, the particular problem of ring-stiffened cylinders in torsion was chosen for study. The present report presents the theoretical and experimental results of that study.

SYMBOLS

d	stiffener spacing $\left(\frac{L}{n + 1}\right)$
i,j,m,p,q	integers
n	number of stiffener rings
r	radius of cylinder
t	thickness of cylinder skin
w	displacement of point on middle surface of cylinder in radial direction, positive outward
x	axial coordinate
y	circumferential coordinate
A_{ij}	elements of a determinant
D	flexural stiffness of cylinder skin $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$
E	Young's modulus
μ	Poisson's ratio
I	effective cross-sectional moment of inertia of stiffener ring
L	length of cylinder
Z	curvature parameter $\left(\frac{L^2}{rt}\sqrt{1 - \mu^2}\right)$
a_m,b_m,a_q,b_q	coefficients in deflection function
k_s	critical-shear-stress coefficient $\left(\frac{\tau_{cr} tL^2}{\pi^2 D}\right)$
$\beta = \frac{L}{\lambda}$	
γ	ratio of ring stiffness to plate stiffness $\left(\frac{EI}{Dd}\right)$

$\delta(x - id)$ Dirac delta defined such that $\int_{-\infty}^{\infty} f(x)\delta(x - id)dx = f(id)$

δ_{ij} Kronecker delta $\left(\delta_{ij} \text{ is} \begin{cases} 1 \text{ if } i \text{ is equal to } j; \\ 0 \text{ if } i \text{ is not equal to } j \end{cases}\right)$

$$\delta_{npq} = \frac{2}{n + 1} \sum_{i=1}^{n} \sin \frac{\pi ip}{n + 1} \sin \frac{\pi iq}{n + 1}$$

λ half wave length of buckles in circumferential direction

τ_{cr} critical shear stress

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

∇^{-4} inverse of ∇^4 defined by $\nabla^{-4}\left(\nabla^4 w\right) = \nabla^4\left(\nabla^{-4} w\right) = w$

RESULTS AND DISCUSSION

When the shear-stress coefficient k_s for a ring-stiffened cylinder in torsion is known, the critical shear stress τ_{cr} can be found by the formula

$$\tau_{cr} = k_s \frac{\pi^2 D}{L^2 t}$$

where τ_{cr} is related to the total torque T by the formula

$$\tau_{cr} = \frac{T}{2\pi r^2 t}$$

Details for the theoretical solution for the critical-shear-stress coefficient k_s of a ring-stiffened cylinder in torsion are presented in the appendix. The equation of equilibrium of cylindrically curved plates (references 2 and 3) was altered to include an additional term that represents the bending stiffness of the rings (reference 4); the equation was then solved by means of the Galerkin method (reference 5) for the buckling-stress coefficient k_s. The equation of equilibrium used applies to a wide range of curvature from the limiting case of a flat plate to a slender tube which may buckle into as few as three waves in the circumferential direction. The limiting case of a tube buckling into two waves (buckling into an elliptical cross section) was not considered since the equations give unconservative results in this range and the dimensions for such stiffened shell structures are not ordinarily encountered in aircraft design.

The ring stiffeners are assumed to have no torsional stiffness and to act along a line in the median surface of the plate. The assumption of zero torsional stiffness usually applies with little error when open-section stiffeners are used. The assumption that the stiffeners may be considered to act along a line is applicable when the width of the attached flange is small in comparison with the stiffener spacing. The assumption that the stiffeners act at the median surface of the plate is applicable when the attached stiffeners are symmetrically located with respect to the median surface of the plate, otherwise the assumption is conservative.

