REPORT No. 114

SOME NEW AERODYNAMICAL RELATIONS

BY

MAX M. MUNK
National Advisory Committee for Aeronautics

REPORT No. 114

SOME NEW AERODYNAMICAL RELATIONS

By Max M. Munk

RÉSUMÉ

This report was prepared for the National Advisory Committee for Aeronautics, and is a series of three notes, designed to extend the modern theory of aerodynamics and to develop it so that it may be applied to certain special problems in some later papers.

The motion of solid bodies in contact with each other is influenced by friction; but, nevertheless, it is often desirable to neglect this, and to make the necessary corrections later. Similarly, in treating the motion of a solid body through a fluid, it is desirable to begin with the case of a motion in which friction is neglected—i. e., with motion in a nonviscous fluid.

When two bodies are moving in a fluid, the disturbance in the fluid produced by one causes forces to act upon the other; and similarly, when there is but a single body in motion, any portion of it experiences a force due to the disturbances produced in the fluid by the other portions of the body. When the fluid is a nonviscous one, these forces may be calculated; and, from analogy with the phenomena of electrodynamics, they are called "induced" forces.

In the diagram let F be the *air* force acting upon the aerofoil; let v be the relative velocity of a particle of air close to the aerofoil; let V be the velocity of flight. Drawing the last two as vectors, it may be proved that F is perpendicular to the vector v and that the downwash w is in the line of F, so that $\bar{w}=\bar{V}-\bar{v}$, as shown. Further, I have proved that

$$w=kv$$

where $k=\frac{F}{\frac{1}{2}\pi b^2 \rho v^2}$, b being the span of the aerofoil and ρ the density of the fluid.

Remembering that the dynamical pressure $q=\frac{1}{2}\rho v^2$, this may be written $k=\frac{F}{\pi b^2 q}$.

The lift is the component of F perpendicular to V; and the drag is the component parallel to V. That is:

$$L=F\frac{v}{V}$$

$$D=F\frac{w}{V}=\frac{w}{v}L=k\,L$$

As a first approximation in the calculation of D let us write $L=F$—i. e., identify v and V; hence $k=\frac{L}{\pi b^2 q}$; and therefore

$$D=\frac{L^2}{\pi b^2 q}$$

where $q=\frac{1}{2}\rho V^2$.

This formula for the induced drag is the one used by Prandtl and all recent writers on the subject of aeronautics.

In the first part of this paper use will be made of this relation and others derived, using the same approximation; in the second part the error introduced by making this approximation will be discussed; and in the third part a new theorem will be developed and its usefulness shown.

THE RELATIVE ABSORPTION OF POWER OF AIRPLANE WINGS AND PROPELLERS IN A NONVISCOUS FLUID

This note has particular reference to the comparison of the induced forces acting on a single pair of wings with those acting on a propeller having a large number of blades. In order to arrive at a proper basis for comparison it is convenient to consider power, instead of force. The wings require a certain amount of externally supplied power for a given performance. All of this power, which is to be considered as "induced," is transmitted to the air acted upon by the wings if the motion is steady. The power absorbed by the propeller, however, is partially recovered in the power utilized to propel the airplane. Only the difference between the power absorbed and the power utilized is transmitted to the surrounding air, and only this difference is to be considered as the induced power of the propeller.

Let L be the lift of the wings (or the thrust of the propeller); V the velocity of the airplane (or of the propeller), relative to the surrounding air; b the span of the wings (or the diameter of the propeller); and ρ the density of the air. As stated above, the induced drag corresponding to a velocity V is $D=\frac{L^2}{\pi b^2 q}$, hence the induced power absorbed by the wings is given by

$$P_{\text{ind}}=\frac{VL^2}{\pi b^2 q} \tag{1}$$

q being the dynamical pressure; that is, $q=1/2\rho V^2$. Similarly, the induced power of the propeller is given by

$$P_{\text{ind}}=\tfrac{1}{2}\,VL\left[\sqrt{1+\frac{L}{b^2 q\,\frac{\pi}{4}}}-1\right] \tag{2}$$

If the term $\frac{L}{b^2 q\,\frac{\pi}{4}}$ be less than unity, the expression within the brackets may be developed in a series. On this condition we have for the propeller—

$$P_{\text{ind}}=\frac{VL^2}{\pi b^2 q}-\frac{1}{16}\,\frac{VL^3}{\left[b^2 q\,\frac{\pi}{4}\right]^2}+-+ \tag{3}$$

Hence, if the thrust per unit area of the propeller disk be small in comparison with the dynamical pressure, the first term of the series gives a close approximation to the induced power. It therefore appears that, on this condition, the induced powers absorbed by wings and propeller agree.

