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THEORY OF WING SECTIONS OF ARBITRARY SHAPE

BY THEODORE THEODORSEN

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SUMMARY

This paper presents a solution of the problem of the theoretical flow of a frictionless incompressible fluid past airfoils of arbitrary forms. The velocity of the 2-dimensional flow is explicitly expressed for any point at the surface, and for any orientation, by an exact expression containing a number of parameters which are functions of the form only and which may be evaluated by convenient graphical methods. The method is particularly simple and convenient for bodies of streamline forms. The results have been applied to typical airfoils and compared with experimental data.

INTRODUCTION

The theory of airfoils is of vital importance in aeronautics. It is true that the limit of perfection as regards efficiency has almost been reached. This attainment is a result of persistent and extensive testing by a large number of institutions rather than of the fact that the important design factors are known. Without the knowledge of the theory of the air flow around airfoils it is well-nigh impossible to judge or interpret the results of experimental work intelligently or to make other than random improvements at the expense of much useless testing.

A science can develop on a purely experimental basis only for a certain time. Theory is a process of systematic arrangement and simplification of known facts. As long as the facts are few and obvious no theory is necessary, but when they become many and less simple theory is needed. Although the experimenting itself may require little effort, it is, however, often exceedingly difficult to analyze the results of even simple experiments. There exists, therefore, always a tendency to produce more test results than can be digested by theory or applied by industry. A large number of investigations are carried on with little regard for the theory and much testing of airfoils is done with insufficient knowledge of the ultimate possibilities. This state of affairs is due largely to the very common belief that the theory of the actual airfoil necessarily would be approximate, clumsy, and awkward, and therefore useless for nearly all purposes.

The various types of airfoils exhibit quite different properties, and it is one of the objects of aerodynamical science to detect and define in precise manner the factors contributing to the perfection of the airfoil. Above all, we must work toward the end of obtaining a thorough understanding of the ideal case, which is the ultimate limit of performance. We may then attempt to specify and define the nature of the deviations from the ideal case.

No method has been available for the determination of the potential flow around an arbitrary thick wing section. The exclusive object of the following report is to present a method by which the flow velocity at any point along the surface of a thick airfoil may be determined with any desired accuracy. The velocity of the potential flow around the thick airfoil has been expressed by an exact formula, no approximation having been made in the analysis. The evaluation for specific cases, however, requires a graphical determination of some auxiliary parameters. Since the airfoil is perfectly arbitrary, it is, of course, obvious that graphical methods are to some extent unavoidable.

Curiously enough, the theory of actual airfoils as presented in this report has been brought into a much simpler form than has hitherto been the case with the theory of thin airfoils. In the theory of thin airfoils certain approximations have restricted its application to small cambers only. This undesirable feature has been avoided, and the results obtained in this report have a complete applicability to airfoils of any camber and thickness.

The author has pointed out in an earlier report that another difficulty exists in the theory of thin airfoils. It consists in the fact that in potential flow the velocity at the leading edge is infinite at all angles except one. This particular angle at which the theory actually applies has been defined as the ideal angle of attack. In the present work we shall not go any further into this theory, since it is included in the following theory as a special case of rather limited practical importance.

THEORY OF THICK AIRFOILS

In the theory of functions there is a theorem by Riemann1 which shows that it is always possible to transform the potential field around any closed contour into the potential field around a circle. The direct transformation of an airfoil into a circle may,

for analytical purposes, conveniently be performed in two steps. The first step is to transform the airfoil into a curve which ordinarily does not differ greatly from a circle by the transformation

$$\xi = z' + \frac{a^2}{z'}$$  \hspace{1cm} (I)

where $\xi$ is a complex quantity defining the points in the plane describing the flow around the airfoil and $z'$ is another complex quantity defining the points in the plane describing the flow around the almost circular curve. The constant $a$ is of dimension length and is merely a geometrical scale factor. In the following theory, attention is directed to the fact that the shape of the curve resulting from transformation (I) is arbitrary, since the airfoil shape is arbitrary. At a later point we shall transform this curve into a circle.

The $z'$ and the $\xi$ planes are shown superposed in Figure 1. It will be noticed that at great distances from the origin $z'\to \xi$; that is, both flows are similar at infinity. In particular, the "angle of attack," defined as the direction of flow at infinity with respect to some fixed reference line in the body, is identical in both flows. Near the origin the two flows are entirely different; one value of $z'$ is, however, uniquely associated with a given value of $\xi$ by the relation (I).

