# REPORT No. 214

## WING SPAR STRESS CHARTS AND WING TRUSS PROPORTIONS

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### INTRODUCTION

Although the coming of the thick airfoil section has somewhat decreased the number of airplanes designed with continuous wing spars externally supported at several points, that type of construction has not by any means disappeared. The truss continuous through two or three bays is still commonly used, and the calculation of continuous beams is still making heavy inroads upon the time of the designer. With the objects of reducing the labor involved in such calculations and of deriving some general conclusions on the properties of continuous beams, the curves described in this report have been prepared for publication by the National Advisory Committee for Aeronautics. In presenting them to the public, the writer takes the opportunity of acknowledging the assistance of Mr. Otto C. Koppen, who has done a very considerable proportion of the work of preparation of the material.

### SUMMARY

In order to simplify calculation of beams continuous over three supports, a series of charts have been calculated giving the bending moments at all the critical points and the reactions at all supports for such members. Using these charts as a basis, calculations of equivalent bending moments, representing the total stresses acting in two bay wing trusses of proportions varying over a wide range, have been determined, both with and without allowance for column effect. This leads finally to the determination of the best proportions for any particular truss or the best strut locations in any particular machine. The ideal proportions are found to vary with the thickness of the wing section used, the aspect ratio, and the ratio of gap to chord.

### BENDING MOMENT CHARTS

Of all the wing cells built with spars continuous over three or more supports, at least 75 per cent of the total number involve calculation for three supports only. If the loading per unit length of spar be assumed uniform in such a case there are only three variables which affect the bending moments, reactions, and bending stresses for unit loading. Those quantities are dependent only on the length of the inner bay, the length of the outer bay, and the length of the effective overhang, and if all results be reduced to a common total length, as can easily be done, one of these three variables disappears and curves of moment, reaction, and stress can be plotted in terms of the remaining two.

With the object of simplifying the calculation of two-bay continuous trusses and of making it apparent at a glance what gain or loss can be expected from a change of arrangement of the wing bracing, a number of continuous beams have been calculated and curves have been plotted from which it is possible to read off at once the results for any case. The calculations were all based on the assumptions of a uniform loading of 1 pound per inch length of spar and of a total length of spar of 100 inches. The spar was assumed to be held by a horizontal pin at its inner end, so that the bending moment there was zero. The bending moments for all cases on these assumptions are plotted in Figures 1 and 2, the choice between the two sets of curves in any case depending on their relative convenience for the particular problem in hand. In Figure 1, curves of the absolute values of the bending moments (signs being ignored) at the outer support, at the middle support, and at the point of maximum moment in the middle of each bay have been plotted against the length of overhang, each curve relating to a particular assumed value of the length of the inner bay. In Figure 1, as everywhere else in this text, $A$ denotes the outer support, $B$ the middle one, and $C$ the innermost. $M_B$ therefore represents the bending moment at the inner strut of a two-bay wing truss, $M_{CB}$ that in the middle of the inner bay. In Figure 2 the same thing has been done, but with the length of the inner bay used as the abscissa and with a separate set of curves for each length of overhang (the curves being separated by intervals of 5 per cent of the total length, or 5 inches in a 100-inch spar, in both cases).

The simplicity of the application of these charts can best be illustrated by immediate solution of an illustrative problem. Supposing a spar to have an inner bay 27 inches long, an outer bay 52 inches long, and a 21-inch effective overhang, the bending moments are read off from Figure 1 by going up along an ordinate at the abscissa corresponding to the length of over-

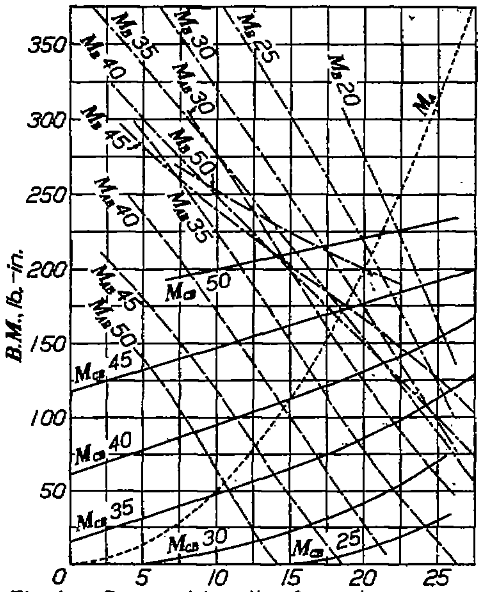


Fig. 1

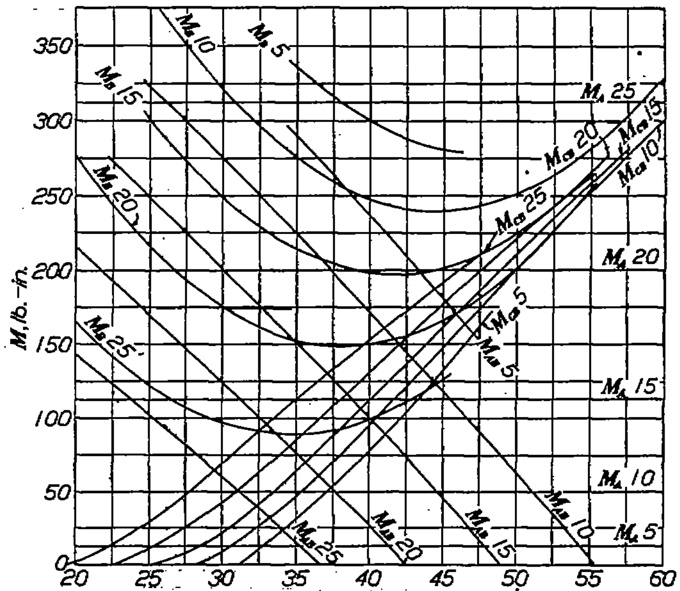


Fig. 2

hang until a point two-fifths of the way from the 25 per cent to the 30 per cent curve is reached. The bending moments are found to be:

222 lb. ft. at the outer support;
179 lb. ft. at the middle support;
140 lb. ft. in the outer bay;
27 lb. ft. in the inner bay.

The moments in the bays are, of course, of opposite sign from those at the supports. Exactly the same results can be obtained from Figure 2 by running up along the 27 per cent line to an interpolation between the 20 per cent and 25 per cent curves.

As a rule, of course, the loading is not equal to unity, and the length of the beam is not 100 inches. In more general cases the bending moments read from the curves can be corrected by multiplying by the actual intensity of loading and by the square of the ratio of the length to 100 inches.

In Figures 3 and 4 the same work has been done for the reactions at the three supports. Taking again the problem just solved, the reactions can be read off directly as 48 pounds at the outer support, 45 pounds at the middle one, and 7 pounds at the innermost. The corrections to be applied are the same as before, except that the index values of the reactions are multiplied by the direct ratio of the lengths instead of the square of the ratio.