Table 1 gives calculated theoretical values for the critical-shear-stress coefficient and the corresponding values of stiffness and wave-length ratio for ring-stiffened cylinders. Figure 1 is based on the calculated values given in table 1. The critical-shear-stress coefficient is given in figure 1 as a function of the curvature parameter Z, the number of rings, and the ratio of ring stiffness to the plate stiffness γ. Figure 1 shows that, with increasing ring stiffness, the buckling stress of a ring-stiffened cylinder will increase appreciably up to a limit that depends on the number of ring stiffeners present. These upper limits are represented by the horizontal cut-off lines in figure 1 and correspond to buckling with a node at the stiffeners. Below the limiting values, the buckling-stress coefficient k_s of a cylinder with two or more rings is essentially independent of the number of rings present. The values of k_s for $Z = 0$ are those given by reference 6 for longitudinally stiffened infinitely long flat plates in shear. The values of k_s for $\gamma = 0$ are those given by reference 2 for unstiffened circular cylinders in torsion.

For the experimental check on the theory, data on five cylinders with enough different ring stiffeners to make fifteen combinations were obtained. The nominal cross-sectional dimensions of these stiffeners are given in figure 2. The experimental results are given in table 2 in terms of the parameters Z, γ, τ_{cr}, and k_s. As indicated in the table, data for four of the combinations obtained from one cylinder are taken from reference 7. Data are given in table 2 for eleven additional combinations obtained from four additional cylinders, the results of which heretofore have not been published.

Comparison of the experimental and theoretical results presented in figure 3 reveals that the ring-stiffened cylinders buckled at a stress about 15 percent below the theoretical values of buckling stress presented in this report. In reference 2, a comparison between theoretical and experimental results indicated that simply supported unstiffened cylinders buckle at stresses averaging about 10 percent below the theoretical critical stress. Since this reduction, which is based on a large number of tests on unstiffened cylinders, is of the same order as the reduction indicated from experiments on ring-stiffened cylinders, a reasonable conclusion is that ring-stiffened cylinders in torsion buckle on the average about 15 percent below the theoretical critical stress calculated by the present method.

CONCLUDING REMARKS

A cylinder in torsion can be appreciably strengthened by the use of ring stiffeners. Up to certain limiting values the buckling stress coefficient for a simply supported cylinder stiffened with two or more identical equally spaced rings of zero torsional stiffness is independent of the number of rings present and dependent only on the ratio of the ring stiffness to the plate stiffness. Comparison of the present theoretical results with experimental results indicates that ring-stiffened cylinders buckle, on the average, at a buckling stress about 15 percent below the theoretical buckling stress obtained by the present method. Proper account should be taken of this difference if the theoretical results are used for design purposes.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., September 14, 1949

APPENDIX

THEORETICAL SOLUTION

The critical shear stress at which buckling occurs in a ring-stiffened cylindrical shell may be found by solving the equation of equilibrium.

Equation of equilibrium.- The equation of equilibrium of a ring-stiffened cylinder in torsion with small radial displacements is

$$D\nabla^4 w + \frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + EI\sum_{i=1}^{n}\frac{\partial^4 w}{\partial y^4}\delta(x - id) + 2\tau_{cr}t\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{A1}$$

in which x is in the axial direction and y is in the circumferential direction. This equation is the equation of reference 3 for torsion of a cylinder with the third term added to represent the radial restoring force due to the bending stiffness of the stiffeners. (See reference 4.) The stiffeners are assumed to act in the median surface of the cylinder and to have no torsional rigidity.

After division by D and introduction of the relations

$$Z = \frac{L^2}{rt}\sqrt{1 - \mu^2}$$

$$k_s = \frac{\tau_{cr}tL^2}{\pi^2 D}$$

$$\gamma = \frac{EI}{Dd}$$

equation (A1) becomes

$$\nabla^4 w + \frac{12Z^2}{L^4}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + \gamma d\sum_{i=1}^{n}\frac{\partial^4 w}{\partial y^4}\delta(x - id) + 2k_s\frac{\pi^2}{L^2}\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{A2}$$

This equation of equilibrium may be solved by the Galerkin method in a manner similar to that of references 3 and 5.

