REPORT No. 100

ACCELEROMETER DESIGN

By F. H. NORTON and EDWARD P. WARNER

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INTRODUCTION.

To carry out the work on accelerometry for the Bureau of Construction and Repair, Navy Department, the first step necessary was to study all previous types of accelerometers, with the result that an instrument was developed by the technical staff of the National Advisory Committee for Aeronautics at the Langley Memorial Aeronautical Laboratory for the purpose of recording more accurately than had been done before the accelerations experienced in flight. The errors due to accelerations acting in other than the required direction and the errors due to angular accelerations were studied and as far as possible eliminated. The response of the instrument to shocks of short duration and the damping of the free vibration were analyzed mathematically, as well as the possibility of resonance with the vibration of the plane or engine. The results of this work are included in the present report, together with a description of the actual instrument.

THE PROPERTIES DESIRED IN AN ACCELEROMETER.

The ideal accelerometer should have a natural period very high compared with that of any shocks that it could experience. It should have a large enough deflection to be read with an error of not more than 0.1 per cent of the maximum acceleration, and it should be so damped that it will follow the actual acceleration in the closest manner. It may be contended that such accuracy is unnecessary, as any given maneuver can not be duplicated within several per cent, and the engineer does not need loads to better than 5 per cent. On the other hand, when it is desired to obtain the difference of accelerations in various parts of the airplane in order to calculate the rotary motions, an accuracy of 0.1 per cent is none too great. The accelerometer should only record linear accelerations along one axis, and should be unaffected by any other accelerations. Besides these qualities it should have compactness, ruggedness, and be simple to operate, and the record should be clear and strong and easy to reproduce.

PREVIOUS TYPES OF INSTRUMENTS.

A type of accelerometer developed by Dr. Zahm[1] consists of a number of styluses held against stops a short distance above a moving strip of paper, by springs of different tensions, as shown in figure 1. Each spring is adjusted so that its stylus is brought into contact with the paper whenever the acceleration exceeds a certain amount, so that by having a number of graded springs a curve of acceleration can be traced, as shown in figure 2. This type of instrument will not trace a continuous curve, and it is not practical to have a sufficient number of springs to trace the small and rapid vibrations; but, on the other hand, it has no lag, no natural period, and none of the errors inherent in the free spring instrument. This type of instrument is very valuable for studying the maximum values of landing shocks, but will not, of course, give a continuous curve of acceleration against time.

The R. A. F. accelerometer consists of a semicircular quartz fiber, illuminated by a small incandescent lamp. The deflection of this illuminated fiber is magnified and projected on to a moving film, as shown in figure 3, thus giving a curve of acceleration against time.[2] The natural

[1] Development of an Airplane Shock Recorder, by A. F. Zahm: Jnl. Franklin Inst., August, 1919.
[2] R. and M. No. 376 of British Advisory Committee for Aeronautics, September, 1917.

period of the fiber is about one-twentieth second, and the damping is solely by air friction. This instrument gives a continuous record, but its period and damping are low and the deflection on the record is small, so that it can not be read to better than 0.1 g. Also, as will be shown later, a rather serious error is introduced by components of acceleration acting at other than normal to the plane of the fiber. However, the instrument is very compact and simple to operate, and has been used very successfully to determine the loads on airplanes in flight, but is unsatisfactory for landing shocks.

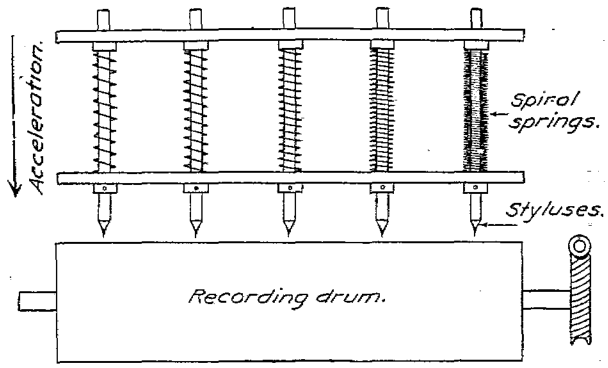

Fig. 1. ZAHM ACCELEROMETER

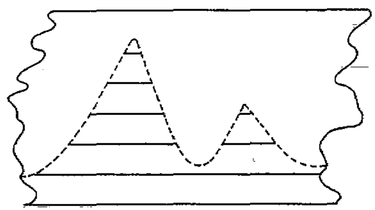

Fig. 2 RECORD FROM ZAHM ACCELEROMETER

A type of accelerometer[3] designed on the principle of a seismograph has been used to determine the properties of automobile springs. This instrument (fig. 4) records displacements so that the curve must be differentiated twice in order to obtain accelerations. This is an inaccurate and laborious method of obtaining accelerations, and it could obviously not be used for anything except short periods. The instrument is of value, however, for obtaining the period and amplitude of small high frequency vibrations.

The National Advisory Committee for Aeronautics' accelerometer consists of a flat cantilever spring, the deflection of whose end rotates a small mirror, thus reflecting a beam of light on to a moving film, as shown diagrammatically in figure 5. A more complete description of this instrument will be given later in this report.

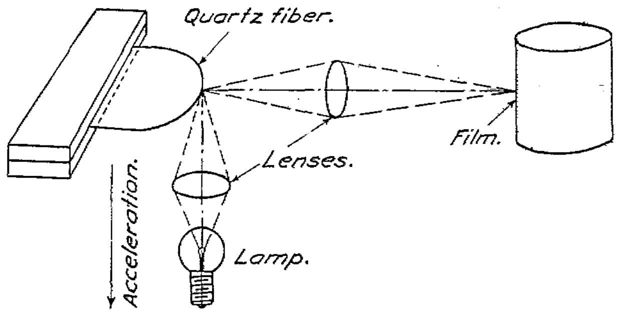

Fig. 3. R. A. F. ACCELEROMETER

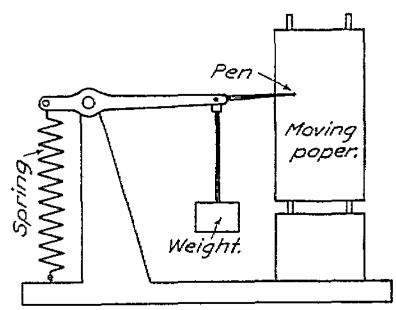

Fig. 4. SEISMOGRAPH TYPE OF ACCELEROMETER

DEFLECTIONS AND NATURAL PERIODS OF SPRINGS.

