REPORT No. 95

DIAGRAMS OF AIRPLANE STABILITY

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INTRODUCTION.

§1. This report was prepared by Dr. H. Bateman for publication by the National Advisory Committee for Aeronautics. The theory of small oscillations about a state of steady motion which was developed many years ago by E. J. Routh[1] has been applied with marked success in aerodynamics, the desired simplicity of the equations being secured by the introduction of the resistance coefficients by G. H. Bryan.[2] This simplification of the equations is based on the assumption that in a slight departure from a state of steady motion the increments in the component aerodynamical forces and couples can be expressed in terms of the increments of the component velocities of translation and rotation alone without any additional terms depending, for instance, on the increments of the accelerations. This assumption seems to give a good approximation to the truth in the case of an airplane, but in the case of a balloon the additional terms are required. When a flying machine is treated as a rigid body the general type of steady motion is one in which the center of gravity describes a helix and the algebraic equation which determines the temporal characteristics of the oscillations is of the eighth degree, but this equation can be simplified in certain cases. In the case of an airplane having a plane of symmetry, the oscillations about a state of steady rectilinear flight can be regarded as built up from longitudinal and lateral oscillations which are practically independent of one another. When certain resistance coefficients are assumed to be zero each set of oscillations is associated with an algebraic equation of the fourth degree.[2]

A notable simplification also occurs in the case of a body like a parachute which has an axis of symmetry, when the steady motion is rectilinear and in the direction of the axis of symmetry.

In a recent report on the dynamical analysis,[3] Messrs. Klemin, Warner and Denkinger have studied the effect on the period and rate of subsidence of the pitching oscillations of an airplane of a change in one of the resistance derivatives when all the others are kept constant. It occurred to the author that it might be worth while to continue this work by considering simultaneous variations in two or three of the derivatives and to pay special attention to cases in which a slight change in some of the derivatives has (1) no effect, and (2) a marked effect in the characteristics of the long oscillation.

The results which have been obtained are exhibited by means of diagrams in which two of the resistance coefficients are used as coordinates, and curves are drawn along which the modulus of decay of a long oscillation has a constant value, numbers being given to indicate the value of the period at various points of each curve. At Mr. Warner's suggestion numbers have been given in some cases to indicate the ratio of the period to the time of damping to half the initial amplitude, as this quantity is adopted as a measure of stability in the report to which we have just referred.

[1] The Stability of a Given State of Motion (Adams Prize Essay). London, 1877. Advanced Rigid Dynamics. Chaps. III and VI.

[2] Stability in Aviation. Macmillan & Co. (1911).

[3] Third Annual Report of the National Advisory Committee for Aeronautics. No. 17, p. 330. (1917.) This report will be referred to subsequently as 17.

The method has also been adapted to the lateral oscillations of a symmetrical airplane and to the oscillations of a parachute. The graphical method used here is inferior in many respects to the beautiful one devised by L. Bairstow and J. L. Nayler, British Advisory Committee's Report No. 116, 1915, but the work was completed before this report came to the writer's notice.

§2. *Pitching oscillations.*—When the pitching oscillations are regarded as independent of those in roll and yaw, the biquadratic equation which determines the temporal characteristics of the oscillations may be written in the form

$$\lambda\,(A\lambda+y)(\lambda+z)(\lambda+w)+x\left[\lambda(\lambda+z)+\lambda\frac{g}{V}\sin\theta_o\right]+\xi\eta\lambda(A\lambda+y)$$

$$(1)\qquad +(\xi x+\zeta w)(\lambda\delta+\cos\theta_o)+(\eta\zeta-zx)\frac{g}{V}(\lambda\delta-\sin\theta_o)+\lambda\zeta(\cos\theta_o-\eta)=o$$

where
$$x=-VM_w,\ y=-M_q,\ z=-X_u,\ w=-Z_w,$$

$$\xi=\frac{g}{V}Z_u,\ \eta=-\frac{V}{g}X_w,\ \zeta=-gM_u,\ \delta=\frac{1}{g}X_q,$$

$$(2)\qquad V=U+Z_q+X_q$$

and the notation is the same as in the reports of Hunsaker, Klemin, Denkinger, and Warner.

Let $\lambda^2-2\alpha\lambda+\gamma$ be one factor of the expression on the left hand side of equation (1). Replacing λ^2 by $2\alpha\lambda-\gamma$ we can reduce the above expression to one which is linear in λ and equate to zero the terms with and without λ. This gives us the two equations

$$(3)\quad \gamma[y+A(w+z)+4A\alpha]=(y+2A\alpha)(w+2\alpha)(z+2\alpha)+\xi\eta(y+2A\alpha)$$
$$+x\left[2\alpha+z+\frac{g}{V}\sin\theta_o+\delta(\xi-\frac{g}{V}z)\right]+\zeta\left[\delta(w+\frac{g}{V}\eta)+\cos\theta_o-\eta\right],$$

$$(4)\quad A\gamma^2-\gamma[(y+2A\alpha)(w+z+2\alpha)+A(wz+\xi\eta)+x]+\cos\theta_o(\xi x+\zeta w)-\frac{g}{V}\sin\theta_o(\eta\zeta-zx)=o,$$

which generally determine x and y uniquely [4] when α and γ are given, that is, when the period $p=\frac{2\pi}{\sqrt{\gamma-\alpha^2}}$ and the co-efficient of subsidence $=\alpha$ of the oscillation are given. Instead of the latter quantity it is convenient to use the time

$$t=\frac{\log_e 2}{-\alpha}=\frac{.69}{-\alpha},$$

which represents the time which it takes for the amplitude of a simple oscillation to fall to half value.

With the aid of equations (3) and (4) the curves t=constant (α=constant) have been drawn in the (x, y) plane for various values of z, w, ξ, η, ζ, δ, and θ_o, the value $A=100$ being adopted in each case. We can use the same diagrams for any other value of A by simply altering the scales for x, y, and ζ. It should be noticed in fact that equation (1) is still satisfied if we replace A, x, y, ζ by κA, κx, κy, $\kappa\zeta$, respectively, keeping the other quantities the same.

In diagrams I–V there are two sets of curves corresponding respectively to the values $\eta=1$ and $\eta=2$. Each set of curves is made up of three pairs, the two curves of each pair correspond

[4] They may fail to do this for certain particular values of α and γ when the two linear equations in x and y are the same. It should be noticed however, that when $\theta_0=o$, $\delta=o$, $\zeta=o$, the equations give

$$(2\alpha+w+z)[w^2+wz(\eta-1)+\xi(\eta-1)^2]=o$$

When $2\alpha+w+z=o$ we have $\gamma=\xi\eta+wz$ and the equation of the line along which both α and γ are constant is $x=o$. This is one of the boundaries of the region of stability in the (x, y) plane. The second factor $w^2+wz(\eta-1)+\xi(\eta-1)^2$ is generally positive with the values of the resistance coefficient usually found for an airplane. If it could vanish there would be an infinite number of straight lines along which α and γ are constant and connected by the relation

$$\gamma=\xi+\frac{w}{\eta-1}(2\alpha+z).$$

to the same value of t, but the upper curve corresponds to the value $\zeta=1$ and the lower curve to the value $\zeta=o$.

The values of z, w, ξ, in diagram I are roughly those found for the Curtiss J. N. 2 in Report 17. A few slight changes have been made to facilitate the calculations, but these do not affect the general conclusions. The values of η in diagram I are greater than the number derived from the value of X_w given in Report 17. This number is about 0.6 and it is remarked on page 332 that X_w decreases as the angle of incidence increases and the speed decreases. Since X_w may actually become negative as the critical speed is approached, curves have also been drawn for the values $\eta=o$ and $\eta=-1$.

In studying these diagrams it should be noticed in the first place that the lines $t=\infty$ and $x=0$ limit the region of stability. The first of these lines is curved. When η is decreased this line rises and the region of stability becomes more restricted. On this account alone we can expect a decrease in η to be unfavorable when the other resistance derivatives remain constant. This agrees with the result found in Report 17 but it is worth while to consider the matter more fully.

If we take any value of η and begin to draw the curves $t=$constant, starting from $t=\infty$ and gradually decreasing t, we find that a certain minimum value t_0 is reached below which the curve no longer lies in the part of the region of stability shown on the diagram. It is clear from the diagrams that t_0 increases when η decreases. This increase in the time of damping is partly offset by an increase in period; but, if the other resistance derivatives remain constant, the ratio of the period to the time of damping apparently decreases with η. This is seen more easily by looking at diagrams Ic and Ib.

If some of the other resistance coefficients alter at the same time as η the unfavorable effect of a decrease in the value of η may be partially or completely offset. Thus if, when we decrease η from 2 to 1, we increase y so as to keep t constant, we increase the period p and so improve the stability as far as the long oscillation is concerned. With small values of η a considerable increase in y may be needed to keep t constant when η is decreased, but a much smaller increase in y may be sufficient to offset the unfavorable effect of the decrease of η. The effect of decreasing η may also be offset by decreasing x; for, if we decrease x so as to keep t constant, the period p is seen to be increased. This may be seen very clearly in diagram I if we start with the point $t=13.8$, $p=20.1$, $\eta=2$, and pass first to the point with the same coordinates in the part of the diagram corresponding to $\eta=1$ and then proceed along a line $y=$constant until the curve $t=13.8$ is reached. A comparison of the different diagrams indicates that the general form and arrangement of the curves is roughly the same for the different sets of values of z, w, and ξ. An exception occurs in the case of the curves for $\eta=-1$ in diagram Ia. It will be seen that in this case the curves $\zeta=0$ cross the curves $\zeta=1$ while in the other diagrams a curve $\zeta=0$ remains below the corresponding curve $\zeta=1$.

It is generally assumed that $\zeta=0$, since the moment about the center of gravity of the airplane due to air forces is zero in horizontal flight and therefore will not be affected by a change in speed. It is easy, however, to imagine some arrangement which will make M_u and therefore ζ different from zero. The inclination of a flap held by a spring and exposed to the air will vary with the speed of the airplane, consequently the force on it will not be proportional to the square of the speed, and so ζ will be either positive or negative.

It will be seen from the diagrams that a positive value of ζ is generally unfavorable to stability. When η is negative, a machine with a moderately large value of x may be an exception to this rule; but, when X_w is positive, the stability can apparently be improved by making ζ negative.

If this is done with the aid of a flap held by a spring,[5] changes may also be produced in the other resistance derivatives, particularly X_u and Z_u. We must therefore also study the effect on stability of changes in ξ and z. This may be done with the aid of diagrams I–V.

[5] The stability of an aeroplane which has springs in the control connections has been discussed in a more rigorous manner by L. Bairstow and R. Jones. British Advisory Committee for Aeronautics. (R—M No. 210.)

It appears that an increase in ξ from 2 to 3 has practically no effect on the position of the curves t = constant, but it does lower the period; hence an increase in ξ seems to be unfavorable.

A comparison of diagrams IV and V indicates that an increase of z from 0.1 to 0.2 lowers the curves t = constant and is on the whole favorable to stability.

The general conclusion then is that, if some kind of a flap held by a spring is to be used to improve the stability, as far as the long oscillation is concerned, it should be chosen so as to decrease Z_u, increase $-X_u$, and make M_u positive.