Solution for simply supported cylinders with equally spaced ring stiffeners.- The solution in this case is given by the infinite series

$$w = \sin\frac{\pi y}{\lambda}\sum_{m=1}^{\infty} a_m \sin\frac{m\pi x}{L} + \cos\frac{\pi y}{\lambda}\sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L} \qquad \text{(A3)}$$

in which the coefficients a_m and b_m are to be determined. When this series is substituted in the equation of equilibrium, the Galerkin method yields the following two sets of equations to determine the coefficients

$$\left.\begin{aligned}
&\frac{a_p}{k_s} M_p + \frac{\pi\beta^3\gamma}{8k_s}\sum_{q=1}^{\infty}\delta_{npq}\, a_q + \sum_{m=1}^{\infty} b_m \frac{mp}{m^2 - p^2} = 0 \\
&\frac{b_p}{k_s} M_p + \frac{\pi\beta^3\gamma}{8k_s}\sum_{q=1}^{\infty}\delta_{npq}\, b_q - \sum_{m=1}^{\infty} a_m \frac{mp}{m^2 - p^2} = 0 \qquad (p = 1, 2, \ldots)
\end{aligned}\right\} \qquad \text{(A4)}$$

where

$$M_p = \frac{\pi}{8\beta}\left[(\beta^2 + p^2)^2 + \frac{12}{\pi^4}\frac{Z^2p^4}{(\beta^2 + p^2)^2}\right]$$

$$\beta = \frac{L}{\lambda}$$

n is the number of stiffeners

$p + m$ is odd

$\delta_{npq} = +1$ if $p - q$ is a multiple of $2(n + 1)$

$= -1$ if $p + q$ is a multiple of $2(n + 1)$

$= 0$ if neither or both are true

These equations are consistent if $\left|A_{ij}\right|$, the determinant of the coefficients, vanishes

$$\left|A_{ij}\right| = 0 \qquad \text{(A5)}$$

The determinantal equation (A5) gives the criterion for buckling where the elements of the determinant are

$$A_{ij} = \frac{1}{k_s} M_i \delta_{ij} + \frac{\pi\beta^3\gamma}{8k_s} \delta_{nij} + (-1)^i \frac{ij}{i^2 - j^2} \tag{A6}$$

where δ_{ij} is the Kronecker delta. The last term of equation (A6) is zero unless $i + j$ is odd.

Sixth-order determinants for the cases $n = 1, 2, 3,$ and ∞ are given in the following. The case $n = 1$ corresponds to one ring stiffener, the case $n = 2$ corresponds to two ring stiffeners, and so on. Thus,

$n = 1$

$$\begin{vmatrix} \frac{1}{k_s}\left(M_1 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{2}{3} & -\frac{\pi\beta^3\gamma}{8k_s} & \frac{4}{15} & \frac{\pi\beta^3\gamma}{8k_s} & \frac{6}{35} \\ \frac{2}{3} & \frac{M_2}{k_s} & -\frac{6}{5} & 0 & -\frac{10}{21} & 0 \\ -\frac{\pi\beta^3\gamma}{8k_s} & -\frac{6}{5} & \frac{1}{k_s}\left(M_3 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{12}{7} & -\frac{\pi\beta^3\gamma}{8k_s} & \frac{2}{3} \\ \frac{4}{15} & 0 & \frac{12}{7} & \frac{M_4}{k_s} & -\frac{20}{9} & 0 \\ \frac{\pi\beta^3\gamma}{8k_s} & -\frac{10}{21} & -\frac{\pi\beta^3\gamma}{8k_s} & -\frac{20}{9} & \frac{1}{k_s}\left(M_5 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{30}{11} \\ \frac{6}{35} & 0 & \frac{2}{3} & 0 & \frac{30}{11} & \frac{M_6}{k_s} \end{vmatrix} = 0 \tag{A7}$$