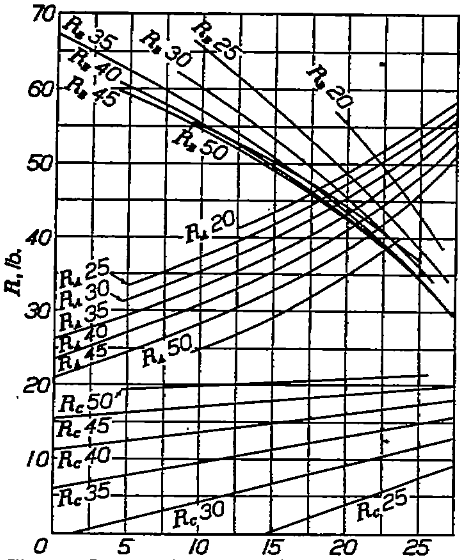


Fig. 3 Per cent length of overhang

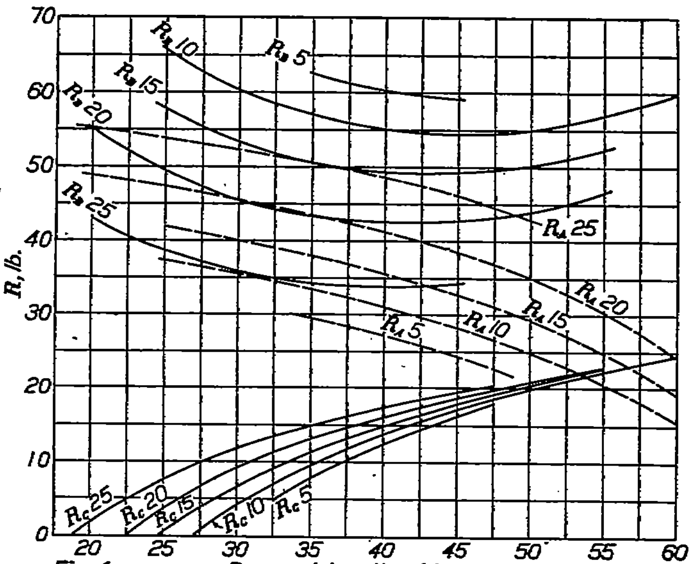


Fig. 4 Per cent length of inner bay

In Figures 5 and 6 the moments and reactions are similarly given for the case of complete fixity of the spar at the inner end. They correspond, in the method of plotting, to Figures 1 and 3. Although it is very rare for a fitting to be used which holds the spar so firmly that the slope is actually unchanged under load, partial fixity is common, and its effect can readily be determined by comparing the figures for complete fixity and complete freedom and taking an intermediate value.

In addition to facilitating the calculation of bending moments and reactions, such charts serve as the basis for calculations of total stress and for a study of the effect of a change in the spacing of interplane struts, as the compressive or tensile stress may readily be thrown in with that due to bending.

## TOTAL STRESS CHARTS

The total stress in a spar is given by the familiar formula:

$$f=\frac{My}{I}+\frac{P}{A}$$

which can also be written for sections symmetrical about the neutral axis in the form

$$f=\frac{1}{A}\left(\frac{Md}{2k^2}+P\right)$$

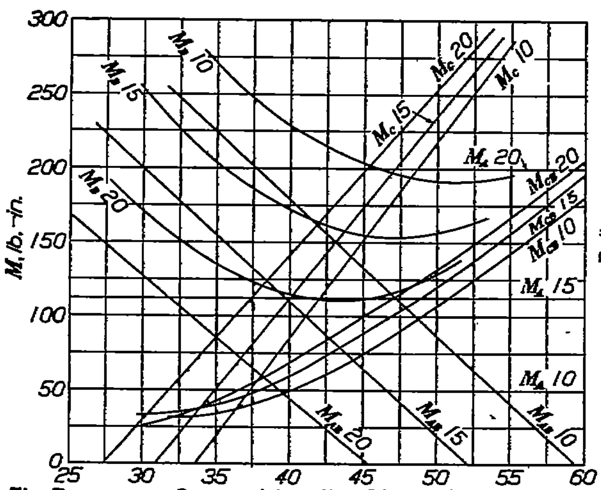


Fig. 5 Per cent length of inner bay

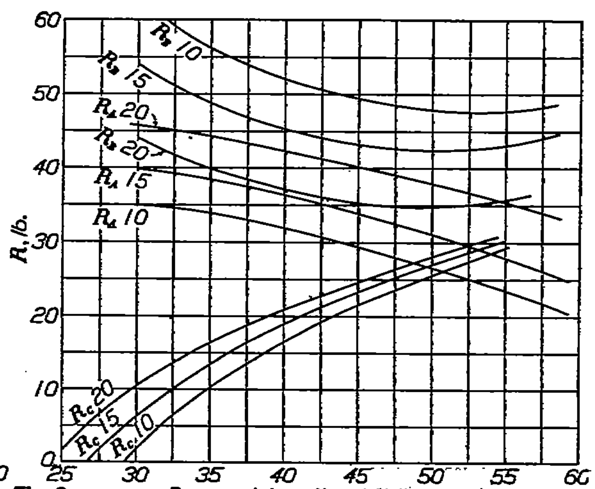


Fig. 6 Per cent length of inner bay

where $d$ is the total depth of the spar. It has been shown by the writer [1] that the radius of gyration about the neutral axis for a spar of conventional section is in the neighborhood of $.36d$. Substituting that value, the stress equation becomes:

$$f=\frac{1}{A}\left(\frac{M}{.26d}+P\right)=\frac{y}{I}(M+.26Pd)$$

The direct stress in a spar, for any given loading and arrangement of strut locations, is inversely proportional to the gap. The total stress is therefore made up of two components, one of which varies inversely as the spar depth and the other inversely as the gap, and their sum, for any given area of section, strut arrangement, and loading, is a function of the ratio of gap to depth of spar, a ratio which may range in magnitude from 6.5, with a thick airfoil section and a low gap-chord ratio, to 24 at the other extreme of design practice. Usually, however, it lies between 12 and 20. For any given value of that ratio, curves of total stress times section modulus, or of equivalent bending moment, can be plotted as those for actual bending moment have already been plotted without regard to the proportions of the wing truss in any respect other than strut spacing.

As an incident to the calculation of equivalent bending moments the compressions in the spar in the two bays were of course calculated, and the compressions in the outer bay of the upper spar (numerically equal to the tensions in the inner bay of the lower spar if the interplane struts are vertical) are plotted in Figure 7. The figures there given must be multiplied by the total length of the spar, by the ratio of the length of the spar to the gap, and by the unit loading. Furthermore, they are based on an assumption of equal area and similar strut location in the upper and lower wings, and biplane loading correction factors were ignored in calculating them, so that the reactions of the upper and lower spars at a given strut point were taken as identical. This method is sound if the struts are vertical and if the unit loading used as a coefficient for the plotted values is the mean of the loadings on the two wings.

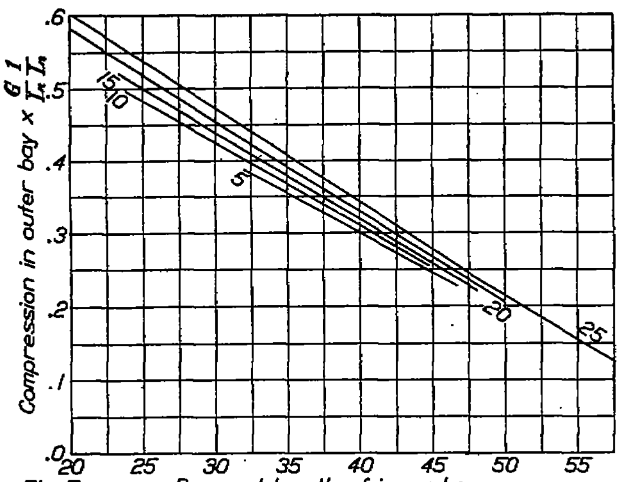

Fig. 7

It is unnecessary to plot the compression in the inner bay, as its value is independent of strut location if the spars are pin jointed at the inner end. The moment of the compression in the upper spar about the lower hinge pin must be numerically equal to the total moment about the same axis of the air loads applied on the two spars, and it is therefore a constant, wherever the struts may be placed. The compression in the inner bay with a unit loading is always equal to the product of the length of the spar by the ratio of spar length to gap, the coefficient analogous to that plotted in Figure 7 being unity.