We shall, at a later point, determine the flow in the $z'$ plane. At present we shall determine the appearance of the airfoil when the almost circular curve $B$ is given, or what amounts to the same thing, we shall determine the curve $B$ when the airfoil is given. In Figure 1, $C$ is a circle of unit radius. Since the matter of dimensions is rather important, we shall avoid confusion in the following by adhering to this length as unity. The curve $B$ is uniquely given by the relation $z' = ae^{i\theta}$ where $\psi$ is a known or unknown real function of the angle $\theta$ where $\theta$ varies from zero to $2\pi$ and $i$ is the imaginary unit. Since the airfoil surface corresponds to the surface of the curve, the former is given from relation (I) as

$$\xi = a(e^{i\theta} + e^{-i\theta}) \cos \theta + i a(\dot{\theta} - e^{-i\theta}) \sin \theta$$

This relation may further be conveniently expressed in hyperbolic functions

$$\xi = 2a \cosh \psi \cos \theta + 2ia \sinh \psi \sin \theta$$

Since $\xi = x + iy$, the coordinates of the airfoil $(x, y)$ are given by

$$x = 2a \cosh \psi \cos \theta$$
$$y = 2a \sinh \psi \sin \theta$$  \hspace{1cm} (II)

We obtain a relation between $\theta$ and the coordinates of the airfoil as follows:

$$\cosh \psi = \frac{x}{2a \cos \theta}$$
$$\sinh \psi = \frac{y}{2a \sin \theta}$$

and since $\cosh^2 \psi - \sinh^2 \psi = 1$

$$(\frac{x}{2a \cos \theta})^2 - (\frac{y}{2a \sin \theta})^2 = 1$$

or developed

$$2 \sin^2 \theta = p + \sqrt{p^2 + (\frac{y}{a})^2}$$  \hspace{1cm} (III)

where

$$p = 1 - (\frac{x}{2a})^2 - (\frac{y}{2a})^2$$

Similarly we obtain a relation between $\psi$ and the coordinates of the airfoil by using the equation

$$(\frac{x}{2a \cosh \psi})^2 + (\frac{y}{2a \sinh \psi})^2 = 1$$

or developed

$$2 \sinh^2 \psi = -p + \sqrt{p^2 + (\frac{y}{a})^2}$$  \hspace{1cm} (IV)

Since $\psi$ is generally small for wing sections it may be more conveniently expressed for purposes of calculation as a series in terms of $\frac{y}{2a \sin \theta}$ as follows:

We have

$$e^\psi = \sinh \psi + \cosh \psi$$
$$= \sinh \psi + \sqrt{1 + \sinh^2 \psi}$$
$$= 1 + \sinh \psi + \frac{1}{2} \sinh^2 \psi + \ldots$$

$$\psi = \log \left(1 + \sinh \psi + \frac{1}{2} \sinh^2 \psi + \ldots\right)$$
$$= \sinh \psi - \frac{1}{6} \sinh^3 \psi + \ldots$$
$$= \frac{y}{2a \sin \theta} - \frac{1}{6} \left(\frac{y}{2a \sin \theta}\right)^3 + \ldots$$  \hspace{1cm} (IVa)

[for $\psi < \log_e 2$]

We are now in a position to reproduce the conformal representation of an airfoil in the $z$ plane, since for each point of the airfoil $(x, y)$ both $\psi$ and $\theta$ have been determined.
The curves $\psi = \text{constant}$ are ellipses in the $\xi$ plane
\[
\left(\frac{z}{2a \cosh \psi}\right)^2 + \left(\frac{y}{2a \sinh \psi}\right)^2 = 1.
\]
The foci are located at $(\pm 2a, 0)$. The radius of curvature at the end of the major axis is $\rho = \frac{(2a \sinh \psi)^2}{2a \cosh \psi}$ or
\[
\rho = \frac{(\sinh \psi)^2}{2a} \cosh \psi = \frac{\psi^2}{2a} \quad \text{(for small } \psi)\]
This relation is useful for the determination of $\psi$ near the nose and the tail.

The leading edge, corresponding to $\theta = 0$, is located at $z = 2a \cosh \psi \left(1 + \frac{\psi^2}{2}\right) = 2a + a \psi^2 = 2a + \frac{1}{2} \rho$.

Thus we see that the length $2a$ corresponds to the point midway between the nose and the center of curvature of the leading edge to the point midway between the tail and the center of curvature of the trailing edge.

To establish the magnitude of the velocity at any point $(x, y)$ on the airfoil, we start in customary manner with the velocity around a circle in 2-dimensional flow. Contrary to usual practice we will, however, make the radius of the circle equal to $ae^{i \theta}$ where $\psi$ is a small constant quantity. This quantity is shown later in this report (equation (e)), to represent the average value of $\psi$ taken around the curve $C$.