It is obvious that for an accelerometer the greatest deflection with the highest natural period is desired. Unfortunately the deflection and period of a uniformally loaded flat spring under static load, are connected by the formula:

$$D = K\,T^2$$

where D = deflection

T = time for complete oscillation

K = a constant varying slightly with the type of spring.

[3] The Journal of the Society of Automobile Engineers, January, 1920, p. 17.

This shows that to double the frequency it is necessary to quarter the deflection, so that it is obvious that the lowest frequency should be chosen consistent with accurately following the highest period accelerations to be encountered.

The worst landing shocks on the JN4H rise from zero to a maximum in 0.23 second, and on a DH4 in 0.37 second. On seaplanes the landing shocks will be sharper, but will probably not rise to a maximum in less than 0.02 of a second. On land airplanes the deceleration in landing obviously can not reach its maximum value until the shock absorbers are fully extended. The natural period of the airplane in flight may reach, on light machines, a frequency of 25 vibrations per second, so that in order to avoid resonance the natural frequency should be well above or below this figure. From these figures it seems that a natural frequency of 50 vibrations per second will be ample except for seaplane landing shocks, which may require a frequency of 100, but that the natural period of 20 in the R. A. F. instrument is in some cases too low.

If a frequency of 50 is assumed, the deflection under an acceleration of 1 g. will be in the neighborhood of 0.005 inch—an amount much too small for direct recording. This motion may be directly magnified as in the R. A. F. instrument; but as the object is magnified in the same amount as the motion it does not pay to use a large multiplication and therefore the record is small. In order to magnify this deflection in the N. A. C. A. instrument, a relatively heavy spring is used, so that the mass and friction of the rocking mirror will have no

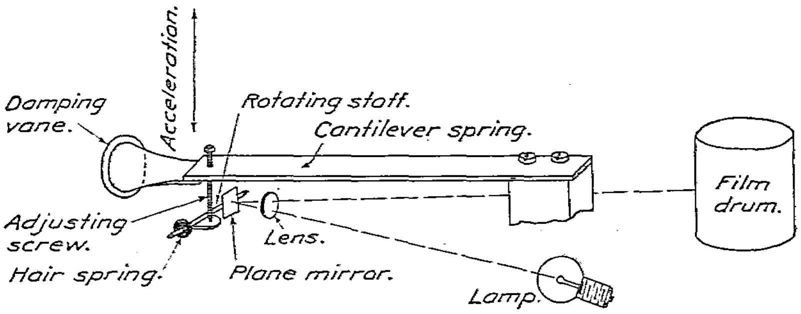

Fig. 5. DIAGRAM OF N. A. C. A. ACCELEROMETER

appreciable effect on the properties of the spring. In this manner the motion of the spring can be easily multiplied 400 times.

In an instrument used in airplanes it is desirable to use as compact a spring as possible; that is, the shortest spring for a given deflection. This leads to the use of a cantilever type of spring, the static deflection of which under gravity is determined by the load distribution, usually a combination of distributed and concentrated loads. The deflection due to weight of the spring is

$$f=\frac{W_1L^3}{8\,EI}$$

and the deflection due a concentrated load at the end is

$$f=\frac{W_2L^3}{3\,EI}$$

where f = the static deflection.

W_1 = the total weight of overhanging portion of spring.

W_2 = concentrated weight at the end of spring.

L = free length of spring.

E = modulus of elasticity of the spring.

I = moment of inertia spring section.

The total deflection is then:

$$f=\frac{W_1L^3}{8\,EI}+\frac{W_2L^3}{3\,EI}.$$

The frequency of any cantilever spring is given by the formula:[4]

$$n=\frac{1}{2\pi}\sqrt{\frac{3\,EIg}{l^3(W+\frac{33}{140}wl)}}\text{ per second}$$

where W = concentrated load at end of spring.
w = weight of spring per unit length.
l = length of spring.
g = acceleration of gravity.
n = the number of vibrations per second.

In the R. A. F. instrument a semicircular quartz fiber is used as a cantilever spring, the deflection being given by the formula:[4]

$$D=Kg\frac{\gamma^4}{A^2}$$

where γ = radius of the semicircle.
A = radius of fiber section.
K = material constant.

The period is given by [5]

$$T=\frac{B\gamma^2}{A}$$

where T = time of one complete period.
B = material constant.

If the deflection of the spring under $1g$ is D_1, and the deflection under Xg is D_2, then the unknown quantity X is given by

$$\frac{D_1}{D_2}=\frac{1g}{Xg}$$

or

$$X=\frac{D_2}{D_1}.$$

The material of the spring should be chosen to give the least hysteresis and the most constant zero. Fused quartz probably has the nearest to any material the desired qualities, but it is hard to obtain in large sizes. Spring steel, hardened and tempered, seems to be very satisfactory, the creep of the zero being negligible. It is quite evident that the spring and the base on which it is mounted should have the same coefficient of expansion when used as on the N. A. C. A. instrument, otherwise the end of the spring will move relative to the mirror axle with varying temperature, thus changing the scale and possibly the zero of the instrument.

RESPONSE OF SPRINGS TO SHOCKS OF VARIOUS DURATIONS.

In order to study the motion of an accelerometer spring when acted on by varying forces, it is necessary to apply the general equation of harmonic motion:

$$\frac{d^2y}{dx^2}+\frac{2K\,dy}{dt}+p^2y=f\sin n\,t$$

the complete solution of which is:

$$y=\frac{f\sin\delta}{2\,Kn}\sin\,(nt-\delta)+ae^{-Kt}\sin\,(qt+\epsilon) \qquad (1)$$

$$\text{where, }\tan\delta=\frac{2\,Kn}{p^2-n^2}$$

$$q=\sqrt{p^2-K^2}$$

a and ϵ are arbitrary constants determined by the initial conditions.

