REPORT No. 80

STABILITY OF THE PARACHUTE AND HELICOPTER

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INTRODUCTION.

This paper was submitted to the National Advisory Committee for Aeronautics by Professor H. Bateman, of the California Institute of Technology, and its publication duly authorized by the committee as Technical Report No. 80.

The mathematical theory of the stability of a parachute which is symmetrical with respect to a plane is very similar to the well-known theory of the stability of an airplane, but the values of the resistance coefficients are naturally different. When the parachute does not rotate, it may be compared with an airplane in a straight dive. There are oscillations corresponding to the longitudinal and lateral oscillations, and these are practically independent of one another. Thus, when the parachute swings like a pendulum as it descends, we have an oscillation corresponding to the pitching of an airplane, while there is another type of oscillation which corresponds to the Dutch roll.

When the parachute is symmetrical about an axis, so that it has the form of a solid of revolution, there may be no banking in the oscillation corresponding to the Dutch roll, and the oscillation then becomes a simple swing like the one which corresponds to the pitching of an airplane. In this case a combination of two simple swings in perpendicular planes may give rise to a compound oscillation, which may be likened to the motion of a conical pendulum falling under gravity. In the general case the combination of the simple swing and Dutch roll gives rise to a compound motion of a complicated character.

ANALYTICAL DISCUSSION.

To discuss the matter analytically we shall write down the equations of motion in the notation used by G. H. Bryan [1] and S. Brodetsky.[2] We shall take the axis of y as axis of symmetry and assume that in the steady state the parachute is falling with velocity V in a vertical direction, which coincides with the axis of y, and is rotating with angular velocity Q around the axis of y. This last assumption is made for the sake of generality, so as to cover the case of a rotating solid of revolution and to obtain a condition corresponding to some extent with the case of the helicopter where the longitudinal and lateral oscillations are not independent. The equations of motion may be written in the form

$$\frac{W}{g}\left[\frac{du}{dt}+(Q+q)w-r(V+v)\right]=W\sin\theta-X,$$

$$\frac{W}{g}\left[\frac{dw}{dt}+p(V+v)-u(Q+q)\right]=-W\cos\theta\sin\phi-Z,$$

$$\frac{A}{g}\frac{dp}{dt}+\frac{A-B}{g}r(Q+q)=-L,$$

$$\frac{A}{g}\frac{dr}{dt}+\frac{B-A}{g}p(Q+q)=-N,$$

$$\frac{W}{g}\left[\frac{dv}{dt}+ru-pw\right]=W\cos\theta\cos\phi-Y,$$

$$\frac{B}{g}\frac{dq}{dt}=-M.$$

[1] Stability in Aviation.

[2] The Tôhoku Mathematical Journal, vol. 14, August, 1918, p. 116.

Here W is the weight of the parachute and passenger, X, Y, Z, L, M, N the component air forces and couples referred to axes fixed in the parachute, B is the moment of inertia about the axis of symmetry, A the moment of inertia about a perpendicular axis through the center of gravity, both measured in gravitational units, $U+u$, v, w, p, $Q+q$, r are component velocities and spins in the disturbed motion. To obtain the directions of the axes in space in a disturbed position of the parachute we suppose the parachute rotated first about the axis of y (from z to x) through an angle ψ, then about the axis of z (from x to y) through an angle θ, and finally about the axis of x (from y to z) through an angle ϕ.

Let us write $u+iw=a$, $p+ir=b$, $\phi+i\theta=\chi$ where i denotes $\sqrt{-1}$, then since u, w, p, r, θ, and ϕ are small in a small oscillation and the parachute is symmetrical about the axis of y, we may write to a first approximation

$$p=\frac{d\phi}{dt},\ r=\frac{d\theta}{dt},\ b=\frac{d\chi}{dt}$$

$$X+iZ=aE+bF$$
$$L+iN=aJ+bK$$

where the generalized resistance coefficients E, F, J, K are complex quantities.