In plotting the equivalent moments, instead of drawing separate curves for the two supports and the two bays, as in the case of the actual moments, only the largest absolute value has been retained for each strut arrangement, and curves of constant value of maximum moment have then been drawn with length of overhang as ordinate and length of inner bay as abscissa. Such curves are more useful, for this particular purpose, than the type previously drawn, for the plots of equivalent moment are intended to serve as a guide to the securing of maximum structural efficiency in design by the choice of an optimum strut location, rather than as a direct aid in routine calculation. The equivalent moment curves for the upper spar

[1] The Design of Wing Spar Sections, by Edward P. Warner, Aviation, May 29, 1922.

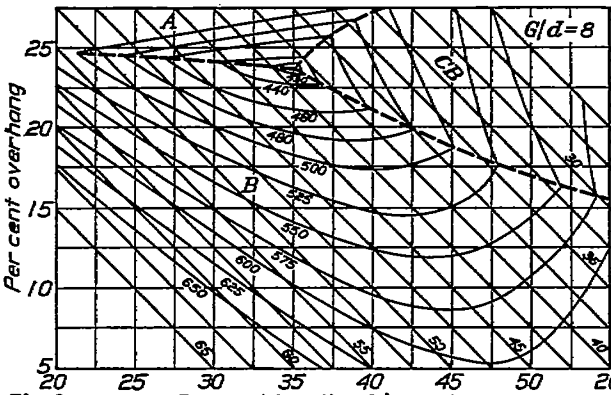


Fig. 8 Per cent length of inner bay

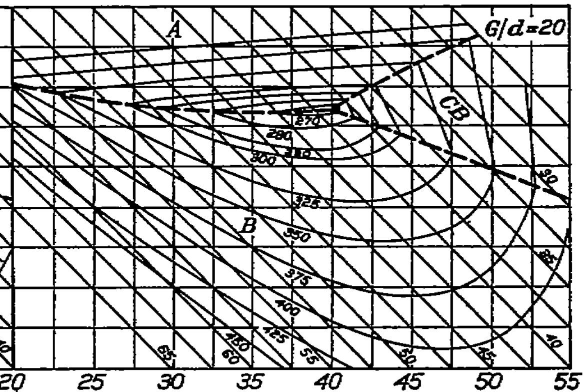


Fig. 9 Per cent length of inner bay

at two values of $G/d$ are given in Figures 8 and 9, and those for the lower spar for a single value in Figure 10. The diagonals sloping downward and to the right give the length of the outer bay.

It will be observed that each of the three charts of equivalent bending moment is divided into three parts by dotted lines. The lines represent the transfer of worst stress from one point in the spar to another, and the point in the spar at which the worst stress is found is indicated by a symbol in each zone of each chart. An overhang length of 22 per cent combined with an inner bay of 35 per cent, for example, would fall in the zone marked $A$ in Figure 9 and in that marked $B$ in Figure 8, signifying that the maximum equivalent bending moment in the upper spar falls at the outer strut when $G/d$ is 20 but at the inner strut point when $G/d$ is only 8. There is, of course, an abrupt break in the form and slope of each envelope curve where it passes from one zone to another.

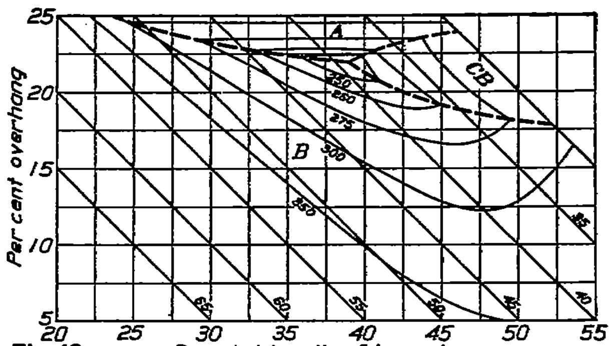


Fig. 10 Per cent length of inner bay

Spruce, the material most commonly used in wing spar construction, shows an exceptionally large difference between ultimate stress in straight compression and modulus of rupture. For rectangular specimens the ratio is about 1.8, the bending strength of course being the larger, but with I and box spars of the proportions ordinarily used the inclusion of a form factor for bending causes the ratio to fall off to about 1.4 on the average. An increase of the proportion of bending stress will then increase the total allowable stress, and an increase of 140 pounds per square inch in bending stress can be balanced by a reduction of 100 pounds per square inch in compression, leaving the factor of safety unchanged. This can be allowed for in drawing charts of equivalent bending moment by multiplying the compressive stress by 1.4, and that has been done in Figures 11 and 12, which otherwise correspond to Figures 8 and 9. Figures 8 and 9 hold most nearly for metal spars; Figures 11 and 12 for spruce.

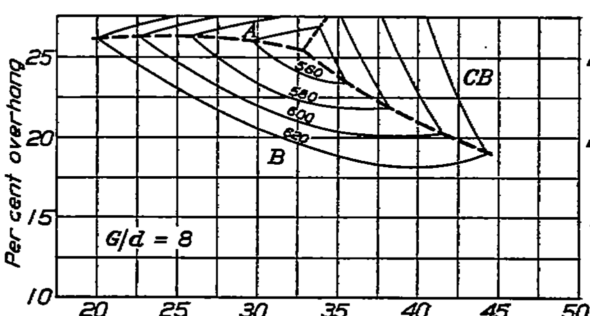


Fig. 11 Per cent length of inner bay

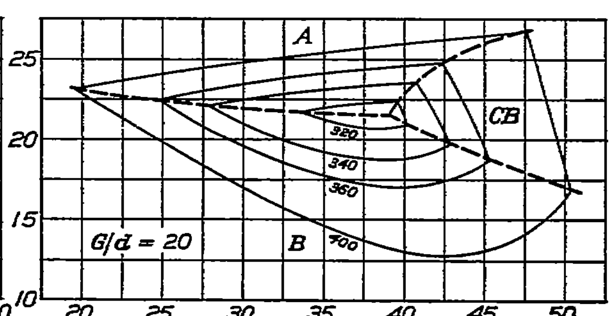


Fig. 12 Per cent length of inner bay

## ALLOWANCE FOR COLUMN EFFECT

In making all these calculations the actual values of all four sets of moments have been taken as on a parity in finding the maximum equivalent moment to enter in the chart. When the distance between bays is great in proportion to the depth of the spar, however, the liability of buckling becomes an important factor, and a bending moment of given magnitude in the middle of a bay is much more serious than one equally large at a strut point. The exact effect of buckling in increasing the liability to failure in the middle of a span is not susceptible of simple treatment, but a satisfactory approximation for most cases can be made by the use of Perry's formula,

$$M' = M \times \frac{P_e}{P_e - P}$$

where $M'$ is the corrected bending moment, $M$ the original bending moment due to lateral loading alone and without allowance for column effect, $P$ the compression in the spar, and $P_e$ the collapsing load under pure compression as calculated by Euler's formula, the length of the column being taken as the distance between points of inflection in the spar and the ends being considered as pin jointed.