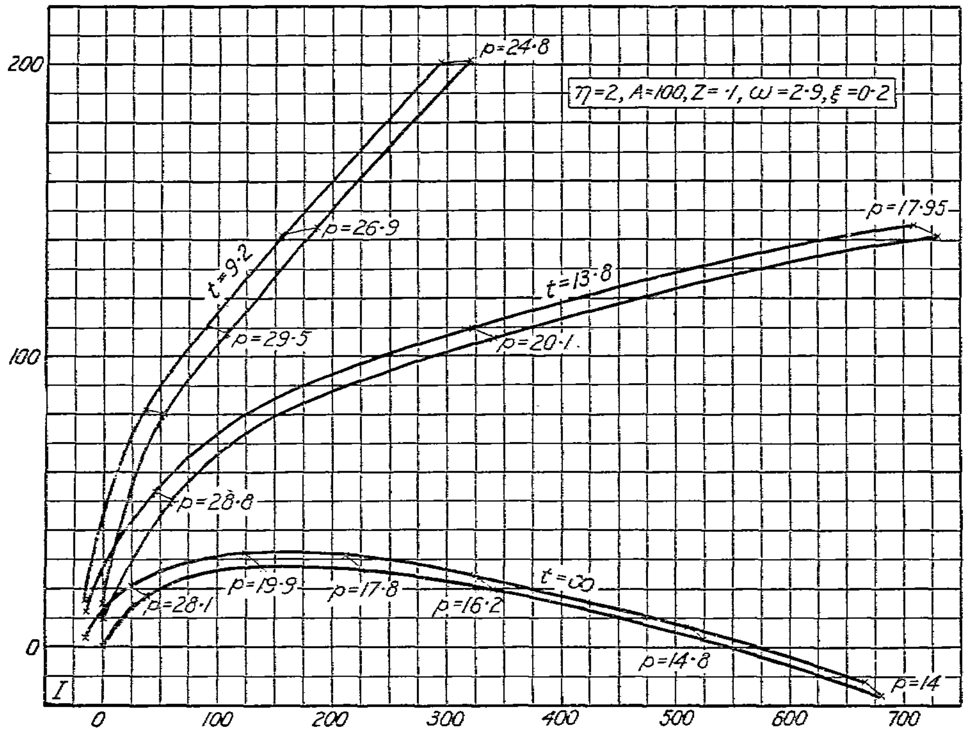
The diagrams may also be used for other purposes; in particular, they may be used to confirm some of the conclusions in Report 17. It will be noticed that as a point moves toward the origin along a curve t = constant, the period p increases, and so the stability is improved.[6] It may then be advantageous to decrease both VM_w and $-M_q$, but $-M_q$ must not be decreased too rapidly. There is a limit to the ratio of $-dy$ to $-dx$ if we wish the change to be favorable to stability, and this limiting value may be shown clearly on the diagram by drawing the curves for which $\frac{p}{t}$ is constant, in accordance with a suggestion made by Mr. Warner. These curves are shown in diagrams Ib and Ic. It will be noticed that the curves $\frac{p}{t}$ constant are steeper than the curves t = constant, but are not quite as steep as the curves p = constant.

A comparison of diagrams I and II indicates that an increase of w from 2.9 to 3.9 increases the period p and has more effect on the time of damping t when η is negative than when η is positive. When $\eta = 2$, the time t apparently increases with w; but when $\eta = 1$, t increases for small values of t, e. g., those less than 13.8, and decreases for large values of t. When $\eta = 0$ and $\eta = -1$, t decreases as w increases, the effect being quite marked when $\eta = -1$.

In diagram VI the effect is shown of making δ different from zero. It appears that by making δ positive we decrease the time of damping and produce very little change in the period. Other things being equal, an increase in δ seems to be favorable to stability. It should be noticed, however, that X_q occurs in the expression for V, consequently unless an increase in X_q is balanced by a decrease in Z_q the numerical value of V will be lowered and the values of the quantities depending on V altered. It appears then that the effect of an increase in X_q may be offset by the change in V. It should be observed, however, that, when X_q is about 3, the percentage change in the value of V is not large, and so the effect of the increase in δ should predominate over the effect of the change in V. In diagram VI σ is written in place of δ.

In diagram VII the effect is shown of a change in θ_o, other things remaining the same. It appears that the stability is improved by making θ_o positive, for the curves t = constant are lowered. By making θ_o negative, the time of damping is increased. In particular, if the speed were not increased, the time of damping would be greater in a vertical dive than in horizontal flight. This may be seen when the stability of the parachute and helicopter are studied. (See Report No. 80.)

[6] It should be noticed that the nearer the representative point is to the origin of coordinates, the more marked is the effect on p and t of small changes in x and y.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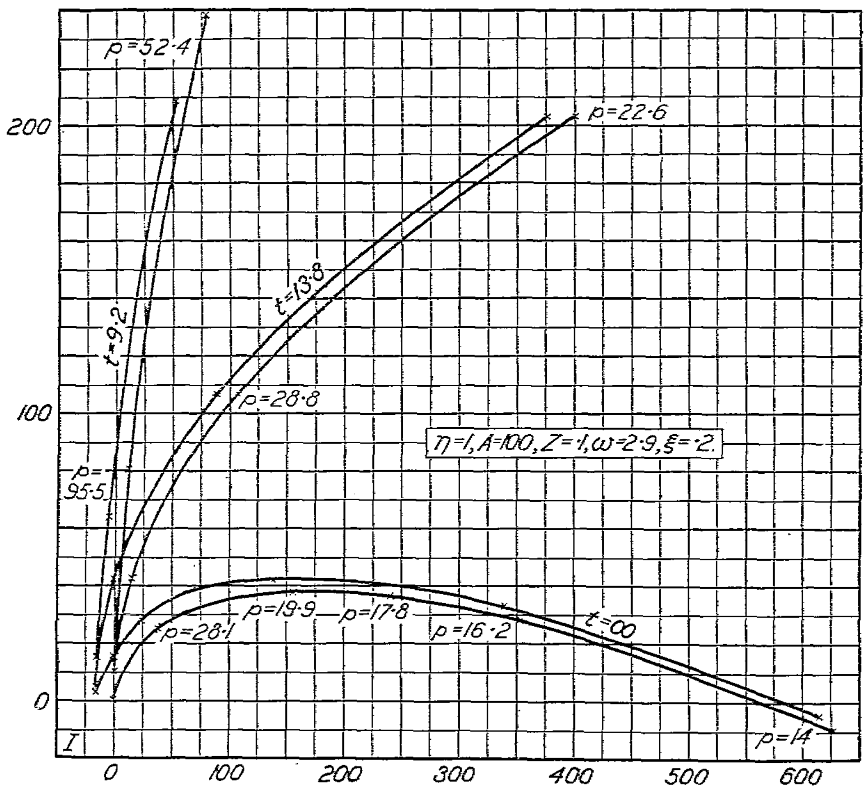
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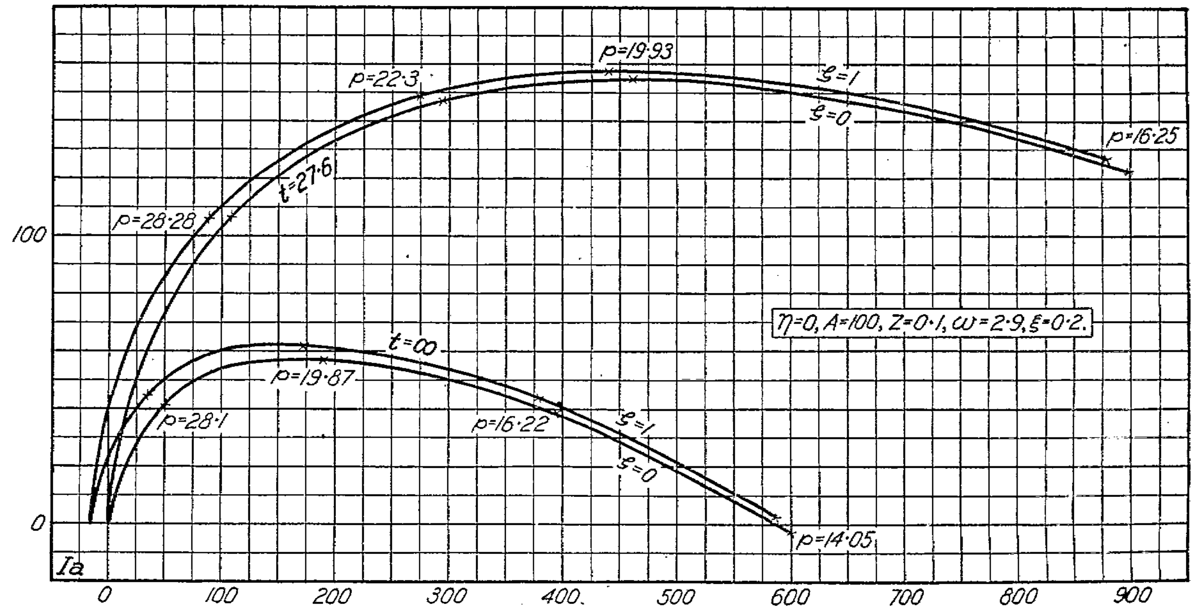
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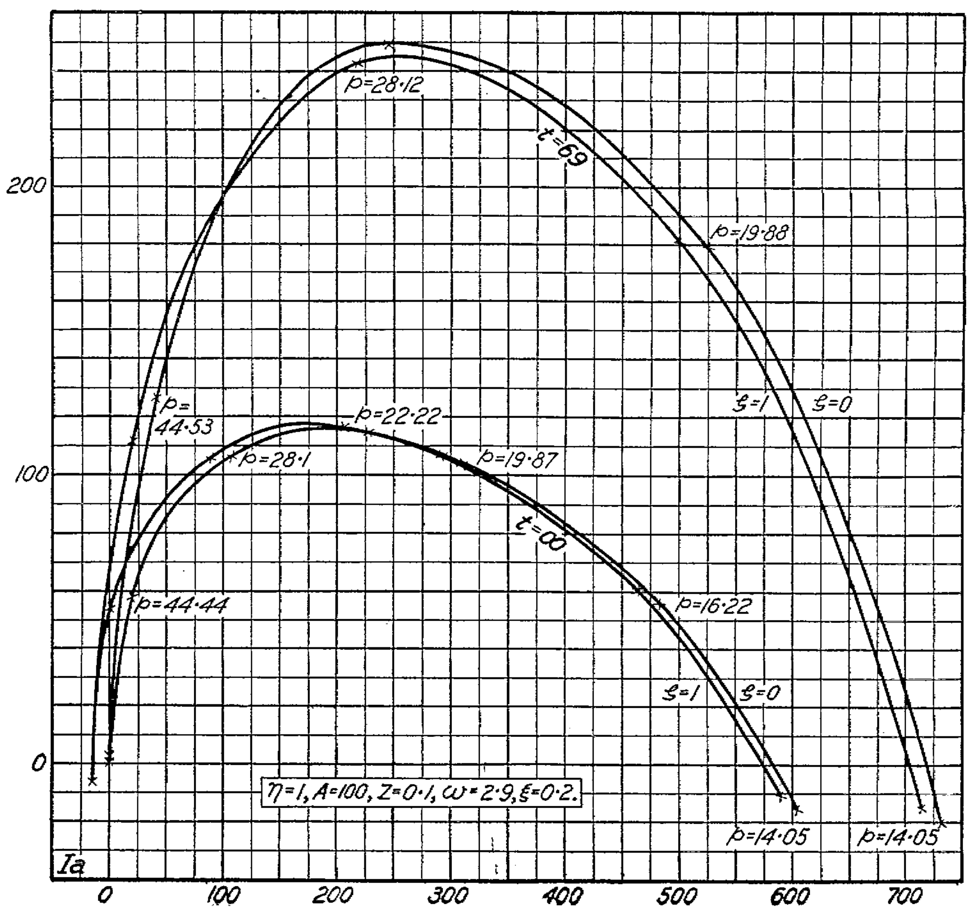
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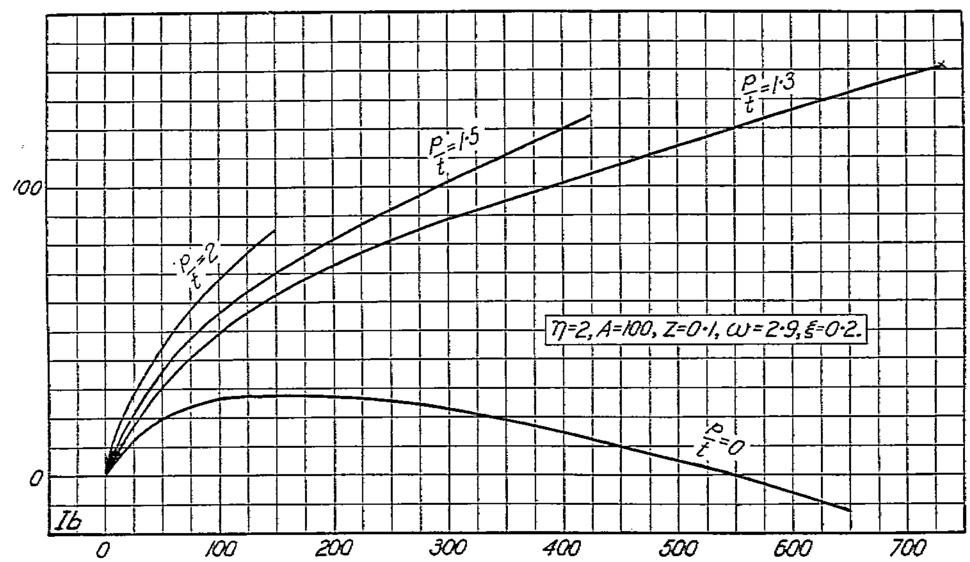
NOTE: $\eta = -1$.



