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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

REPORT No. 180

THE INFLUENCE OF THE FORM OF A WOODEN BEAM ON ITS STIFFNESS AND STRENGTH—I.

DEFLECTION OF BEAMS WITH SPECIAL REFERENCE TO SHEAR DEFORMATIONS

By J. A. NEWLIN and G. W. TRAYER

WASHINGTON
GOVERNMENT PRINTING OFFICE
1924

AERONAUTICAL SYMBOLS.

1. FUNDAMENTAL AND DERIVED UNITS.

	Symbol.	Metric.		English.	
		Unit.	Symbol.	Unit.	Symbol.
Length	l	meter	m.	foot (or mile)	ft. (or mi.).
Time	t	second	sec.	second (or hour)	sec. (or hr.).
Force	F	weight of one kilogram	kg.	weight of one pound	lb.
Power	P	kg.m/sec		horsepower	HP
Speed		m/sec	m. p. s.	mi/hr	M. P. H.

2. GENERAL SYMBOLS, ETC.

Weight, $W=mg$.

Standard acceleration of gravity, $g=9.806\text{m/sec.}^2=32.172\text{ft/sec.}^2$

Mass, $m=\frac{W}{g}$

Density (mass per unit volume), ρ

Standard density of dry air, 0.1247 (kg.-m.-sec.) at 15.6°C. and 760 mm. = 0.00237 (lb.-ft.-sec.)

Specific weight of "standard" air, 1.223 kg/m.^{3} = 0.07635 lb/ft.^{3}

Moment of inertia, mk^2 (indicate axis of the radius of gyration, k, by proper subscript).

Area, S; wing area, S_w, etc.

Gap, G

Span, b; chord length, c.

Aspect ratio $=b/c$

Distance from $c.\ g.$ to elevator hinge, f.

Coefficient of viscosity, μ.

3. AERODYNAMICAL SYMBOLS.

True airspeed, V

Dynamic (or impact) pressure, $q=\frac{1}{2}\rho V^2$

Lift, L; absolute coefficient $C_L=\frac{L}{qS}$

Drag, D; absolute coefficient $C_D=\frac{D}{qS}$

Cross-wind force, C; absolute coefficient $C_C=\frac{C}{qS}$.

Resultant force, R

(Note that these coefficients are twice as large as the old coefficients L_c, D_c.)

Angle of setting of wings (relative to thrust line), i_w

Angle of stabilizer setting with reference to thrust line i_t

Dihedral angle, γ

Reynolds Number $=\rho\frac{Vl}{\mu}$, where l is a linear dimension.

e. g., for a model airfoil 3 in. chord, 100 mi/hr., normal pressure, 0°C: 255,000 and at 15.6°C, 230,000;

or for a model of 10 cm. chord, 40 m/sec., corresponding numbers are 299,000 and 270,000.

Center of pressure coefficient (ratio of distance of $C.P.$ from leading edge to chord length), C_p.

Angle of stabilizer setting with reference to lower wing. $(i_t-i_w)=\beta$

Angle of attack, α

Angle of downwash, ϵ

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DEFLECTION OF BEAMS WITH SPECIAL REFERENCE TO SHEAR DEFORMATIONS.

By J. A. Newlin and G. W. Trayer.

INTRODUCTION.

This publication is one of a series of three reports prepared by the Forest Products Laboratory of the Department of Agriculture for publication by the National Advisory Committee for Aeronautics. The purpose of these papers is to make known the results of tests to determine the properties of wing beams of standard and proposed sections, as conducted by the Forest Products Laboratory and financed by the Army and the Navy.

Many of the mathematical operations employed in airplane design are nothing more than the solution of equations which are either empirical or are based on assumptions which are known to be inaccurate, but which have been adopted because of their simplicity. These inaccuracies of the formulas were not of primary consideration as long as the stresses used for design were obtained by the test of specimens of the same form as those to be used, and great refinement was not necessary.

The advent of the airplane and the impetus given to its development by the recent war has created a demand for more definite knowledge of the limitations and proper application of the common theory of flexure. There is probably no other field in which greater refinement in the design of wooden members is required than in that of aircraft construction. The ever-present problem of weight reduction has led to the use of comparatively small load factors and the introduction of such shapes as are not commonly used for other construction purposes. Formulas which give comparable results when applied to wooden beams of rectangular section have been found to be considerably in error when applied to wooden beams of other shapes.

The tests were made at Madison, Wis., in cooperation with the University of Wisconsin. An analysis of the results of these tests has furnished information which, when correlated with that from other studies conducted by the Forest Service for the past 18 years, provided a more exact method of computing the stiffness of wood beams and led to the development of formulas for estimating the strength of beams of any cross section, using the properties of small rectangular beams as a guide.

For convenience, the report of this investigation has been divided into three parts. The first part deals with the deflection of beams with special reference to shear deformation, which usually has been neglected in computing deflections of wood beams. The second part has to do with stresses in beams subjected to transverse loading only, with a subdivision on nonsymmetrical sections; and the third part, with stresses in beams subjected to both longitudinal thrust and bending stresses.

SUMMARY.

In addition to the deflection due to the elongation and compression of fibers from bending stresses, there is a further deflection due to the shear stresses and consequent strains in a beam. This is not usually considered in computing deflections of wood beams, though the modulus of elasticity in shear for wood is relatively low, being but approximately one-sixteenth the modulus of elasticity in tension and compression, whereas for steel, for example, it is about two-fifths the ordinary modulus.

By neglecting the deformation due to shear, errors of considerable magnitude may be introduced in determining the distortion of a beam, especially if it is relatively short, or has comparatively thin webs as the box or **I** beams commonly used in airplane construction. A great many tests were made to determine the amount of shear deformation for beams of various sections tested over many different spans. As the span over which the beam is tested is increased the error introduced by neglecting shear deformations becomes less, and the values obtained by substituting measured deflections in the ordinary formulas approach more nearly the modulus of elasticity in tension and compression. For short spans, however, the error is considerable, and increases rapidly as the span is reduced. This variation is illustrated in Figures 3 and 4.

Two formulas were developed for estimating the magnitude of shear deformations, both of which have been verified by tests. It is known that the distribution of stress assumed in both formulas does not exactly represent the actual distribution of stress in a beam. Both formulas check experimental results very closely when the calculations are made with great refinement. It is not known which is the more accurate formula under these conditions, since the difference in results obtained by the two is only a small part of the normal variation of the material. The first formula, with its high powers and numerous factors, will obviously lead one into inaccuracies due to the ordinary approximations used in calculations more readily than will the second, or similar formula. In both formulas the deformation due to shear is equal to $\frac{KPl}{F}$, where P is the load on a beam of length l, F is the modulus of elasticity in shear, and K is some coefficient depending upon the shape of the beam and upon the loading. The formulas differ only in the determination of the coefficient K. Under the heading "Analysis of Results" K by the first formula is shown and also by the second, or more simple formula.

The modulus of elasticity in shear was found to vary greatly according to the direction of the grain of the ply wood in webs of box beams. It was found to be over three and one-half times as great for beams having ply-wood webs with the grain at 45° to the length as for beams having webs the face grain of which was perpendicular to the length of the beam.

Although the tests showed conclusively that shear stresses are present in the overhang, the change in deformation on this account did not prove to be of sufficient importance to take overhang into account even with the most heavily routed **I** sections.

These tests show that the values of modulus of elasticity for small beams given in Bulletin 556[1] are approximately 10 per cent lower than the true modulus of elasticity in tension and compression. However, when substituted in the usual deflection formula they will give correct values for the deflection of solid beams with a span-depth ratio of 14, which is about the average found in most commercial uses. The bulletin values are therefore recommended for use in the ordinary formulas when no corrections are to be made. For solid beams with spans from 12 to 28 times the depth of beam the maximum error introduced by substituting these values in the ordinary formulas is about 5 per cent. For very short spans it would be well to use the more exact formulas, which take into account shear distortions, using for the true modulus a value 10 per cent greater than that given in the bulletin.

But in **I** and box beams, however, which have a minimum of material at the plane of maximum horizontal shear stress, very considerable errors will be introduced if shear distortions are neglected even for relatively large span-depth ratios.