The above equations imply that if $E=E_1+iE_2$, etc.,

$$\begin{array}{ll}
X_u=Z_w=E_1, & Z_u=-X_w=E_2, \\
X_p=Z_r=F_1, & Z_p=-X_r=F_2, \\
L_u=N_w=J_1, & N_u=-L_w=J_2, \\
L_p=N_r=K_1, & N_p=-L_r=K_2, \\
X_v=O,\ X_q=O, & Z_v=O,\ Z_q=O, \\
L_v=O,\ L_q=O, & N_v=O,\ N_q=O.
\end{array}$$

It is on account of these relations that the analysis for the symmetrical parachute is simpler than that for an airplane in a straight drive. If, in fact, we neglect squares of small quantities and use D to denote the operatore $\frac{d}{dt}$ the first four equations may be written in the form

$$\left[\frac{W}{g}(D-iQ)+E\right]a+\left[\left(\frac{WV}{g}-iF\right)D+W\right]i\chi=o,$$

$$Ja+\left[\frac{A}{g}D^2+\left(K+iQ\,\frac{B-A}{g}\right)D\right]\chi=o.$$

Seeking a solution of the form $a=a_0e^{\lambda t}$, $\chi=\chi_0e^{\lambda t}$ we obtain the period equation $\left[\frac{W}{g}(\lambda-iQ)+E\right]\left[\frac{A}{g}\lambda^2+\left(K+iQ\frac{B-A}{g}\right)\lambda\right]-iJ\left[\left(\frac{WV}{g}-iF\right)\lambda+W\right]=o$. This is a cubic equation with complex coefficients for the determination of λ. When $Q=o$ it reduces to a cubic equation with real coefficients for we have the relations $E_2=o$, $F_1=o$, $J_1=o$, $K_2=o$.

Remembering that in the general case of a rigid airplane the oscillations about a state of steady motion are determined by an algebraic equation of the eighth degree, we infer that in the present degenerate case our real cubic is a double factor of the octic and the periods and decrements are the same for simple swings in two perpendicular planes through the axis of symmetry of the parachute.

When $Q=o$ a root $\lambda=\alpha+i\beta$ for which α is negative and β positive implies the existence of a damped oscillation in which the center of gravity of the parachute describes a curve on a vertical cylinder in such a way that it moves in a counterclockwise direction when the axes are left handed. To see this we notice that $u+iw$ is of the form

$$Re^{\alpha(t-t_0)}\,[\cos\beta(t-t_0)+i\sin\beta(t-t_0)]$$

where R and t_0 are real quantities. This equation indicates that $\frac{w}{u}$ increases with t and since the axis of z is to the left of the axis of x, the axis of y being downwards, this means that the center of gravity moves on the cylinder in the counterclockwise direction.

The equation $\phi+i\theta=X_0e^{(\alpha+i\beta)t}$ implies that at the same time the axis of symmetry precesses around the vertical in the counterclockwise direction. If we disregard the vertical motion the character of the oscillation may be pictured by imagining a cone representing the parachute to partly roll and partly slide on the outside of another cone whose axis is vertical.

Let us call this a *positive oscillation* and use the term *negative* oscillation to denote one in which β is negative and the two motions take place in the clockwise direction.

Either a positive or negative oscillation can be regarded as built up from two simple swings in perpendicular planes, the two swings having the same period and damping time, but differing in phase by a quarter period. By combining two such swings with an arbitrary difference in phase a type of elliptic oscillation is obtained.

It should be noticed than an *undamped* positive oscillation may correspond to a new type of steady motion of the parachute. If such a motion exist it is characterized by $v=o$, $q=o$ and the fact that the cubic equation has two purely imaginary roots.

Thus cubic equation may be written in the form

$$\lambda^3+y\lambda^2+xw\lambda+x=o$$

where

$$x=\frac{g^2N_u}{A},\ y=g\left(\frac{N_r}{A}+\frac{X_u}{W}\right),\ w=\frac{V}{g}-\frac{1}{W}\left(X_r-\frac{X_uN_r}{N_u}\right).$$

Let $\lambda^2-2\alpha\lambda+\theta$ be a factor of the cubic, then α and θ are determined by the equations

$$4\alpha^2-\theta+2\alpha y+xw=o$$

$$-2\alpha\theta-y\theta+x=o.$$

With the aid of these equations we can plot the curves $t=$constant in the (x, y) plane. Since

$$(4\alpha^2+2\alpha y+xw)\ (y+2\alpha)=x$$

a curve $t=$constant ($\alpha=$constant) is a hyperbola. When $\alpha=o$ it reduces to $x=o$, $y=\frac{1}{w}$ and this is a boundary of the region of stability in the (x,y) plane. The condition for undamped oscillations is, moreover, either $x=o$ or $y=\frac{1}{w}$.