The ratio of distance between points of inflection to total length of bay and the ratio of compression stress to total stress both vary widely with interplane strut spacings. Taking account of these variations, the formula for corrected bending moment can be written

$$M' = M \times \frac{1}{1 - \dfrac{\frac{f_c}{f'_t} \times f'_t A l^2}{\pi^2 E k^2 A}} = \frac{M}{1 - \dfrac{\frac{f_c}{f'_t} \times f'_t l^2}{\pi^2 E k^2}} \qquad \text{(A)}$$

where $l$ is the length between the points of inflection, $A$ the cross-sectional area of the spar, $k$ the radius of gyration of the spar section, $f_c$ the compressive stress in the material due to direct compressive load, and $f'_t$ the total stress. In the case of a spruce spar the total stress may be assumed to be 5,500 pounds per square inch at failure, taking a form factor of approximately .8, assuming a 15 per cent moisture content, and on the further assumption that the ratio of bending stress to compressive stress is approximately two to one. The ultimate stress in the material will of course vary with this ratio, but if the attempt is made to deal separately with the strengths in compression and bending the expression becomes somewhat complex. If a value of 5,500 then be assumed, and $E$ be taken as 1,600,000, the expression for corrected bending moment can be reduced to

$$M' = \frac{M}{1 - \dfrac{\frac{f_c}{f'_t}\left(\frac{l}{L}\right)^2 L^2}{2{,}900\ (k/d)^2\ d^2}}$$

where $L$ is the total length of bay, $d$ the depth of the spar, and the other symbols have the same significance as before. The writer has previously shown[2] that $\frac{k}{d}$, for typical spar sections, s about .36. If this value be used, the expression becomes

$$M' = M \times \frac{1}{1 - \dfrac{\frac{f_c}{f'_t}\left(\frac{l}{L}\right)\left(\frac{L}{d}\right)^2}{375}}$$

The total stress as used in these formulas is, of course, that due to the final corrected value of the bending moment, so that

$$f'_t = \frac{M'y}{I} + \frac{P}{A}$$

[2] Aviation, loc. cit.

It is desirable, however, that the solution for $M'/M$ should be direct and simple, and should involve only quantities dependent on the geometrical properties of the spar alone. $M'$ should appear only in the final result. This end could be attained if it were assumed that $f'_t=f_t\,\frac{M'}{M}$ where $f_t$ is the total stress which would exist in the spar if there were no buckling effect, or

$$f_t=\frac{My}{I}+\frac{P}{A}$$

The general equation for column effect correction would then become

$$\frac{M'}{M}=\frac{1}{1-\dfrac{\frac{f_c}{f_t}\frac{M}{M'}f'_t\,l^2}{\pi^2\,Ek^2}}$$

or

$$\frac{M'}{M}=1+\frac{\frac{f_c}{f_t}f'_t\,l^2}{\pi^2\,Ek^2}$$

The correction factors given by this formula are somewhat too low, while those obtained from the form (A), using $\frac{f_c}{f_t}$ in place of $\frac{f_c}{f'_t}$, are too high, in some cases very much too high. The obvious solution is to use a formula intermediate between the two, such as

$$\frac{M'}{M}=\frac{1+\dfrac{\frac{f_c}{f_t}f'_t\,l^2}{2\pi^2\,Ek^2}}{1-\dfrac{\frac{f_c}{f_t}f'_t\,l^2}{2\pi^2\,Ek^2}}$$

The mathematical justification of this procedure need not be given. It is sufficient to say that the compromise formula finally arrived at, although admittedly only an approximation, is found by trial to give results satisfactorily close to the truth in the typical cases to which it has been applied, and for which its results have been directly compared with those obtained by actual calculation from a particular set of figures. Such error as does exist is almost always in the direction of safety, the formula giving too large a correction factor.

In the particular case of a spruce spar, the formula becomes

$$\frac{M'}{M}=\frac{750+\left\{\frac{f_c}{f_t}\left(\frac{l}{L}\right)^2\left(\frac{L}{d}\right)^2\right\}}{750-\left\{\frac{f_c}{f_t}\left(\frac{l}{L}\right)^2\left(\frac{L}{d}\right)^2\right\}}$$

The values of $\frac{f_c}{f_t}$ and of $\frac{l}{L}$ have been worked out for all of the cases of interplane strut spacings covered by the extent of Figures 1 to 4 and have been found to vary through exceedingly wide limits. When the ratio of gap to spar depth is 8, for example, the value of $\frac{f_c}{f_t}$ at the worst stressed point in the middle of a bay in the upper spar ranges from .33 to .93, while, when the gap is twenty times the depth of spar, the corresponding spread is from .16 to .65. The ratio $\frac{l}{L}$ varies as shown by Figure 13, reaching a maximum value of about .86.

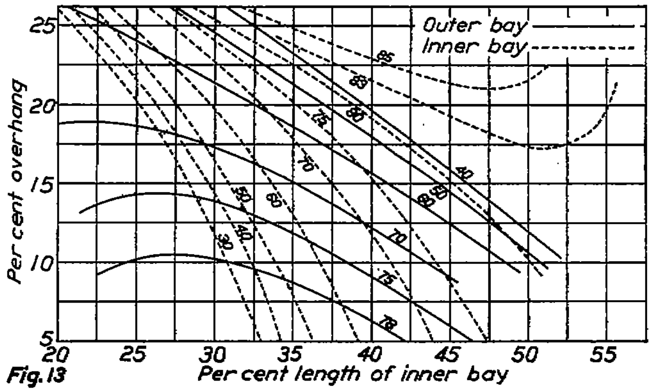


Fig. 13

$f_c$ = compressive stress.

$f_t$ = total stress.

$l$ = distance between points of inflection.

$L$ = distance between supports.

Fortunately, however, it happens that the variation of $\frac{f_c}{f_t} \times \left(\frac{l}{L}\right)^2$ is no larger than that of one factor alone. When $\frac{G}{d}$ is 8 the product for the worst stressed bay ranges from .18 to .55, with the highest values reached when the overhang is long and the inner and outer bays are of equal length. The products are plotted in Figure 14. Similar curves are given in Figure 15 for a gap/depth ratio of 20, the extreme range in that case being from .09 to .41. The division of the curves of each sheet into two seemingly independent groups, separated by dotted lines, corresponds to the transition of the point of worst stress from one bay to the other (the worst stress being in the outer bay for points to the left of the dotted lines). While it is, of course, possible that the worst stress with allowance for buckling may come in the bay other than that in which it would occur when no such allowance had to be made, that is unlikely except when the truss is so proportioned that the equivalent stresses in the two bays are very nearly equal in any case, so that it will make little difference which one is used. The dotted line was located without reference to any difference between compressive and bending strengths of the spar material.

The ratio of $L$ to $d$ is limited by the necessity of keeping the angle between the lift wires and the wing spars above a certain minimum to provide rigidity to the structure. Neither in a Pratt nor in a Warren truss is it likely that the length of any single bay of a two-bay arrangement will ever exceed twice the gap. When $G/d$ is 8, therefore, the maximum probable value of $L/d$ will be 16. The equation of equivalent moment would then become approximately

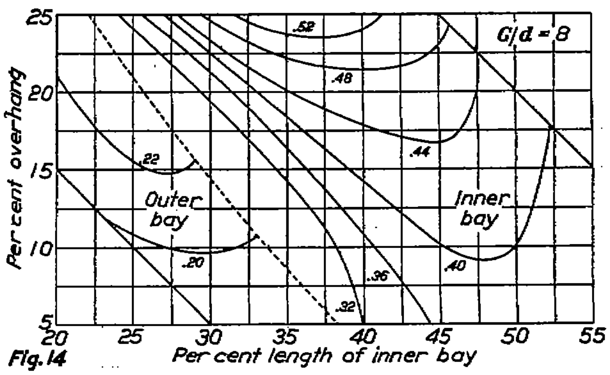


Fig. 14

$$M' = M \frac{1 + \frac{1}{3}\frac{f_c}{f_t}\left(\frac{l}{L}\right)^2}{1 - \frac{1}{3}\frac{f_c}{f_t}\left(\frac{l}{L}\right)^2}$$

For the largest value of the product plotted in Figure 14, this would give $M' = 1.45\,M$. The corresponding maximum when $G/d$ is 20 is about $14\,M$, a value so large as merely to signify the impracticability of designing a spar with the length of a single bay equal to forty times the spar depth. In fact, it seems unlikely, with a spar so shallow in proportion to the gap, that the value of $\frac{M'}{M}$ will ever fall below 1.35 for the worst stressed bay in actual practice.