PURPOSE.

The purpose of the tests was to determine to what extent ordinary deflection formulas, which neglect shear deformations, are in error when applied to beams of various sections and to develop reasonably accurate yet comparatively simple formulas which take into account such deformations.

[1] Bulletin No. 556, United States Department of Agriculture, "Mechanical Properties of Woods Grown in the United States," by J. A. Newlin and T. R. C. Wilson.

MATERIAL.

The beams were made of either Sitka spruce or Douglas fir wing-beam material conforming to standard specifications and had either box I, double I, or solid rectangular sections as shown in Figure 1. The box and I beams, which were made of Sitka spruce, were either 14 or 18 feet in length. The double I beams had Sitka spruce flanges and $\frac{1}{24}$–$\frac{1}{12}$–$\frac{1}{24}$ inch yellow poplar ply-wood webs with the grain of face plies in some cases perpendicular and in other cases at 45° to the length of the beam. The flanges were $2\frac{3}{32}$ inches wide and 2 inches deep, the depth over all was $8\frac{3}{8}$ inches, and the length 14 feet 6 inches. All the beams of solid rectangular section were made of Douglas fir. They were $2\frac{3}{4}$ inches wide, 5 inches deep, and 14 feet 6 inches long.

It must not be construed that the beams were tested only in the lengths given above. As tests for modulus of elasticity were kept well within the elastic limit, the length of the beams could be reduced after each test and another test run over a new span.

Torsion specimens were 24 inches long and $2\frac{1}{2}$ inches of each end were 2 inches square. For 18 inches the section was reduced to a circular section $1\frac{1}{4}$ inches in diameter, the square ends and circular center portion being connected by a circular fillet of $\frac{1}{2}$-inch radius.

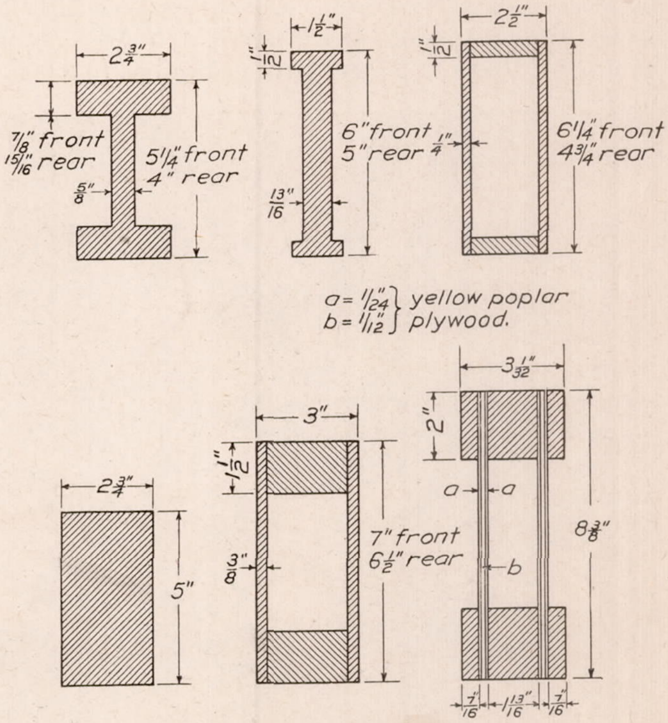


FIG. 1.—Sections of beams used for modulus of elasticity tests.

OUTLINE OF TESTS.

A. Beam tests:
  1. Test for modulus of elasticity—
     (*a*) Center loading.
     (*b*) Symmetrical 2-point loading.
  2. Moisture determinations.

B. Tests of minor specimens matched with the beams:
  1. Static bending tests of 30-inch specimens.
  2. Compression-parallel-to-grain specimens 8 inches long.
  3. Compression-perpendicular-to-grain specimens 6 inches long.
  4. Specific gravity determinations specimens 6 inches long.
  5. Moisture determinations. Disks cut from all minor specimens.

C. Torsion tests:
  1. Test for modulus of rigidity.
  2. Moisture determination.

METHODS OF TESTS.

MODULUS OF ELASTICITY TESTS.

In order to eliminate the variability of material in our comparison of different spans, the same beam was tested several times, the span being changed for each test. Since the relation of modulus of elasticity in shear to the ordinary modulus of elasticity is not the same for different beams and species, several beams were tested that we might learn something of its range. In some cases the ends were cut off to maintain a constant overhang and in other cases the total length was kept constant as the span was changed. The accompanying tables show how spans up to 18 were reduced by either 1 or 2 foot intervals to either 2 or 3 foot spans. Deflections were read by referring a scale, attached at the center of the beam, to a fine wire drawn between nails over the supports, or when greater precision was required, by observing the movement of a pointer on a dial attached to a light beam resting on nails driven in the test beam over the supports. A fine silk line attached to a nail at the center of the test beam passed around the drum of the dial and carried a weight to keep it taut. Movements of the test beam were so multiplied that the pointer gave deflections to 0.0001 inch, whereas by the first method deflec-

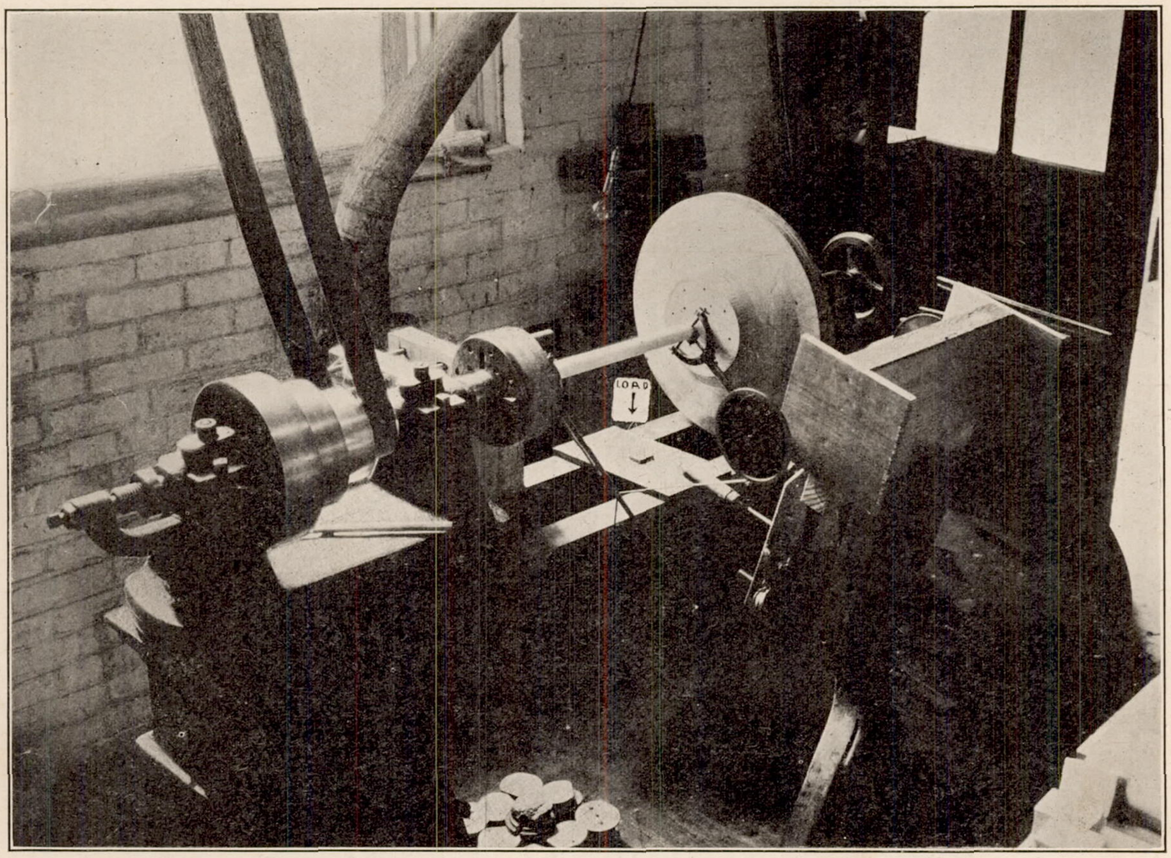


FIG. 2.—Torsion apparatus.

tions could only be read to 0.01 inch. The two methods were never interchanged during a series of tests on any one beam.