The potential function of the flow past this circle is
\[
w = -V \left(z + \frac{ae^{i \theta}}{z}\right) - \frac{i \Gamma}{2\pi} \log \frac{z}{ae^{i \theta}} \quad \text{(V)}
\]
and the velocity \footnote{The choice of axes is entirely arbitrary. It is a matter of convenience only to choose the axes so that the airfoil appears as nearly elliptical as possible, thereby making the "almost circular" curve $B$ as nearly circular as possible by means of the single transformation $L$. It will be seen that the evaluation of the important integral appearing in the appendix is then most easily accomplished. In fact, the transformation $L$ itself is only a matter of convenience to permit the ready evaluation of this integral.}
\[
\frac{dw}{dz} = -V \left(1 - \frac{ae^{i \theta}}{z^2}\right) - \frac{i \Gamma}{\pi z^2} \quad \text{(VI)}
\]
where $\Gamma$ is the circulation. This expression must vanish at the rear stagnation point \footnote{The velocity field actually equals $u-iw$, the image of the velocity vector about the z-axis.} (Kutta condition) whose coordinate is $z = -ae^{i \theta + i \alpha}$, where $\alpha$ is the angle of attack and $\epsilon_r$ is shown to be the angle of zero lift.

We obtain $\Gamma = -\frac{2Vz}{\pi} \left(1 - \frac{ae^{i \theta}}{z^2}\right)$
\[
= 4\pi V ae^{i \theta} \left(e^{-ia \epsilon_r} - e^{-i(a + \phi)}\right)
\]
\[
= 4\pi V ae^{i \theta} \sin(a + \epsilon_r) \quad \text{(VII)}
\]

This flow around the circle $a \epsilon_r$ now be transformed into the flow around any other body. In the particular case in which the flow at infinity is not altered the circulation will not be altered and the force experienced by a body at the origin will remain at the fixed value $L = \rho \sqrt{V \Gamma}$.

We will now transform this circle, defined as $z = ae^{i \theta + i \phi}$ into our curve $B$ defined by the relation $z' = ae^{i \theta}$. For this purpose we employ the general transformation $z' = ae^{it} + \frac{1}{a} \frac{\partial}{\partial \theta}$ which leaves the flow at infinity unaltered, the constants being determined by the boundary conditions. By definition
\[
z' = ae^{i \theta - \phi + i (\theta + \phi)}.
\]

Consequently
\[
\psi - \psi_0 + i(\theta - \phi) = \sum \frac{A_n}{n} (A_n + i B_n) \frac{1}{r^n} \sin n \phi \cos n \phi - i \sin n \phi
\]
and by De Moivre's theorem
\[
\frac{1}{r^n} = \frac{1}{a^n} (\cos n \phi - i \sin n \phi)
\]

Equating the real and imaginary parts we obtain the two Fourier expansions:
\[
\psi - \psi_0 = \sum \frac{A_n}{r^n} \cos n \phi + \frac{B_n}{r^n} \sin n \phi \quad \text{(a)}
\]
and
\[
\theta - \phi = \sum \frac{A_n}{r^n} \cos n \phi - \frac{B_n}{r^n} \sin n \phi \quad \text{(b)}
\]

The values of the coefficients $A_n, B_n$, as well as the quantity $\psi_0$, may be determined from (a) as follows:
\[
A_n = \frac{1}{\pi} \int_0^{2\pi} \psi \cos n \phi \, d\phi \quad \text{(c)}
\]
and
\[
B_n = \frac{1}{\pi} \int_0^{2\pi} \psi \sin n \phi \, d\phi \quad \text{(d)}
\]

and
\[
\psi_0 = \frac{1}{2\pi} \int_0^{2\pi} \psi \, d\phi \quad \text{(e)}
\]

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The quantity \( \theta - \varphi \) is necessary in the following analysis. Let us eliminate the coefficient \( \frac{A}{r^n} \) and \( \frac{B}{r^n} \) in (b) by means of (c) and (d). We obtain

\[
(\theta - \varphi) = n \int_0^{2\pi} \psi \sin n\varphi d\varphi
\]

The subscript \( c \) is added to indicate that the angles so distinguished are kept constant while the integrations are performed. The expression may be simplified

\[
(\theta - \varphi) = \frac{1}{n} \int_0^{2\pi} \psi \sin n(\varphi - \varphi_c) d\varphi
\]

But

\[
\sum_n \sin n (\varphi - \varphi_c) = \frac{1}{2} \cot \left( \frac{\varphi - \varphi_c}{2} \right) - \frac{\cos (2n+1) \left( \frac{\varphi - \varphi_c}{2} \right)}{2} \sin \frac{\varphi - \varphi_c}{2}
\]

Therefore,

\[
(\theta - \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \psi \cot \left( \frac{\varphi - \varphi_c}{2} \right) d\varphi
\]

The latter integral is identically zero. (See Wilson, E. B. Advanced Calculus, p. 388. Follow method of exercise 10.) Then

\[
(\theta - \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \psi \cot \left( \frac{\varphi - \varphi_c}{2} \right) d\varphi \quad (VIII)
\]

For purposes of calculation this integral is expressed in convenient form in the appendix.

We shall now resume the task of determining the velocity at any point of the surface of the airfoil.