These figures, of course, relate only to the inner bay, where the values of $\frac{f_c}{f_t}\times\left(\frac{l}{L}\right)^2$ reach their maximum because of the high compressions. If the values for the outer bay be lifted from the sections of Figures 12 and 13 to the left of the dotted lines $\frac{M'}{M}$ when the worst conditions are in that bay is found never to exceed 1.17 with $G/d$ equal to 8, or 1.90 when $G/d$ is 20. As already noted, however, it is unlikely that the actual percentage correction for buckling in a given truss, the proportions of which bring it near to the dotted line of transition, would be materially larger for the inner than for the outer bay, and it is correspondingly unlikely that the introduction of the buckling correction would appreciably shift the transition line. The actual extent of the change can best be shown by a couple of examples. Suppose, for instance, that a wing truss for which $G/d$ is 20 has its spar length divided into an inner bay of 36 per cent, an outer bay of 49 per cent, and an overhang of 15 per cent, proportions which correspond to a point on the dotted line in Figure 15. The values of $\frac{f_c}{f_t}\times\left(\frac{l}{L}\right)^2$ (found by interpolation from the curves) are then .30 in the inner bay and .12 in the outer. If the length of the outer bay be taken as twice the gap, that of the inner bay will be 1.47 times the gap. The values of $L/d$ are 40.0 and 29.4, and those of

$$\frac{\frac{f_c}{f_t}\times\left(\frac{l}{L}\right)^2\times\left(\frac{L}{d}\right)^2}{750}$$

are .26 and .35 for the outer and inner bays, respectively, corresponding to correction factors of 1.70 and 2.08 to be applied to the bending moments. While the difference between these quantities is considerable, the problem is based on a truss of extreme proportions, and the correction factors would hardly be likely ever to reach such values in practice. If the length of the outer bay had been taken as one and a half, instead of two. times the gap, the corrections would have been only 1.32 and 1.47. A similar problem for a point on the dotted line in Figure 14, $G/d$ being 8 and the lengths being 33 per cent in the inner bay, 57 in the outer, and 10 in the overhang, gives correction factors of 1.15 in the outer bay and 1.06 in the inner if the outer length be twice the gap. Furthermore, the basic bending moment in the inner bay will be smaller than that in the outer if the proportions of the spar are chosen for uniform stress at the supports as the direct compression is largest in the inner bay, and the larger relative correction applied to $M$ in the inner bay may therefore be little or no larger in its absolute effect on total stress. In general, therefore, it appears that the difference in the factors along the transition line is not great and that no shift of that line need be made. In almost all cases the worst stress in the middle of a bay with made allowance for buckling effect will occur in the same bay where it would be found if buckling were nonexistent or neglected.

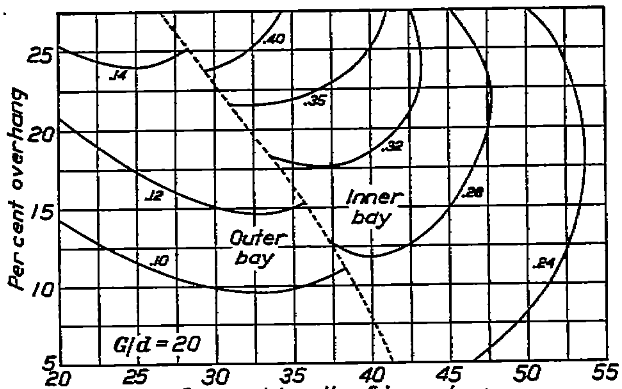

Fig. 15

The equivalent bending moments with allowance for buckling have been calculated for both gap-spar depth ratios used in the preceding work and for two ratios of length of spar to gap, and envelope curves have been plotted, just as they were plotted in Figures 8 and 9, without the allowance for column effect. Figures 16 and 17 give the equivalent moments for the two spar depths on the assumption that the total effective length of spar is 4.5 times the gap (surely as large a ratio as would ever be reached in a two-bay machine in practice), while Figure 18 presents similar data for a spar length of 3 times the gap and a gap-spar depth ratio of 20. When $L_t/G$ is 3 and $G/d$ is 8 the column effect is so small as to be negligible. In calculating these curves both bays have been taken into account in all cases. Any shifting of the transition lines of worst conditions from the positions shown in Figures 12 and 13 has therefore been allowed for.