Two of the types of beams tested showed a decided tendency to buckle during test. This was overcome by using pin-connected horizontal ties, which prevented bending in more than one plane.

Loads were applied by a 30,000-pound capacity testing machine, which was fitted with auxiliary wings to accommodate spans up to 18 feet.

Center loading was used in all except two series of tests. The first of these series consisted of tests of the same beam over different spans, center and third point loading being applied for each span, in order to determine the relation between the moduli of elasticity as computed by the formulas for each condition. In the second series of tests the span was kept constant and the distance between symmetrical loads changed in order to determine what effect, if any, the distance between loads had on the modulus of elasticity as computed by the usual formula for symmetrical loading.

There were matched with all I and box beams, static bending specimens approximately 2 by 2 inches in section and 30 inches long, compression parallel test pieces 2 by 2 inches by 8 inches long, and compression perpendicular specimens 2 by 2 inches by 6 inches long. These minors were tested and specific gravity and moisture determinations made in accordance with standard laboratory methods.

A simple torsion apparatus was set up in an ordinary wood lathe. Figure 2 is a photograph of the machine. Load was applied in 25 inch-pound increments and the angle of twist read for each increment over a 16-inch gauge length. All torsion specimens were matched with standard 2 by 2 inch specimens which were tested in bending over a 28-inch span. For further description of the test see Description of figures and tables.

DESCRIPTION OF FIGURES AND TABLES.

Figure 1.—This figure shows sections of all beams used in modulus of elasticity tests Such dimensions as "7 inches front" and "6½ inches rear" indicate that two beams of that type were tested, the words front and rear designating their position in the wing.

Figure 2.—This is a photograph of a simple torsion apparatus set up in an ordinary wood lathe. The right-hand wooden disk is set on ball bearings and has a wire passing around it to a tray marked "load." The smaller wooden disk at the left is fixed. The specimen is square at the ends, which fit into the two wooden disks. The angle of twist was measured by the two troptometer arms, each of which carries a string which passes around the drum of a dial.

Figure 3.—This shows the typical variation of the quantity $\frac{Pl^3}{48\Delta I}$ with span for a beam of solid rectangular section loaded at the center.

Figure 4.—This shows a similar variation before and after routing a solid section. The amount of shear deformation is considerably increased by reducing the thickness at the plane of maximum horizontal shear.

Figure 5.—This figure shows the same variation. The $\frac{Pl^3}{48\Delta I}$ values, which are the average from tests of three beams, are expressed as per cent of the true modulus of elasticity in tension and compression.

Figure 6.—Curve A shows the distribution of shear stress in a beam of rectangular section, and curve B the distribution in an I beam with square corners which was used as a basis for the developments of the shear deformation formulas presented in this report.

Figure 7.—This figure shows the superiority of 45° ply wood as regards rigidity. Shear distortion being less the values of $\frac{Pl^3}{48\,\Delta\,I}$ are closer to the true modulus of elasticity for the beam with 45° ply wood.

Figure 8.—In this dual figure is represented the variation of $\frac{Pl^3}{48\,\Delta\,I}$ with span for various standard wing-beam sections as well as for a solid section. The beams were all made of Sitka spruce and tested under center loading. The values of $\frac{Pl^3}{48\,\Delta\,I}$ are expressed as per cent of the true modulus of elasticity in tension and compression. The dimensions of these beams are shown in Figure 1. In the upper row, from left to right, is the *F–5–L*, Loening, and *TF*, and in the center of the lower row, the *NC*.

Table I.—In this table is given the measured and computed deflections of Douglas-fir beams of solid rectangular section loaded at the center. The formula used takes into account shear deformations usually neglected in such calculations. The differences in the two values are expressed as errors in per cent of the measured deflection.

Table II.—Here we have measured and computed deflections for standard sections. For description of these sections see description of Figure 8. The computed deflections are from two formulas, one taking shear into account and the other neglecting it. Errors are expressed in per cent of the measured deflections.

ANALYSIS OF RESULTS.

If a solid beam is tested over different spans, load being applied at the center and measured deflections substituted in the expression $\frac{Pl^3}{48\,\Delta\,I}$ the resulting values for spans greater than 20 or 25 times the depth of beam will be fairly constant, approaching the true modulus of elasticity in tension and compression, while for spans below this ratio there will be a rapid decrease. Figure 3 shows the results of just such a test. The beam was of Douglas fir, 2.75 inches wide, 4.97 inches deep, and was tested over spans starting at 14 feet and reduced by 2-foot intervals after each test to a span of 10 feet and then by 1-foot intervals to a span of 2 feet. Evidently the constant value which this curve would approach with longer spans is about 1,600,000 pounds per square inch.

In this test a constant overhang of 3 inches was maintained for all spans. For some of the comparisons described below this was impossible since it was necessary to maintain a constant

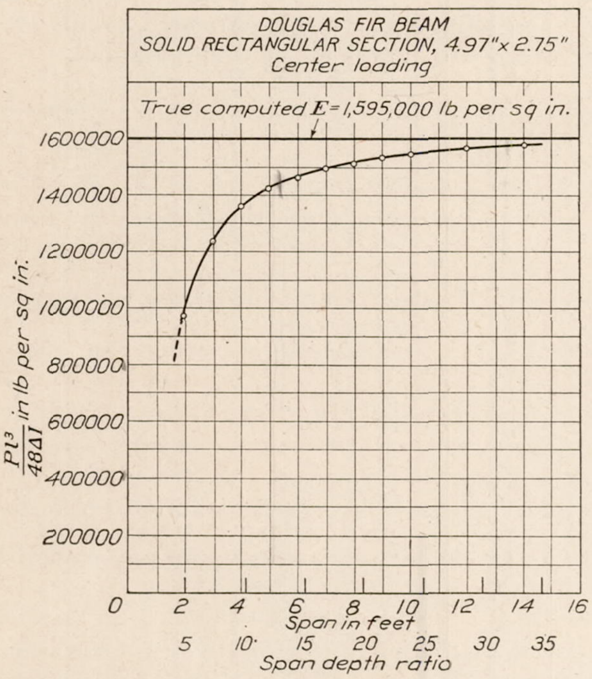


FIG. 3.—Relation of span to value obtained by substituting deflections in $\frac{Pl^3}{48\Delta I}$.

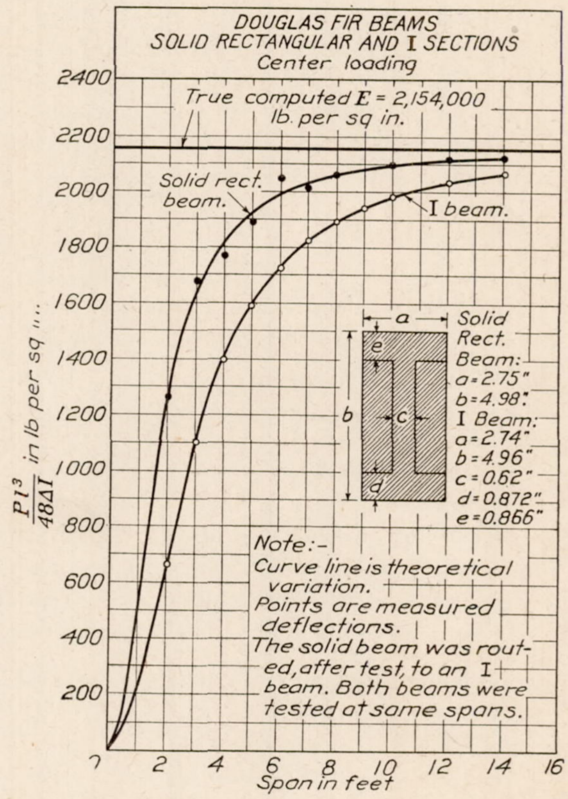


FIG. 4.—Relation of span to value obtained by substituting measured deflections in $\frac{Pl^3}{48\Delta I}$.

over-all length with a consequent variation in overhang as the span was changed. Observations proved conclusively that shear strains crept out into the overhang, but the change in deflection at the center due to this influence was too small to be measured.