$n = 2$

$$\begin{vmatrix}
\frac{1}{k_s}\left(M_1 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{2}{3} & 0 & \frac{4}{15} & -\frac{\pi\beta^3\gamma}{8k_s} & \frac{6}{35} \\
\frac{2}{3} & \frac{1}{k_s}\left(M_2 + \frac{\pi\beta^3\gamma}{8}\right) & -\frac{6}{5} & -\frac{\pi\beta^3\gamma}{8k_s} & -\frac{10}{21} & 0 \\
0 & -\frac{6}{5} & \frac{1}{k_s}M_3 & \frac{12}{7} & 0 & \frac{2}{3} \\
\frac{4}{15} & -\frac{\pi\beta^3\gamma}{8k_s} & \frac{12}{7} & \frac{1}{k_s}\left(M_4 + \frac{\pi\beta^3\gamma}{8}\right) & -\frac{20}{9} & 0 \\
-\frac{\pi\beta^3\gamma}{8k_s} & -\frac{10}{21} & 0 & -\frac{20}{9} & \frac{1}{k_s}\left(M_5 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{30}{11} \\
\frac{6}{35} & 0 & \frac{2}{3} & 0 & \frac{30}{11} & \frac{1}{k_s}M_6
\end{vmatrix} = 0 \qquad \text{(A8)}$$

$n = 3$

$$\begin{vmatrix}
\frac{1}{k_s}\left(M_1 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{2}{3} & 0 & \frac{4}{15} & 0 & \frac{6}{35} \\
\frac{2}{3} & \frac{1}{k_s}\left(M_2 + \frac{\pi\beta^3\gamma}{8}\right) & -\frac{6}{5} & 0 & -\frac{10}{21} & -\frac{\pi\beta^3\gamma}{8k_s} \\
0 & -\frac{6}{5} & \frac{1}{k_s}\left(M_3 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{12}{7} & -\frac{\pi\beta^3\gamma}{8k_s} & \frac{2}{3} \\
\frac{4}{15} & 0 & \frac{12}{7} & \frac{1}{k_s}M_4 & -\frac{20}{9} & 0 \\
0 & -\frac{10}{21} & -\frac{\pi\beta^3\gamma}{8k_s} & -\frac{20}{9} & \frac{1}{k_s}\left(M_5 + \frac{\pi\beta^3\gamma}{8}\right) & \frac{30}{11} \\
\frac{6}{35} & -\frac{\pi\beta^3\gamma}{8k_s} & \frac{2}{3} & 0 & \frac{30}{11} & \frac{1}{k_s}\left(M_6 + \frac{\pi\beta^3\gamma}{8}\right)
\end{vmatrix} = 0 \qquad \text{(A9)}$$

$$\lim_{n\to\infty}\begin{vmatrix}
\frac{1}{k_s}\left(M_1+\frac{\pi\beta^3\gamma}{8}\right) & \frac{2}{3} & 0 & \frac{4}{15} & 0 & \frac{6}{35} \\
\frac{2}{3} & \frac{1}{k_s}\left(M_2+\frac{\pi\beta^3\gamma}{8}\right) & -\frac{6}{5} & 0 & -\frac{10}{21} & 0 \\
0 & -\frac{6}{5} & \frac{1}{k_s}\left(M_3+\frac{\pi\beta^3\gamma}{8}\right) & \frac{12}{7} & 0 & \frac{2}{3} \\
\frac{4}{15} & 0 & \frac{12}{7} & \frac{1}{k_s}\left(M_4+\frac{\pi\beta^3\gamma}{8}\right) & -\frac{20}{9} & 0 \\
0 & -\frac{10}{21} & 0 & -\frac{20}{9} & \frac{1}{k_s}\left(M_5+\frac{\pi\beta^3\gamma}{8}\right) & \frac{30}{11} \\
\frac{6}{35} & 0 & \frac{2}{3} & 0 & \frac{30}{11} & \frac{1}{k_s}\left(M_6+\frac{\pi\beta^3\gamma}{8}\right)
\end{vmatrix} = 0 \qquad \text{(A10)}$$

In addition to satisfying equation (A5) for given values of the curvature parameter Z and the relative-stiffness parameter γ, k_s must be minimized with respect to the wave-length parameter β. Thus equation (A5) is solved for k_s for several values of β until the minimum k_s is obtained. Sixth-order determinants were used to obtain the results given in table 1. These results are plotted in figure 1. Values for $Z = 0$ (flat plate) were obtained from reference 6.