The asymptotes of the hyperbola $t=$constant are

$$y+2\alpha=\frac{1}{w}\text{and } 2\alpha y+xw+4\alpha^2+\frac{2\alpha}{w}=o$$

hence when $\theta=-\frac{2\alpha}{w}$ we must have $x=\infty\,y=\infty$. This means that we can not have a value of θ which is smaller than $-\frac{2\alpha}{w}$, consequently when the time of damping is given the period p can not be greater than a certain limiting value P.

If α be given it is clear that P increases with w and so it is desirable that w should be made as large as possible. This can be done either by making V large or by making $X_r-\frac{X_uN_r}{N_u}$ negative and fairly large.

Now V is generally less than $\frac{1}{2}g$ and ought not to be increased beyond this value, consequently the most hopeful method of improving the stability of the parachute is to make $X_r-\frac{X_uN_r}{N_u}$ negative. It is probable that in the Calthrop parachute this condition has been secured.

In Figure I the curves $t=$constant have been drawn for two different values of w. It will be seen that when $w=.25$ the time of damping t is generally greater than the period. This is sometimes the case even when $w=.5$, but by choosing x and y in a suitable manner it is possible to make p greater than t. The importance of a fairly large value of w is thus manifest.

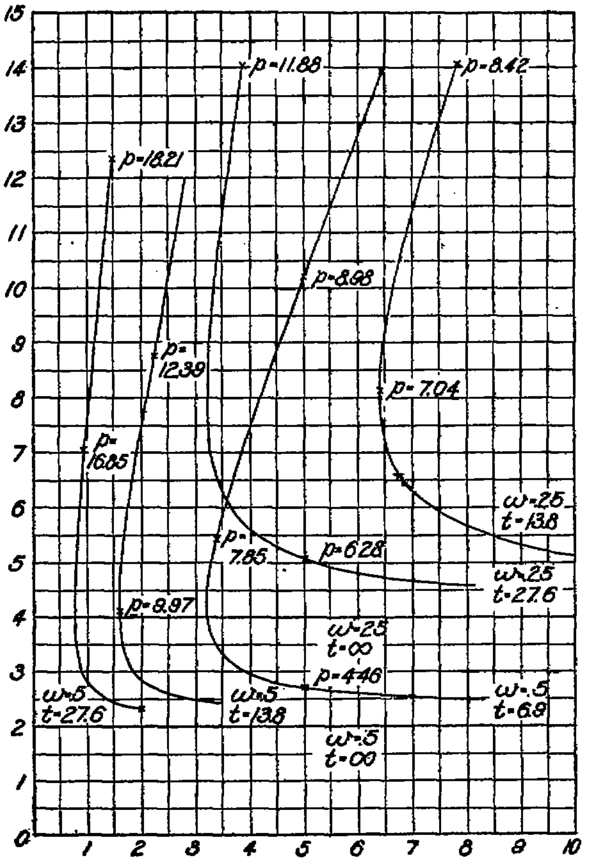

FIG. I.

In his study of the stability of the parachute, Brodetsky treats the parachute as a circular disk attached to a weighted stick and finds from a condition equivalent to $y>\frac{1}{w}$ that the center of gravity must lie between certain limits. The center of gravity must naturally be fairly low, and there seems no way of avoiding this except perhaps by attaching a ring-shaped balloon to the rim of the disk. The upward thrust on the balloon and the downward force through the center of gravity would then have as their resultant a force acting through a point lower than the center of gravity, and this point would act as the center of gravity of the parachute in the usual theory. Unless it could be combined with a helicopter in some way, the combination of a parachute and toroidal balloon would be of theoretical interest only.