Figure 4 shows the results of tests of a solid beam tested over various spans, after which it was routed out to an **I** beam and again tested over the same spans. Both apparently are approaching the same asymptote, but for all spans within practical limits the **I** beam is considerably below the solid beam, showing that the shear deformations are greater for such a section than for the solid one. When we measure the deflection of a beam in test we measure not only the deflection due to the lengthening of the tension fibers and the shortening of the compression fibers but the deflection due to all other distortions of the fibers. If we substitute this measured value in a formula which does not take into account all such distortions we can not expect a constant result for all spans and forms of beams but something like what is shown in Figures 3 and 4.

While it is recognized that any distortion due to a force producing bending moment is reflected in the deflection of a beam, the only distortions that appear to be of a magnitude to justify consideration are those resulting from the lengthening of the tension fibers and shortening of the compression fibers and from shear stresses.

The asymptote or constant value which these curves of Figures 3 and 4 approach is the true modulus of elasticity in tension and compression, which we will call E_T. If we assume that the

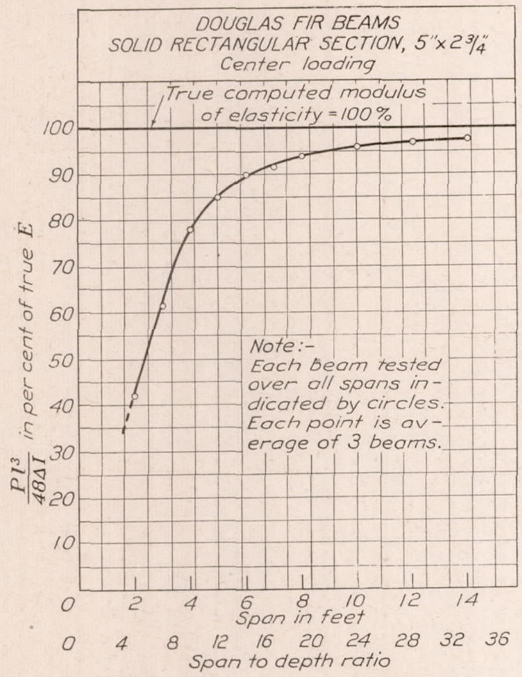


FIG. 5.—Relation af span-depth ratio to value obtained by substituting measured deflections in $\frac{Pl^3}{48\Delta I}$.

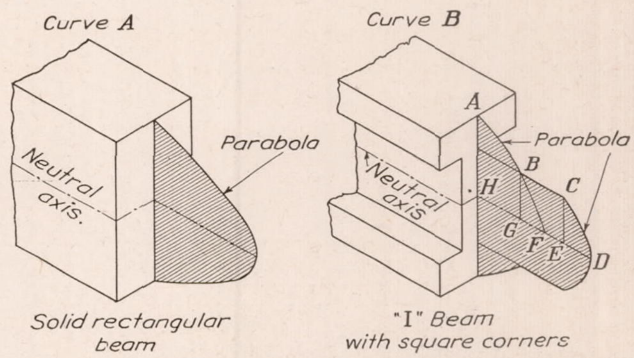


FIG. 6. Distribution of shear stress in beams.

deformation due to shear is proportional to the moment, a point which will be proved later, we may write

$$\Delta_1 = \frac{P_1 l_1^3}{48\,E_T I} + \frac{KP_1 l_1}{F}$$

where,

Δ_1 = the deflection of a beam of span l_1 loaded at the center with a load P_1, and
F = the modulus of elasticity in shear.

For a span l_2 with a load P_2 at the center of the same beam we have

$$\Delta_2 = \frac{P_2 l_2^3}{48 E_T I} + \frac{KP_2 l_2}{F}$$

These two equations contain the two unknown quantities E_T and F, and hence the solution of the two equations will furnish values of the true modulus E_T and the shearing modulus F. By making many experiments on the same beam instead of two and writing an equation for each it is possible to obtain reliable values for these two moduli for that particular beam. From the results shown in Figure 3 the true modulus of elasticity was found in this way to be 1,595,000 pounds per square inch and from the results shown in Figure 4 it was found to be 2,154,000 pounds per square inch. Figure 5 shows results similar to those of Figures 3 and 4. They are expressed, however, in per cent of the true computed E_T taken as 100 per cent. In this case each point represents the average of three beams rather than the results of a single beam.

Since for ordinary spans the deformation due to shear is small in comparison with the deflection due to elongation and compression of the fibers, it was difficult to obtain reliable values for F by the solution of simultaneous equations as outlined above, since the slightest errors in measuring deflections for ordinary spans were reflected in F more than in E_T. Torsion tests were made for the purpose of checking on this value, which showed F for spruce to be about 1/15 E_T and for Douglas fir about 1/17 or 1/18 E_T.

Assuming a parabolic distribution of shear stress, as shown in Figure 6, expressions for shear deformation can be determined by setting up an expression for internal work and equating it to the external work done in producing shear distortions.

In this way, for a beam of solid rectangular section loaded at the center, we get:

$$f=\frac{0.3Pl}{AF}$$

and for an **I** or box beam with square corners similarly loaded:

$$f=\frac{Pl}{8FI^2}\left[t_2\left(\frac{8}{15}K_2^5-K_2^4K_1+2K_2^2K_1^3-\frac{23}{15}K_1^5\right)+\frac{t_2^2}{t_1}(K_2^4K_1-2K_2^2K_1^3+K_1^5)+t_1\left(\frac{8}{15}K_1^5\right)\right]$$

which may be written

$$f=\frac{KPl}{F}$$

where,

$$K=\frac{1}{8I^2}\left[t_2\left(\frac{8}{15}K_2^5-K_2^4K_1+2K_2^2K_1^3-\frac{23}{15}K_1^5\right)+\frac{t_2^2}{t_1}(K_2^4K_1-2K_2^2K_1^3+K_1^5)+t_1\left(\frac{8}{15}K_1^5\right)\right]$$

where f = the deformation due to shear.
F = modulus of elasticity in shear.
P = load at the center.
l = span.
A = area of cross section.
I = moment of inertia of the section.
K_2 = distance neutral axis to extreme fiber.
K_1 = distance neutral axis to flange.
t_2 = width of flange.
t_1 = thickness of web; in box beams combined thickness of webs.

The development of the above expressions is given in the appendix, together with expressions for other conditions of loading.

The above formula assumes the parabolic distribution of shear stress on a cross section of a beam, and the deflection due to shear is determined by the ordinary method of equating external work to internal energy. It involves high powers and numerous factors which may lead to inaccuracies when the ordinary approximations in calculations are employed. Consequently a more simple formula was sought.

The development of the second, a more simple formula, follows. In the two formulas the same shear distribution is assumed, but in the second formula the fundamental assumption is that deflections due to shear in any two beams of the same length, height, and moment of inertia, which are similarly loaded, are proportional to the summations of the shear stresses on their respective vertical sections.

Let us assume that we have an **I** beam of a given length, depth, and moment of inertia, and a rectangular beam of the same length, depth, and of a width to make its moment of inertia equal to that of the **I** beam. The shear stress distribution would be as indicated in Figure 6. Let us further assume that the shear deformations will be proportional to the areas under the stress curve. Knowing the shear deflection of the rectangular beam to be $\frac{0.3Pl}{bdF}$ when supported at the ends and loaded at the center, we can determine f for an **I** beam similarly loaded by multiplying this value by the ratio of the area under the shear stress curve of the **I** beam to the area under the stress curve of the rectangle, which ratio is:

$$\frac{\frac{VK_2^3}{3I}+\frac{V}{2I}\,(K_2^2-K_1^2)\,K_1\left(\frac{t_2}{t_1}-1\right)}{\frac{VK_2^3}{3I}}$$

Referring to curve B, Figure 6

$$HF=\frac{VK_2^2}{2I} \text{ and since } ABF \text{ is a parabola the area } ABFH=2/3\,K_2\times\frac{VK_2^2}{2I}=\frac{VK_2^3}{3I}$$

the total area $ABCDH=$ area $ABFH+$ area $BCEG$

$$\text{Area } BCEG=\frac{V}{2I}\,(K_2^2-K_1^2)\,K_1\left(\frac{t_2}{t_1}-1\right) \text{ and the total area}$$

$$ABCDH=\frac{VK_2^3}{3I}+\frac{V}{2I}\,(K_2^2-K_1^2)\,K_1\left(\frac{t_2}{t_1}-1\right).$$

The area under the stress curve of the rectangular beam from the extreme fiber down to the neutral axis, must necessarily be $\frac{VK_2^3}{3I}$.