$p=\frac{2\,\pi}{T}$, where T is the natural period of the spring.

$n=\frac{2\,\pi}{\tau}$, where τ is the period of the forcing vibration.

f is the amplitude of the forced vibration.
K is the damping coefficient.

[4] Morley, Strength of Materials, page 454. [5] R&M=No. 376 of British Advisory Committee for Aeronautics, September, 1917.

Assuming that

$$y=0 \text{ when } t=0.$$

then $a \sin \epsilon = \dfrac{f (\sin \delta)^2}{2\,Kn}$ (2)

Differentiating,

$$\frac{dy}{dt}=\frac{f \sin \delta}{2\;K} \cos\;(nt-\delta)+a[-Ke^{-Kt} \sin\;(qt+\epsilon)+e^{-Kt}\,q \cos\;(qt+\epsilon)] \tag{3}$$

Assuming that

$$\frac{dy}{dt}=0 \text{ when } t=0$$

$$\frac{f \sin \delta}{2\;K} \cos \delta - Ka \sin \epsilon + aq \cos \epsilon = 0 \tag{4}$$

$$\text{or } \frac{f \sin \delta}{2\;K} \cos \delta - K\,a \sin \epsilon + q\;\;a \sin \epsilon\;\cot \epsilon = 0 \tag{5}$$

as $a \sin \epsilon$ is known from (2) the value of ϵ and a can be found, thus determining all the constants of the equation.

The magnitude of the dynamical force acting on the spring is given by,

$$F = f \sin\;n\,t \tag{6}$$

By substituting values of t in these equations, the motion of the spring can be plotted. By the use of Fourier's series the effect of a nonharmonic force can be studied in the same manner; but as the variations of the accelerations usually follow sine curves very closely, it was thought that nothing of value would be learned by this method.

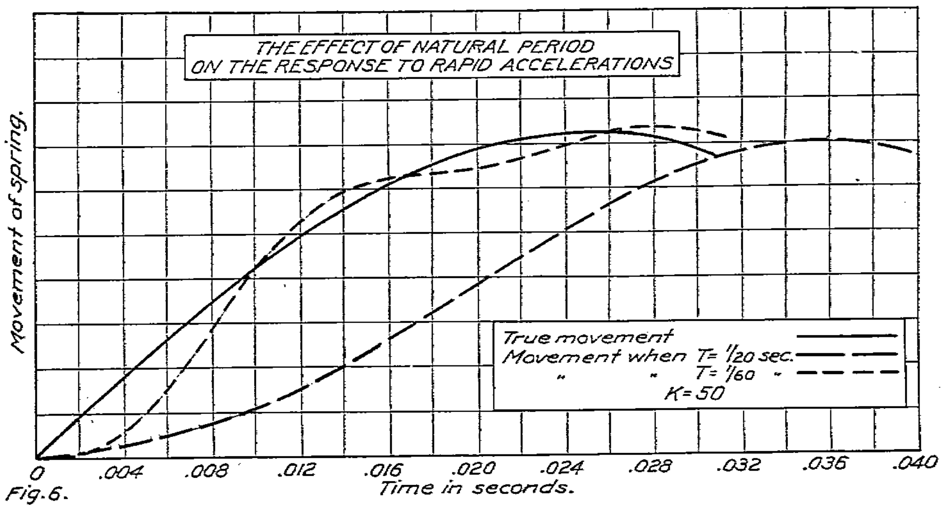

Fig. 6.

THE EFFECT OF NATURAL PERIOD ON RESPONSE TO SHOCKS.

In figure 6 is plotted a curve of acting force rising to a maximum harmonically in 0.025 second. On the same sheet are plotted the motions of two similar springs, except that one has a period of one-twentieth second and the other a period of one-sixtieth second. The higher period spring with six times the period of the acting force oscillates about the true curve, reaching a maximum 0.003 second later and 1.8 per cent higher. The slower spring, with a period of twice the acting force, rises very slowly, reaching a maximum 0.010 second later and 3 per cent lower. These curves show that a spring with a natural period of six times that of the forced vibration will give the height of the maximum within 2 per cent and its time of occurrence within 10 per cent. The curve can, however, be corrected so that it should be possible to obtain the true maximum within 0.5 per cent in magnitude and 1 per cent in time. If a spring of relatively longer period is used the lag increases, but if proper damping is used the recorded maximum will be very nearly correct, until the natural period of the spring exceeds the period of the forced vibration. Whenever possible a natural frequency of at least five times the frequency of the variation of the acceleration should be used.

THE EFFECT OF DAMPING ON RESPONSE TO SHOCKS.

In figure 7 are plotted, first, the same acting force as before rising to a maximum in 0.025 second, and the curves of motion of three springs having a natural period of one-sixtieth second and damping coefficient of 5, 50, and 300. With the lowest damping the oscillation about the true curve is very marked and the maximum reached is 10.3 per cent too high and 0.003 second

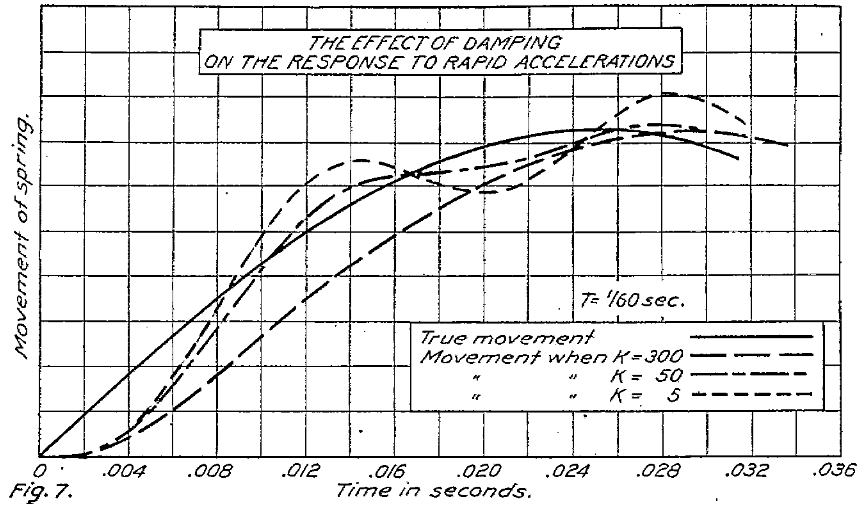

Fig. 7.

too late. With a damping of 50 the oscillations are less marked and the maximum is 1.8 per cent too high. With a damping coefficient of 300, the motion starts slowly, but reaches a maximum of the correct height 0.005 second late. A damping coefficient between 50 and 200 would seem to give the best results. The present N. A. C. A. instrument has a damping factor in the neighborhood of 5, and the R. A. F. instrument has probably a coefficient of even lower magnitude, so that in the new instruments the damping will be increased at least 10 times by using more efficient and more powerful magnets or by employing a liquid dashpot.