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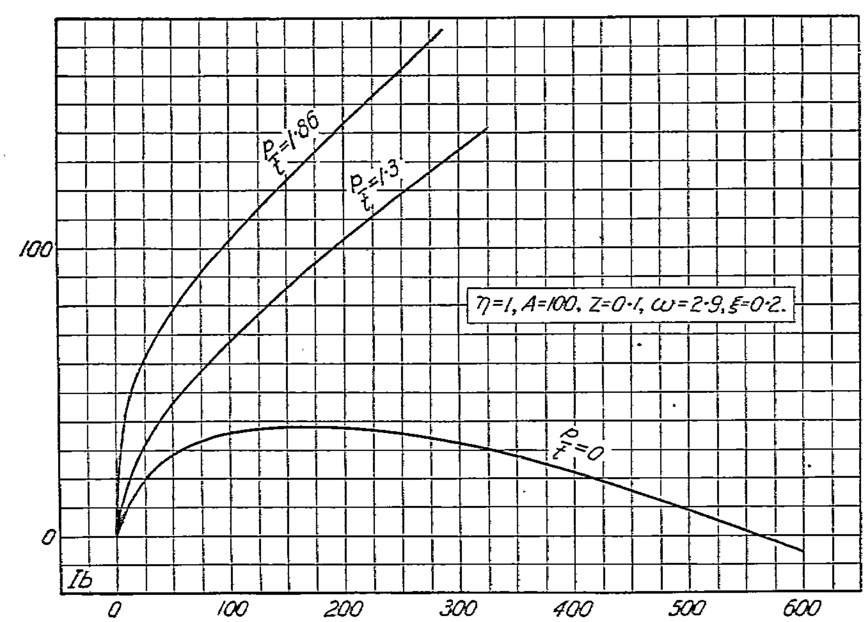

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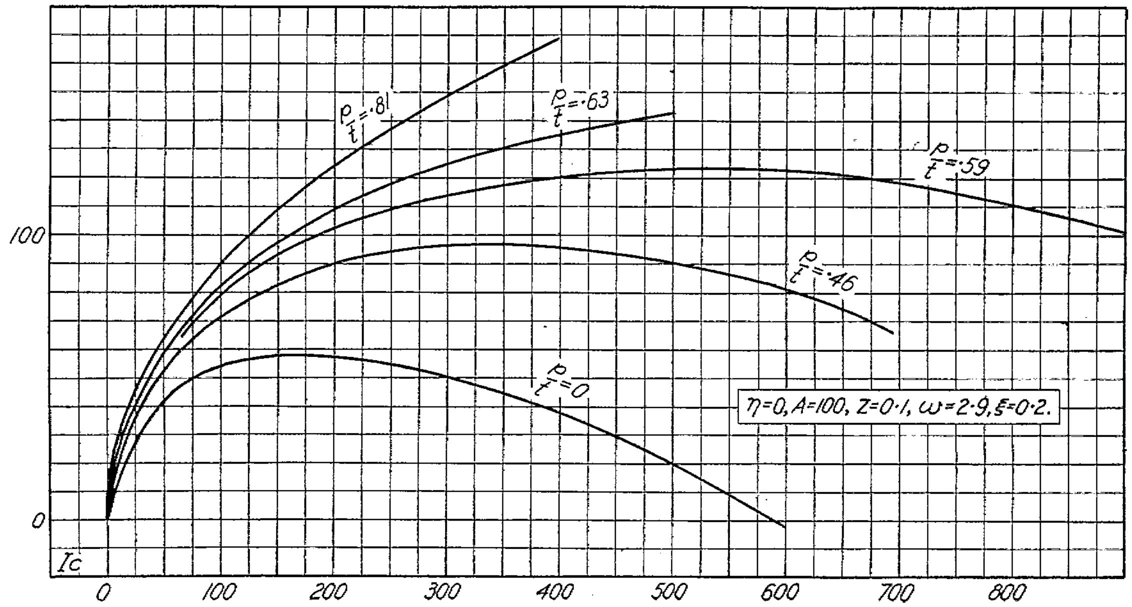
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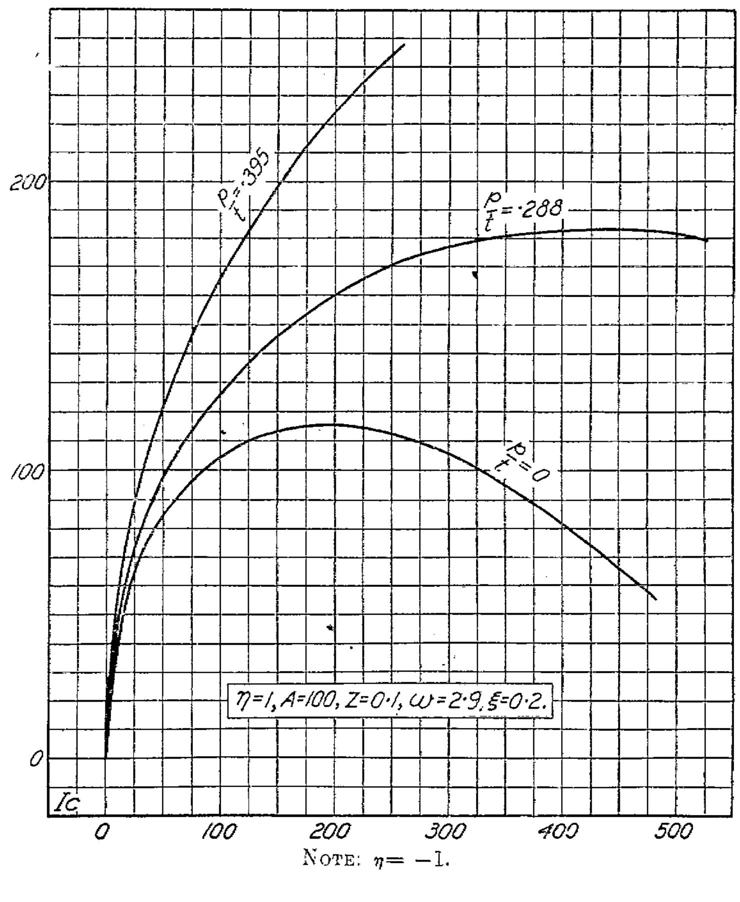
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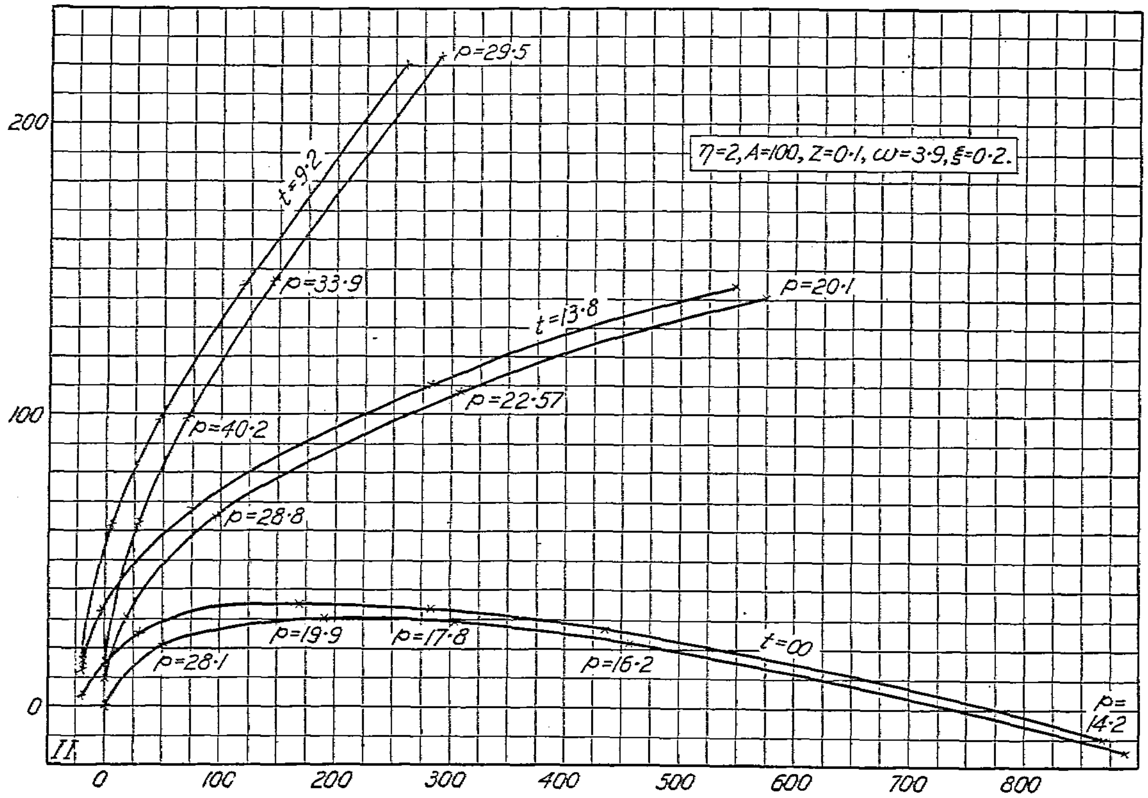
NOTE: $\eta = -1$.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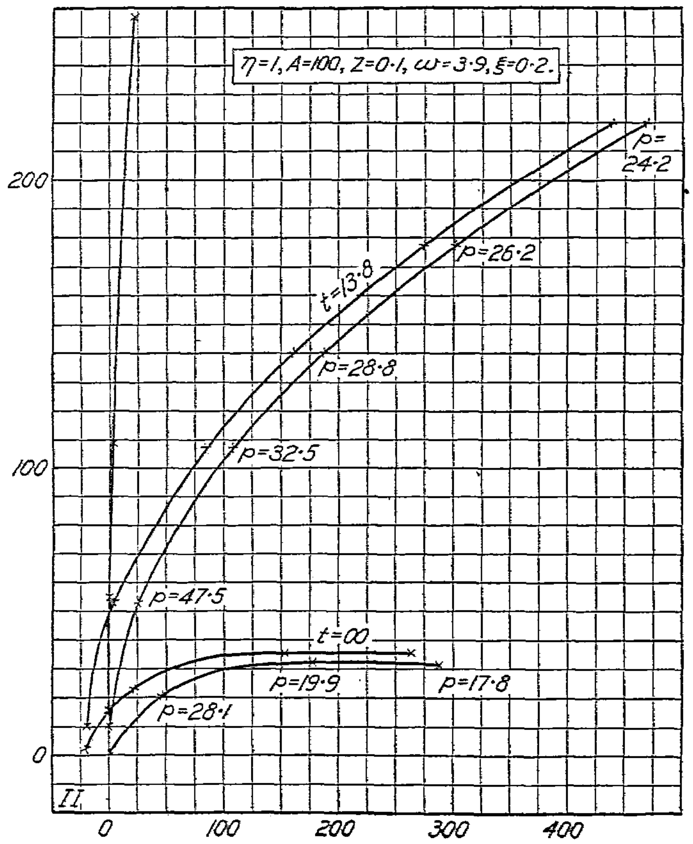
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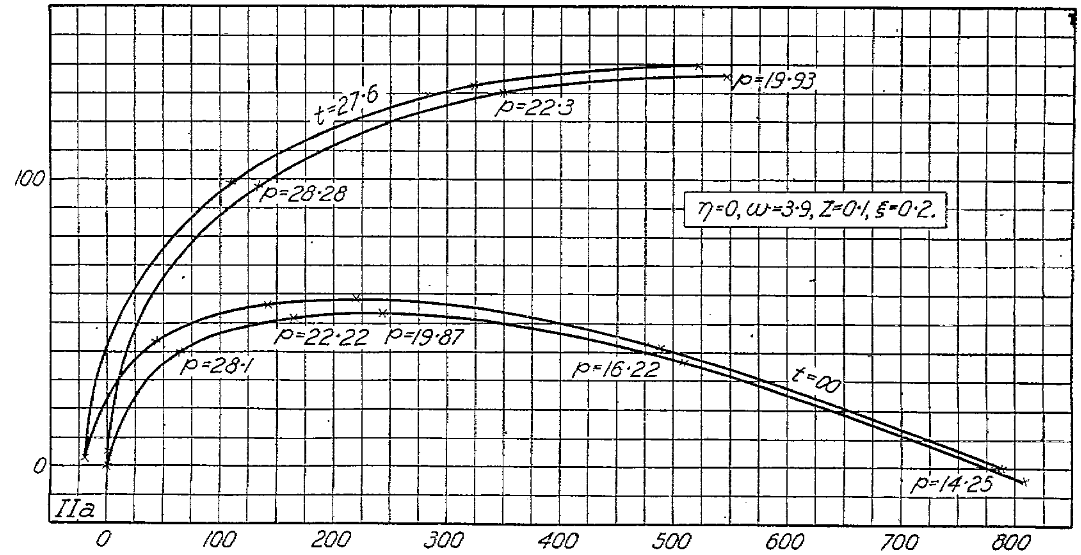