The calculations showed that for cyliners with two or more ring stiffeners k_s may be obtained with sufficient accuracy by solving equation (A10) which applies for an infinite number of stiffeners. Equation (A7) should be used, however, in the case of one stiffener.

The values of the shear-stress coefficient k_s derived in this appendix are to be used up to limiting values which depend on the number of ring stiffeners present. These limiting values correspond to the buckling load of a simply supported cylinder having the same length as the part of the cylinder contained between ring stiffeners.

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TABLE 1.- CRITICAL-SHEAR-STRESS COEFFICIENTS AND CORRESPONDING VALUES OF STIFFNESS RATIO AND WAVE-LENGTH RATIO FOR RING-STIFFENED CYLINDERS IN TORSION

$n = 1$

Z	k_s	γ	β
5	21	187.4	0.325
	15	47.0	.443
	10	7.71	.648
10	21.5	53.4	.585
	17	20.9	.71
	12	4.7	.90
30	26.5	8.20	1.95
	21	2.87	1.55
	15	.49	1.75
10^2	46	2.79	3.15
	40	1.14	2.90
	33	.36	2.85
10^3	216	.92	6.85
	190	.389	6.36
	170	.16	6.25
10^4	1170	.608	12.75
	1100	.405	12.25
	950	.130	11.75

$n = 2$

Z	k_s	γ	β
5	48	2958.8	0.14
	32	586.1	.21
	15	27.67	.43
10	48	640.9	.265
	32	133.0	.385
	20	22.05	.59
30	51	38.0	.90
	40	16.2	1.03
	25	3.18	1.35
10^2	66	2.97	2.50
	53	1.50	2.54
	40	.551	2.75
10^3	257	.650	6.00
	215	.335	6.00
	180	.138	6.10
10^4	1400	.4925	12.50
	1200	.2835	11.75
	1000	.1090	11.70

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TABLE 1.- CRITICAL-SHEAR-STRESS COEFFICIENTS AND CORRESPONDING VALUES OF STIFFNESS RATIO AND WAVE-LENGTH RATIO FOR RING-STIFFENED CYLINDERS IN TORSION - Concluded

$n = 3$

Z	k_s	γ	β
10^2	100	6.7	2.04
10^3	310	.74	5.80
10^4	1600	.48	11.0

$n \rightarrow \infty$

Z	k_s	γ	β
5	85.2	86,900	0.078
	55	15,100	.12
	25	647.98	.264
10	85.2	17,725	.15
	55	3,154	.23
	25	150.71	.485
30	85.2	649	.55
	55	132.2	.8
	30	17.1	1.2
10^2	100	25.3	2.0
	80	13.38	2.25
	45	2.50	2.57
10^3	320	3.07	5.5
	260	1.73	6.0
	200	.71	6.0
10^4	1600	1.87	11.0
	1350	1.175	11.0
	1100	.541	11.4

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TABLE 2.- EXPERIMENTAL RESULTS GIVEN IN TERMS OF THE

PARAMETERS OF THE PRESENT THEORY

[Nominal dimensions of cylinders: L = 43 in., r = 15 in., t = 0.032 in.; dimensions of ring stiffeners are given in figure 2.]

Cylinder	Number of rings	Z	γ	τ_{cr} (psi)	k_s
a1	1	3490	79.2	3040	520
b2	3	3490	158.4	3880	665
c3	3	3490	61.2	3880	665
d4	7	3490	122.4	5790	991
5	1	3560	2.56	2710	476
6	3	3560	5.12	3310	582
7	1	3600	4.04	2750	504
8	1	3600	1.2	2620	479
9	1	3600	58.4	2420	443
10	1	3670	61.3	3220	608
11	1	3590	.059	2310	417
12	3	3590	.12	2470	446
13	3	3590	14.2	3960	709
14	3	3590	1.64	3100	657
15	7	3590	3.27	4100	760

[a]Cylinder 2, of reference 7.

[b]Cylinder 3a of reference 7.

[c]Cylinder 3b of reference 7.

[d]Cylinder 4 of reference 7.