Turning now to the case in which $Q \pm O$, we write

$$E=E_0+Q\ (e_1+ie_2) \qquad F=iF_0+Q\ (f_1+if_2)$$

$$J=iJ_0+Q\ (j_1+ij_2) \qquad K=K_0+Q\ (k_1+ik_2)$$

The additional terms involving Q give the forces and couples arising from the Magnus effect or its inverse [1] and from related phenomena.[2] The cubic equation may now be written in the form

$$\Psi(\lambda)\equiv\Phi(\lambda)+Q[(G+iH)\lambda^2+(I+iR)\lambda-iW(j_1+ij_2)]+Q^2[\quad]=o,$$

where

$$\Phi(\lambda)=\frac{WA}{g^2}\lambda^3+\frac{E_0A+WK_0}{g}\lambda^2+\left(E_0K_0+J_0F_0+J_0\frac{WV}{g}\right)\lambda+J_0W.$$

$$G=\frac{Ae_1+Wk_1}{g}$$

$$H=\frac{W}{g^2}(B-2A)+\frac{1}{g}(Ae_2+Wk_2),$$

$$I=E_0k_1+K_0e_1+J_0f_2+F_0j_2+\frac{WV}{g}j_2,$$

$$R=E_0k_2+K_0e_2-J_0f_1-F_0j_1-\frac{W}{g}K_0-\frac{WV}{g}j_1+E_0\frac{B-A}{g}.$$

To study the effect of rotation upon the oscillations let λ_0, $\alpha_1+i\beta_1$, $\alpha_1-i\beta_1$ be the roots of the equation $\Phi(\lambda)=o$, λ_0 and α_1 being supposed negative and let $\sigma_0+i\tau_0$, $\sigma_1+i\tau_1$, $\sigma_2+i\tau_2$, be the quantities which must be added to λ_0, $\alpha_1+i\beta_1$, $\alpha_1-i\beta_1$, respectively, to give the corresponding roots of the equation $\Psi(\lambda)=o$. If σ and τ are both positive it means that the effect of rotation is to diminish the period of a positive oscillation and to increase the time of damping t.

[1] For this effect see A. Lafay. Comptes Rendus. Paris. Vol. 151 (1910), p. 867; vol. 153 (1911), p. 1472.

[2] For the effect of sideslip on a propeller see the Technical Reports of the British Advisory Committee for Aeronautics 1912-13, 1913-14, and 1918.

The effect on a negative oscillation is to increase both the period and the time of damping. The effect on a simple subsidence is to make it periodic and increase the time of damping. The effect of rotation when σ and τ are not both positive is easily inferred.

In order that σ_0, σ_1, σ_2 may all be negative and the damping times for the three disturbances associated with λ_0, $\alpha_1+i\beta_1$, $\alpha_1-i\beta$, all increased, it is necessary, but not sufficient, that QG, QI, and Qj_2 should all be positive. Now $N_u=J_0+Qj_2$, hence it is necessary in the first place that the effect of the rotation should be to increase N_u. The effect on the period is of some interest because in general the single period and time of damping associated with the two conjugate roots $\alpha_1+i\beta_1$ and $\alpha_1-i\beta_1$, will give rise to two different periods and two different times of damping; also a new positive or negative oscillation with a long period will arise from the simple subsidence associated with the root λ_0. The phenomenon of the division of one period into two is somewhat analogous to that which occurs in the Zeeman effect, especially as we may have a compound oscillation built up from positive and negative circular oscillations of nearly equal period. The curves in Figure II give the horizontal projection of the path of the center of gravity of the parachute or helicopter in two particular cases of compound oscillations of the above type. In the first case the two component circular oscillations are undamped, in the second case they are both damped, the rate of subsidence being the same in each case. A complete discussion of the effect of rotation on stability is out of the question at present owing to our lack of knowledge of the values of the various resistance coefficients, but it may be worth while to ascertain the conditions for stability by finding the conditions that a cubic equation with complex coefficients may have roots whose real parts are all negative.