It may therefore be stated that: If the thrust per unit area of the propeller disk be small n comparison with the dynamical pressure, then a wing and a propeller of equal span (or diameter), lift, and dynamical pressure require the same induced power.

This agreement is not accidental, but is due to the similar mechanical conditions of the two cases. The thrust of a propeller is produced by giving momentum to the air which passes through its disk. If the indraft substantially equal the velocity, then the mass of air passing through the disk per second is

$$V\,b^2\,\frac{\pi}{4}\,\rho$$

The lift of the wing is, however, produced by imparting momentum to the mass of air passing its plane in a direction perpendicular to the direction of flight, at that particular moment. But the share of each particle of air in the production of lift varies from point to point according to its position. With reference to the flow in the infinite plane perpendicular to the wing, it behaves like a single line in a two-dimensional fluid accelerated perpendicular to its length. Let such a line have the length b, and let ρ be the density of the fluid. Then such an infinite two-dimensional fluid motion around the line has the same kinetic energy as the mass

$$b^2 \rho \frac{\pi}{4}$$

moving with the same velocity as the line. That is to say, the whole infinite disk of air has the same mechanical effect as a circle with the diameter b cut out of the fluid, in the two-dimensional case, and moving like a rigid body with the line of length b.

Hence the equivalent mass of air flowing past the wing per second and producing the lift is

$$V b^2 \frac{\pi}{4} \rho.$$

In the two cases the same mass of air produces the same lift (or thrust), and this is why the same induced power is required.

If the thrust be large, or if the velocity of the propeller be small, the theorem no longer holds true. Otherwise a stationary rotating propeller, as in a helicopter, would require an infinite amount of power. The original expression (instead of the development into series) for induced power must now be used, because the conditions which permitted the expansion in series are not maintained. If $V=O$, expression (2) becomes

$$P_{ind} = L \sqrt{\frac{L}{b^2 \pi \frac{\rho}{2}}}$$

and it appears that the induced power of a propeller without forward velocity is equal to the induced power required by a single wing of span equal to the diameter of the propeller and moving with a velocity corresponding to a dynamical pressure of

$$q = \frac{L}{b^2 \pi}, \text{ i. e., } V^2 = \frac{2L}{\pi b^2 \rho}.$$

This expression is the fourth part of the thrust per unit area of the propeller disk. Therefore, a helicopter requires the same induced power as a wing of the same load and span, moving with a speed corresponding to a dynamical pressure equal to the fourth part of the thrust per unit area of disk. At the same time this required power equals the induced power absorbed by a wing with twice the load, the same span and a velocity corresponding to a dynamical pressure equal to the thrust per unit area of disk

MORE ACCURATE CALCULATION OF DRAG

Referring to the introduction, it is known that

$$L = F\frac{v}{V}$$

$$D = F\frac{w}{V} = kL, \text{ where } k = \frac{F}{\frac{1}{2}\pi b^2 \rho v^2}$$

in which F is the force acting upon a wing—

v is the velocity of a particle of air near the wing.
V is the velocity of flight.
w is the down-wash velocity.
b is the span of the wing.

In the deduction of these formulæ it has been assumed that the effective parts of the longitudinal vortices, which give rise to the lift, may be considered to be straight. This assumption certainly is allowable, and is approximately true.

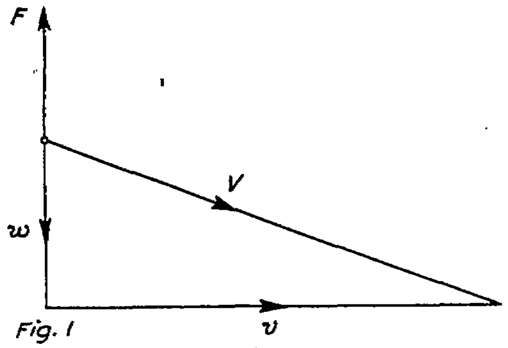


Fig. 1

Referring to the diagram,

$$V^2 = v^2 + w^2$$

and, since $w = kv$, $V^2 = v^2 (1 + k^2)$. Therefore, since k is a small quantity, $V = v\ (1 + \frac{1}{2}k^2)$, or $v = V\ (1 - \frac{1}{2}k^2)$. Hence