The velocity at the surface of the circle is \( \frac{dw}{dz} \) (see equation (VI) and footnote). For corresponding points on the curve \( B \) in the \( z' \) plane and on the airfoil in the \( \zeta \) plane the velocities are respectively \( \frac{dw}{dz} \cdot \frac{dz}{dz'} \) and \( \frac{dz'}{dz} \cdot \frac{dw}{dz} \cdot \frac{dz'}{dz} \).

The quantities \( \zeta \) and \( z' \) are related by the expression

\[
\zeta = z' + \frac{a^2}{z'}
\]

Hence

\[
\frac{dz}{dz'} = 1 - \frac{a^2}{z'} = \frac{1}{2 \pi} \left( \frac{z' - a^2}{z'} \right) = \frac{1}{2 \pi} \left( a e^{\psi - \varphi} - a e^{\psi + \varphi} \right)
\]

\[
= \frac{1}{\pi} \left[ a (e^{\psi - \varphi}) \cos \theta + ia (e^{\psi + \varphi} \sin \theta) \right]
\]

\[
= \frac{1}{\pi} \left[ 2a \sinh \psi \cos \theta + 2ia \cosh \psi \sin \theta \right]
\]

Using the relations (II),

\[
2a \sinh \psi = \frac{y}{\sin \theta} \text{ and } 2a \cosh \psi = \frac{x}{\cos \theta}
\]

we obtain

\[
\frac{dz}{dz'} = \frac{1}{z'} \left( y \cot \theta + i x \tan \theta \right). \quad (IX)
\]

It now remains to find the ratio \( \frac{dz}{dz'} \). From the relation

\[
z' = z e^{\left( A_i + iB_n \right)^{1/2}}
\]

we obtain

\[
\frac{dz'}{dz} = z' \left[ \frac{1}{z} + \frac{1}{z} \sum (A_n + i B_n) \frac{1}{z^n} \right]
\]

or

\[
\frac{dz'}{dz} = z' \left[ \frac{1}{z} + \frac{1}{z} \left[ (\psi - \varphi) + i (\theta - \varphi) \right] \right]
\]

\[
= z' \frac{dz}{dz} (\psi + i (\theta - \varphi) + \log z)
\]

But

\[
\frac{dz}{dz'} = \frac{z}{z'} \left( \psi + i (\theta - \varphi) \right)
\]

from which,

\[
\frac{1}{z'} = \frac{dz}{dz} (\log z) = \frac{dz}{dz} (\log a + \psi + i\varphi) = \frac{dz}{dz} (i\varphi)
\]

Therefore

\[
\frac{dz'}{dz} = z' \frac{dz}{dz} (\psi + i (\theta - \varphi) + i\varphi)
\]

This expression may be written

\[
\frac{dz'}{dz} = z' \frac{dz}{dz} (\psi + i\theta)
\]

But we have

\[
\frac{dz}{dz} = i \frac{dz}{d\theta} \frac{d\varphi}{d\theta}
\]

or

\[
\frac{dz}{dz} = \frac{i}{z} \frac{d\varphi}{d\theta} = i \frac{d(\varphi - \theta)}{d\theta}
\]

and

\[
\frac{dz}{dz} \frac{dz}{d\theta} = i z \left( 1 + \frac{d(\varphi - \theta)}{d\theta} \right)
\]
The theory of wing sections of arbitrary shape

\[ \frac{dz'}{dz} = \frac{z'}{z} \frac{d\theta}{d\theta} \left(-i\psi + \theta\right) \frac{1}{1 + \epsilon'} \]

where

\[ \epsilon = \varphi - \theta \]

or

\[ \frac{dz'}{dz} = \frac{z'}{z} 1 - i\psi' \frac{1}{1 + \epsilon'} \]

where \( \epsilon' \) and \( \psi' \) indicate \( \frac{de}{d\theta} \) and \( \frac{d\psi}{d\theta} \), respectively.

Equations (IX) and (X) give now

\[ \frac{dz'}{dz} = \frac{z'}{z} \left( y \cot \theta + ix \tan \theta \right) \frac{1}{1 + \epsilon'} \]

Because we are interested more in the magnitude than in the direction of the velocity we will write for the numerical value of this expression

\[ \left| \frac{dz'}{dz} \right| = \frac{\sqrt{(y^2 \cot^2 \theta + x^2 \tan^2 \theta) (1 + \psi'^2)}}{\alpha \epsilon' (1 + \epsilon')} \]

The quantity \( \left( \frac{y}{2a} \right)^2 \cot^2 \theta \) is readily seen to be equal to (by relation (II))

\[ \left( \frac{y}{2a \sin \theta} \right)^2 + \sin^2 \theta \]

or also

\[ \sinh^2 \psi + \sin^2 \theta \]

Hence

\[ \left| \frac{dz'}{dz} \right| = 2 \sqrt{\left( \frac{y}{2a \sin \theta} \right)^2 + \sin^2 \theta} \]

The numerical value of the velocity at the surface of the circle is obtained by equations (VI) and (VII) as follows:

Substituting the general point \( z = a e^{i \psi + (\alpha + \epsilon)} \), where \( \alpha \) is the angle of attack as measured from the coordinates, in equation (VI)

\[ \frac{dw}{dz} = -V(1 - e^{-2i(\alpha + \epsilon)}) - 2i V \sin(\alpha + \epsilon) e^{-(\alpha + \epsilon)} \]

\[ = -V[1 \sin 2(\alpha + \epsilon) + 2 \sin(\alpha + \epsilon) \sin(\alpha + \epsilon)] \]

\[ + i \sin 2(\alpha + \epsilon) \cos(\alpha + \epsilon)] \]

\[ \frac{dw}{dz} = V[4 \sin^2(\alpha + \epsilon) + 8 \sin(\alpha + \epsilon) \sin(\alpha + \epsilon)] \]

\[ + 4 \sin^2(\alpha + \epsilon)] \]

\[ \frac{dw}{dz} = 2 V \sin(\alpha + \epsilon) + \sin(\alpha + \epsilon) \]

Replacing \( \varphi \) by \( \theta + \epsilon ' \), the angle of zero lift, is the value of \( \varphi - \theta \) at the \( \epsilon = 0 \), we have

\[ \left| \frac{dw}{dz} \right| = 2 V \sin(\alpha + \theta + \epsilon) + \sin(\alpha + \epsilon)' \]

For a point on the airfoil we have, then,

\[ v = \frac{1}{2} \frac{d \omega}{dz} \]

and from (XI), finally

\[ v = \frac{V(\sin(\alpha + \theta + \epsilon) + \sin(\alpha + \epsilon)')(1 + \epsilon') e^{i \epsilon}}{\sqrt{(\sinh^2 \psi + \sin^2 \theta) (1 + \psi'^2)}} \]

where the various symbols have the following significance:

\( v \) is the velocity at any point \((x, y)\) of the airfoil.

\( V \) is the uniform velocity of flow at infinity.

\( y \) is the ordinate of the airfoil as measured from the \( x \)-axis, while the system of coordinates \((2a, 0)\) is the point midway between the nose and center of curvature of the nose, and \((-2a, 0)\) is the point midway between the tail and center of curvature of the tail.

\( \alpha \) is the angle of attack as measured from the \( x \)-axis as indicated in Figure 6.

\( y, \theta, \psi, \epsilon, \) and \( \epsilon' \) are all functions of \( x \).

Equation (XII), expressing the value of the velocity at any point of any airfoil of any shape, is surprisingly simple when the complex nature of the problem is considered. It has the distinct advantage of being exact; no approximations have been made in the preceding analysis.

We shall note some of the properties of this important relation. Because \( y \) is generally small, the term \( \frac{y}{2a \sin \theta} \) is of influence chiefly near the leading edge, where \( \sin \theta \) is small. It is noticed, however, that if \( y = 0 \) for \( \theta = 0 \), equation (XII) yields in all cases \( v = \infty \). This means that the velocity at the nose becomes infinite for \( \sinh \psi = 0 \) (thin airfoils). This fact has been pointed out in an earlier report. (Reference 2.) The quantity \( \frac{y}{2a \sin \theta} \) or \( \sinh \psi \) is thus of considerable significance in the theory of thick airfoils.

The velocity near the tail is obtained by putting \( \theta = x + \Delta \theta \) and \( \epsilon = \epsilon' + \epsilon' \Delta \theta \). Where \( \Delta \theta \) is a small angle, in equation (XII)

\[ \frac{v}{V} = \frac{e^{i \epsilon} (1 + \epsilon')} \left[ \sin(\theta + \alpha + \epsilon) + \sin(\alpha + \epsilon)' \right] \]

we get

\[ \frac{v}{V} = \frac{e^{i \epsilon} (1 + \epsilon')} \left[ -\Delta \theta + \alpha + \epsilon + \epsilon' \Delta \theta + \alpha + \epsilon \right] \]

\[ \frac{\sqrt{(\psi^2 + \Delta \theta^2)} (1 + \psi'^2)}{\sqrt{(\psi^2 + \Delta \theta^2)} (1 + \psi'^2)} \]

\[ \frac{\epsilon'^2}{\sqrt{1 + \psi'^2}} \]

It should be pointed out that the rear stagnation point is shown to be on the \( x \)-axis at \( \theta = -\pi \). The curvature at the tail is, as far as the specification of the ideal circulation is concerned, to be considered as a mechanical imperfection.
The quantity \( \frac{\psi}{\Delta \theta} \) is infinite if \( \psi \) is different from zero at \( \Delta \theta = 0 \). The velocity is in this case zero, indicating the presence of the rear stagnation point. If, on the other hand, \( \psi \) is zero, that is, if the tail is perfectly sharp,

\[
\frac{\psi}{\Delta \theta} = \psi' \text{ for } \Delta \theta = 0
\]

and the velocity at the tail is

\[
v_r = V e^{\psi'(1 + \epsilon')^2} \frac{(1 + \psi'^2)^{1/2}}{(1 + \psi'^2)^{1/2}}
\]