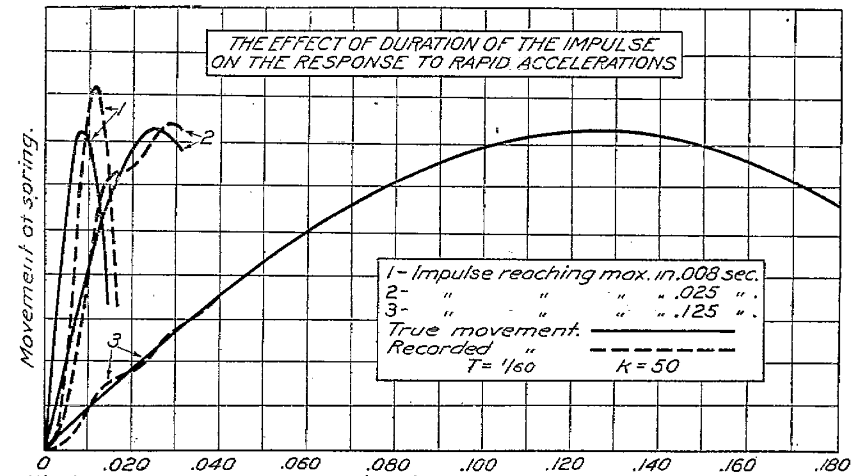

Fig. 8

THE EFFECT OF THE DURATION OF THE SHOCK ON THE MOTION OF THE SPRING.

In order to determine the error in recording shocks of various periods, the motion of a spring of one-sixtieth second natural period and a damping of 50 is plotted when acted upon by impulses reaching a maximum in 0.008, 0.025, and 0.125 second (fig. 8). The height of the recorded maximum increases from the true value for slow impulses to higher and higher peaks as the shock becomes sharper. In order to show this effect more clearly a curve of error in maximum reading is plotted against ratio of natural to forced vibration (fig. 9).

RESONANCE WITH MOTOR AND AIRPLANE VIBRATIONS.

As shown in technical report No. 99, an airplane in the air has a certain fundamental period which is independent of the motor vibration. This period would probably never exceed 25 or 30 vibrations per second for the structure as a whole. It is quite probable, however, that certain portions of the structure would have a secondary vibration of a shorter period. In recording the accelerations on an airplane there are really two problems: The first and the only one that has been dealt with is the acceleration of the center of gravity of the complete machine, due either to air or landing loads, and in this case it is desirable to damp out as completely as possible the high frequency vibrations by means of shock-absorbing supports. On the other hand these high-period vibrations set up local stresses in the structure, and it would be of interest to determine their period and amplitude for different portions of the machine. For this purpose the accelerometer must be rigidly attached to the vibrating part, and, what is rather difficult to accomplish at the present time, the mass of the instrument must be small compared with the mass of the vibrating part.

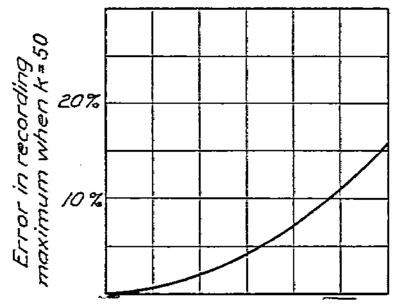

FIG. 9.—Ratio of $\frac{\text{Free period.}}{\text{Forced period.}}$

To return to the problem of measuring the slow accelarations, it is evidently impossible to construct a mounting that will absorb the rapid vibrations, and yet hold the instrument closely to a given position in respect to the machine, so that a compromise is made, and some of the high-frequency vibrations are necessarily transmitted to the instrument while a small part of the maximum shock applied to the airplane as a whole is absorbed by the shock absorbing mounting and so are not recorded.

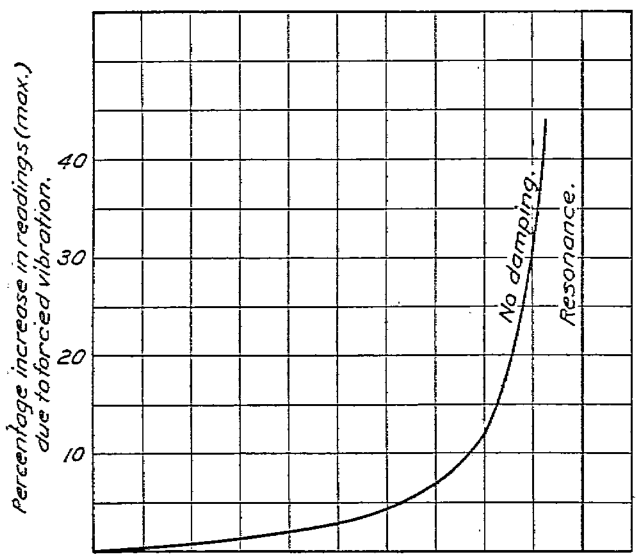

FIG. 10.—Ratio of $\frac{\text{Free period.}}{\text{Forced period.}}$

If there is no damping the errors in the recorded maxima depend on the ratio of the natural to the imposed period, and increase rapidly as the periods approach each other. The nature of the variation of these errors is shown in figure 10. With a damped spring (and in any actual case there is always some damping) the amplitude would not reach infinity at resonance. A simple expression can not be found for the amplitude in terms of the damping, but the error due to an approach to resonant conditions will be decreased nearly proportionally as the damping coefficient is increased. In order to show the effect clearly a curve of acting force and the motion of the spring are shown in figure 11. With a natural period of one-sixtieth second, a forced period of one-thirtieth second, and a damping coefficient of 50, the maximum recorded is not in error by more than 1 per cent; so that it may be concluded that if the natural frequency is not less than twice the forced frequency, no appreciable error will result.