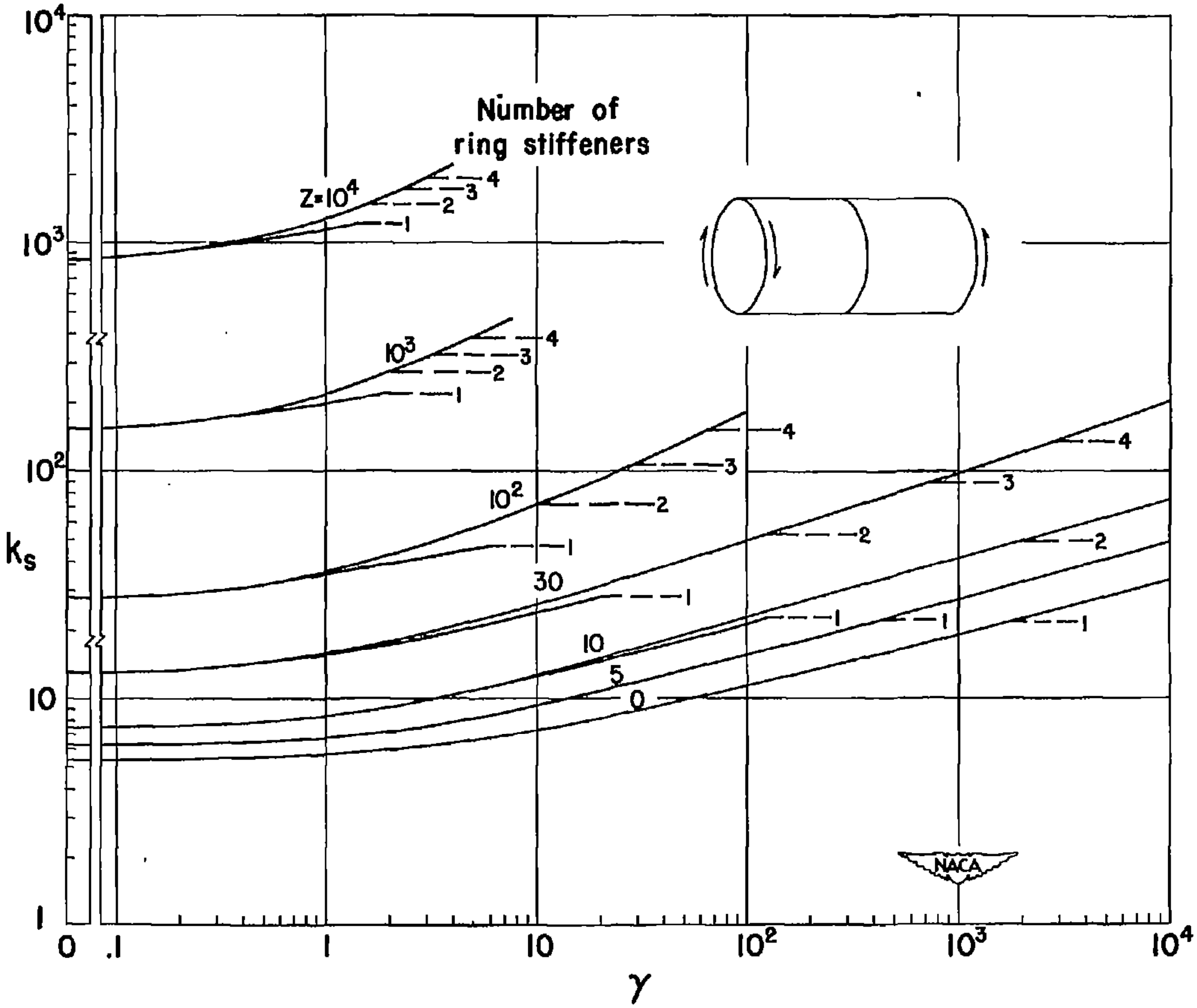

Figure 1.- Theoretical critical-stress coefficients of a ring-stiffened cylinder in torsion.