$$L = F\ (1 - \tfrac{1}{2}k^2)$$
$$D = kF\ (1 - \tfrac{1}{2}k^2).$$

Let us now introduce a new coefficient, $k' = \dfrac{L}{\pi b^2 q^1}$ where $q^1 = \frac{1}{2}\rho V^2$—i. e., is the dynamical pressure corresponding to the velocity of flight. Hence, neglecting terms of high orders,

$$V = v + \tfrac{1}{2}k'^2 V$$
$$L = F - k'^2 L$$

and, therefore,

$$D = k'L\ (1 + 1.5k'^2).$$

In Part I we used

$$D = \frac{L^2}{\frac{1}{2}\pi b^2\ \rho V^2};\ \text{i. e., } D = k'\ L.$$

It is now seen that, using the second approximation, the induced drag is greater than this. Since k' is small, $1.5\ k'^2$ is small compared with unity; and therefore the error made in using the first approximation is not large. This is the reason why the simpler formula gives results agreeing well with experience.

Let us calculate the error for a practical case; e. g., span $b=35$ feet; lift $L=2{,}000$ pounds; velocity $V=100$ mi/hr. Then $k'=0.0203$; $k'^2=0.0004$, and the error $1.5\ k'^2=0.06$ of 1 per cent. Even if the density is decreased to one-half the normal value and the velocity to two-thirds of the previous value, k' therefore becoming 4.5 greater, the error is only 1.3 per cent.

As a consequence, the original simple formula,

$$D=\frac{L^2}{\pi b^2 q}, \text{ where } q=\frac{1}{2}\rho V^2$$

may be considered to be sufficiently exact for ordinary purposes. For extraordinary cases the more exact form may be necessary.

THE FORCES ACTING ON BODIES MOVING IN A PERFECT FLUID

The force acting on an aerofoil moving in a perfect fluid is calculated by taking into consideration the mutual effect of the velocity in the immediate neighborhood of the aerofoil and the intensity of the vortices imagined in the space occupied by the aerofoil. This leads to the result stated in the introduction. The resulting force on the aerofoil equals the product of the velocity referred to and the intensity of the vortex, and its direction is perpendicular to the velocity and the vortex, as shown.

It is not generally known, however, that these are not the only forces between a body and a perfect fluid moving around it; and that forces between them may exist even if the effect of the body must not necessarily be considered as due to hypothetical vortices occupying the space of the body. I shall deduce the following theorem: If the body may be replaced by sources and sinks, a source of intensity J will give rise to a force $-\bar{v}\, J\, \rho$, in which $\bar{v}$ is the velocity of the fluid and ρ its density, $\bar{v}$ indicating the vector character of the velocity.

This theorem is analogous to the one which states that if there is a hypothetical vortex of intensity J at a point in a fluid where the velocity is v, the force per unit length of the vortex is the vector product $\rho\, [\bar{v}\, \bar{J}]$.

In both theorems the fluid is considered to be incompressible, to be flowing irrotationally, and to be in a steady state. They can be proven for more general assumptions also.

In order to prove this theorem I shall first use general considerations and then shall develop the mathematical formulæ. Let us consider a liquid flowing in any manner inside a closed boundary, which may be thought of as a solid shell. There will be pressures acting on each element of the boundary, and these will be equal and opposite to the pressures on the outside if the same liquid is flowing around the immersed body, and has the same velocity at each point of the boundary as the fluid inside has. This flow inside may be due to vortices and to sources; but for these to give rise to a steady condition and also to exist as forming a distinct isolated system, certain conditions must be satisfied. In particular, for a vortex to remain stationary there must be certain applied forces, according to the well-known theorem of Kutta. If the vorticity at any point has the angular velocity w, and if at that point the velocity of flow is V, the force required to maintain a steady state is $\rho\, [\overline{V w}]\, d\tau$, where ρ is the density and $d\tau$ is an element of volume. (This may be written $\rho\, [\overline{V J}]\, dl$, where J is the strength of the vortex.) This force must be due to the action of the solid boundary; and, if the system is to be self-contained, this force must be equal and opposite to the force of the liquid acting on the boundary; i. e., must be equal to the force acting on the boundary due to the liquid flowing on the *outside*.