By our assumption the $V's$ and $I's$ will cancel and the deflection of the **I** beam will be:

$$f=\left[1+\frac{3}{2}\,\frac{(K_2^2-K_1^2)\,K_1}{K_2^3}\left(\frac{t_2}{t_1}-1\right)\right]\frac{0.3Pl}{A_r F}$$

where,

$A_r=$ area of rectangle. This value is readily expressed in dimensions of the **I** beam for, since I of **I** beam $=I$ of rectangle $=2/3\,bK_2^3$,

$$b=\frac{3I}{2K_2^3} \text{ and } A_r=\frac{3I}{2K_2^3}\times 2K_2=\frac{3I}{K_2^2}$$

and

$$f=\left[1+\frac{\frac{3}{2}\,(K_2^2-K_1^2)\,K_1}{K_2^3}\left(\frac{t^2}{t_1}-1\right)\right]\frac{PlK_2^2}{10FI}$$

which may be written

$$f=\frac{KPl}{F} \text{ where } K=\left[1+\frac{\frac{3}{2}\,(K_2^2-K_1^2)\,K_1}{K_2^3\,.}\left(\frac{t^2}{t_1}-1\right)\right]\frac{K_2^2.}{10I}$$

The formula $\Delta=\frac{Pl^3}{48EI}+\frac{KPl}{F}$ can be applied to **I** and box sections of irregular shape by first reducing the given section to one of equivalent section, which is one whose height equals the mean height of the beam and whose flange areas equal those of the beam. By using K for the equivalent beam only a slight error will be introduced in the results.

TABLE I.—*Showing deflections determined by test compared with values computed by the formula.*

$$\Delta=\frac{Pl^3}{48EI}+\frac{0.3\,Pl}{A\,F}$$

DOUGLAS FIR BEAMS—NOMINAL 2¾ BY 5 INCHES—CENTER LOADING.

Span.	R G.			R H.			R K.			R L.			R M.		
	E=2105000.			E=1693000.			E=1595000.			E=2227000.			E=1968000.		
	Computed Δ	Test Δ	Error (per cent).	Computed Δ	Test Δ	Error (per cent).	Computed Δ	Test Δ	Error (per cent).	Computed Δ	Test Δ	Error (per cent).	Computed Δ	Test Δ	Error (per cent).
2 feet	0.0501	0.046	+8.9	0.0410	0.0405	+1.2	0.0421	0.0420	+0.2	0.0413	0.0410	+0.7	0.0275	0.0305	−9.8
3 feet	.0989	.0925	+6.9	.1072	.1058	+1.3	.0835	.0838	−0.3	.1026	.1037	−1.1	.0560	.0616	−9.0
4 feet	.1877	.190	−1.2	.2284	.2272	+0.5	.1789	.1805	−0.9	.1772	.1766	+0.2	.0982	.1023	−4.0
5 feet	.1888	.190	−0.6	.4228	.425	−0.5	.2652	.2705	−1.9	.226	.2287	−1.1	.1799	.1820	−1.2
6 feet	.2856	.275	+3.8	.3544	.354	+0.1	.3712	.3778	−1.7	.322	.3245	−0.8	.3004	.3062	−1.9
7 feet	.4432	.450	−1.5	.5525	.556	−0.6	.5793	.5900	−1.8	.4575	.4620	−1.0	.467	.474	−1.4
8 feet	.5211	.520	+0.2	.8146	.819	−0.5	.8549	.869	−1.6	.5495	.5485	+0.1	.549	.553	−0.7
9 feet	....	....	....	....	....	....	.8439	.854	−1.2	.6008	.594	+1.1	.677	.682	−0.8
10 feet	.625	.625	0	1.254	1.254	0	1.153	1.161	−0.8	.8176	.812	+0.7	.924	.927	−0.3
12 feet	.9612	.960	+0.1	1.343	1.343	0	1.411	1.417	−0.4	1.206	1.196	+0.8	1.354	1.353	−0.1
14 feet	1.5184	1.520	−0.1	1.486	1.486	0	1.563	1.566	−0.3	1.577	1.575	+0.1	1.425	1.429	−0.2

NOTE.—Each beam was tested over all the indicated spans. The error is expressed in per cent of the measured deflection. In the above formula—

Δ=deflection in inches.
P=load in pounds applied at the center.
I=moment of inertia of the section.
l=span in inches.
A=area of the cross section in square inches.
E=true computed modulus of elasticity.
F=the shearing modulus of elasticity taken in the computation as one-fifteenth the average true modulus of elasticity.

Let us now see how measured deflections compared with those computed by the formulas. Table I shows the results of tests on five rectangular Douglas-fir beams approximately 2¾ by 5 inches in section. True moduli of elasticity in bending were computed as outlined in this analysis and the average found to be 1,918,000 pounds per square inch. The modulus of elasticity in shear F was taken as one-fifteenth of this value, or 127,900 pounds per square inch. The beams were supported near the ends and loaded at the center. Computed deflections were obtained by substituting in the formula

$$\Delta=\frac{Pl^3}{48EI}+\frac{0.3\,Pl}{AF}$$

where A = area of the cross section.

The errors are expressed in percentage of the measured deflections. The average F was used for all beams, but in using E its value for each particular beam was substituted. An examination of the table shows that test and computed values agree remarkably well.

In Table II are given measured deflections for the **I** and box beams, sections of which are shown in Figure 1.

Deflections were computed by the usual formula

$$\Delta=\frac{Pl^3}{48EI}$$

and by the more exact formula

$$\Delta=\frac{Pl^3}{48EI}+\frac{KPl}{F}$$

where,

$$K \text{ is the quantity } \left[1+\frac{\frac{3}{2}(K_2{}^2-K_1{}^2)\,K_1\left(\frac{t_2}{t_1}-1\right)}{K_2{}^3}\right]\frac{K_2{}^2}{10I}.$$

The true modulus of elasticity in tension and compression was used in both formulas. The shearing modulus F was taken as 99,000 pounds per square inch, or about one-eighteenth the average true modulus of elasticity. Errors by the two formulas are expressed in per cent of the measured deflections. An examination of the table will show at a glance how much more closely the deflections can be estimated by the exact formula. For example, estimated values for a 3-foot span by the exact formula check test results within 0 to 12.1 per cent, whereas values by the ordinary formula are in error from 34.6 to 65.7 per cent.

TABLE II.—*Showing deflections determined by test compared with values computed by the two formulas.*

$$(1)\quad \Delta=\frac{Pl^3}{48EI}+\frac{KPl}{F}$$

$$(2)\quad \Delta=\frac{Pl^3}{48EI}$$

STANDARD I AND BOX BEAMS—CENTER LOADING—SITKA SPRUCE.