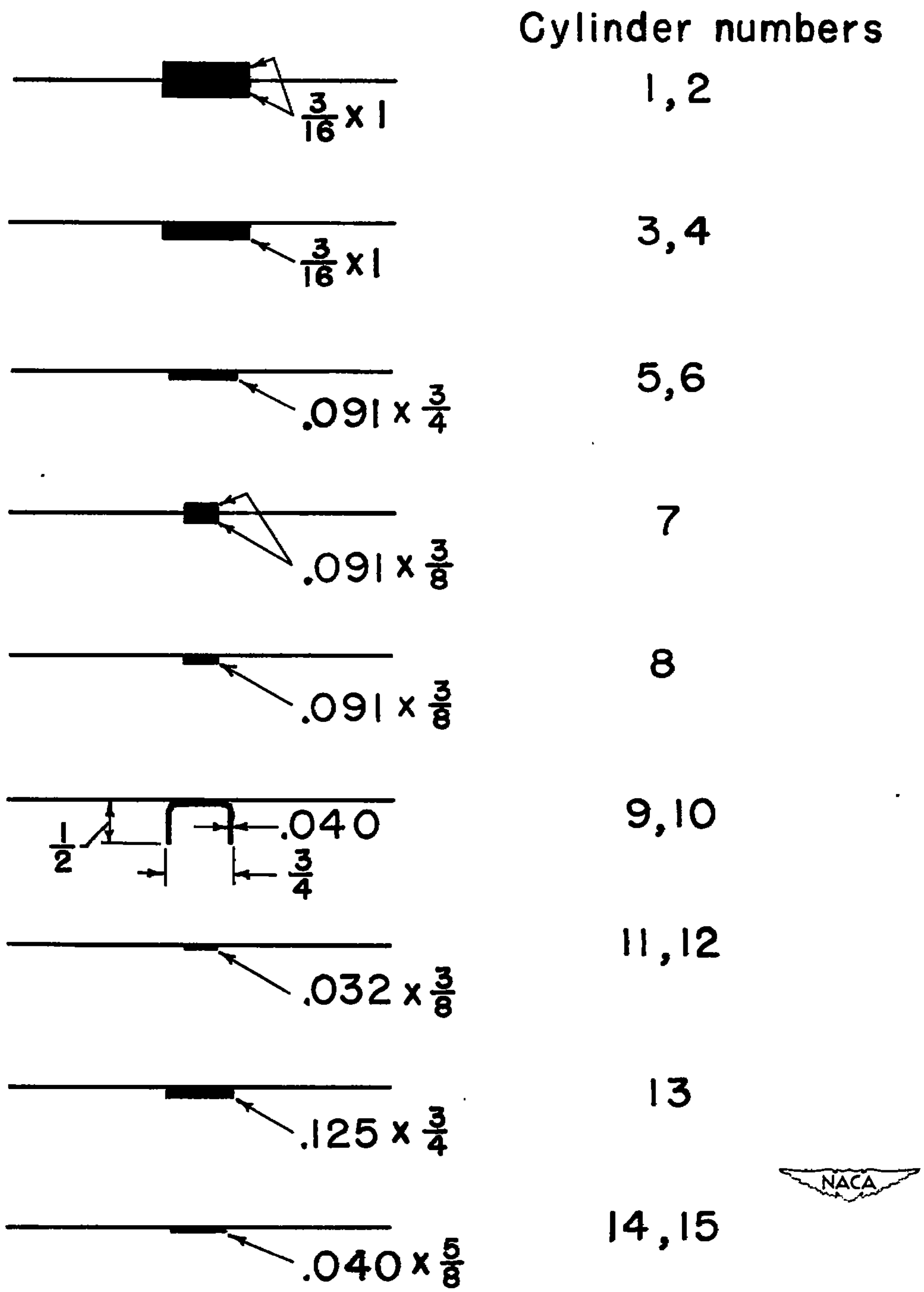

Figure 2.- Nominal dimensions of ring stiffeners of cylinders tested. (One-sided ring stiffeners are located on inside of cylinder.)