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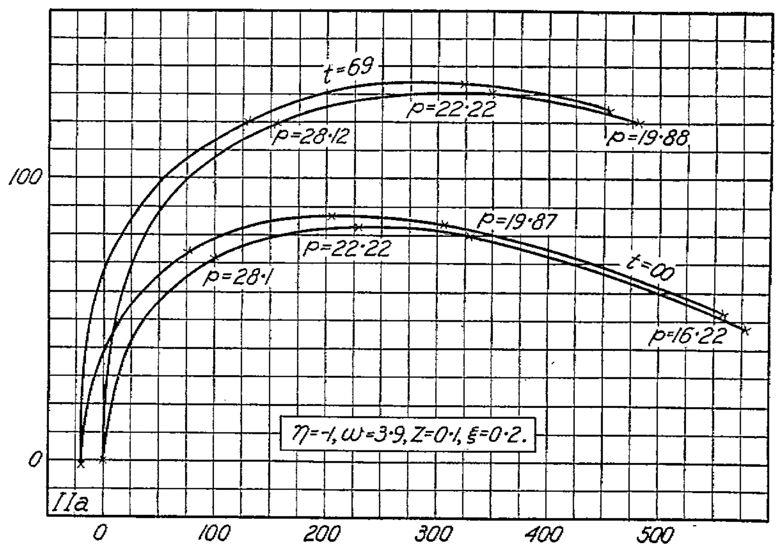