(For the Clark Y, \( v_r^2 \) is about 0.58 \( V^2 \) near the tail.) We obtain the front stagnation point by letting \( v = 0 \) in equation (XII). Hence

\[
\alpha + \theta + \epsilon_n = - (\alpha + \epsilon_r)
\]

\[
\theta = - (2\alpha + \epsilon_n + \epsilon_r)
\]

In a previous report (reference 2)

\[
\alpha = - \frac{\epsilon_x + \epsilon_r}{2}
\]

has been defined as the ideal angle of attack. It is seen that, for this angle of attack, \( \theta \) is zero or the stagnation point occurs directly at the nose.

Equation (XII) may also be applied to strut forms, and for such symmetrical shapes takes even a simpler form.

**PRACTICAL APPLICATION OF RESULTS**

We will now apply Formula (XII) to the typical case of the Clark Y airfoil and calculate the velocities at points of the airfoil surface. The detailed method of procedure is as follows.

1. The axis of coordinates is drawn through the points \((2a, 0)\) and \((-2a, 0)\) located respectively at the point midway between the nose and the center of curvature of the nose and the point midway between the tail and the center of curvature of the tail. (See fig. 6.) The radius of curvature at the leading edge is 1.75 per cent chord.

2. The points \((x, y)\) of the upper and lower surfaces of the airfoil are determined with respect to this axis.

3. \( \sin^2 \theta, \sin \theta, \) and \( \theta \) are determined by the relation

\[
2 \sin^2 \theta = p \times \frac{y}{a} - \frac{y}{2a}
\]

4. \( \psi \) is given by the relation

\[
\psi = \frac{-y}{2a \sin \theta} - \frac{1}{6} \left( \frac{y}{2a \sin \theta} \right)^3
\]

5. \( \psi \) is plotted as a function of \( \theta \)

\[
\psi = \frac{1}{2\pi} \int_0^{2\pi} \psi \sin \theta \, d\theta
\]

6. Determine \( \epsilon_\psi \) by the relation

\[
\epsilon_\psi = - \frac{1}{\pi} \left[ 0.628 \psi_e + 1.065 (\psi_1 - \psi_e) + 0.445 (\psi_2 - \psi_e) + 0.231 (\psi_3 - \psi_e) + 0.104 (\psi_4 - \psi_e) \right]
\]

where \( \psi_e \) is the slope of the \( \psi \) curve at \( \varphi = \varphi_1 \), \( \psi_1 \) the value of \( \psi \) at \( \varphi = \varphi_1 + \frac{\pi}{2} \), \( \psi_2 \) at \( \varphi = \varphi_2 + \frac{\pi}{2} \), etc.

7. From the \( \epsilon \) versus \( \theta \) curve and from the \( \psi \) versus \( \theta \) curves \( \epsilon' \) and \( \psi' \) are determined.

8. Determine \( F \) by the relation

\[
F = \frac{(1 + \epsilon')e^{\psi'}}{\sqrt{\left[ \frac{y}{2a \sin \theta} \right]^2 + \sin^2 \theta}} (1 + \psi')
\]

9. \( (\theta + \epsilon) \) is determined in radians and degrees.

10. \( \sin(\theta + \alpha \pm \epsilon) \) is now calculated where \( \alpha \) is the angle of attack as measured from the axis of coordinates.

11. \( \frac{v}{V} = F \times [\sin (\theta + \alpha \pm \epsilon) + \sin (\alpha \pm \epsilon)] \)

12. \( \frac{P}{q} = 1 - \left( \frac{v}{V} \right)^2 \) (pressure)

The entire calculation, properly arranged, can be quite accurately obtained in a very short time.

**COMPARISON WITH EXPERIMENTAL RESULTS**

In order to compare the theory with experimental results, the geometric angle of attack \( \alpha_0 \) as measured in the wind tunnel must be corrected for a number of items, such as finite span and effect of wall interference. We may, however, obtain approximately the apparent or effective angle of attack \( \alpha_e \) (in radians as measured from the angle of zero lift) by taking the quotient of the area of the pressure-distribution curve and 3.5, since it is known that this value of the lift coeffi-
THEORY OF WING SECTIONS OF ARBITRARY SHAPE

Figure 2.—Pressure-distribution curves along z-axis of Clark Y; $P/q$ against percent chord.
(a) $\alpha = 9^\circ 33'$. (b) $\alpha = 2^\circ 10'$. (c) $\alpha = -1^\circ 10'$. (d) $\alpha = -3^\circ 15'$
cient is very nearly realized in most cases. This has been done in Table III, and the angle of attack $\alpha$, which should be substituted in the Equation (XII), is given in the last column. The pressure distribution curves, Figures 2a, b, c, d, and 3a, b, c, d, were obtained by application of Equation (XII) to the Clark Y airfoil. Numerical results are shown in Tables I, II, and III. The experimental values are from original data sheets for N. A. C. A. Technical Report No. 353, and are not entirely consistent due to difficulties experienced in these experiments. After the theoretical pressure distribution curves have been obtained, the moments about any required axis may be found. Table IV gives some of these results and Figure 4 shows the comparison with experimental data taken from N. A. C. A. Technical Report No. 312.