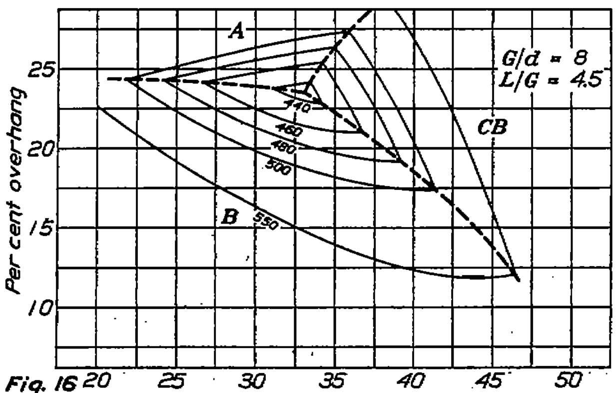

Fig. 16

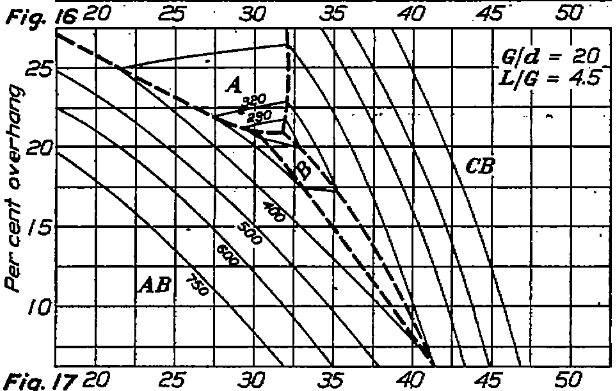

Fig. 17

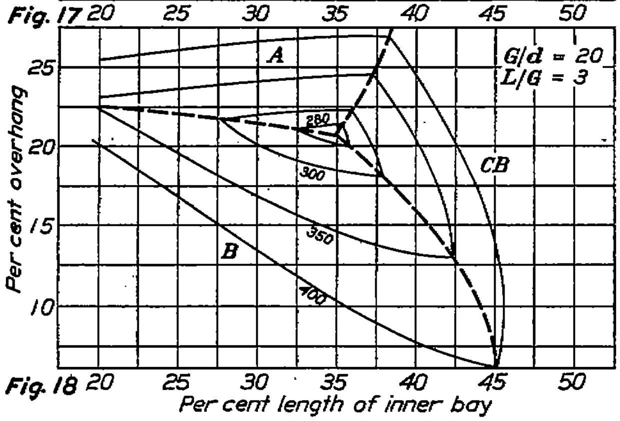

Fig. 18

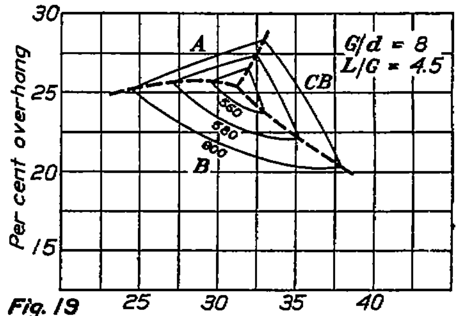

Fig. 19

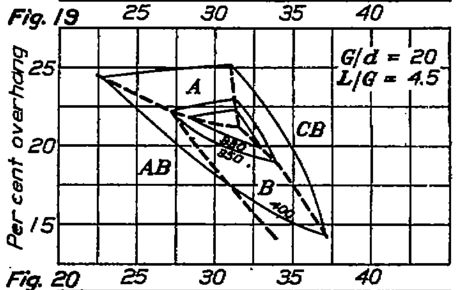

Fig. 20

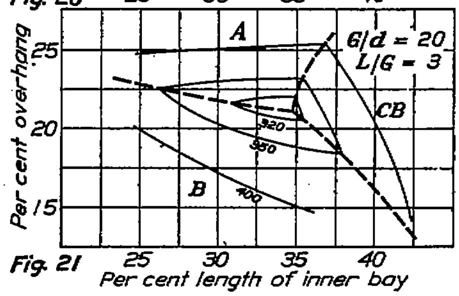

Fig. 21

Figures 19, 20, and 21 stand in the same relation to those just discussed as do Figures 11 and 12 to 8 and 9. They are drawn to include allowance for the difference between the bending and compressive strengths of spruce and for the change in allowable stress with variation in the proportion of direct compressive stress to total stress.

## DISCUSSION OF CURVES—BENDING MOMENTS

Inspection of the curves in Figures 1 and 2 reveals certain interesting characteristics of the variation of bending moment with the proportions of the truss which are not at once evident from the three-moment equation, nor even from a consideration from a purely physical point of view of the conditions under which the beam works. The first point of interest is the behavior of the moment at the middle support, which has a minimum value for each length of overhang, the minimum being very nearly a linear function of the three-halves power of the length of overhang.

$$M_{B_{min.}}=296-1.66\ (l_3)^{3/2}$$

where $l_3$ is the length of effective overhang as a percentage of the total effective length of spar. Furthermore, it appears that the minimum value of $M_B$, for a given overhang is reached when the outer bay is longer than the inner by approximately one-sixth the length of the overhang. If either the inner or the outer bay be held to a fixed length $M_B$ decreases steadily, and roughly along a straight line, as the overhang is lengthened at the expense of the other bay, but that, of course, is what would have been expected.

As for the moments in the middle of the bays, when the overhang is held constant the variation in both bays is almost exactly linear, the maximum in each bay, of course, increasing as the length of that bay itself is increased. The rate of change is approximately 10 pounds inches of moment for every inch of length of the bay in which that moment occurs, the total effective length of the two bays and overhang still being taken as 100 inches.

When the outer bay is held constant, instead of the overhang, both $M_{AB}$ and $M_{CB}$ increase as the inner bay increases. The variation still approximates to the linear, but only roughly, the moment in the inner bay tending toward a minimum as that bay becomes very short, while that in the outer bay appears to approach a maximum as the overhang approaches zero. With the inner bay held constant, linear relationships are again comparatively roughly observed, the moment going up in the outer bay and down in the inner as the outer bay is lengthened at the expense of the overhang.

Since all the variations of bending moments in the bays with changing distribution of the points of support follow straight-line laws at least approximately, it is possible to express them to a first approximation by a pair of very simple equations

$$M_{AB}=708-10l_1-14l_3$$

or alternatively,

$$M_{AB}=10l_2-4l_3-292$$

and

$$M_{CB}=9l_1+3.6l_3-296$$

where $l_1$, $l_2$, and $l_3$ are the percentages of total spar length in the inner bay, outer bay, and overhang, respectively. The equation for $M_{AB}$ gives results correct within 7 pounds inches for every point within the range of the curves, while that for $M_{CB}$ is good within 9 pounds inches except under the most extreme conditions. Either is useful as an approximation when the curves are not available. For the sake of completeness a similar equation, necessarily somewhat more complex in form but fitting the curves even more accurately, has been obtained for the bending moment at the middle support.

$$M_B=292-1.66\ (l_3)^{3/2}+.39\ (l_1-50+\frac{7}{12}\,l_3)^2.$$

This is considerably more simple than the direct solution from the three-moment equation and gives a result correct within 5 pounds inches at every point. The last moment, that at the outer support, of course, depends only on the effective length of the cantilevered overhang, and is given rigorously by

$$M_A=\frac{l_3^2}{2}$$

## REACTIONS

Although the variation of the reactions is comparatively simple in form, it does not lend itself to elementary analytic representation so well as does that of the moments, the curves not running parallel to each other. When the overhang length is kept constant and the bays varied, the reactions at the outer and inner supports of course change in magnitude in the same sense as the lengths of the bays to which they are adjacent. The middle reaction remains virtually constant, reaching a minimum when the inner bay is longer than the outer by about one-eighth the length of the overhang and increasing very gradually with change from that distribution in either direction. The curves for reaction at the middle support with fixed overhang are, in fact, very similar in form to the curves of bending moment at the same point under the same conditions.

With the inner bay fixed in length the reactions at both the outermost and the innermost supports increase with increasing overhang, the former rapidly and substantially uniformly, the latter very slowly, especially when the inner bay is long. The reaction at the middle support drops off as the overhang grows. For all proportions within the range of ordinary design practice, the outer and middle reactions are within 25 per cent of the same magnitude and the inner reaction is less than half as large as either of the others.

## EFFECT OF FIXITY AT THE INNER END

As already remarked, a hinge fitting with a vertical pin puts partial restraint on the change of slope of the spar at its inner end, a degree of fixity which may conceivably lie anywhere between zero and 100 per cent, but which in practice probably is seldom less than 20 per cent or more than 60. (This, of course, does not apply to cantilever wings of thick section, where the fixity must necessarily be complete.)

Comparisons of Figures 2 and 5 show little alteration in the general form of the moment curves but considerable changes in detail. As would be expected, the minimum bending moment at the middle support is obtained with a considerably longer inner bay when the inner end is fixed than when it is free. Whatever the length of overhang, the length of the outer bay for a minimum value of $M_B$ remains virtually constant at 38 per cent. By the time this point of minimum $M_B$ has been reached, however, the inner support has become the critical location, $M_C$ increasing approximately lineally and very rapidly as the inner bay is lengthened. For a fixed length of inner bay, $M_C$ goes up with increasing overhang.

The bending moments in the middle of the inner bay are much decreased, while those in the outer bay are slightly increased, by fixity. Since it is in the inner bay, where the largest compressions are found, that failure by buckling is most likely to occur, the use of a fitting giving partial fixity would be particularly useful when a long inner bay has to be used with a shallow spar. So far as maximum moment at a support is concerned, however, fixity is of comparatively little use, since, for a given overhang, the value of $M_B$ and $M_C$ at the point where they are equal is only about 18 per cent less than the minimum reached by $M_B$ with the inner end of the spar perfectly free in slope. If the comparison of the two conditions be made on the basis of the strut location which gives the lowest value for the bending moment at a support (this is equivalent to making $M_B$ and $M_C$ equal when the inner end is fixed) keeping the overhang fixed, the average decrease of $M_{CB}$ by the fixity is 40 per cent, while $M_{AB}$ is increased by an average of only 5 per cent, and is actually slightly decreased if the overhang be short. It is interesting to note, also, that the proportions which make $M_C$ and $M_B$ equal when there is complete fixity at the end also make $M_{CB}$ and $M_{AB}$ very nearly the same. This of course means that the inner bay would always be the critical one, as the larger compression there makes the column effect much more serious than it can be in the outer bay.