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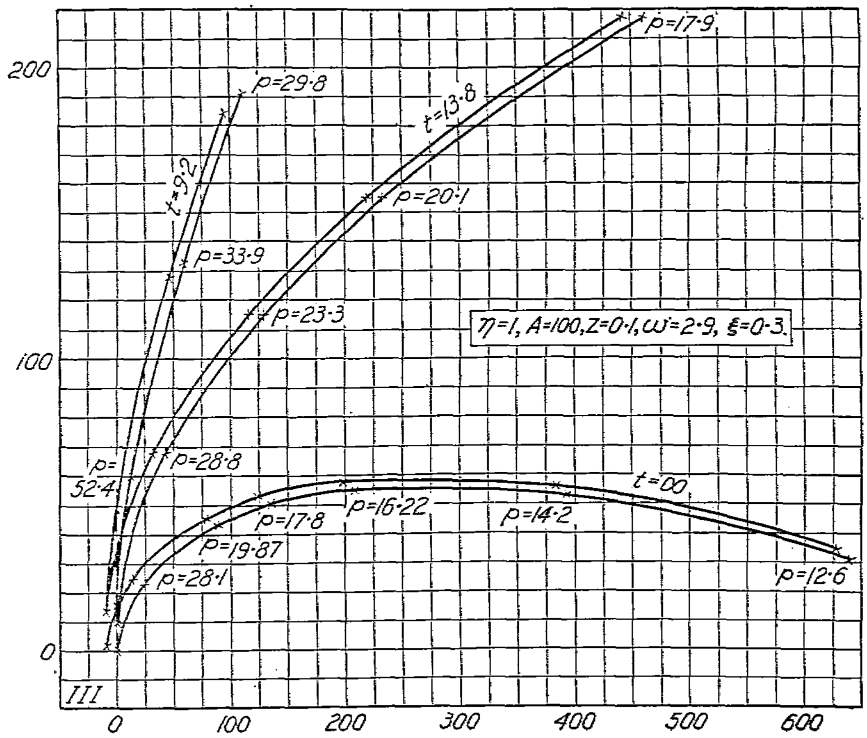
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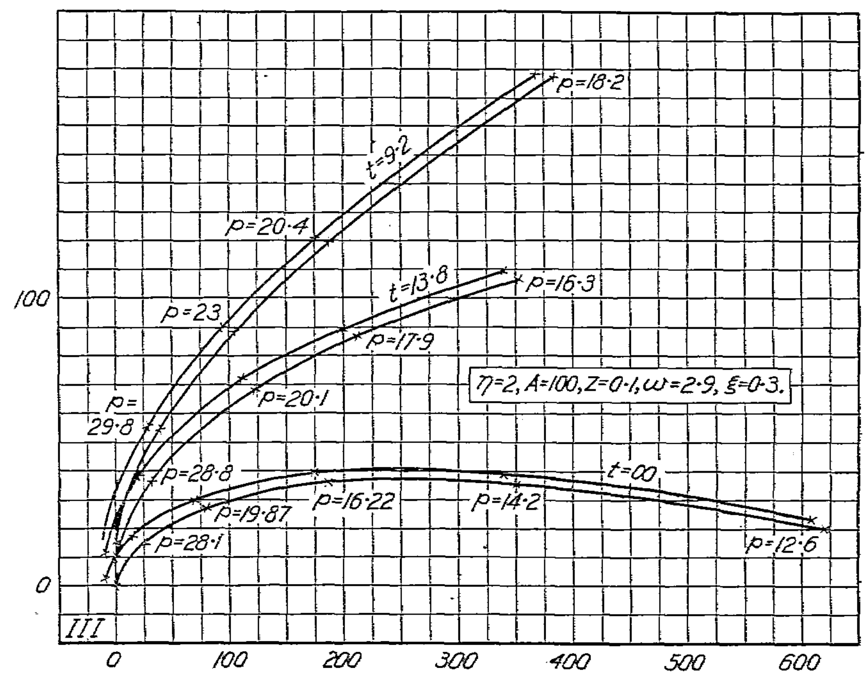
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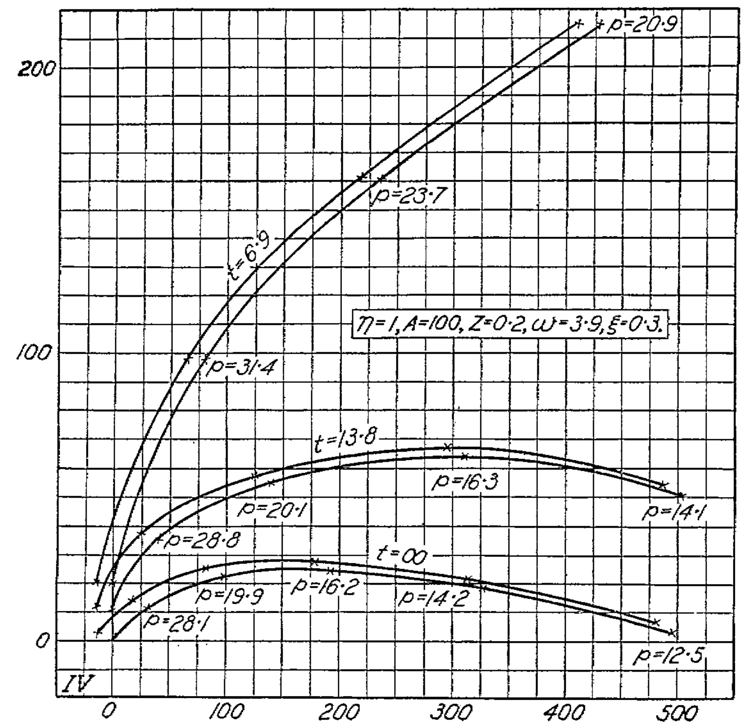
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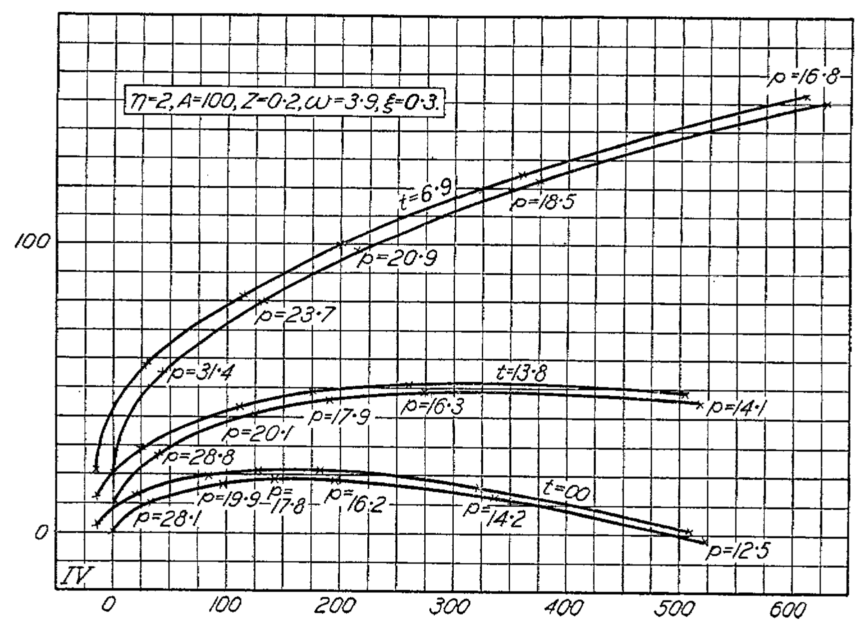
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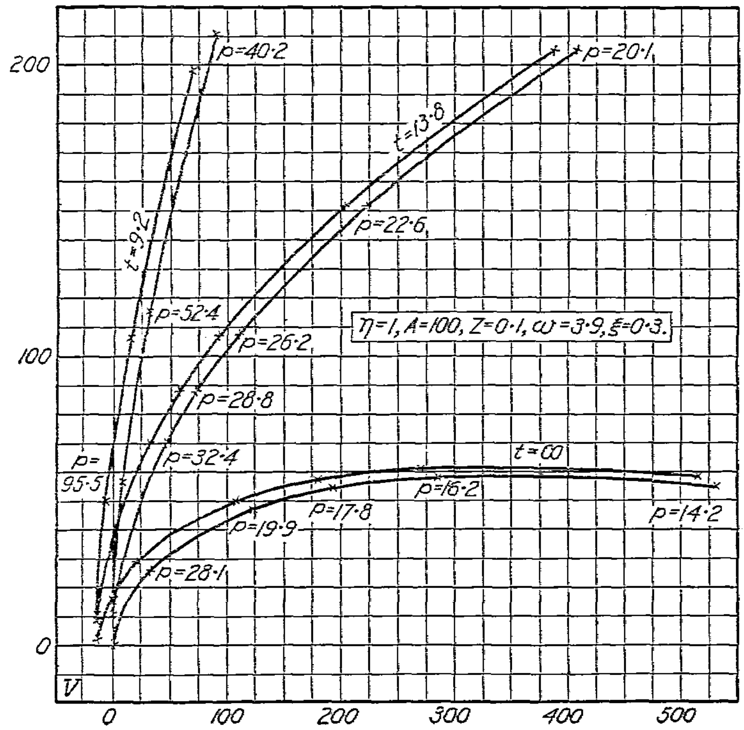
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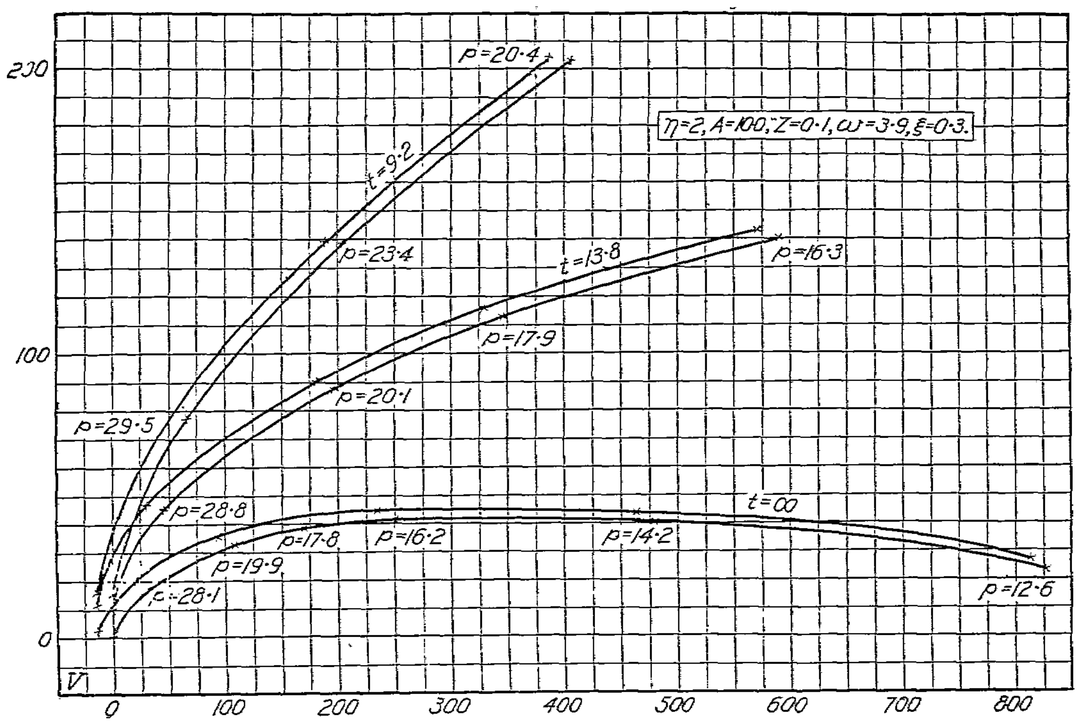
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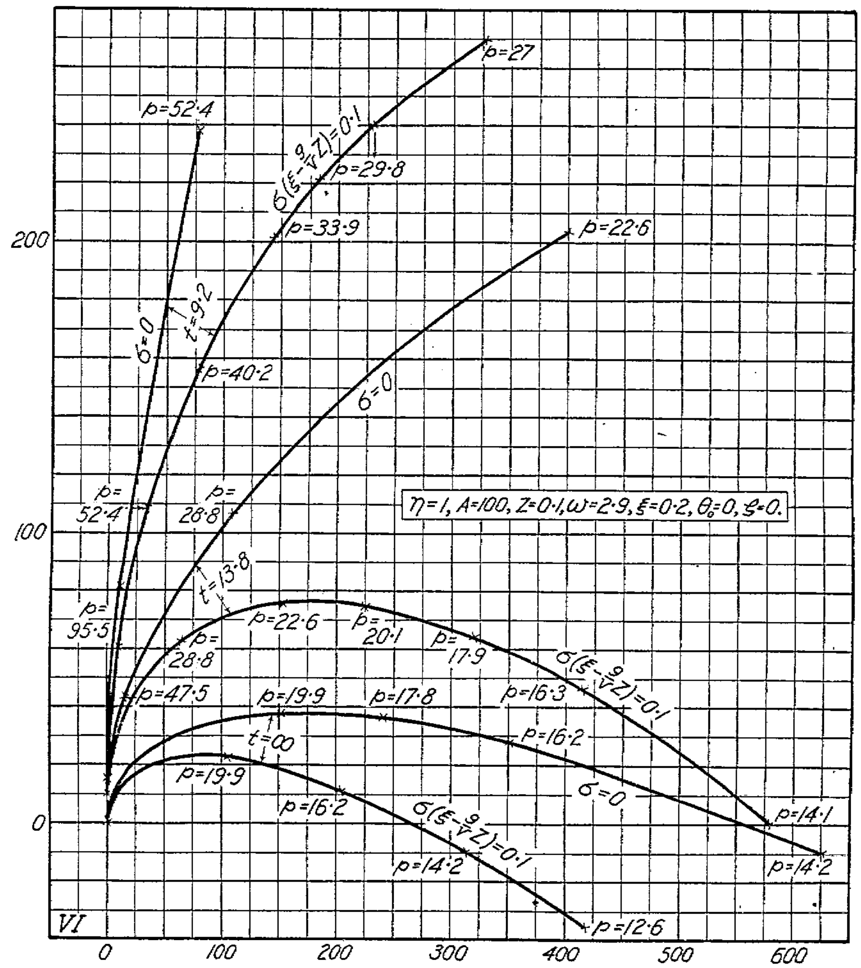
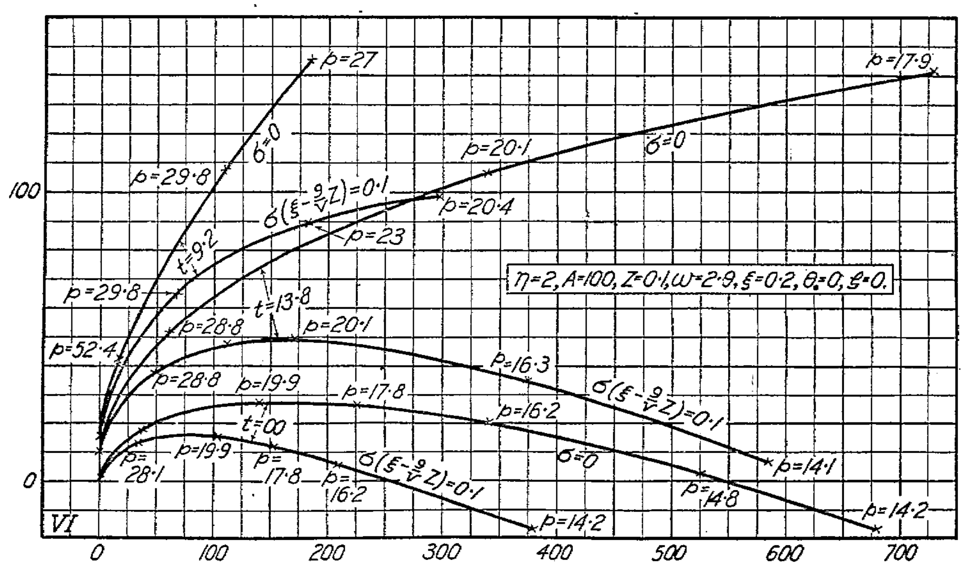
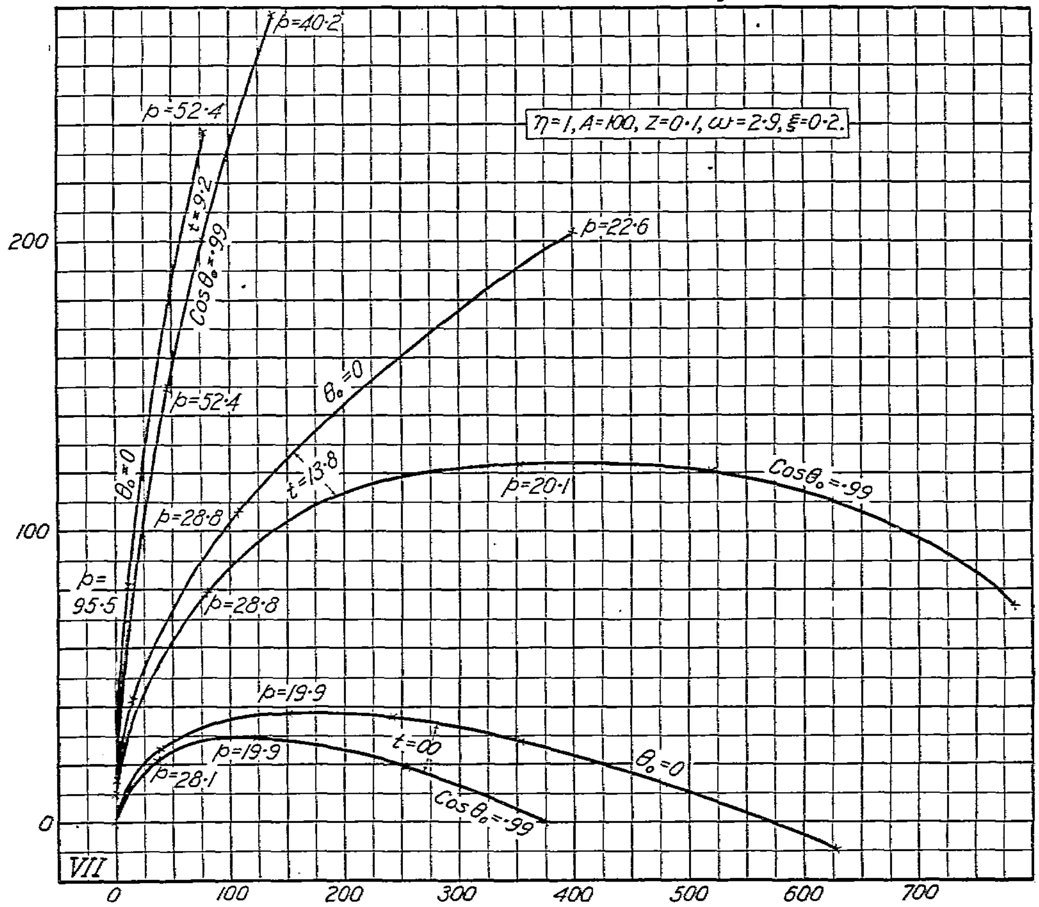
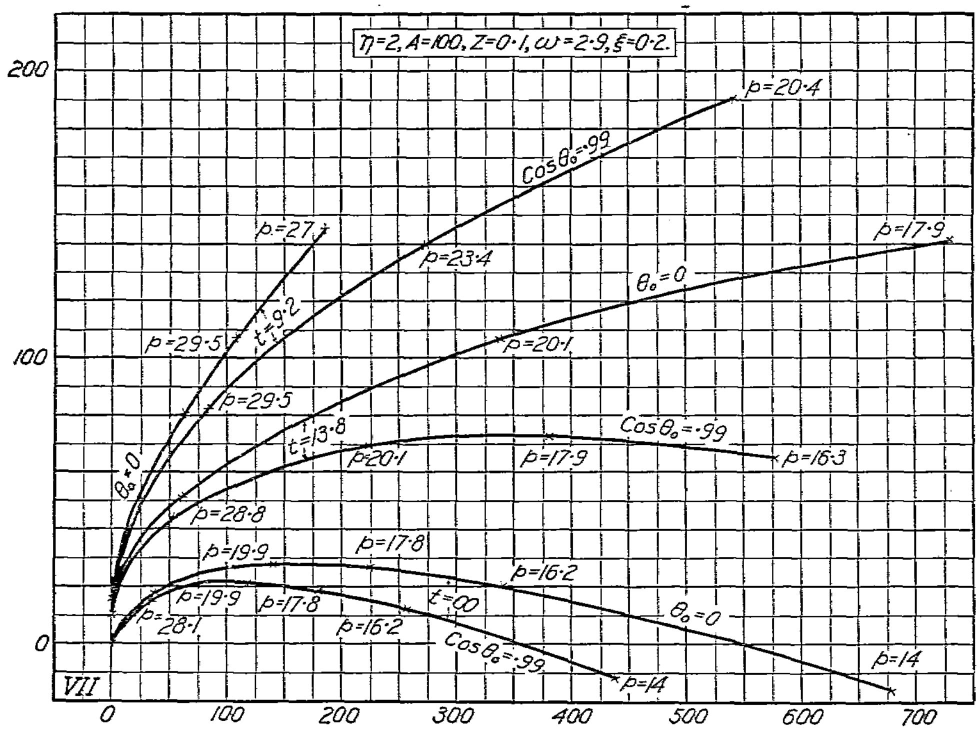
t=9·2
p=23·4
t=13·8
p=16·3
p=17·9
p=20·1
p=29·5
p=28·8
t=∞
p=17·8
p=16·2
p=14·2
p=19·9
p=28·1
p=12·6
200
100
0
V
0
100
200
300
400
500
600
700
800