ERRORS DUE TO ACCELERATIONS ACTING AT OTHER THAN NORMAL TO THE PLANE OF THE SPRING.

Flat spring accelerometers are subject to errors due to forces acting along the axis of the spring. Assuming that the spring is normally straight, homogeneous, of constant section, and submitted to an unvarying load, the bending moment due to normal acceleration is:

$$M_x = \frac{-K_x W x^2}{2} \tag{8}$$

where W = the weight per unit length of spring.
K_x = the acceleration, in terms of g, acting normal to the plane of the spring.
x = the distance from the free end of the spring.

Integrating (8) twice gives the deflection at any point as:

$$y_x = \frac{K_x W}{6 E I}\left(l^3 x - \frac{x^4}{4}\right) \tag{9}$$

where E = modulus of elasticity of the spring.
I = moment of inertia of the spring section.
l = free length of the spring.
y_x = normal deflection under K_x, measured relative to the free end of the spring.

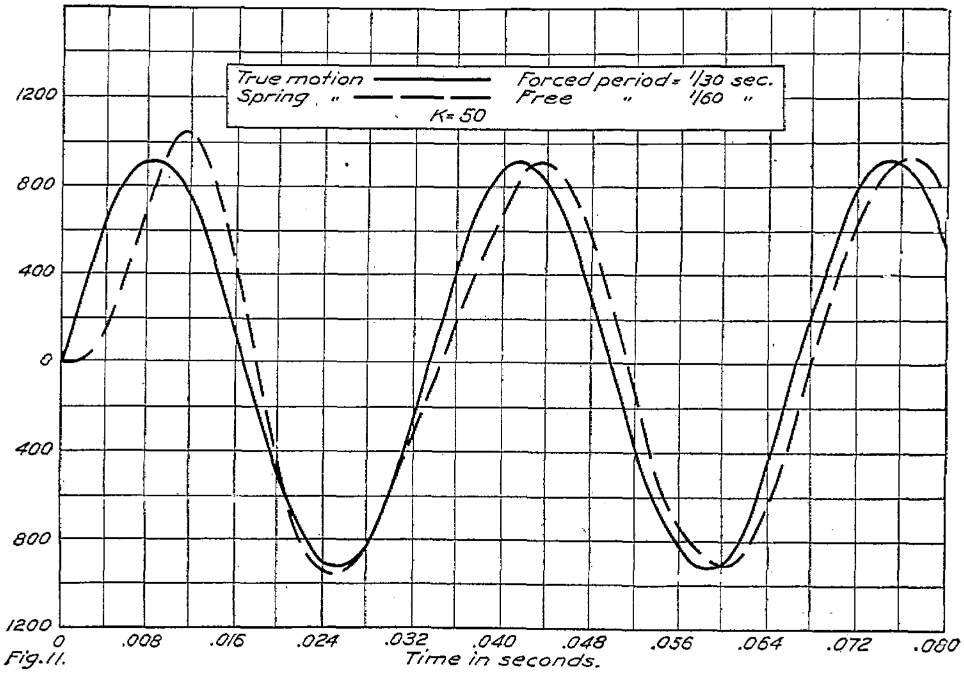

Fig. 11.

The bending moment due to accelerations along the axis of the spring would be zero if the spring was in an undeflected position. If the spring is deflected, the moment arm of an element of mass at any point with respect to an axis of bending at any other point is equal to the difference between the normal deflections of the two points. The bending moment is, then:

$$M_y = \int_0^{x_0} K_y W (y_0 - y)\, dx \tag{10}$$

where K_y = the longitudinal acceleration in terms of g.
Substituting in this equation in the values of y and y_0 found in (9)—

$$M_y = \frac{K_x K_y W^2}{6 E I} \int_0^{x_0} \left(l^3 x_0 - \frac{x_0^4}{4} - l^3 x + \frac{x^4}{4}\right) dx$$

$$= \frac{K_x K_y W^2}{6 E I}\left[l^3 x_0 x - \frac{x_0^4 x}{4} - l^3 \frac{x^2}{2} + \frac{x^5}{20}\right]_0^{x_0}$$

$$= \frac{K_x K_y W^2}{6 E I}\left(l^3 x_0^2 - \frac{x_0^5}{4} - l^3 \frac{x_0^2}{2} + \frac{x_0^5}{20}\right)$$

$$= \frac{K_x K_y W^2}{6 E I}\left(\frac{l^3 x_0^2}{2} - \frac{x_0^5}{5}\right)$$

Dropping the subscript, since x_o can have any value, and integrating twice:

$$y_y = \frac{K_x K_y W^2}{36\ E^2\ I^2}\left(\frac{l^3 x^4}{4} - \frac{x^7}{35} - \frac{4l^6 x}{5}\right)$$

Substituting l for x in the expressions for y_x and y_y, it appears that the deflections of the free end of the spring due to the accelerations in the two directions are:

$$y_x = \frac{K_x W l^4}{8\ E I}$$

$$y_y = \frac{-9 K_x K_y W^2}{560\ E^2\ I^2}$$

$$\frac{y_y}{y_x} = \frac{-9\ W\ l^3 K_y}{70\ E\ I} = \frac{72\ K_y y_x}{70\ K_x \cdot l}$$