[Front F5L beams.]
F5AB 1997=mote.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	0.792	0.798	0.745	+0.8	−6.0
12	.630	.643	.586	+2.0	−7.0
10	.459	.464	.407	+1.1	−11.3
9	.451	.464	.396	+2.9	−12.2
8	.425	.424	.348	−0.2	−18.1
7	.354	.358	.279	−1.0	−21.2
6	.242	.244	.176	+0.8	−27.3
5	.232	.238	.153	+2.5	−34.1
4	.156	.162	.087	+3.8	−44.2
3	.091	.102	.040	+12.1	−56.1

F 5 CE 1742=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
18	1.518	1.521	1.47	+0.2	−3.3
16	1.079	1.076	1.031	−0.2	−4.4
14	.727	.731	.692	+0.5	−4.8
12	.659	.665	.617	+0.9	−6.3
10	.445	.444	.399	−0.2	−10.3
9	.212	.209	.184	−1.4	−13.1
8	.308	.303	.258	−1.6	−16.2
7	.287	.283	.230	−1.4	−19.8
6	.214	.214	.163	0	−23.3
5	.086	.084	.058	−2.3	−32.6
4	.081	.082	.048	+1.2	−40.7
3	.049	.051	.023	+4.0	−53.1

[Front Loening beams.]
L A A 1627=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	1.265	1.266	1.225	+0.1	−3.2
12	.730	.704	.674	−3.5	−7.6
10	.630	.594	.558	−5.6	−11.3
9					
8	.390	.378	.343	−3.1	−12.0
7	.570	.564	.498	−1.0	−12.6
6	.450	.456	.386	+1.3	−14.2
5	.305	.316	.251	+3.5	−17.7
4	.200	.201	.143	+0.5	−28.5
3	.069	.073	.042	+5.7	−39.1

L A B 1711=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	0.910	0.909	0.878	−0.1	−3.5
12	.870	.867	.828	−0.4	−4.8
10	.630	.626	.586	−0.6	−7.0
9					
8	.465	.453	.409	−2.6	−12.0
7	.470	.458	.402	−2.5	−14.5
6	.370	.356	.299	−3.8	−19.2
5	.249	.238	.187	−4.4	−24.9
4	.180	.175	.123	−2.8	−31.6
3	.112	.121	.069	+8.0	−38.4

[Rear F5L beams.]
F5AC 2093=mote.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	0.800	0.802	0.771	+0.2	−3.6
12	.792	.768	.728	−3.0	−8.0
10	.798	.758	.703	−5.0	−11.9
9	.580	.561	.511	−3.2	−11.9
8	.498	.484	.431	−2.8	−13.4
7	.347	.335	.289	−3.5	−16.7
6	.298	.296	.243	−0.6	−18.4
5	.235	.231	.176	−1.6	−25.1
4	.162	.161	.108	−0.6	−33.4
3	.098	.099	.053	+1.0	−45.9

F 5 CF 1586=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
18	2.131	2.124	2.08	−0.4	−2.4
16	1.517	1.499	1.464	−1.2	−3.4
14	.767	.759	.736	−1.0	−4.0
12	.976	.966	.926	−1.0	−5.1
10	.666	.665	.626	−0.2	−6.0
9	.391	.385	.358	−1.5	−8.4
8	.354	.351	.320	−0.8	−9.6
7	.246	.242	.215	−1.6	−12.6
6	.185	.181	.154	−2.1	−16.8
5	.249	.278	.223	+11.5	−10.4
4	.077	.079	.057	+2.5	−26.0
3	.055	.057	.034	+3.6	−38.2

[Rear Loening beams.]
L A C 1640=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	1.800	1.803	1.761	+0.1	−2.2
12					
10	.960	.961	.918	+0.1	−4.4
9					
8	.500	.505	.470	+1.0	−6.0
7	.680	.690	.629	+1.4	−7.4
6	.400	.404	.357	+1.0	−10.8
5	.340	.354	.298	+4.0	−12.3
4	.170	.167	.129	−1.8	−24.1
3	.136	.136	.089	0	−34.6

L A B H 2315=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	1.212	1.218	1.182	+0.4	−2.5
12	.783	.776	.745	−0.9	−4.8
10	.908	.914	.862	+0.6	−5.0
9	.669	.675	.628	+0.9	−6.2
8	.473	.482	.441	+1.9	−6.8
7	.433	.444	.395	+2.5	−8.8
6	.428	.435	.372	+1.6	−13.0
5	.214	.223	.180	+4.1	−15.9
4	.142	.152	.110	+7.0	−22.6
3	.095	.104	.062	+9.4	−34.8

[Front TF beams.]
T F Y 1620=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	1.000	1.004	0.942	+0.4	−5.8
12	.823	.807	.741	−1.9	−10.0
10	.584	.581	.515	−0.5	−11.8
9	.512	.507	.438	−0.9	−14.5
8	.424	.422	.352	−0.5	−17.0
7	.368	.371	.294	+0.8	−20.1
6	.196	.201	.148	+2.5	−24.5
5	.153	.162	.107	+5.8	−30.1
4					
3					

[Front NC beams.]
N C Y 1728=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	0.665	0.676	0.614	+1.6	−7.6
12	.530	.549	.483	+3.6	−8.9
10	.391	.402	.336	+2.7	−14.1
9	.346	.355	.285	+2.6	−17.6
8	.365	.375	.286	+2.7	−21.6
7	.260	.269	.192	+3.4	−26.2
6	.186	.187	.121	+0.5	−35.5
5	.182	.187	.105	+2.7	−42.3
4	.117	.120	.054	+2.5	−53.9
3	.067	.072	.023	+7.4	−65.7

[Rear TF beams.]
T F Z 1640=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	0.919	0.914	0.875	−0.5	−4.8
12	.747	.731	.688	−2.1	−7.8
10	.354	.347	.319	−2.0	−9 9
9	.389	.386	.348	−0.7	−10.5
8	.372	.372	.328	0	−12.4
7	.257	.258	.219	+0.4	−14.8
6	.175	.171	.138	−2.2	−21.1
5	.096	.094	.070	−2.1	−27.1
4	.096	.095	.061	−1.0	−36.4
3	.071	.077	.039	+8.4	−45.0

T F D G 1954=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
18	0.783	0.788	0.763	+0.6	−2.5
16	1.115	1.117	1.072	+0.1	−3.8
14	.659	.663	.629	+0.5	−4.5
12	.416	.425	.396	+2.2	−4.8
10	.425	.435	.393	+2.3	−7.5
9	.264	.270	.238	+2.2	−9.8
8	.188	.196	.167	+4.2	−11.1
7	.208	.219	.180	+5.3	−13.4
6	.169	.184	.141	+8.8	−16.6
5	.095	.105	.074	+10.5	−22.1
4	.078	.091	.054	+16.6	−30.8
3					

[Rear NC beams.]
N C Z 1368=M of E.

Span.	Deflection. Measured.	Deflection. By (1).	Deflection. By (2).	Error (per cent). By (1).	Error (per cent). By (2).
14	1.006	1.026	0.962	+2.0	−4.4
12	.791	.825	.756	+4.3	−4.4
10	.478	.495	.438	+3.5	−8.3
9	.430	.445	.383	+3.5	−11.0
8	.412	.433	.359	+5.0	−12.8
7	.370	.381	.301	+2.9	−18.6
6	.249	.258	.189	+3.6	−24.2
5	.173	.184	.120	+6.4	−30.6
4	.143	.153	.084	+7.0	−41.3
3					

NOTE.—Each beam was tested over all the indicated spans. The error is expressed in per cent of the measured deflection. An accompanying figure shows sections of the beams. In the above formulæ—

Δ = deflection in inches.
P = load in pounds applied at the center.
I = moment of inertia of the section.
l = span in inches.
A = area of the cross section in square inches.
E = true computed modulus of elasticity.
F = the shearing modulus of elasticity taken in the above computations as 99,000 pounds per square inch.

The great difference in the shearing modulus of elasticity of ply-wood webs with the grain at 45° to the length of a beam and with the grain of face plies perpendicular to the length of the beam is well illustrated in Figure 7. The section of the beam is that of the double **I** shown in Figure 1. A pair of beams were matched throughout, the only difference in the two being in the direction of the grain of the ply-wood webs. Both were tested over spans from 2 to 14

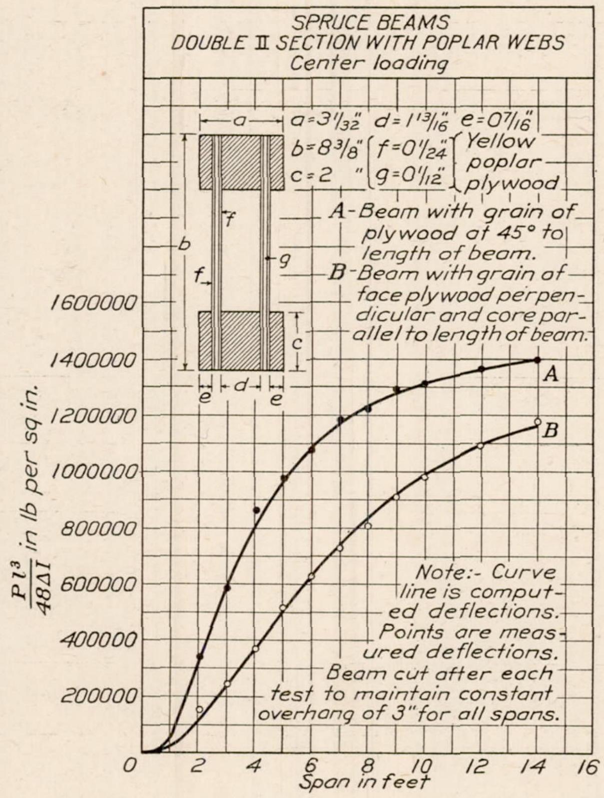


FIG. 7.—Relation of span to value obtained by substituting deflections in $\frac{Pl^3}{48\Delta I}$.