Again, as far as the flow inside the boundary is caused by sources, it is evident that, in order that the system may be self-contained, there must be both sources and sinks whose intensities are equal, and we must picture these as connected by thin tubes, so that the fluid disappearing at a sink may be brought to the source out of which it flows. Further, if these positive and negative sources are to be stationary, and thus give rise to a steady motion, certain forces must be applied. If the fluid entering any sink has the velocity V_1, and if, on emerging from the tube at the source, it has the velocity V_2, the strength of each being J (i. e., the volume output per unit of time), the momentum has been increased in a unit of time by an amount $\rho J (V_2 - V_1)$, consequently there must be a force applied; and, as before, this force must be due to the boundary, and must be equal to the force acting on the boundary due to the liquid flowing outside.

Thus, if in any problem the effect of an actual solid immersed in a stream of fluid is considered as due to an "equivalent" distribution of vortices or of sources in the space occupied by the solid, we have the value of the force acting on the solid.

The exact proof may be deduced mathematically. If V is the velocity at any point inside the boundary, the pressure is $-\frac{\rho}{2}V^2$; therefore the force against the boundary from within is $-\frac{\rho}{2}\int \bar{n}_1 V^2 dS$, where $\bar{n}_1$ is a unit vector normal to an element of surface dS, and drawn outwards. Therefore, if the velocity of the fluid outside is the same at each point as that inside, the force acting on the boundary from *without* is $+\frac{\rho}{2}\int \bar{n}_1 V^2 dS$. This may be transformed into an integral through the volume, viz:

$$\frac{\rho}{2}\int \bar{n}_1 V^2 dS = \frac{\rho}{2}\int \operatorname{grad} V^2 d\tau = \rho\int \nabla \bar{V} \cdot \bar{V} d\tau \text{ using Gibbs's symbols.}^1$$

This holds true whether the velocity at the boundary has a component normal to it or not.

We must now introduce the condition that no fluid passes the surface, and therefore no momentum is transferred through it. If momentum were flowing through the boundary, the flow per unit of time would be $\rho\int \bar{V} V_n dS$. Thus must equal zero, therefore; and, as before, the integral may be transformed into a volume one, viz:

$$\rho\int \bar{V} V_n d S = \rho\int (\bar{V} \operatorname{div} \bar{V} + \bar{V} \nabla \bar{V})\, d\tau, \text{ using Gibbs's symbols.}^2$$

Since this equals zero, it may be subtracted from the previous integral giving the force, and hence

$$F = \frac{\rho}{2}\int \bar{n}_1 V^2 d S \begin{array}{l} = \rho\int(\nabla V . V - V \nabla V - \bar{V} \operatorname{div} \bar{V})\, d\tau \\ = \rho\int(\bar{V} \times \operatorname{rot} \bar{V} - \bar{V} \operatorname{div} \bar{V})\, d\tau. \end{array}$$

This is a formal statement of the theorem.

The conclusions derived from this theorem agree with other results of hydromechanics and with the general laws of mechanics. A source and a vortex are the two elementary solutions of the partial differential equation determining the motion. Each of these solutions can be derived from the other, and it is possible to pass continuously from one to the other. For these reasons it would be impossible that for one of the solutions a theorem exists without there being a corresponding theorem for the other. The continuous transformation may begin with two distant two-dimensional vortices of opposite sign but of equal intensity (fig. 2). The distance may grow less and less till at last the two vortices form a double vortex, the intensity of each increasing at the same time in such a way that the product of the intensity and the distance is constant. The doublet vortex is identical with a doublet source, consisting of a rectilinear source and equal sink, whose axis is perpendicular to that of the vortex doublet. This source doublet may be continuously transformed into two separate sources, by increasing the distance of the source and sink and at the same time decreasing their intensity, so that the product of the intensity of each and the distance remains constant as before.

The velocity of the fluid in the neighborhood of these vortices or sources may be supposed to have a constant rate of change in the direction of the axis of the doublet vortices. The rate of change in the direction of the axis of the doublet sources is perpendicular to the first mentioned rate, if the velocity is irrotational and without divergence, and both have equal absolute magnitude. Then the force acting on the two vortices is constant during the whole transformation. It certainly would be absurd to expect this constant force suddenly to vanish either on proceeding from the doublet vortex to the doublet source, which is only another expression for the same phenomenon, or on transforming the doublet source to two separate sources. We are led therefore to attribute to each source the force given by the theorem.