The ratio $\frac{y_x}{K_x}$ is a constant for any given instrument, and is equal to the static deflection of the free end of the spring. The ratio of the deflections is, then:

$$\frac{y_y}{y_x} = -\frac{36}{35}\frac{\delta_o}{l}\ K_y$$

where δ_o = the static deflection $\frac{y_x}{K_x}$

Since any change in deflections perpendicular to the plane of the spring gives rise to a change in the deflection due to forces acting parallel to that plane, there is a secondary effect which modifies y_y. If, for example, K_y acts toward the free end of the spring, so that y_y and y_x are in the same direction, the increase of y due to the addition of y_y will itself produce a further increase, and the total effect of longitudinal acceleration will be greater than that given by the first approximation written above. If the two deflections are opposed, on the other hand, the actual value of y_y will be less than that given by the approximate formula. These effects can be allowed for by substituting y_t, the total deflection, for y_x in the above equation, writing:

$$\frac{y_y}{y_t} = \frac{y_y}{y_x \pm y_y} = -\frac{36}{35}\frac{\delta_o}{l} K_y$$

$$\frac{y_y}{y_x} = \frac{-\frac{36}{35}\frac{\delta_o}{l} K_y}{1 \pm \frac{36}{35} \cdot \frac{\delta_o}{l}\ K_y}$$

In the glass fiber instrument devised and used at the Royal Aircraft Establishment l is 1.3 cm. and δ_o ranges from 0.05 to 0.08 cm. For an acceleration of 1g along the axis of the spring $\frac{y_y}{y_x}$ would therefore be 0.05. The accelerometer may be placed with the axis of the spring coinciding either with the X or the Y axis of the airplane. The accelerations along the Y axis certainly never exceed 1g, whereas the computation of the behavior of a JN2 during a loop[6] shows that the longitudinal deceleration of that machine when pulling out of a dive may be as great as 1.92 g. The conditions assumed in this problem were unduly severe, and 1g may be taken as the maximum acceleration along the spring axis to which the accelerometer will be submitted. In the R. A. E. accelerometer an acceleration of this magnitude would produce an error of +5.43 per cent or −4.89 per cent in the determination of the normal acceleration, or a maximum error of 0.23 g for the largest normal acceleration so far recorded. This is considerably greater then the sensitivity of the instrument, and shows that it is not safe to rely on its indications to within 0.1 g.

In the N. A. C. A. accelerometer as originally designed l is 12.7 cm., and δ_o is 0.006 cm. The maximum error due to a longitudinal acceleration of g. under these conditions would be 0.05 per cent, the plus and minus errors being practically identical. This is 0.002 g for the maximum normal acceleration. It is evident that in this instrument the errors due to accelerations at right angles to the one to be measured will be small enough to be neglected with perfect safety.

[6] Forces in Dive and Loop: Bulletin Airplane Engineering Department, U. S. A. June, 1918.

As has already been noted, the assumption of a uniform distribution of weight along the spring does not accord closely with the facts. If it be assumed, as an alternative, that all the weight is concentrated at the free end of the spring, the bending moment due to the normal acceleration is:

$$M_x = - W \cdot K_x \cdot x$$

The deflection due to normal acceleration is found by integrating twice:

$$i = \frac{1}{EI} W . K_x \left(\frac{-x^2}{2} + \frac{l^2}{2} \right)$$

$$y_x = \frac{1}{EI} W . K_x \left(\frac{l^2 x}{2} - \frac{x^3}{6} \right).$$

Proceeding in the same manner as for a distributed load—

$$M_y = K_y W y_x = \frac{W^2 K_x K_y}{EI} \left(\frac{l^2 x}{2} - \frac{x^3}{6} \right)$$

$$i_y = \frac{W^2 K_x K_y}{E^2 I^2} \left(\frac{l^2 x^2}{4} - \frac{x^4}{24} - \frac{5 l^4}{24} \right)$$

$$y_y = \frac{W^2 K_x K_y}{E^2 I^2} \left(\frac{l^2 x^3}{12} - \frac{x^5}{120} - \frac{5 l^4 x}{24} \right).$$

The deflection at the free end, due to longitudinal acceleration, is:

$$y_{y_0} = \frac{- W^2 K_x K_y l^5}{E^2 I^2} \cdot \frac{2}{15}.$$

Then

$$\frac{y_{y_0}}{y_{x_0}} = - \frac{W K_y l^2}{EI} \cdot \frac{2}{5} = - \frac{K_y}{K_x} \cdot \frac{y_{x_0}}{l} \cdot \frac{6}{5}.$$

The error due to longitudinal acceleration in this case is therefore about 17 per cent greater than in the case of a uniformly distributed loading. These cases are the extreme antitheses of each other and the true value of the error in either the R. A. E. or the N. A. C. A. instrument will lie somewhere between the two values found, as both these accelerometers have a tendency to concentrate the active mass near the free end of the springs.

Instruments of the Zahm type, using helical springs, are free from these types of error due to longitudinal accelerations, except in so far as the friction of the stylus is increased. This effect certainly can be neglected if the mounting is carefully made.

THE ERRORS DUE TO ANGULAR ACCELERATIONS.

The effect of angular acceleration appears in two ways. In the first place the spring, no matter where it may be placed, is affected by the angular acceleration as such. Secondly, if the origin of coordinates in the spring does not coincide with the center of gravity of the airplane an angular acceleration about the center of gravity will give a linear acceleration to the spring.

The origin will be taken at the base of the spring as a first assumption, being shifted later to a more convenient and logically chosen location. An angular acceleration of K_a radians per second per second, the base of the spring being assumed to remain stationary, imposes upon every element of length d_x a load—

$$\frac{x . K_a . W . dx}{g}$$

where W is the weight per unit length. The shear at a distance x from the base is, integrating from the free end of the spring to the point in question—

$$S_a = \int_x^l \frac{x . K_a . W . dx}{g} = - \frac{K_a . W}{g} \left(\frac{l^2 - x^2}{2} \right)$$

and the bending moment is—

$$M_a = \frac{-K_a . W}{g}\left(\frac{l^2x}{2} - \frac{x^3}{6} - \frac{l^3}{3}\right).$$

Integrating twice more—

$$i_a = \frac{K_a \cdot W}{gEI}\left(\frac{-l^2x^2}{4} + \frac{x^4}{24} + \frac{l^3x}{3}\right)$$

$$y_a = \frac{K_a \cdot W}{gEI}\left(\frac{x^5}{120} + \frac{l^3x^2}{6} - \frac{l^2x^3}{12}\right)$$

The deflection at the free end is, then:

$$y_{ao} = \frac{K_a \cdot W}{gEI}\left(\frac{11\, l^5}{120}\right) = \frac{11}{15} \times \frac{l}{g} \times K_a \times$$

The direct error arising from angular accelerations is therefore directly proportional to the length of the spring, and the R. A. E. instrument, with its very short spring, would seem to have a marked advantage in this particular. It is, however, evident that a judicious location of the origin of coordinates with respect to the center of gravity of the airplane will introduce linear accelerations, resulting from angular accelerations, which will counterbalance the direct effect of the accelerated rotational motion.