The effect of fixity on reactions is comparatively slight. The minimum reaction at the middle support occurs, in the case of complete fixity, with the length of the inner bay in excess of that of the outer by approximately 16 per cent, and the minima are lower than with a freely hinged end by an average of 7 pounds, or 15 per cent of the mean reaction. The reaction at the inner support is increased by an average of 3 pounds, while that at the outer support goes up about a pound. The compression in the outer bay, for a truss of given proportions is therefore almost entirely independent of the degree of fixity, but fixity will obviously reduce the inner bay compression very materially. That affords another reason, additional to the shortening of the distance between points of inflection in the inner bay, for using a hinge which will fix the spar at least partially when there is danger of trouble from buckling because of the use of a long bay in a shallow spar. Complete fixity may easily have the effect, everything taken into account, of more than doubling the factor of safety in the inner bay considered as a column.

## DIRECT LOADS IN SPARS

The compression in the outer bay of the upper spar is primarily a function of the length of the inner bay, being almost entirely independent of the distribution of the remaining length between the outer bay and the overhang. The effect of a given shift in the location of the inner strut has approximately five times as much effect as a corresponding change in the position of the outer.

The change of compression with a change in the position of either strut, the other being held fixed, is very nearly linear. The straight lines tend to diverge when plotted, however, and the single equation which expresses the force for all proportions has therefore to be complicated to the form

$$F_t \times \frac{G}{L_t^2} = .7 - .015l_1 + .000107\,(60 - l_1)\,l_3$$

which gives the compression in the outer bay (always on the assumption that the upper and lower wings are similarly supported, with struts at the same points, and that the mean loading of the two is used in calculation) to within .006 for every condition, $L_t$ being the total effective length of the spar.

The compression in the inner bay is, as has already been noted, quite independent of the proportions of the truss if the spars are freely hinged at their inner ends, but is reduced by fixity there, the amount of the reduction being directly proportional to the fixing moments and so being largest when the inner bay is longest and there is most need for some cutting down of the compression and stiffening of the spar.

The tension in the inner bay of the lower spar is of course numerically equal to the compression in the outer bay of the upper member for the type of truss to which these curves relate.

## LOCATION OF POINTS OF INFLECTION

The distances between the two points of inflection within a bay have been given, both for the inner and the outer bays, by the curves of Figure 13. It should be noted in using these values that they were calculated without taking into account the effect of buckling in increasing the bending moment within the bay. The effect of increasing that moment while leaving the values at the supports constant is, of course, to shift the points of inflection outward toward the supports. The effective length of column is therefore a function of the depth of the spar and of the amount of compression, as well as of the distribution of the interplane struts, but the shift of the points of zero bending moment is not likely to be great enough to be of serious importance except in spars which would approach very closely to failure by pure lateral instability in any case, and when spars answer to that description no approximations such as these can be of much avail. It is necessary then to apply the generalized theorem of three moments rigorously.

In general, lengthening one bay at the expense of the other tends to increase the relative separation of the points of inflection in that bay, as would be expected. There are, however, exceptions to this general rule. The relative separation in the inner bay of a spar with a long overhang decreases when the length of the inner bay is increased beyond about 50 per cent of the total, and the same holds true in the outer bay when the overhang is short and the outer bay forms more than 60 per cent of the whole effective length of the spar. The diagonal lines representing constant lengths of outer bay have been omitted, to avoid confusion of the figure, but can readily be inserted if desired. With a constant length of inner bay the points of inflection in that section of the spar shift constantly farther apart as the overhang is increased, while for the outer bay the reverse is true and the largest separations always correspond to short overhangs.

## EQUIVALENT BENDING MOMENTS AND SPAR PROPORTIONS

The curves in Figures 8 and 9 relate to the case of a spar in a wing of small aspect ratio, in which the column effect is of practically no importance. Specifically, they can be considered as applying with sufficient accuracy to all spars having a total length of less than thirty times their depth, a condition which is sometimes complied with in using thick airfoils. With a

Göttingen 387, for example, a section in which the mean spar depth is likely to be about 9 per cent of the chord, this permits an aspect ratio (figured on the total length of wing, including the part dropped off for tip loss correction) of about 5.8, while with an R. A. F. 15 the corresponding limit is about 3.7. Such proportions are of course unusual, and Figures 8 and 9 are useful as defining a limit rather than as applying directly to actual airplanes.

It will be observed in both figures that the minimum equivalent bending moment is found under the conditions which give equal moments at the two struts and in the middle of the inner bay. The proportions of the truss which give equality of these three moments change somewhat with $G/d$, the ideal length of inner bay, as shown by Figures 8 and 9, being 35 and 40 per cent, respectively, when $G/d$ is 8 and when the ratio rises to 20, while the outer bay is 41 and 39 per cent and the effective overhang 24 and 21 per cent under the same sets of conditions. It is rather astonishing to find that the outer bay should actually be shorter than the inner for best results if a thin section is used with a large gap and a small aspect ratio. The effect of the location of the inner strut in a truss of that form is, however, small; and a reduction of the length of inner bay from 40 to 32 per cent, the overhang being held constant and $G/d$ being 20, increases the equivalent moment only 5 per cent from its minimum. A reduction

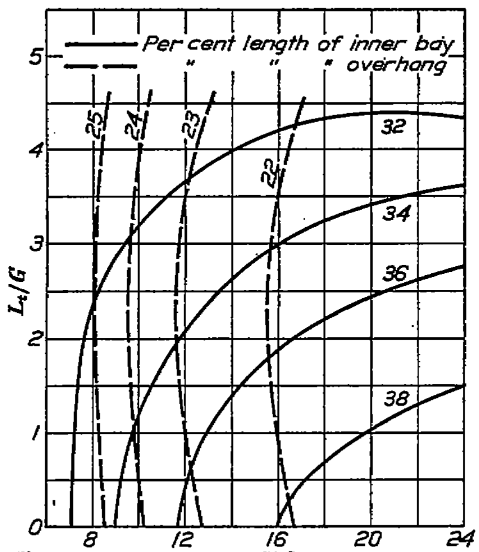


Fig. 22

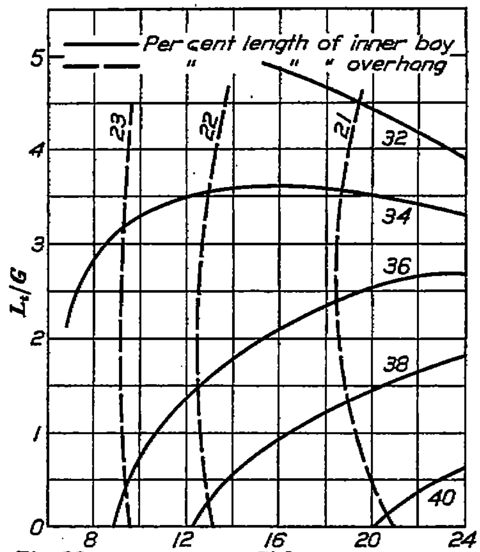


Fig. 23

of only 1 per cent, or an increase of one-half of 1 per cent, in overhang has as much effect. When $G/d$ is 8, the triangular figures of equal moment are more nearly equilateral, and the locations of inner and outer struts are of more nearly equal importance.

Comparing Figures 9, 17, and 18, all of which relate to the same value of $G/d$ and to materials capable of sustaining equal maximum stresses in bending and compression, it is apparent that the minimum equivalent moment is relatively little affected by column action, but that the ideal proportions for the truss are considerably modified. That is shown in the tabulation below, and the result of a comparison of Figures 12, 20, and 21, relating to spruce spars, would be much the same.

| $\frac{L_t}{G}$ | 0 (No column effect.) | 3 | 4.5 |
|---|---|---|---|
| Minimum equivalent moment, | 269 | 275 | 285 |
| Inner bay, | 40 | 35 | 32 |
| Outer bay, | 39 | 44 | 47 |
| Overhang, | 21 | 21 | 21 |

When $G/d$ is only 8 the effect is still less. The column effect with $L_t/G$ equal to 4.5, corresponding to an aspect ratio of nearly 10 if the gap is equal to the chord, increases the minimum equivalent moment only from 429 to 436, while making it advisable to shorten the inner bay from 35 per cent to 33, leaving the overhang unchanged.