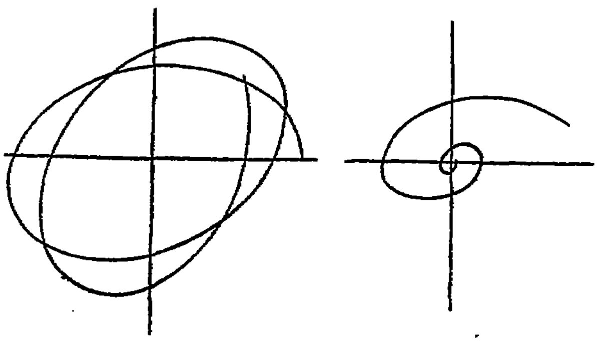

FIG. II.

The conditions that a cubic equation with complex coefficients may have the real parts of all its roots negative.

The method given by E. J. Routh[1] for finding the conditions that an algebraic equation with real coefficients may have the real parts of all its roots negative may be extended to the case of an equation with complex coefficients.

Let us consider the case of the cubic

$$(x+\alpha_1)\ (x+\alpha_2)\ (x+\alpha_3)\equiv x^3+px^2+qx+r\equiv x^3+(p_1+ip_2)x^2+(q_1+iq_2)x+r_1+ir_2$$

Let $\alpha_1=x_1+iy_1$, $\alpha_2=x_2+iy_2$, $\alpha_3=x_3+iy_3$, and let the equation whose roots are $-x_1+iy_1$, $-x_2+iy_2$, $-x_3+iy_3$, be

$$(x+\beta_1)\ (x+\beta_2)\ (x+\beta_3)\equiv x^3+Px^2+Qx+R.$$

In Routh's method the first step is to write iy for x and to separate the real and imaginary parts of the cubic, thus obtaining two expressions

$$y^3+p_2y^2-q_1y-r_2=f_1(y)$$
$$p_1y^2+q_2y-r_1=f_2(y)$$

The process of finding the greatest common measure of $f_1\ (y)$ and $f_2\ (y)$ must now be carried out, the sign of the remainder being changed at each step just as in Sturm's theorem. In this way we obtain a series of polynomials whose first coefficients are

$$1,\ p_1,\ \frac{H}{p_1^{\,2}},\ \text{and}$$
$$K\equiv\frac{1}{H^2}\{q_2HI-p_1I^2+r_1H^2\}$$

[1] Loc. cit.

respectively, where

$$H = p_1 p_2 q_2 - q_2^2 - p_1 r_1 + p_1^2 q_1,$$
$$I = p_1^2 r_2 + r_1 q_2 - r_1 p_1 p_2.$$

In order that x_1, x_2, and x_3 may all be positive it is necessary and sufficient that p_1, H, and K should all be positive. The quantity K becomes zero when any one of the quantities x_1, x_2, x_3 becomes zero, it corresponds, therefore, to Routh's discriminant in the case of the real quartic. In terms of x_1, x_2, x_3, y_1, y_2, y_3, the expressions for H and K are[1]

$$H = Ax_2x_3 + Bx_3x_1 + Cx_1x_2,$$
$$H^2 K = x_1x_2x_3\,[H^2 - (lC - nA)\,(mA - lB) - (mA - lB)\,(nB - mC) - (nB - mC)\,(lC - nA)]$$

where

$$A = (x_2 + x_3)\,(2x_1 + x_2 + x_3) + (y_2 - y_3)^2,$$
$$B = (x_3 + x_1)\,(2x_2 + x_3 + x_1) + (y_3 - y_1)^2,$$
$$C = (x_1 + x_2)\,(2x_3 + x_1 + x_2) + (y_1 - y_2)^2,$$
$$l = x_1(y_2 - y_3),\ m = x_2(y_3 - y_1),\ n = x_3\,(y_1 - y_2).$$

It is not evident from these expressions that x_1, x_2, and x_3 are positive when p_1, H, and K are positive;[2] also $H^2 K$ is of the eleventh degree in the quantities x_1, x_2, x_3, y_1, y_2, y_3, and is not the simplest function of the coefficients which becomes zero when one of the quantities x_1, x_2, x_3 is zero; consequently it will be worth while to find an alternative set of conditions which will make x_1, x_2, and x_3 positive.