The normal acceleration required to produce a deflection equivalent to that produced by the angular acceleration, K_a would be of the magnitude—

$$K_x = \frac{11}{15} \times \frac{l}{g} \times K_a$$

where K_x is expressed in terms of feet per second per second. It is then evident that if the center of gravity of the airplane lies in the plane of the spring, and eleven-fifteenths of its length from its base there will be no deflection of the free end of the spring due to angular accelerations, and the two manners in which the effects of such accelerations appear just canceling each other. If the weight is concentrated at the tip of the spring, instead of being uniformly distributed along its whole length, the free end should obviously be at the center of gravity. Compromising between the two conditions it may be said that, for the accelerometers now in use, the location of the mounting should be such that the center of gravity lies from 75 per cent to 80 per cent of the way out along the spring.

If the center of gravity is not at the point thus defined there are, as has already been pointed out, two possible sorts of error. The first of these is the error due to angular acceleration when the center of gravity lies in the axis of the spring. By properly choosing the origin in the spring the direct effect of angular acceleration can be eliminated, and the total effect can be reduced to that of a linear acceleration given (in terms of g) by the expression:

$$K_x = \frac{K_a \times d}{g}$$

where d is the distance from the center of gravity of the airplane to a point in the spring and 75 per cent of its length from the base.

The analysis of the "loop problem," already mentioned, showed that, under the conditions assumed, the angular acceleration about the Y axis has a maximum value of 10.5 radians per second per second at the instant when the elevator was pulled up, the angular acceleration falls to 2.4 radians per second per second in 0.3 second, and that it never rises above 1.0 radian per second per second after 0.9 second until the loop is completed. The assumption made in this analysis, that the elevator is pulled up instantaneously, is, of course, much too severe, and it is probable that 6.5 radians per second per second is the largest acceleration in pitch that an airplane would ever have to undergo. Experiments on the rolling moment due to the ailerons suggest that the acceleration about the X axis has a maximum value, on small and medium sized airplanes, of about 5 radians per second per second.

If d_x, d_y, and d_z be the projections on the three axes of the distance from the origin of coordinates in the spring to the center of gravity of the airplane, it is evident that the maximum error due to acceleration in roll is 0.15 g when d_y is 1 foot, and that the similar error arising from the pitching motion is 0.20 g when d_x is 1 foot. Corrections can be made, using estimated values for the angular accelerations, which can be relied upon to reduce these errors by about 60 per cent. In order that the normal accelerations, thus corrected, may be accurate within 0.05 g., the value of d_x must be less than $7\frac{1}{2}$ inches and d_y must not exceed 10 inches.

Another source of error is the centripetal acceleration due to angular velocity. In the usual case, where the accelerations along the Z axis are being measured, centripetal accelerations arise whenever there is any rolling or pitching motion if the active mass of the instrument is above or below the center of gravity. The acceleration is, of course,

$$K_x = \frac{\omega^2 d_z}{g}.$$

The loop analysis showed a maximum angular velocity of 1.81 radians per second occuring 0.4 second after the elevator was pulled up. Since the theoretical time for completing the loop was about a third less than actual measurement shows to be required, this maximum is about 50 per cent too high, and the true maximum may be taken as 1.2 radians per second. Since it is reported that an airplane can be rolled onto its back in 3 seconds, the maximum rolling velocity must be about 1.5 radians per seconds. The error when d_z is 1 foot would then be 0.07 g. As in the case of angular accelerations, approximate corrections can be made, and the error reduced by at least 50 per cent. To keep the corrected value of the normal acceleration within 0.05 g. of the truth, d_z must not exceed 16 inches. This condition is easy to realize, and it will usually be found that the best place for mounting an accelerometer is directly above or below the center of gravity.

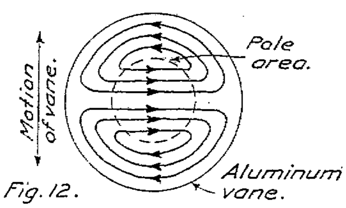

Fig. 12.
EDDY CURRENTS IN DAMPING VANE

MECHANICAL CONSTRUCTION.

As it is desirable to obtain as large a deflection on the film as possible with a high frequency, some device must be used for magnifying the of the end of the spring. In the N. A. C. A. instrument a fairly heavy spring is used, and the motion of its end is transmitted to a very light staff, mounted in hardened-steel sockets. The staff has a small horizontal platform on which a pointed screw from the end of the spring rests, and a thin plane mirror is mounted on the staff and reflects a beam of light through a lens onto a moving film. A watch hair-spring attached to the staff holds the platform tightly against the pointed screw, as shown in figure 5. The moment of inertia of the moving parts are so low that they do not appreciably affect the period of the spring, but they do increase the damping due to pivot and air friction. The mirror staff sockets are mounted on a steel base, which runs under the spring and is rigidly fastened to it at its fixed end.

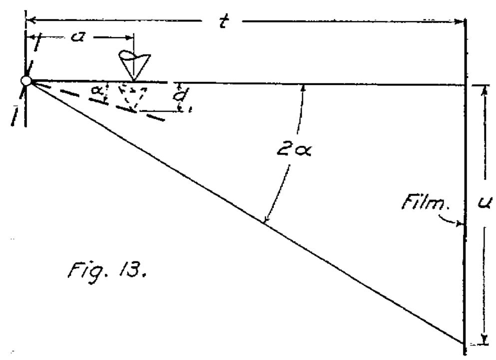

Fig. 13.