---

[1] This is written in ordinary notation: $\rho\int\left(V_x \operatorname{grad} V_x + V_y \operatorname{grad} V_y + V_z \operatorname{grad} V_z\right)d\tau$.

[2] This is written in ordinary notation: $\rho\int \{\bar{V} \operatorname{div} \bar{V} + (i\bar{V} \operatorname{grad} \bar{V}_x + jV \operatorname{grad} \bar{V}_y + V \operatorname{grad} \bar{V}_z)\}\, d\tau$.

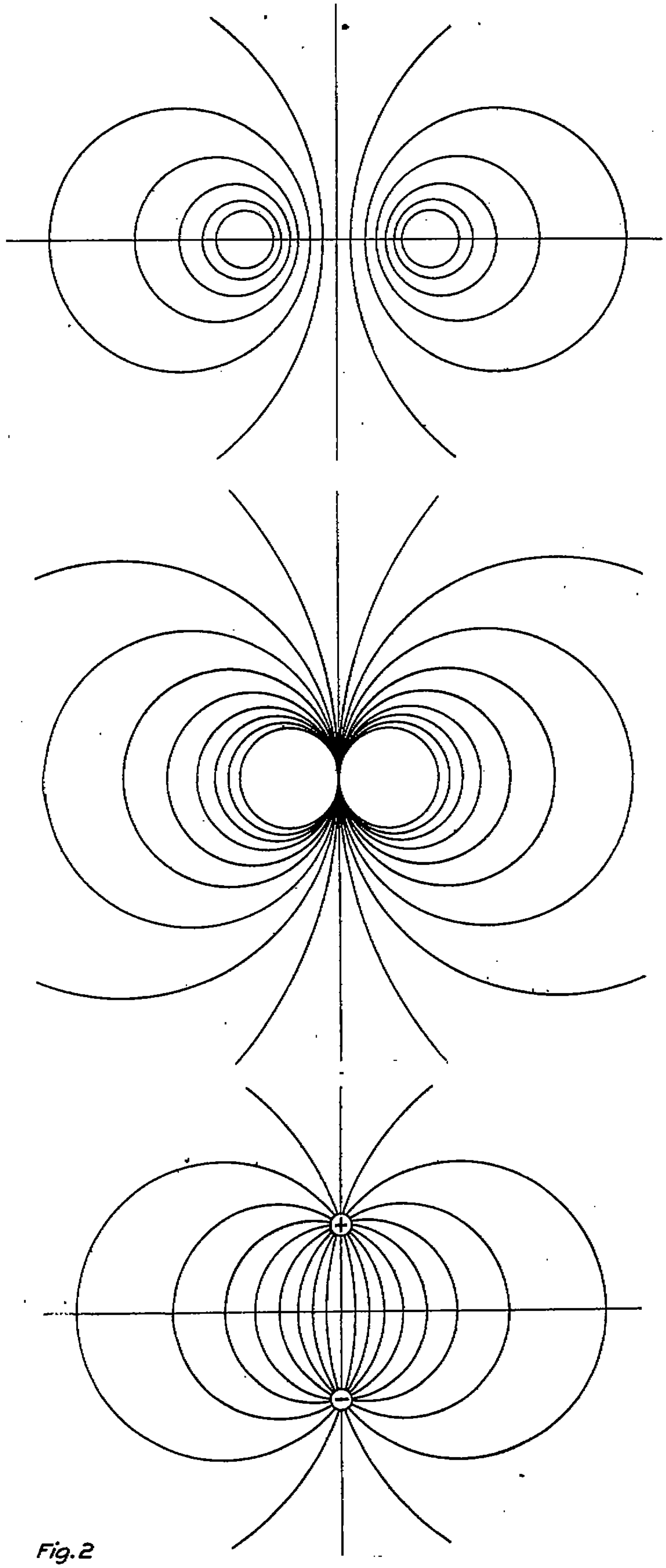


Fig. 2

The vortex and the source are the elements of the two possible kinds of discontinuities. If the fluid streams along a boundary with constant velocity on one side and with a different constant velocity on the other side (fig. 3) Euler's theorem states that the pressure on the two sides of the boundary is different. The difference is $\frac{1}{2}(V^2_2 - V^2_1)\rho$, the pressure being higher on the side of the smaller velocity. This boundary is to be represented by vortices perpendicular to the discontinuity of velocity and uniformly distributed over the boundary. They can be considered to be in fluid having the velocity $\frac{1}{2}(V_1 + V_2)$. Their intensity per unit of area is $(V_2 - V_1)$. The difference of pressure equals the product of the intensity per unit area, the velocity and the density, and is directed at right angles to the velocity and to the vortex. The force on the vortex is directed toward the space of the greater velocity.