feet, and the points indicate the results of these tests. The full lines were obtained by substituting in the formula

$$\Delta=\frac{Pl^3}{48EI}+\frac{KPl}{F}.$$

For the beam having ply-wood webs with the grain at 45° to the length of the beam, 353,000 pounds per square inch was used for F, and for the beam in which the face grain of the ply wood was perpendicular to the length of the beam, 99,000 pounds per square inch was used, the shearing modulus in the former case being over three and one-half times that required in the latter case.

With the aid of the complete deflection formula we can determine the error for any span introduced by neglecting shear deformations.

Now, in substituting measured deflections in $\frac{Pl^3}{48\Delta I}$, the ordinary formula for center loading, we get:

$$E_c=\frac{Pl^3}{48I\left(\frac{Pl^3}{48E_TI}+\frac{KPl}{F}\right)}$$

since, as shown above:

$$\Delta=\frac{Pl^3}{48E_TI}+\frac{KPl}{F}.$$

This value E_c has been plotted for various spans in Figure 8 for a rectangular beam and for a few standard **I** and box sections E_T was taken as 100 per cent and F as $\frac{E_T}{17.5}$.

E_T = true modulus of elasticity.

$E_c = \frac{Pl^3}{48\Delta I}$ where Δ = measured deflection.

F = modulus of elasticity in shear.

K = a constant for the section. Taking F for spruce $= \frac{E_T}{17.5}$.

For extremely short spans in which the shear deformation might be as much as one-half the total deformation we might anticipate that deflections of beams loaded at the third point would give considerably different values for E_c when substituted in the usual formula than would deflections for beams loaded at the center. The shear deformation in both cases is proportional

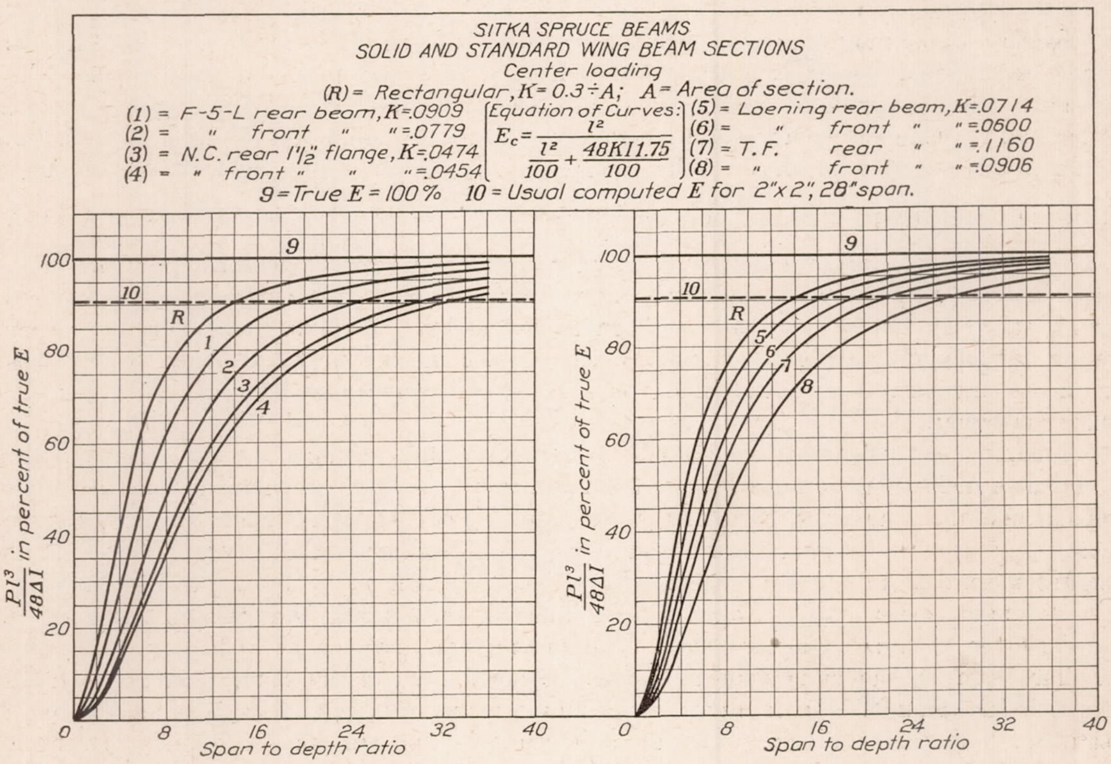


FIG. 8.—Relation of span-depth ratio to value obtained by substituting measured deflections in $\frac{Pl^3}{48\Delta I}$.

to the stress, but for equal stresses the deflection of a beam loaded at the third points is greater by $\frac{23}{18}$. Assuming the deformation due to shear in the case of the beam loaded at the center 0.50 of the total deflection, E_c would be 50 per cent in error. Then for the third-point loading the shear deformation is numerically the same because of equal stress, but the deflection due to change in the length of the fibers is $\frac{23}{18}$ as much as in the former case and our error is now approximately 44 per cent, or a difference of only 6 per cent, and this only in an extreme case. For all practical purposes we could neglect this difference and assume our error equal in the two cases.

An examination of Figures 3, 4, and 8 would indicate that the moduli of elasticity given in our Bulletin 556 for small clear specimens tested over a span 14 times the depth of specimen are about 10 per cent below the true modulus of elasticity in tension and compression. This is true; it is a value obtained by substituting measured deflections in the usual deflection formula neglecting shear deformation. However, if this value is in turn used to estimate the deflection of a solid rectangular beam by substituting in the usual formula we arrive at the correct deflection provided our span is 14 times the depth. For ordinary spans, say from 12 to 28 times the depth, the error would be within 5 per cent. For rectangular beams used in ordinary lengths then we would not vitiate our results to any great extent by using these values of modulus of elasticity in the usual formula.

In the design of box and **I** sections with relatively little material at the plane of maximum horizontal shear, however, very considerable errors will occur even for large span-depth ratios unless the more accurate method of determining the elastic properties of a beam is employed. For some sections tested the error introduced at a span of 14 times the depth was over 35 per cent as against 10 per cent for a solid rectangular beam.

CONCLUSIONS.

Because of the magnitude of shear distortions it is often necessary to calculate the elastic properties of wood beams by formulas which take into account such distortions. This is especially true for box and **I** beams which have the material distributed in a way to take care of maximum tensile and compressive stresses, which means a minimum of material at the plane of maximum longitudinal shear. The shear deformation is proportional to the moment to which the beam is subjected and may be expressed by $\frac{KPl}{F}$, where P is the load on a beam of span l, F is the modulus of elasticity in shear, and K a coefficient depending upon the shape of the cross section and upon the loading. Two formulas for the determination of K have been developed. The first is a rather long formula developed by ordinary methods, the second a simpler formula and more empirical in its nature. Both check experimental results very closely, but the second formula is recommended because its use involves less labor and offers less opportunity for error.

Usually shear deflections are neglected, and deflection determined by test when substituted in the usual deflection formulas will give a modulus of elasticity less than the tension and compression modulus, the error increasing as the span is reduced. The elastic properties given in such tables as are included in Bulletin 556 were determined in this way. These standard bending specimens have a span depth ratio of 14, for which ratio the modulus of elasticity in shear is about 10 per cent below the true modulus in tension and compression.