FIGURE 4.—Moment against angle of attack

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., OCTOBER 15, 1931.
EVALUATION OF THE FORMULA

\[ \varepsilon_e = (\varphi - \theta) = -\frac{1}{2\pi} \int_0^{2\pi} \psi \cot \frac{\varphi - \varphi_0}{2} \, d\varphi \]

Although the above integrand becomes positively and negatively infinite around \( \varphi = \varphi_0 \), it is readily verified that for \( \psi \) finite, throughout the range \( 0 - 2\pi \), the integral remains finite, the positive and negative infinite strips exactly canceling each other.

The value of the integral for any point \( \varphi_e \) may be accurately obtained by the following device. We know that if \( \psi \) is a continuous function and the range \( \varphi_1 \) to \( \varphi_2 \) is not too large

\[ \frac{1}{2} \int_{\varphi_1}^{\varphi_2} \psi \cot \frac{\varphi - \varphi_0}{2} \, d\varphi \]

where \( \psi_A \) is the average value of \( \psi \) in the range \( \varphi_1 \) to \( \varphi_2 \). Also near \( \varphi = \varphi_e \) we may write

\[ \psi = \psi_e + (\varphi - \varphi_e) \psi_e' + (\varphi - \varphi_e)^2 \psi_e'' + \ldots \]

Then for a small quantity

\[ \int_{\varphi_e}^{\varphi_e + \Delta \varphi} \psi \cot \frac{\varphi - \varphi_0}{2} \, d\varphi = \int_{\varphi_e - \Delta \varphi}^{\varphi_e} \psi \cot \frac{\varphi - \varphi_0}{2} \, d\varphi = 4 \psi_e' \Delta \varphi \]

(Since the even powers drop out and the \( \lim_{\varphi \to 0} \varphi \cot \varphi = 1 \).

Let us now divide the interval \( 0 - 2\pi \) into 10 parts, starting with \( \varphi_e \) as a reference point. (See Fig. 5.)

![Figure 5](image)

**APPENDIX**

Then,

\[ \varepsilon_e = -\frac{1}{2\pi} \int_0^{2\pi} \psi \cot \frac{\varphi - \varphi_0}{2} \, d\varphi \]

\[ \approx -\frac{1}{\pi} \left[ \frac{\pi}{5} \psi' + (\psi_1 - \psi') \log \frac{\sin \frac{3\pi}{20}}{\sin \frac{7\pi}{20}} \right. \]

\[ + (\psi_2 - \psi_3) \log \frac{\sin \frac{5\pi}{20}}{\sin \frac{3\pi}{20}} + (\psi_3 - \psi_4) \log \frac{\sin \frac{9\pi}{20}}{\sin \frac{7\pi}{20}} \]

\[ \left. - \frac{9\pi}{20} \right] \]

where \( \psi' \) is the slope of the \( \psi \) curve at \( \varphi = \varphi_e \).

\[ \psi_1 \text{ value of } \psi \text{ at } \varphi = \varphi_e + \frac{\pi}{5}, \psi_2 \text{ at } \varphi = \varphi_e - \frac{\pi}{5}, \psi_3 \text{ at } \varphi = \varphi_e + \frac{2\pi}{5}, \psi_4 \text{ at } \varphi = \varphi_e + \frac{3\pi}{5}, \ldots \]

To evaluate the above integral it is, strictly speaking, necessary to know \( \psi \) as a function of \( \varphi \) rather than of \( \theta \). We have \( \varphi = \theta + \varepsilon \). For all flattened or streamline bodies, however, \( \varepsilon \) is small; for ordinary airfoils it is, in fact, so small that \( \psi(\theta) \) may unconditionally be considered equal to \( \psi(\phi) \). For the sake of mathematical accuracy we will, however, indicate how the problem may be solved also for bodies of more irregular contour by successive approximations. We have

\[ \psi(\phi) = \psi(\theta) + \varepsilon \psi'(\theta) + \ldots \]

As a first approximation we neglect the second and all following terms of this expression. The value of \( \varepsilon \) thus obtained by graphical integration or otherwise is then used in the expression for \( \psi(\varphi) \) and a second integration is performed, etc.