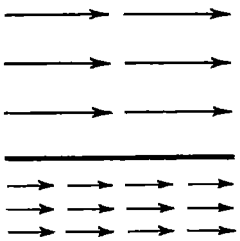


Fig. 3

The other kind of discontinuity occurs if the boundary is perpendicular to the two constant velocities (fig. 4). This is only possible if the boundary is occupied by sources, uniformly distributed, with the intensity $(V_2 - V_1)$ per unit of area. They again are in fluid having the velocity $\frac{1}{2}(V_2 + V_1)$ and the force acting on them is $\frac{1}{2}(V_2 + V_1)(V_2 - V_1)\rho$. This equals the pressure obtained by applying our theorem to the single sources which are distributed over the boundary.

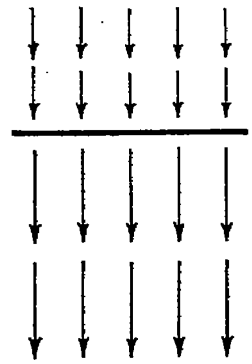


Fig. 4

CONCLUSIONS.

Consider two point sources with the intensities J_1 and J_2 at the distance R apart. The velocity in the neighborhood of the first source due to the second is

$$\frac{J_2\rho}{4\pi R^2}.$$

Hence, according to the theorem:

Two sources of the same sign produce attractive forces between them which are proportional to the product of their intensities, and inversely to the square of their distance apart. The magnitude of the force is

$$\frac{J_1 \cdot J_2 \cdot \rho}{4\pi R^2}.$$

Two sources of opposite signs produce repulsive forces according to the same law.

The consideration of the velocity in the neighborhood of a source due to an element of a vortex or in the neighborhood of this element due to a source show:

A source and an element of a vortex produce a force on each other proportional to the product of their intensities and inversely to the square of their distance. The force is perpendicular to the vortex and to the line of connection, and has the magnitude

$$\frac{J_1 \cdot J_2}{4\pi R^2} \cdot \sin\varphi$$

where φ is the angle between the direction of the vortex and the line joining it to the source.

These two facts are analogous to the theorem known before, which is derived from the relation between the force on a vortex, its velocity, and its intensity:

Two elements of vortices having the same direction produce repulsive forces on each other, which are proportional to the product of their intensities and inversely to the square of their distance apart. The magnitude of the force is

$$\frac{J_1 \cdot J_2}{4\pi R^2} \sin\varphi$$

where φ is the angle between the direction of the vortices and the line joining the elements. Elements having opposite directions produce forces of attraction according to the same law. [N. B.—These forces produced by or on vortices refer to elements of unit length.]

(For two dimensional problems the denominator would always be $2\pi R$.)

These statements are valid for steady motion. If there are no extraneous forces or only such as have a potential, the fluid motion involving vortices can not be steady if the vortex has a finite magnitude. The vortex moves with the fluid, hence it has no velocity relative to the fluid and therefore requires no extraneous force.[1] For sources a similar theorem is valid, for the same reason. It is difficult, however, to connect a physical meaning with it. The motion of the positive source would not even be stable.

A pair consisting of a source and a sink of equal intensities experiences a force proportional to the difference of the velocities at the two points. Passing to the limit, we see that the force on such a doublet is proportional to its intensity and to the rate of change of the velocity in the direction of its axis. These cases are of some practical importance, it being possible to represent moving bodies by such doublets.

If two such doublets are situated one behind the other, the two axes coinciding, the influence of the two nearest sources predominates. Two such doublets repel each other, with a force whose magnitude is

$$\frac{M_1 M_2}{4\pi R^4}$$

where M_1 and M_2 are the strengths of the doublets. The force varies inversely as the fourth power of the distance. It is to be expected, therefore, that it will not be of practical importance.