NOTE: σ is written in place of δ in this diagram.


η=1, A=100, Z=0·1, ω=2·9, ξ=0·2.
p=40·2
p=52·4
t=9·2
Cos θ₀=·99
p=22·6
θ₀=0
p=52·4
θ₀=0
t=13·8
p=20·1
Cos θ₀=·99
p=28·8
p=
95·5
p=28·8
p=19·9
t=∞
p=19·9
θ₀=0
p=28·1
Cos θ₀=·99
VII
200
100
0
0
100
200
300
400
500
600
700


η=2, A=100, Z=0·1, ω=2·9, ξ=0·2.
p=20·4
Cos θ₀=·99
p=27
p=17·9
p=23·4
θ₀=0
p=29·5
t=9·2
p=20·1
p=29·5
t=13·8
Cos θ₀=·99
p=20·1
p=17·9
p=16·3
θ₀=0
p=28·8
p=19·9
p=17·8
p=16·2
p=19·9
p=17·8
t=∞
θ₀=0
p=28·1
p=16·2
Cos θ₀=·99
p=14
p=14
VII
200
100
0
0
100
200
300
400
500
600
700

Lateral oscillations.—When the lateral oscillations are analytically independent of the pitching oscillations the biquadratic equation which determines the temporal characteristics of the oscillations is

$$\lambda(\lambda+x)\ (\lambda+w)\ (\lambda+\eta)+\zeta\lambda(\lambda+x)-\xi y\lambda(\lambda+w)+z(\lambda+\eta)-\xi z\lambda-y\zeta=o \tag{1}$$

where [7]

$$\begin{aligned} &L_p=-xK_A^2, && L_r=-\frac{U}{g}yK_A^2, && L_v=\frac{z}{g}K_A^2 \\ &N_p=-\frac{g}{U}\xi K_C^2, && N_r=-\eta K_C^2, && N_v=\frac{\zeta}{U}K_C^2, \\ &Y_p=o, && Y_r=o, && Y_v=-w. \end{aligned} \tag{2}$$

Writing $\lambda^2-2\alpha\lambda+\theta=o$ and reducing the above equation to a linear one in λ we find, on equating to zero the coefficients of λ and 1, that

$$\begin{gathered} \theta(w+\eta+x+4\alpha)=(x+2\alpha)\ (w+2\alpha)\ (\eta+2\alpha)+\zeta(x+2\alpha)-\xi y(w+2\alpha)+z(1-\xi), \\ \theta^2-\theta[(x+2\alpha)\ (w+\eta+2\alpha)+w\eta+\zeta-\xi y]+z\eta-y\zeta=o. \end{gathered} \tag{3}$$

With the aid of these equations the curves $t=$ constant may be drawn in the (x, y) plane for various sets of values of z, ζ, η, and w. It will be seen from the diagrams that there is a marked difference between the curves for which ξ is positive and those for which ξ is zero or negative.

It has been pointed out recently by Prof. E. B. Wilson [8] that in normal flight at a fairly low angle of incidence the quantity N_p is negative, and that consequently there is no need for a fine adjustment of the values of N_v, L_v, N_p, and L_p in order to secure both spiral stability and stability in the Dutch roll. At larger angles of incidence N_p may be zero and even positive, and then there is more need for a fine adjustment. On this account most of the diagrams have been drawn for zero and positive values of ξ, one diagram being deemed sufficient to indicate the general arrangement of the curves when ξ is negative. It will be noticed that when ξ is negative the period p is rather short but the time of damping t can also be made short and so the short period is not a great disadvantage. By making ξ positive the period can be more than doubled, but this is offset by the lengthening of the time of damping.

It should be noticed that in the diagrams the region of stability is bounded by the curve $t=\infty$ and the line $y=\frac{z\eta}{\zeta}$. If the representative point lies to the left of the curve $t=\infty$, the airplane is unstable in the Dutch roll; while if the point lies above the line $y=\frac{z\eta}{\zeta}$, the airplane is spirally unstable.[9] The shape of the curves $t=$ constant is very much the same for the different values of η, ζ and z used in the diagrams, but the position of the line $y=\frac{z\eta}{\zeta}$ varies considerably.

An examination of diagrams VIII–XII indicates that it should be possible to construct an airplane which, when flying at a large angle of incidence, is spirally stable and has a period for the Dutch roll of about 15 seconds with a fairly short time of damping.

It has been remarked that in a moderately stable airplane the period is not sensitive, comparatively large changes in the airplane having small effect on the period and damping.[10] As an illustration of this phenomenon it is of some interest to consider the lines along which both p and t are constant. Such a line is obtained by choosing α and θ so that the two equations (3) are the same. Writing down the conditions

$$\frac{(w+2\alpha)(\eta+2\alpha)+\zeta-\theta}{-\theta(w+\eta+2\alpha)}=\frac{-\xi(w+2\alpha)}{\theta\xi-\zeta}=\frac{z(1-\xi)-\theta(w+\eta+2\alpha)}{\theta^2-\theta(w\eta+\zeta)+z\eta},$$

[7] Cf. J. C. Hunsaker, Dynamical Stability of Aeroplanes, Smithsonian Miscellaneous Collections (Washington), Vol. 62, No. 5, June (1916), pp. 55–57.

[8] Fourth Annual Report of the National Advisory Committee of Aeronautics. No. 26. Washington (1919).

[9] For these terms see Hunsaker (loc. cit.).

[10] Cf. W. S. Farren, "Full Scale Aeroplane Experiments," The Aeronautical Journal, February, 1919, p. 56.

we have two equations to determine α and θ when z, w, ξ, η and ζ are given. If on the other hand ξ, ζ, w, α, and θ be regarded as given, the resulting equations will determine η and z uniquely. The equation for η is

$$-(\eta+2\alpha)(w+2\alpha)\zeta=\xi\theta^2+\theta[\xi w(w+2\alpha)-\xi\zeta-\zeta]+\zeta^2.$$

If $w+2\alpha$ is small, a comparatively small percentage change in θ means a comparatively large percentage change in the value of η. For example if $w=0.1$, $\zeta=0.6$, $\xi=2$, $w+2\alpha=-.05$,

$$\theta=.27,\ \eta=\ .72,\quad z=1.782$$

$$\theta=.26,\ \eta=\ .97,\quad z=1.794$$

$$\theta=.25,\ \eta=1.233,\ z=1.777$$

In diagram VIII with $\xi=1$, the equation of the straight line along which p and t are constant is $.0884x+.0241y-.3843=o$; and we have $w+2\alpha=-.0241$, $\theta=.4953$, $t=11.12$, $p=8.96$.

It appears from the diagrams that the greater the curvature of the curve $t=$ constant, the more rapid is the variation of p as the representative point moves along it. If the curve is concave to the axis of y, p increases as the point moves up the curve, while, if the curve is convex to the axis of y, p increases as the point moves down the curve.

If we wish an airplane to have resistance derivatives for which the period is not sensitive, a good plan is to bring the representative point as close as possible to the straight line along which p and t are constant. When ξ is negative this line does not seem to appear on the diagram.

A study of diagrams VIII–XII reveals the following properties:

(1) The greater the value of ξ the greater seems to be the effect on the damping of a change in w.

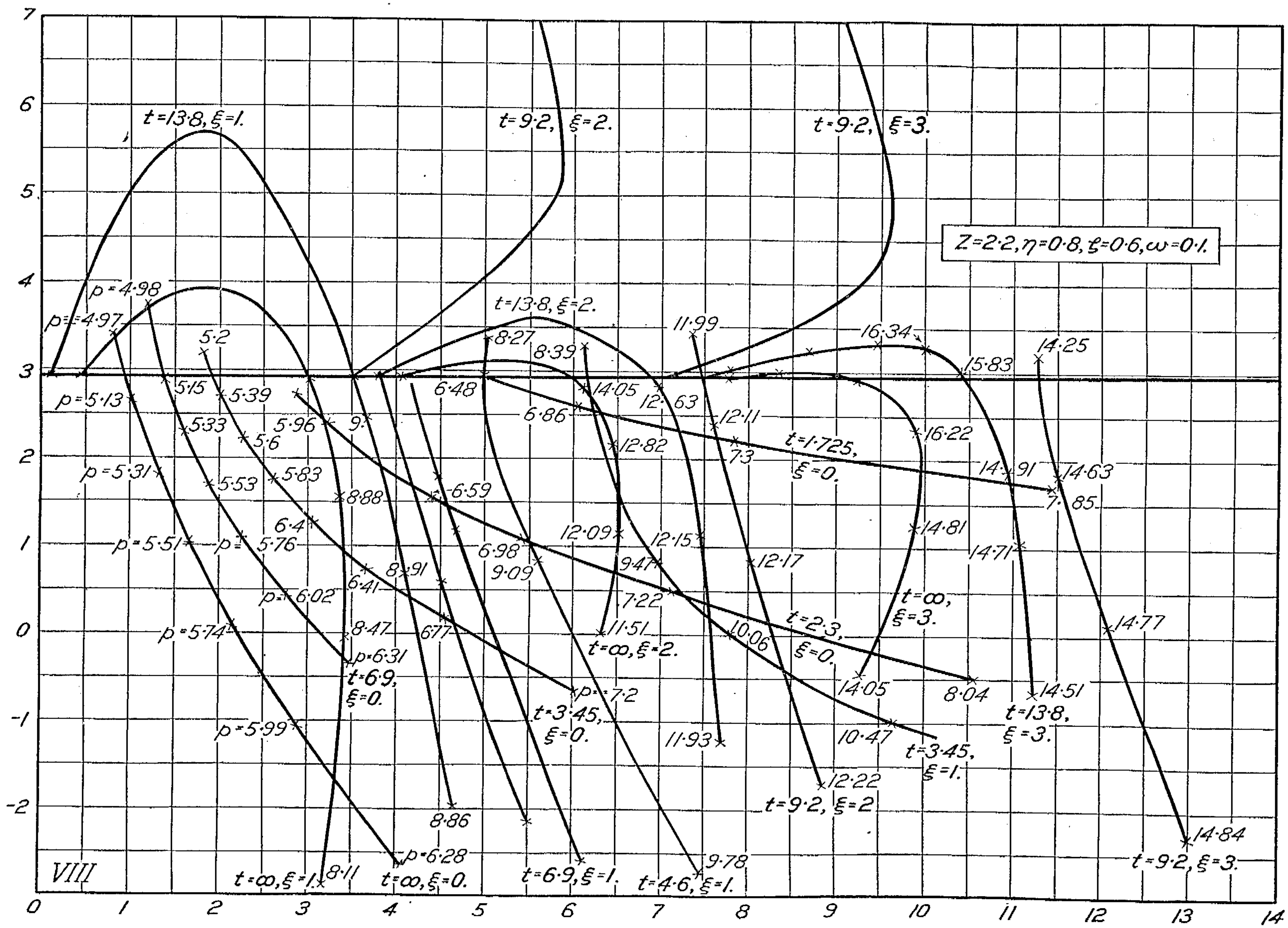
(2) An increase in w decreases the time of damping but does not greatly alter the period.