Let us consider the expressions

$$\begin{aligned}
T &= (\alpha_1+\beta_1)\,(\alpha_1+\beta_2)\,(\alpha_1+\beta_3)\,(\alpha_2+\beta_1)\,(\alpha_2+\beta_2)\,(\alpha_2+\beta_3)\,(\alpha_3+\beta_1)\,(\alpha_3+\beta_2)\,(\alpha_3+\beta_3) \\
&= 8x_1x_2x_3\,[(x_2+x_3)^2+(y_2-y_3)^2]\,[(x_3+x_1)^2+(y_3-y_1)^2]\,[(x_1+x_2)^2+(y_1-y_2)^2], \\
S &= [(\alpha_2+\beta_3)\,(\alpha_3+\beta_3)+(\alpha_3+\beta_3)\,(\alpha_1+\beta_1)+(\alpha_1+\beta_1)\,(\alpha_2+\beta_2)]\times \\
&\quad [(\alpha_2+\beta_3)\,(\alpha_3+\beta_1)+(\alpha_3+\beta_1)\,(\alpha_1+\beta_2)+(\alpha_1+\beta_2)\,(\alpha_2+\beta_3)]\times \\
&\quad [(\alpha_2+\beta_1)\,(\alpha_3+\beta_2)+(\alpha_3+\beta_2)\,(\alpha_1+\beta_3)+(\alpha_1+\beta_3)\,(\alpha_2+\beta_1)] \\
&= 4[x_2x_3+x_3x_1+x_1x_2]\,[\{x_2+x_3+i(y_2-y_3)\}\,\{x_3+x_1+i(y_3-y_1)\} \\
&\quad +\{x_3+x_1+i(y_3-y_1)\}\,\{x_1+x_2+i(y_1-y_2)\}+\{(x_1+x_2)+i(y_1-y_2)\}\,\{x_2+x_3+i(y_2-y_3)\}]\times \\
&\quad [\{(x_1+x_2)-i(y_1-y_2)\}\,\{x_2+x_3-i(y_2-y_3)\}+\{x_2+x_3-i(y_2-y_3)\}\,\{x_3+x_1-i(y_3-y_1)\}+ \\
&\quad \{x_3+x_1-i\,(y_3-y_1)\}\,\{x_1+x_2-i(y_1-y_2)\}], \\
S^1 &= [(\alpha_2+\beta_2)\,(\alpha_3+\beta_1)+(\alpha_3+\beta_1)\,(\alpha_1+\beta_3)+(\alpha_1+\beta_3)\,(\alpha_2+\beta_2)]\times \\
&\quad [(\alpha_2+\beta_3)\,(\alpha_3+\beta_2)+(\alpha_3+\beta_2)\,(\alpha_1+\beta_1)+(\alpha_1+\beta_1)\,(\alpha_2+\beta_3)]\times \\
&\quad [(\alpha_2+\beta_1)\,(\alpha_3+\beta_3)+(\alpha_3+\beta_3)\,(\alpha_1+\beta_2)+(\alpha_1+\beta_2)\,(\alpha_2+\beta_1)] \\
&= [4x_2(x_1+x_3)+(x_1+x_3)^2+(y_3-y_1)^2]\,[4x_1(x_2+x_3)+(x_2+x_3)^2+(y_2-y_3)^2]\times \\
&\quad [4x_3(x_1+x_2)+(x_1+x_2)^2+(y_1-y_2)^2], \\
p_1 &= x_1+x_2+x_3.
\end{aligned}$$

It will be noticed that the second and third factors of S are conjugate complex quantities and so their product is a positive quantity. It is clear then that if p_1, S, and T are all positive, the quantities

$$x_1+x_2+x_3,\ \ x_2x_3+x_3x_1+x_1x_2,\ \ x_1x_2x_3$$

are all positive.[3] Now let x_1, x_2, and x_3 be regarded as the distances of a point from the sides of an equilateral triangle. This is legitimate since $x_1+x_2+x_3$ is positive. Keeping this last quantity constant a point can be used to represent any possible set of values of x_1, x_2, and x_3 whose sum has this constant value.

Now when $x_2x_3+x_3x_1+x_1x_2$ is positive the representative point lies within the circumscribing circle of the triangle and when $x_1x_2x_3$ is positive the representative point lies within

[1] See note 1.