The damping of the spring is accomplished by attaching a light aluminum or copper vane to the free end of the spring and allowing it to vibrate between the poles of an electromagnet. When any conducting plate is moved across the lines of force, as shown in figure 12, a current is induced in the plate in such a direction that it tends to oppose the motion of the plate. In order to obtain the maximum damping from a given weight of magnet the vane must be thick enough to carry a heavy current, and it should have ample area outside the magnetic field for the return flow. The air gaps can be reduced to 0.005 inch if the magnet frame is stiff enough to prevent the poles from drawing together. The damping of the N. A. C. A. instrument is not nearly high enough, and in the new instruments the damping will be increased at least 10 times by larger magnets and improved design of the vane.

The scale of accelerations on the film is not quite uniform, as is made clear in figure 13. Let the pointed screw (s) on the end of the spring rest on the platform (p), which is assumed to be initially horizontal, and the light beam at the same time is horizontal. If the screw (*s*) is now deflected a distance *d* vertically downward the angle of rotation of the staff *a* is given by—

$$\alpha = \tan^{-1}\frac{d}{a}$$

and the deflection on the film in respect to the deflection at the end of the spring is:

$$u = t\tan\left(2\tan^{-1}\frac{d}{a}\right) = \frac{2t\frac{d}{a}}{1-\left(\frac{d}{a}\right)^2}$$

Where d is the deflection of the spring,

a = the moment arm,
t = the distance from the mirror to the film,
u = the deflection on the film,
d = the deflection on the spring.

The ratio of proportionality at any point is then

$$\frac{du}{d\left(\frac{d}{a}\right)} = \frac{2t\left[1-\left(\frac{d}{a}\right)^2+2\left(\frac{d}{a}\right)^2\right]}{\left[1-\left(\frac{d}{a}\right)^2\right]^2} = \frac{2t\left[1+\left(\frac{d}{a}\right)^2\right]}{\left[1-\left(\frac{d}{a}\right)^2\right]^2}$$

It is therefore desirable, in order that the scale may be as uniform as possible, that $\frac{d}{a}$ should be kept as small as possible without making the record inconveniently small.

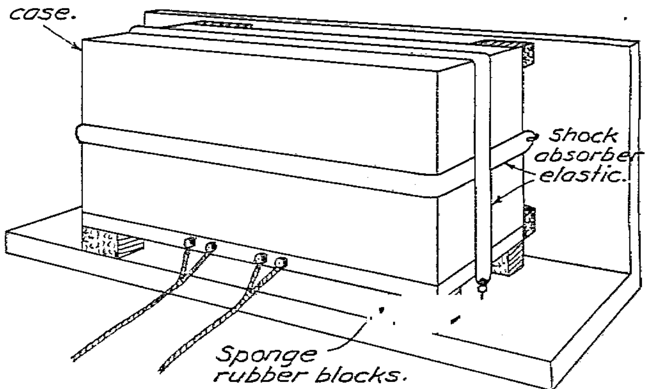

Fig. 14.
METHOD OF MOUNTING ACCELEROMETER.

It is quite essential on an accelerometer record to have an accurate time scale, as in many cases the duration of an acceleration is quite as important as its magnitude. There are two methods of accomplishing this—first, and most satisfactory, is to run the film at constant speed; and second, to run the film at any speed and impress on the record a line or dot at definite time intervals by means of a clock-controlled light or by a tuning fork. This latter method gives a very accurate time record, but it is indirect and awkward to use. Satisfactory methods are discussed in N. A. C. A. Technical Note No. 22 for driving the film at very nearly constant speed, and sufficient synchronization can be obtained between various instruments by connecting all the lamps to one switch, so that the records are started and stopped simultaneously.

It is desirable to either carry a large amount of film, as in the R. A. F. accelerometer or to be able to change light tight film drums in the same way as plate holders on a camera. This latter method is preferable as it separates the records and makes identification simpler, as described in N. A. C. A. Technical Note No. 22.

It is quite important that the accelerometer should be mounted in such a way that the small high-frequency vibrations of the airplane are not directly transmitted to it. The instrument can be very well insulated from shocks by holding the instrument in the hands, but as this is rather a makeshift, and as it can not be easily held near the center of gravity of the machine, a more permanent mounting is desirable. The most satisfactory mounting for an accelerometer is that shown in figure 14. Several blocks of sponge rubber about 1½ inches thick are placed under and at the back of the instrument and shock-absorber elastics hold it firmly against these blocks, thus allowing the instrument to move only a few millimeters, and yet absorbing the shocks satisfactorily.

OTHER USES OF THE ACCELEROMETER.

Up to the present time the accelerometer has been used only in airplane work, but there are many other problems that could be studied to advantage with this type of instrument. Perhaps its most obvious use is in the design of automobile spring suspensions, as the riding comfort of a car can be accurately recorded. A few records of this type have been taken with the N. A. C. A. instrument, and every oscillation of the car springs can be seen clearly on the record. The riding qualities of tires could also be studied by mounting the accelerometer directly on the axle and running over a definite obstacle. The engine vibration can be studied in the same way, and the merits of various types of motors easily compared. A longitudinal accelerometer would be a convenient method of measuring the pickup and braking power of automobiles, and a lateral accelerometer would record the side load on the tires when rounding curves.

In the same way the riding qualities and stresses in steam and electric cars could be studied, particularly as to types of rail joints and switches, and the banking of curves. Another use that would be more interesting than valuable is the study of amusement devices, such as roller coasters, in order to furnish data for new design and for advertising purposes.

If a high-period accelerometer is fastened rigidly to a machine which is out of balance, such as a gasoline motor, a curve will be obtained showing the unbalanced component acting normal to the accelerometer spring. By analyzing this curve not only the amount of the forces can be measured, but also the components that make up the unbalanced force can be separated and studied. The N. A. C. A. accelerometer is so sensitive to slight vibrations that the shock of hammering or a slamming door in a distant part of the building is distinctly recorded, and the effect of anyone walking even lightly in the same room is quite evident. By making the instrument much more sensitive, which could easily be done, it might be used for studying building vibrations due to heavy machinery, such as printing presses, and to detect very slight tremors in any structure.