(3) An increase in η decreases the time of damping and increases the period when $\xi=o$; but, when $\xi=1$, 2, or 3, the effect seems to be reverse.

(4) An increase in z seems to widen the gaps between the curves $t=$constant and to greatly increase the period when $\xi=1$ and $\xi=2$.

It will be noticed that in diagram XII the numbers on the curves $t=$constant indicate the value of $\frac{p}{t}$.

(5) The chief effect of a decrease in ζ seems to be a slight change in curvature of the curves $t=$constant.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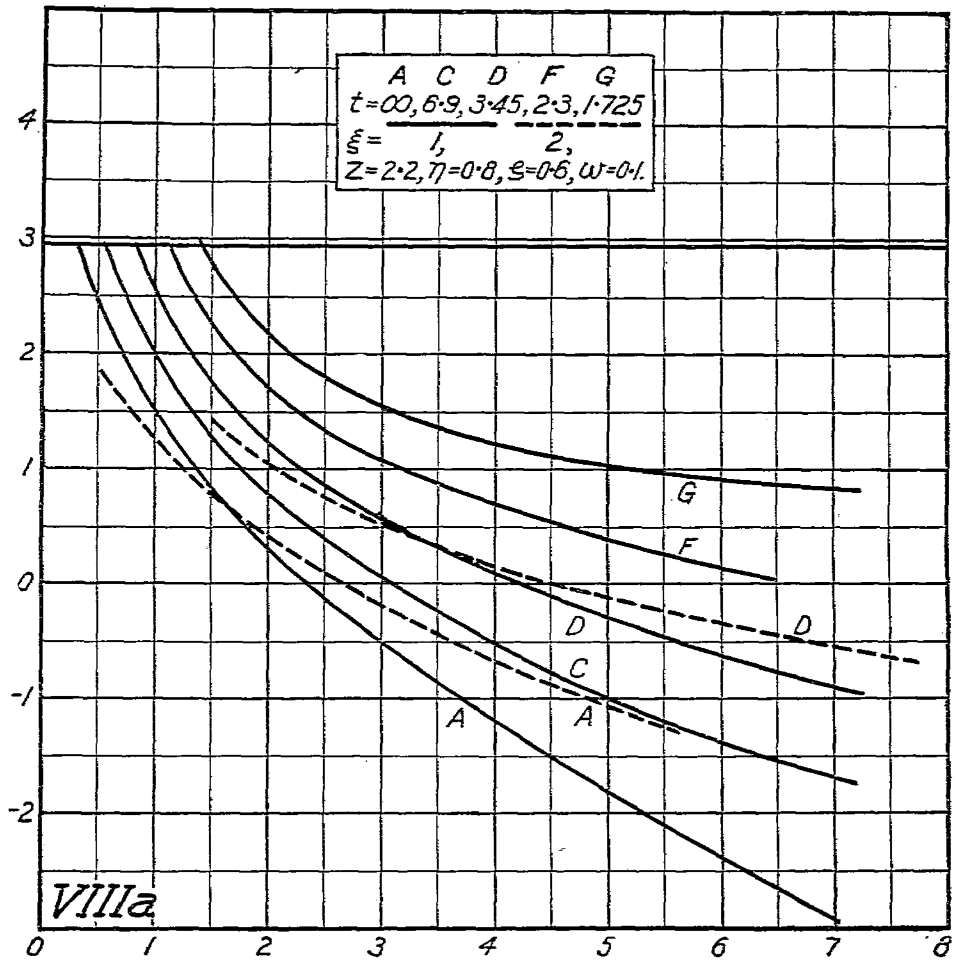
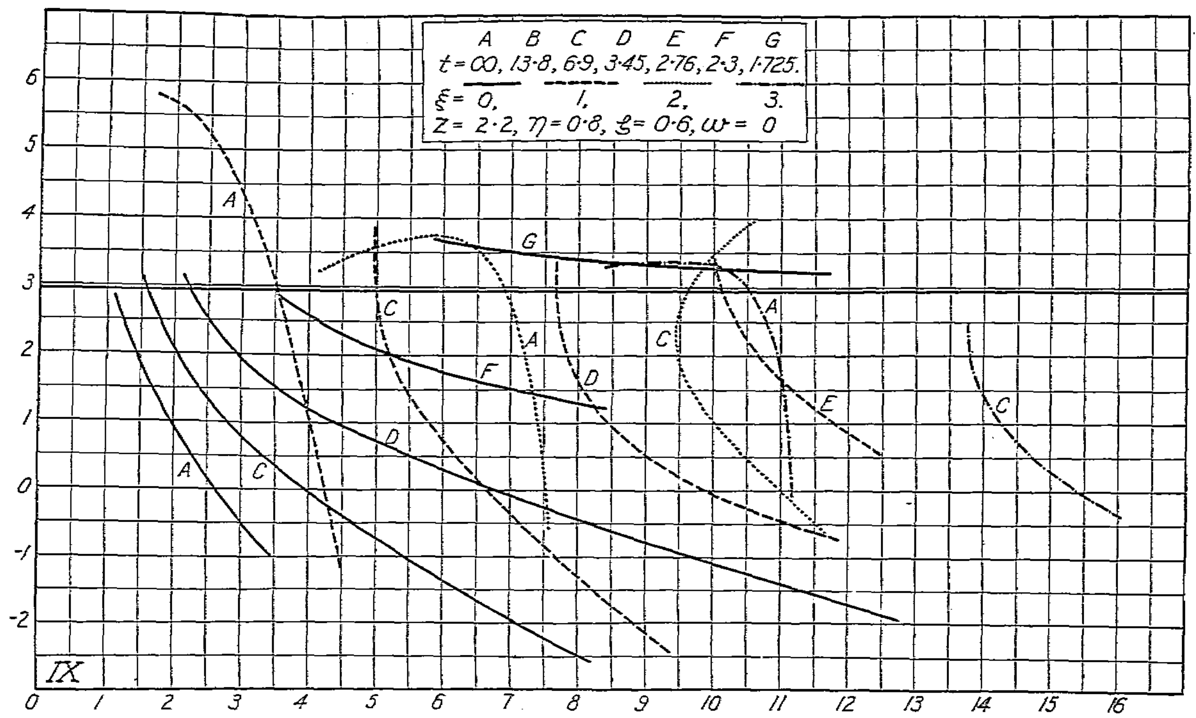
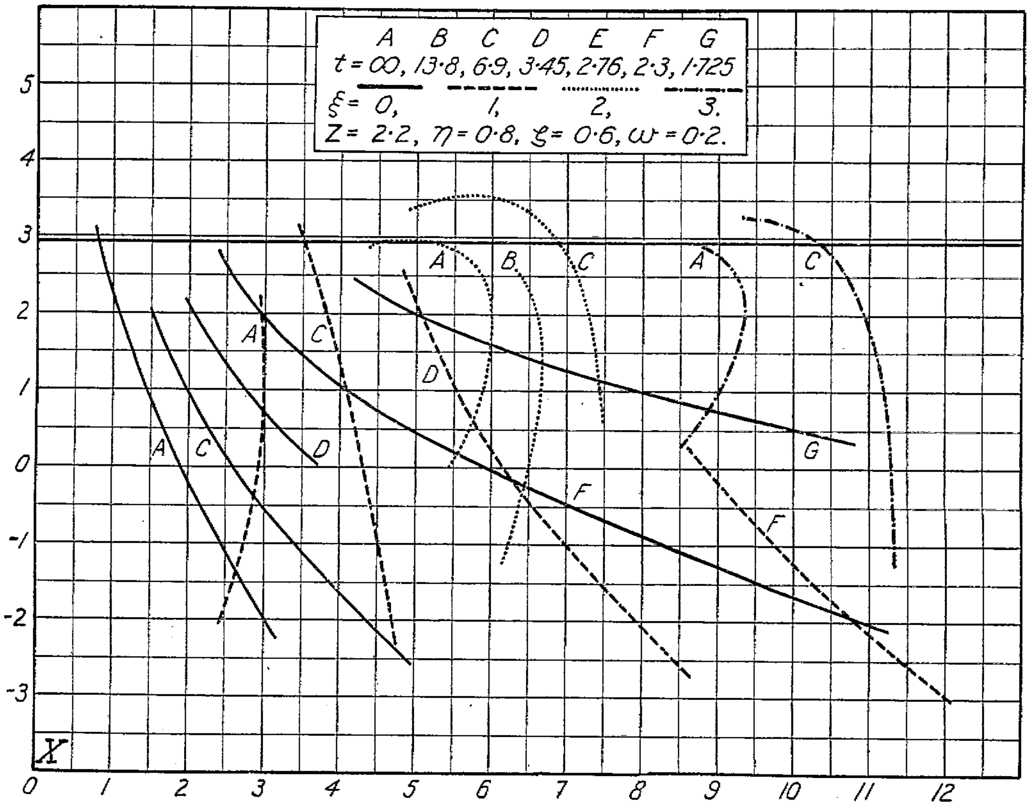
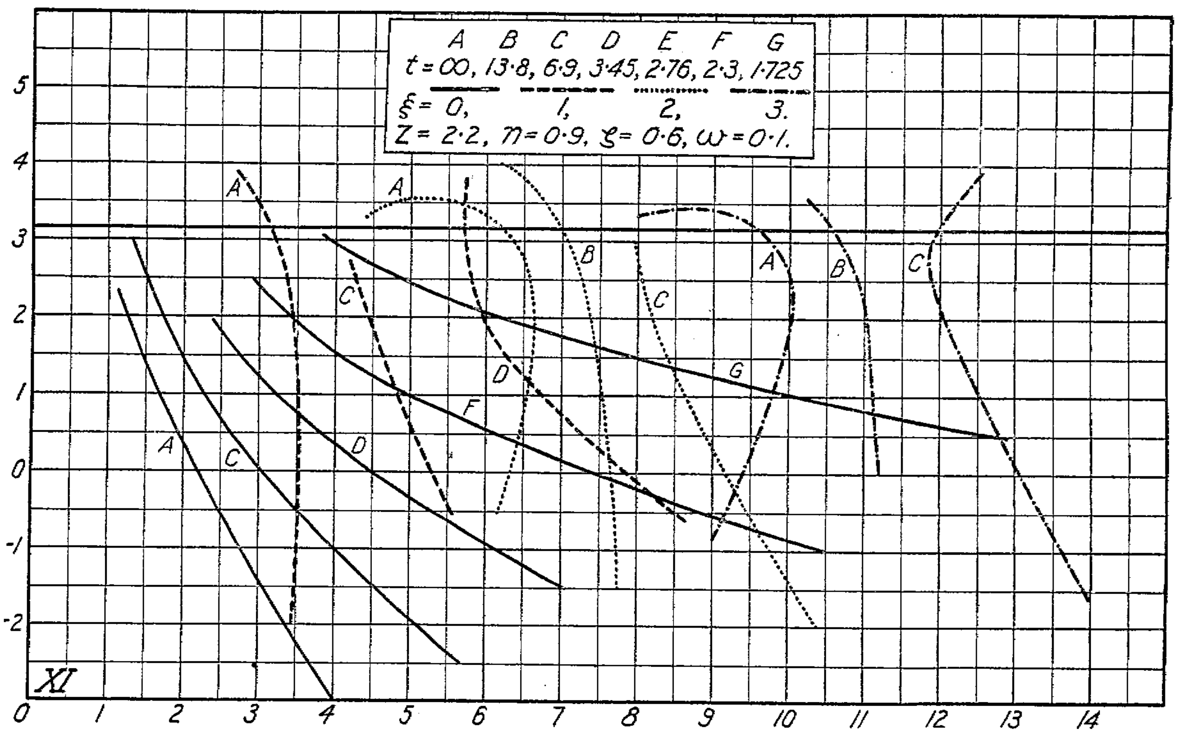
t=9·2, ξ=2.
t=9·2, ξ=3.
t=13·8, ξ=2.
t=1·725, ξ=0.
t=2·3, ξ=0.
t=6·9, ξ=0.
t=3·45, ξ=0.
t=∞, ξ=2.
t=∞, ξ=3.
t=13·8, ξ=3.
t=3·45, ξ=1.
t=9·2, ξ=2
t=9·2, ξ=3.
t=∞, ξ=1.
t=∞, ξ=0.
t=6·9, ξ=1.
t=4·6, ξ=1.
p=4·98
p=4·97
p=5·13
p=5·31
p=5·51
p=5·74
p=5·99
p=6·02
p=6·31
p=6·28
p=7·2
VIII
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
-1
-2