However, if these values are used in design they will give correct deflections for solid rectangular beams of the same span-depth ratio if substituted in the usual formulas with which they were determined. Furthermore, for ordinary spans, say from 12 to 28 times the depth of beam, they will give values correct within 5 per cent. For shorter spans it would be preferable to use the more exact formulas which take into account shear deformations. There is very little difference in the errors for center and third-point loading. For beams of **I** and box section shear distortions are far more pronounced and errors of considerable magnitude will be introduced even for large span-depth ratios unless the exact formulas are employed.

Box beams with ply-wood webs have a greater modulus of rigidity with the grain of the plywood at 45° to the length of the beam than with the grain of the face plies perpendicular to the length. Tests showed the former type to have a modulus of rigidity over three and one-half times the latter type.

APPENDIX.

The development of the formulas for shear deformations.

BEAMS OF SOLID RECTANGULAR SECTION.

Let us assume first a rectangular beam supported near the ends and with a concentrated load at the center.

Let

q = unit shearing stress.
V = total vertical shear.
I = moment of inertia of section.
b = thickness of section.
d = depth of section.
y = distance from neutral axis.
F = modulus of elasticity in shear.
f = deflection due to shear.

We have,

$$q=\frac{V}{Ib}\int bydy,$$

a well-known formula, which gives a distribution as shown in Figure 6, curve A. This gives

$$q=\frac{V}{I}\times\frac{1}{b}\int_{y}^{d/2}bydy=\frac{V}{8I}(d^2-4y^2).$$

Now, the unit shearing stress q produces a deformation $\frac{q}{F}$ in planes at unit distance apart. The work in shear per unit of volume, therefore, is

$$\frac{q}{2}\times\frac{q}{F}=\frac{q^2}{2F}$$

$$\frac{q^2}{2F}=\frac{V^2(d^4-8d^2y^2+16y^4)}{128FI^2}.$$

Multiplying by the element of volume $b\ dy\ dx$ and first integrating with respect to y with limits $-d/2$ and $+d/2$

$$\text{Internal work}=\int\frac{V^2bd^5}{128FI^2}\times\frac{8}{15}dx=\int\frac{3}{5}\frac{V^2dx}{Fbd}.$$

In the case assumed V is a constant and the expression becomes

$$\text{Internal work}=\frac{3}{5}\frac{V^2l}{bdF}$$

Now, for a beam supported near the ends and loaded at the center $V=P/2$ and the external work is $\frac{Pf}{2}$.

We may write therefore:

$$\frac{Pf}{2}=\frac{3P^2l}{5\times4\times bdF}$$

$$f=\frac{0.3Pl}{bdF}.$$

If Δ = the total deflection we then have for a solid rectangular beam loaded at the center

$$\Delta=\frac{Pl^3}{48EI}+\frac{0.3Pl}{AF}\text{ where }A=bd.$$

In the case of a cantilever beam we would have $V=P$ and

$$f=\frac{1.2\,Pl}{bdF}\text{ and }\Delta=\frac{Pl^3}{3EI}+\frac{1.2Pl}{AF}$$

for a solid rectangular beam. For beam supported at the ends and loaded equally at the third points

$$f=\frac{0.4P'l}{bdF}$$

where,

$$P'=\text{load at each third point,}$$

or

$$f=\frac{0.2Pl}{bdF}$$

where,

$$P=\text{total load.}$$

Similarly, we may show that for a uniformly distributed load P

$$f=\frac{0.15Pl}{bdF}$$

So far these expressions for shear deformations apply only to beams of rectangular section.

I OR BOX BEAMS.

Let us now examine an **I** beam or, what is practically the same, a box beam. The following notations will be used in addition to those already given:

K_2 = distance neutral axis to extreme fiber.
K_1 = distance neutral axis to inner edge of flange.
t_2 = width of flange.
t_1 = thickness of web; in box beams combined thickness of webs.

In the flange:

$$q=\frac{V}{It_2}\int_y^{K_2}t_2y\,dy=\frac{V}{2I}\,(K_2^2-y^2).$$

In the web:

$$q=\frac{V}{It_1}\left[\int_{K_1}^{K_2}t_2y\,dy+\int_y^{K_1}t_1y\,dy\right].$$

The distribution of shearing stress will be as shown in Figure 6, curve B.
The internal work per unit volume is

$$\frac{q^2}{2F}\,da\,dx$$

where,

$$da=t\,dy.$$

Assuming a beam of length l, loaded at the center with a load P, the external work $= Pf/2$ and since the external work equals the internal work:

$$Pf/2 = 2 \cdot \frac{l}{2F} \int_0^{K_2} q^2 t\, dy$$

or

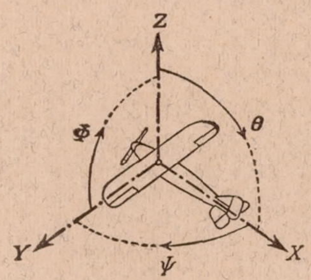
$$\frac{Pf}{2} = \frac{l}{F}\left[\frac{t^2 P^2}{16 I^2}\int_{K_1}^{K_2} (K_2^4 - 2K_2^2 y^2 + y^4)\, dy + \frac{t_1 P^2}{16 I^2 t_1^2}\int_0^{K_1} t_2^2 (K_2 - K_1)^2 dy + 2t_1 t_2 (K_1^2 K_2^2 - K_2^2 y^2 - K_1^4 + K_1^2 y^2)\, dy + t_1^2 (K_1^4 - 2K_1^2 y^2 + y^4)\, dy\right].$$

Integrating with respect to y and substituting the limits and $\frac{P}{2}$ for V we obtain:

$$f = \frac{Pl}{8FI^2}\left[t_2\left(\frac{8}{15} K_2^5 - K_2^4 K_1 + 2K_2^2 K_1^3 - \frac{23}{15} K_1^5\right) + \frac{t_2^2}{t_1}(K_2^4 K_1 - 2K_1^3 K_2^2 + K_1^5) + t_1 \frac{8}{15} K_1^5\right].$$

Note that for the limiting condition when $K_1 = K_2$ and $t_1 = t_2$, we get $f = \frac{0.3\, Pl}{bd\, F}$, which has already been determined for a rectangular beam loaded at the middle.

O



Positive directions of axes and angles (forces and moments) are shown by arrows.

Axis.		Force (parallel to axis) symbol.	Moment about axis.			Angle.		Velocities.	
Designation.	Symbol.		Designation.	Symbol.	Positive direction.	Designation.	Symbol.	Linear (component along axis).	Angular.
Longitudinal....	X	X	rolling.....	L	$Y \longrightarrow Z$	roll.....	Φ	u	p
Lateral..........	Y	Y	pitching...	M	$Z \longrightarrow X$	pitch....	Θ	v	q
Normal..........	Z	Z	yawing.....	N	$X \longrightarrow Y$	yaw.....	Ψ	w	r

Absolute coefficients of moment

$$C_l = \frac{L}{q\,b\,S} \quad C_m = \frac{M}{q\,c\,S} \quad C_n = \frac{N}{q\,f\,S}$$

Angle of set of control surface (relative to neutral position), δ. (Indicate surface by proper subscript.)

4. PROPELLER SYMBOLS.

Diameter, D
Pitch (a) Aerodynamic pitch, p_a
(b) Effective pitch, p_e
(c) Mean geometric pitch, p_g
(d) Virtual pitch, p_v
(e) Standard pitch, p_s
Pitch ratio, p/D
Inflow velocity, V'
Slipstream velocity, V_s

Thrust, T
Torque, Q
Power, P
(If "coefficients" are introduced all units used must be consistent.)
Efficiency $\eta = T\,V/P$
Revolutions per sec., n; per min., N
Effective helix angle $\Phi = \tan^{-1}\left(\frac{V}{2\pi r n}\right)$

5. NUMERICAL RELATIONS.

1 HP = 76.04 kg. m/sec. = 550 lb. ft/sec.
1 kg. m/sec. = 0.01315 HP
1 mi/hr. = 0.44704 m/sec.
1 m/sec. = 2.23693 mi/hr.

1 lb. = 0.45359 kg.
1 kg. = 2.20462 lb.
1 mi. = 1609.35 m. = 5280 ft.
1 m. = 3.28083 ft.