[2] In the case of the quadratic $p_1 = x_1+x_2$, $H = x_1x_2\,[(x_1+x_2)^2+(y_1-y_2)^2] = p_1p_2q_2 - q_2^2 + p_1^2q_1$. When p_1 and H are positive it is evident that x_1+x_2 and x_1x_2 are both positive and that consequently x_1 and x_2 are positive.

[3] When these quantities are all positive S^1 is also positive.

one of the four regions bounded by lines making an acute angle with one another. It is clear then that when $x_1+x_2+x_3$, $x_2x_3+x_3x_1+x_1x_2$ and $x_1x_2x_3$ are all positive the representative point must lie within the triangle and x_1, x_2, x_3, be all positive.

To express S and T in terms of the coefficients we notice that if two of the quantities α_1, α_2, α_3 are interchanged, or if two of the quantities β_1, β_2, β_3 are interchanged, S is transformed into S^1. Hence we are justified in assuming that there is an identity of type

$$S-S^1=k\triangle$$

where

$$\triangle=(\alpha_2-\alpha_3)\ (\alpha_3-\alpha_1)\ \alpha_1-\alpha_2)\ (\beta_2-\beta_3)\ (\beta_3-\beta_1)\ (\beta_1-\beta_2)$$

Comparing coefficients we find that $k=-1$.

Writing S in the form

$$\begin{aligned}S&=(q+Q+pP-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3)(q+Q+pP-\alpha_3\beta_1-\alpha_1\beta_2-\alpha_2\beta_3)(q+Q+pP-\alpha_2\beta_1-\alpha_3\beta_2-\alpha_1\beta_3)\\&=(q+Q+pP)^2(q+Q)+(q+Q+pP)(p^2Q+P^2q-3qQ)-U\end{aligned}$$

where

$$U=(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)\ (\alpha_3\beta_1+\alpha_1\beta_2+\alpha_2\beta_3)\ (\alpha_2\beta_1+\alpha_3\beta_2+\alpha_1\beta_3)$$

we assume that

$$-U+\frac{1}{2}\triangle=ArR+B(rPQ+Rpq)+C(rP^3+Rp^3)+DPQpq+Ep^3P^3+F(p^3PQ+P^3pq)$$

Determining the unknown coefficients by putting

(1) $\alpha_1=o,\ \beta_1=o,$

(2) $\beta_2=o,\ \beta_3=o,$

(3) $\alpha_2=\alpha_3,\ \beta_1=o,$

(4) $\beta_1=\beta_2=\beta_3,$

we find that

$$A=-\frac{27}{2},\ B=\frac{9}{2},\ C=-1,\ D=-\frac{1}{2},\ E=o,\ F=o.$$

Hence finally

$$\begin{aligned}S=(q+Q)(q+Q+pP)^2+(q+Q+pP)(p^2Q+P^2q-3qQ)-\frac{27}{2}rR+\frac{9}{2}(rPQ+Rpq)\\-rP^3-Rp^3-\frac{1}{2}PQpq-\frac{1}{2}\triangle,\end{aligned}$$

$$\triangle^2=(p^2q^2-4p^3r+18pqr-4q^3-27r^2)\ (P^2Q^2-4P^3R+18PQR-4Q^3-27R^2)$$

Writing T in the form

$$T=(\alpha_1^3+P\alpha_1^2+Q\alpha_1+R)\ (\alpha_2^3+P\alpha_2^2+Q\alpha_2+R)\ (\alpha_3^3+P\alpha_3^2+Q\alpha_3+R)$$

we easily find that

$$\begin{aligned}T=r^3+R^3+Pr^2q+pR^2Q+Qrq^2+qRQ^2-2Qpr^2-2qPR^2+3Rr^2+3R^2r+Rq^3+rQ^3-3Rpqr-3rPQR\\+P^2pr^2+p^2PR^2+P^3r^2+p^3R^2+PQpqr+pqPQR+P^2Qqr+p^2qQR+PQ^2pr+pq^2PR+P^2Rq^2\\+p^2rQ^2-2P^2Rpr-2p^2rPR-3r^2PQ-3R^2pq-QRpr-qrPR-2q^2QR-2Q^2qr.\end{aligned}$$

O