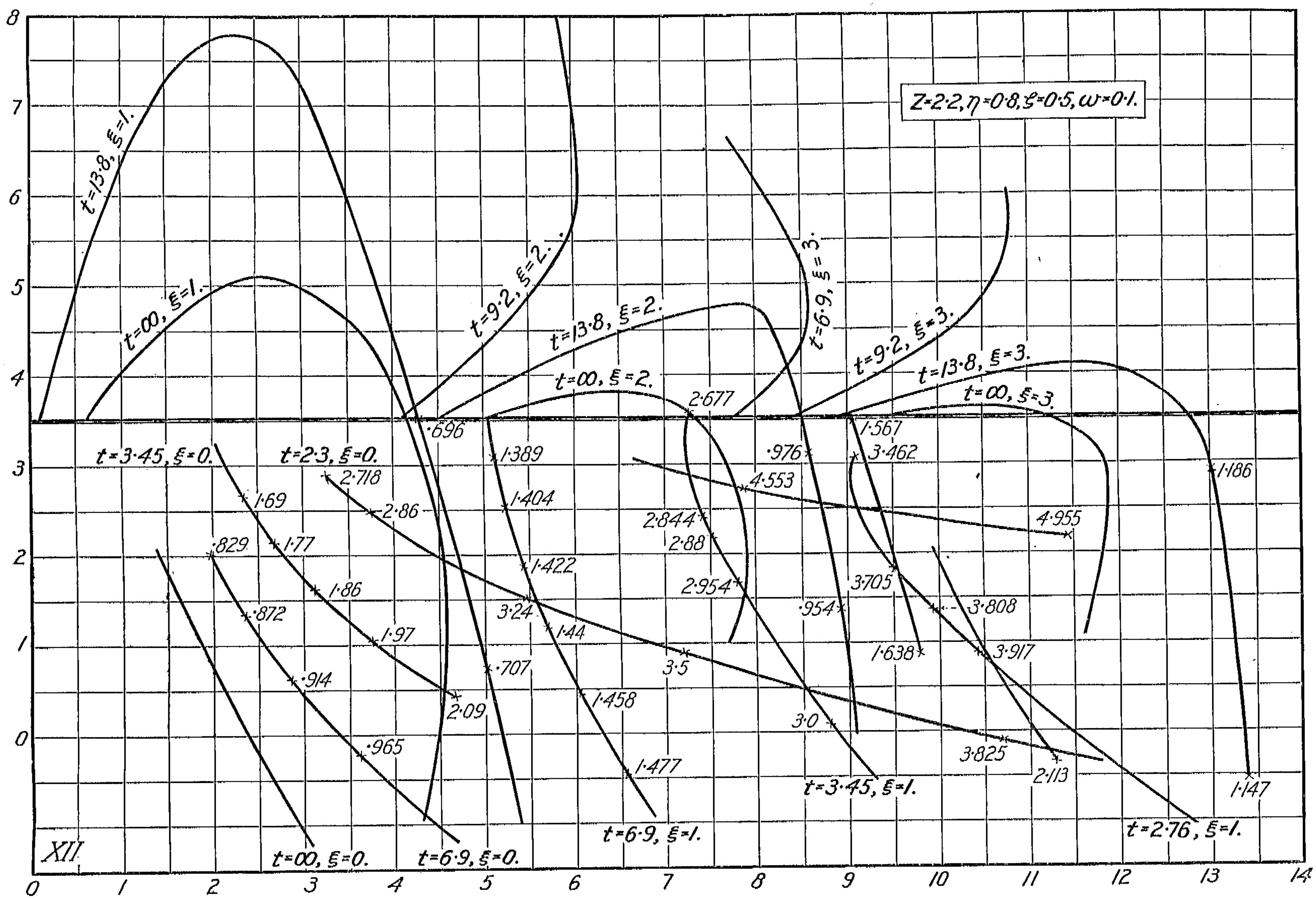

A C D F G
t=∞, 6·9, 3·45, 2·3, 1·725
ξ= 1, 2,
Z=2·2, η=0·8, ς=0·6, ω=0·1.
VIIIa
A C D F G A A C D D

NOTE: ξ=−1; −2.


A B C D E F G
t=∞, 13·8, 6·9, 3·45, 2·76, 2·3, 1·725.
ξ= 0, 1, 2, 3.
Z= 2·2, η=0·8, ς=0·6, ω= 0
IX


A B C D E F G
t = ∞, 13·8, 6·9, 3·45, 2·76, 2·3, 1·725
ξ = 0, 1, 2, 3.
Z = 2·2, η = 0·8, ζ = 0·6, ω = 0·2.
X


A B C D E F G
t = ∞, 13·8, 6·9, 3·45, 2·76, 2·3, 1·725
ξ = 0, 1, 2, 3.
Z = 2·2, η = 0·9, ζ = 0·6, ω = 0·1.
XI


Z=2·2, η=0·8, ς=0·5, ω=0·1.
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t=∞, ξ=1.
t=9·2, ξ=2.
t=13·8, ξ=2.
t=∞, ξ=2.
t=6·9, ξ=3.
t=9·2, ξ=3.
t=13·8, ξ=3.
t=∞, ξ=3.
t=3·45, ξ=0.
t=2·3, ξ=0.
t=∞, ξ=0.
t=6·9, ξ=0.
t=6·9, ξ=1.
t=3·45, ξ=1.
t=2·76, ξ=1.
XII
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

APPENDIX TO REPORT NO. 80.

ABILITY OF THE PARACHUTE AND HELICOPTER.

Note I.

The expressions for H and K in terms of x_1, x_2, x_3, y_1, y_2, y_3 may be found as follows:

Writing

$$f_1(y)=(y+y_1)(y+y_2)(y+y_3)-x_2x_3(y+y_1)-x_3x_1(y+y_2)-x_1x_2(y+y_3),$$

$$f_2(y)=x_1(y+y_2)(y+y_3)+x_2(y+y_3)(y+y_1)+x_3(y+y_1)(y+y_2)-x_1x_2x_3,$$

we find that

$$f_1(y)=\frac{x_1(y+y_1)+x_2(y+y_2)+x_3(y+y_3)}{(x_1+x_2+x_3)^2}f_2(y)-f_3(y),$$

where

$$(x_1+x_2+x_3)^2 f_3(y)=Ax_2x_3(y+y_1)+Bx_3x_1(y+y_2)+Cx_1x_2(y+y_3).$$

The coefficient of y on the right-hand side is the quantity denoted by H.

Again writing

$$f_2(y)=(ay+b)f_3(y)-K$$

where a, b, and K are constants to be determined, we may find the value of K by substituting in the last equation a value of y which makes $f_3(y)=0$. We thus find that

$$H(y+y_1)=x_1(Bn-Cm), \qquad H(y+y_2)=x_2(Cl-An), \qquad H(y+y_3)=x_3(Am-Bl)$$

$$=x_1P \qquad\qquad =x_2Q \qquad\qquad =x_3R$$

$$H^2K=x_1x_2x_3[H^2-QR-RP-PQ].$$

Putting $f_3(y)=o$ in the identical relation connecting $f_1(y)$, $f_2(y)$, and $f_3(y)$ we also find that

$$H^2K(x_1^2P+x_2^2Q+x_3^2R)=[H^2(P+Q+R)-PQR]x_1x_2x_3.$$

Hence

$$H^2K(H+ix_1^2P+ix_2^2Q+ix_3^2R)=x_1x_2x_3(H+iP)(H+iQ)(H+iR)$$

or

$$H^2K\left[H^2+(x_1^2P+x_2^2Q+x_3^2R)^2\right]^{\frac{1}{2}}=x_1x_2x_3(H^2+P^2)^{\frac{1}{2}}(H^2+Q^2)^{\frac{1}{2}}(H^2+R^2)^{\frac{1}{2}}.$$

In this relation the positive signs must be taken for the square roots so that K is positive when x_1, x_2, and x_3 are positive.

Note II.

In the case of a helicopter rising or falling vertically it may be sufficient to take into account the fin action and gyroscopic action of the lifting screw.

Writing

$$Q=o,\ E=X_u,\ F=-iX_r,\ J=iN_u,\ K=N_r+iN_p$$

the period equation becomes

$$\lambda(\lambda+u)(\lambda+y+iz)+x(\lambda w+i)=o$$

where

$$x=\frac{g^2N_u}{A},\ y=g\frac{N_r}{A},\ z=g\frac{N_p}{A},$$

$$u=g\frac{X_u}{W},\ w=\frac{V}{g}-\frac{X_r}{W}.$$

Routh's conditions indicate that for stability the quantities

$$p_1 = y + u,$$

$$H = uyz^2 + (y+u)^2(wx+uy) - x(y+u),$$

and

$$K = \frac{x}{H^2}\left[H^2 - Hyz^2u - (y+u)xy^2z^2\right]$$

must be positive.

On the other hand we find that

$$T = \frac{8x}{(y+u)^2}\left[H^2 - Hyz^2u - (y+u)xy^2z^2\right]$$

so that the conditions $K > o$ and $T > o$ are equivalent.

The effect of gyroscopic action on stability has been estimated for the case of an airplane in rectilinear flight and found to be small.[11] The value adopted for N_p was $N_p - I\Omega$ where I is the moment of inertia of the propeller about the axis of y and Ω its angular velocity about this axis.

With $I = 150$ pounds $-ft^2$ and $\Omega = 2\pi \times \frac{1200}{60} = 125.8$ radians per second, this gave a value of N_p of about $15m$ for an airplane of mass $m = 1{,}300$ pounds. In the case of the helicopter, Ω is smaller than for the airplane propeller but I is very much greater if the diameter of the lifting screw is large. It seems likely, then, that the gyroscopic effect on stability will be greater than in the case of the airplane.

[11] L. Bairstow, B. Melvill Jones, and A. W. H. Thompson, British Advisory Committee's Report. 1912–13, p. 166.