Curves of best length for inner bay and overhang have been plotted in Figure 22 for spruce and in Figure 23 for materials of equal compressive strength and modulus of rupture. The length of inner bay for mimimum equivalent bending moment decreases steadily as $L_t/G$ goes up in both cases, and in general it falls off with increasing thickness of airfoil section. With the aspect ratios and gap-chord ratios most commonly used at the present time, however, the thickness of the section has but little effect on the ideal location of the inner strut.

The best overhang, on the other hand, is independent of aspect ratio, being a function only of wing thickness.

If there is to be any departure from the ideal dimensions, or if there is any doubt about what they are, it is better to err in the direction of making the inner bay too short rather than too long, especially when the spars are long and slender. When $L_t/G$ is 4.5 and $G/d$ is 20, for example, the equivalent bending moment is increased 33 per cent by making the inner bay 3 per cent too long, only 12 per cent by making it too short by a like amount. When $L_t/G$ is 3, the inner bay can be shortened 3 per cent at the expense of an increase of less than 4 per cent in the equivalent bending moment.

The proportions here suggested as best are not in exact agreement with those arrived at in previous investigations, but the difference is small. The United States Army Air Service, Engineering Division, for instance, recommends[2] in all cases an inner bay length of 32 per cent and an overhang of 19½ per cent of the effective spar length. If a single set of proportions were to be picked from this work, on the other hand, as the best average for all conditions, 34 per cent in the inner bay and 21 per cent overhang would appear to be the best choice in metal, 33 per cent and 22 per cent in spruce.

## SPAR WEIGHT

If the equivalent bending moment in the spar is known, the sectional area needed in a given material can easily be calculated. The equivalent bending moment is given by the formula

$$M_e = K\left(\frac{L_t}{100}\right)^2 w$$

where $K$ is the quantity plotted in Figures 8 and 9 and elsewhere, $L_t$ the total effective spar length, as before, and $w$ the load per unit length of spar.

On the assumption that $k=0.36d$, $f=\frac{M_e}{0.26dA}$, $d$ being the depth of the spar and $A$ the sectional area. Then

$$A=\frac{Kw\left(\frac{L_t}{100}\right)^2}{0.26\,df}=\frac{Kw\left(\frac{L_t}{100}\right)^2\frac{G}{d}}{0.26\,Gf}$$

Taking the density of spruce as 26 pounds per cubic foot and the allowable bending stress as 6,400 pounds per square inch, the weight of a spruce spar becomes

$$W_s=\frac{A\,L_t\times 26}{1,728}=\frac{Kw\left(\frac{L_t}{100}\right)^2\frac{G}{d}\times 26}{17.28\times 0.26\times 6,400G}=\frac{KW'\left(\frac{L'_t}{100}\right)\frac{G}{d}\frac{L_t}{G}}{11,060,000}$$

$W'$ being the total load carried by the spar and $L'_t$ the true length of the spar.

As a general rule, the front spar in a wing with two spars carries about two-thirds of the total load on the wing when the center of pressure is in its farthest forward position, while the rear spar carries all the load at the angle of attack arbitrarily chosen for a low-angle analysis.

[2] Structural Analysis and Design of Airplanes, Engineering Division, Air Service, U. S. Army, p. 49. The figures there are given in terms of the actual, not the effective, spar length, and they accordingly differ slightly in absolute value from those quoted in this text.

The ratio of total spar weight to total weight of the airplane, exclusive of the wing structure, is therefore approximately

$$\frac{W_S}{W_N}=\frac{\left(\frac{L'_t}{100}\right)}{11{,}060{,}000}\left\{\left(K\frac{G}{d}\right)_R F_L+\frac{2}{3}\left(K\frac{G}{d}\right)_F F_H\right\}\frac{L_t}{G}$$

where $W_N$ is the weight without the wings, $W_S$ the weight of the spar as before, $F_L$ and $F_H$ are the load factors used in the low angle and high-angle analyses, respectively, and the subscripts $R$ and $F$ relate to the characteristics of the rear and front spars, respectively.

Since $F_L$ is usually very nearly two-thirds of $F_H$, and since $W_N$ is roughly 85 per cent of the total weight, it is possible to simplify further to the form

$$\frac{W_S}{W}=\frac{\left(\frac{L'_t}{100}\right)F_H\frac{L_t}{G}}{9{,}780{,}000}\times\frac{\left\{\left(K\frac{G}{d}\right)_R+\left(K\frac{G}{d}\right)_F\right\}}{2}$$

Values of the product $K\times\frac{G}{d}\times\frac{L_t}{G}$ based on the assumption that the best strut location is used in every case and that the front and rear spars are of the same depth, have been calculated and are tabulated below. The variation of the product with $\frac{G}{d}$, $\frac{L_t}{G}$ being kept equal to unity, is plotted in Figure 24.

| $\frac{G}{d}$ | $\frac{L_t}{G}$ | $K\frac{G}{d}\frac{L_t}{G}$ | $K\frac{G}{d}$ |
|---|---|---|---|
| 8 | 1 | 4, 300 | 4, 300 |
| 8 | 4. 5 | 19, 500 | 4, 340 |
| 20 | 1 | 6, 240 | 6, 240 |
| 20 | 3 | 18, 800 | 6, 280 |
| 20 | 4. 5 | 28, 900 | 6, 420 |

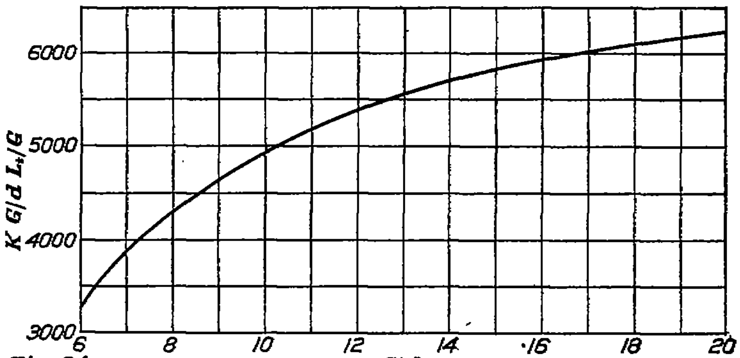

Fig. 24

It will be observed from these figures that both the gap and spar depth have important effect on the weight of the spars. If, for example, the depth of a spar is one-thirty-second of its total length and the gap is reduced from 20 to only 8 times the spar depth the spar weight will be increased by 74 per cent. If the gap be held constant at one-third of the spar length and the spar depth cut from one-eighth to one-twentieth of the distance between the wings the increase of weight will be 45 per cent.

To illustrate the use of the weight formula, it may be applied to the case of a pursuit airplane with a span of 30 feet, a gap of 5 feet, a spar depth of 3.5 inches, and designed for a load factor of 10. $L_t$, the effective length of a single spar, is then approximately 13 feet allowing for tip correction and for the length of the center section, and $L'_t$ 14 feet. $\frac{L_t}{G}$ is 2.6 and $\frac{G}{d}$ is 17.1, and the product of $K$, $\frac{G}{d}$, and $\frac{L_t}{G}$ is given by Figure 24 as 15,800. The ratio $\frac{W_s}{W}$ is then:

$$\frac{W_s}{W}=\frac{\frac{14\times12}{100}\times10\times15{,}800}{9{,}780{,}000}=.0272$$