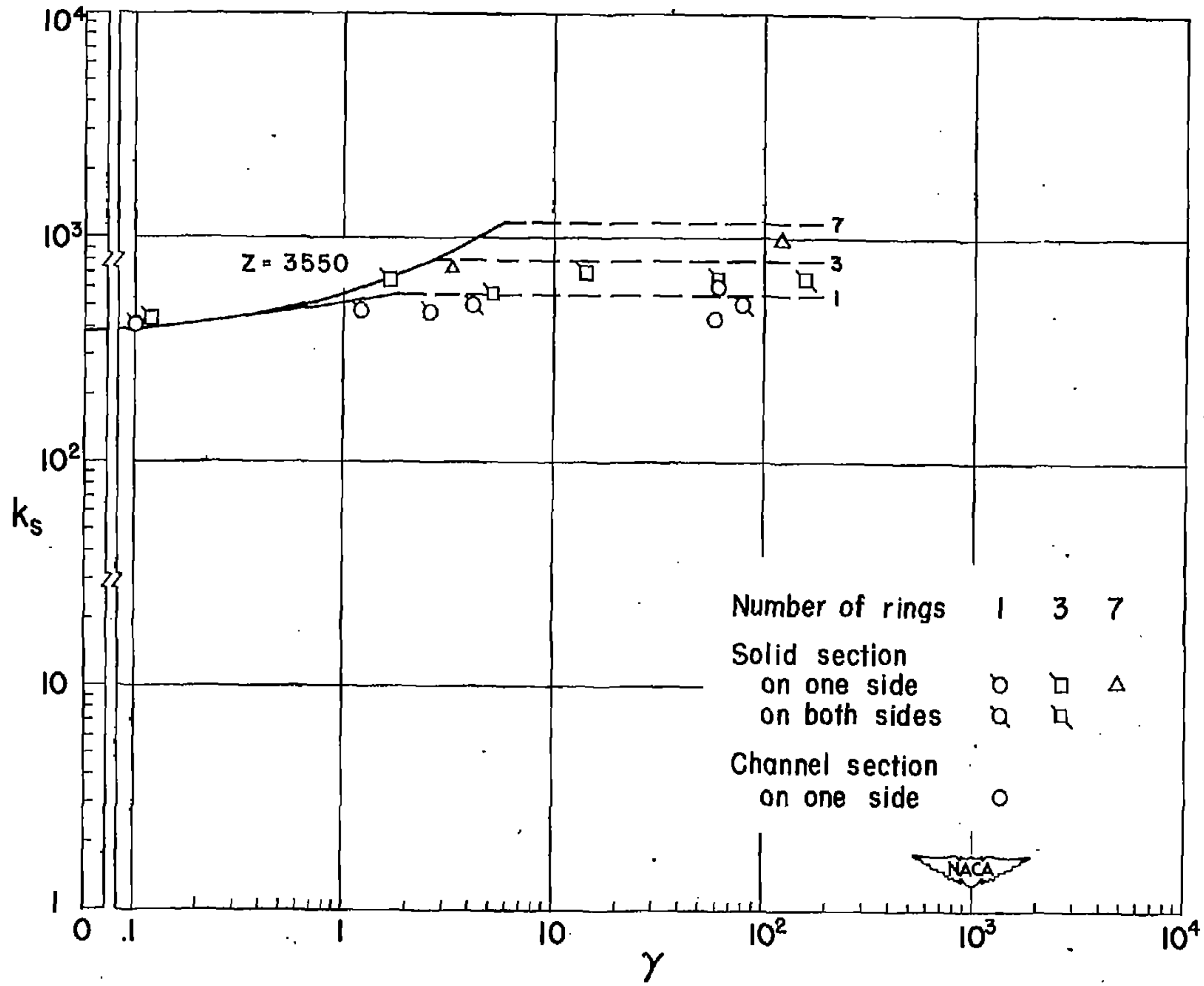

Figure 3.- Comparison of theoretical curves with experimental results for the critical stress of ring-stiffened cylinders in torsion.