It may be mentioned that the forces between a single linear vortex and a doublet are inversely proportional to the square of the distance. The force and its direction also depend on the angle between the vortex and the axis of the doublet.

We will omit the consideration of peculiarities of higher order, as we do not think that we shall be forced to apply them.

APPLICATIONS

It has been noted by several writers[2] that the combination of two equal and opposite sources at a finite distance apart will, when placed in a fluid flowing parallel to the line joining the two sources, give rise to such a distribution of lines of flow that the total effect is the same as that produced by a body of the general shape of an airship.

Thus let the intensities of the sources be $+J$ and $-J$; let their distance apart be a; and let the velocity of flow parallel to the line joining the sources be v. Then the area of the greatest cross-section of the "equivalent airship" is, approximately,

$$S=\frac{J}{v+\frac{J^2}{\pi a^2}}$$

and therefore the greatest diameter is

$$d=\sqrt{S\frac{4}{\pi}}.$$

The length of the equivalent airship is approximately

$$l=a+2\sqrt{\frac{J}{4\pi v}}.$$

If $\frac{J}{v}$ is small compared with a^2, the airship is an elongated one; and it will be sufficient to assume that $S=\frac{J}{v}$, $l=a$; or, if l and S are given, that $J=vS$; $a=l$.

We will now apply this general theory to two problems.

1. The calculation of the force acting between two similar airships in motion in a perfect fluid, side by side, at a distance apart of their axes given by b.

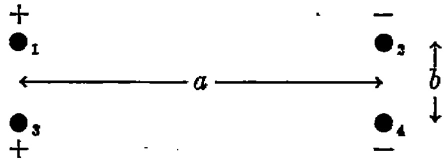
[1] Helmholtz's theorem.

[2] D. W. Taylor: Ship-shaped stream forms. Trans. British Inst. Naval Arch., Vol. 35, p. 385, 1894.

Solid stream forms and the depth of water necessary to avoid abnormal resistance of ship. Ib., Vol. 36, p. 234, 1895.

G. Fuhrmann: Theoretische und experimentelle Untersuchungen an Ballonmodellen. Jahrb. der Motorluftschiff-Studiengesellschaft. 1911-12.

Each airship may be represented by a pair of sources, as just noted.



Let us label them 1, 2, 3, 4; 1 and 3 are positive, 2 and 4 are negative. The attraction between 1 and 3 and between 2 and 4 is

$$f_{13}=f_{24}=\frac{J^2\rho}{4\pi b^2}=\frac{v^2S^2\rho}{4\pi b^2}.$$

The forces between 1 and 4 and between 2 and 3 are repulsions and their components perpendicular to the axes of the airships are

$$f_{14}=f_{23}=-\frac{v^2S^2\rho}{4\pi}\,\frac{b}{(b^2+l^2)^{3/2}}.$$

So the total attractive force on each airship is

$$F=2\frac{v^2S^2\rho}{4\pi}\left(\frac{1}{b^2}-\frac{b}{(b^2+l^2)^{3/2}}\right).$$

In a practical case let $l=600$ feet, $S=2{,}500$ square feet, $v=80$ mi/hr, $b=200$ feet. The calculated force comes out to be 800 pounds. Not only is this force very small, but in practice airships do not come so close together. So the result is to prove that the aerodynamical forces due to displacement are small.

2. The calculation of the longitudinal buoyancy of an airship model in a wind tunnel, due to the pressure gradient in the air stream. The force on the airship described as in the previous problem would be

$$F=vS\,(v_2-v_1)\,\rho=vS\rho\frac{dv}{dx}\cdot a=\frac{dp}{dx}\cdot\ \text{(volume)}.$$

This formula is the one used in practice and is sufficiently accurate for an elongated model. It can not, however, be applied to a short one. Consider, for instance, a sphere of radius r. It is "equivalent" to a doublet of the intensity $2\pi r^3\cdot v=\frac{3}{2}\,v$ (volume). The force acting on it $\frac{3}{2}\,\rho\,v\,\frac{dv}{dx}\cdot$ (volume) $=1.5\,\frac{dp}{dx}\cdot$ (volume). Hence the proper coefficient for a short airship lies between 1 and $1\frac{1}{2}$.

More complicated distributions of sources which are equivalent to different forms of airships may be found in the paper by Fuhrmann. This should be used in the calculation of the exact error, applying the theorem deduced in this paper.