---

1 The equation for \( \varepsilon \) is a nonlinear integral equation and to obtain its exact solution is a difficult matter; fortunately because of the small magnitude of \( \varepsilon \) the solution is obtainable to any desired accuracy by ordinary definite integrals.
APPLICATION OF FLOW FORMULA TO THE SPECIAL CASE OF AN ELLIPTIC CYLINDER

As a matter of interest we will assume the form of the body to be the ellipse \( \frac{x}{2a \cosh \psi} + \frac{y}{2a \sinh \psi} = 1 \)
and find \( \frac{v}{V} \) for zero angle of attack, i.e., we have \( \psi = \psi_0 \) constant, \( \psi' = 0 \), \( \epsilon = 0 \), \( \epsilon' = 0 \), \( \alpha = 0 \).
Equation (XII) becomes
\[
\left( \frac{v}{V} \right) = \frac{\sin \theta \cdot e^\psi}{\sqrt{\sinh^2 \psi + \sin^2 \theta}} = \frac{y \cdot e^\psi}{\sqrt{\sinh^2 \psi + \left( \frac{y}{2a \sinh \psi} \right)^2}}
\]
and
\[
\frac{v}{V} = 1 - \left( \frac{v}{V} \right)^2 = 1 - \frac{e^\psi \cdot (2ay)^2}{(2a \sinh \psi)^2 + (2ay)^2}
\]
This result checks exactly with the form given by Dr. A. F. Zahm in N. A. C. A. Technical Report No. 253, Flow and Drag Formulas for Simple Quadrics, equation 14.

REFERENCES

EXPLANATION OF THE TABLES

The first part of Table I refers to the upper surface or to positive ordinates of the Clark Y, the second part to the lower surface or to negative ordinates. Column 1 gives the location in per cent of the chord; 2 gives the ordinates with respect to the \( x \)-axis in this same unit; 3 and 4 give \( x \) and \( y \) in the present system of coordinates; 5, 6, and 7 give \( \sin^2 \theta \), \( \sin \theta \), and \( \theta \), respectively (Equation (III)); 8 gives \( \psi \) (by equation (IVa)); 9 gives \( \epsilon \) (appendix); 10 and 11 give \( \frac{d\psi}{d\epsilon} \) and \( \frac{d\psi}{d\epsilon} \) as obtained from \( \psi \) against \( \theta \) and \( \epsilon \) against \( \theta \) curves; (See figs. 7 and 8). Column 12 gives the quantity

\[ F = \sqrt{\left( \frac{y}{2 \sin \theta} \right)^2 + \sin^2 \theta} \left( 1 + \psi^2 \right) \]

Column 14 gives \( \theta + \epsilon \) in degrees. The velocity at any point \( x \) and angle of attack \( \alpha \) is given by
\[ v = V [\sin (\alpha + \theta + \epsilon) + \sin (\alpha + \epsilon') - \epsilon] \]
and the pressure, by \( \frac{P}{q} = 1 - \left( \frac{v}{V} \right)^2 \)

It must be noted that \( \alpha \) is measured from the line of flow to the \( x \)-axis as shown in Figure 6, and if otherwise measured, must be reduced to this basis.
### TABLE I

**CLARK Y**

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### LOWER SURFACE

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### TABLE II

**CLARK Y**

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### Table II gives the numerical values for Figure 2a in detail as an example. See also Table I.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Geometric angle as measured from chord line experimentally $\alpha$</th>
<th>Area of experimental pressure distribution curve $A_{\gamma}$</th>
<th>Angle measured from chord line to be used as a basis for comparison with $\alpha$ $\alpha_0$</th>
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### Table III

<table>
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<tr>
<th>Figure</th>
<th>$M_x$ Moment about line $x=25$ per cent chord</th>
<th>$M_y$ Moment about line $y=0$</th>
<th>$M_z$ Moment about point $z=25$ per cent chord, $y=0$</th>
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Theoretical Wing Sections of Arbitrary Shape
Positive directions of axes and angles (forces and moments) are shown by arrows.

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<tr>
<th>Axis</th>
<th>Force (parallel to axis) symbol</th>
<th>Moment about axis</th>
<th>Angle</th>
<th>Velocities</th>
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<td>Symbol</td>
<td>Designation</td>
<td>Symbol</td>
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<td>$N$</td>
<td>$w$</td>
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</table>

Absolute coefficients of moment

\[ C_l = \frac{L}{q \delta S} \quad C_m = \frac{M}{q \delta S} \quad C_n = \frac{N}{q \delta S} \]

Angle of set of control surface (relative to neutral position), $\delta$. (Indicate surface by proper subscript.)

4. PROPELLER SYMBOLS

- $D'$: Diameter.
- $P$: Power, absolute coefficient $C_P = \frac{P}{\rho n^3 D'^3}$.
- $C_{\text{sp}}$: Speed power coefficient $= \sqrt{\frac{V^2}{P_{\text{sp}}}}$.
- $n$: Efficiency.
- $\eta$: Revolutions per second, r. p. s.
- $\phi$: Effective helix angle $= \tan^{-1}\left(\frac{V}{2\pi n}\right)$.

5. NUMERICAL RELATIONS

- 1 hp = 76.04 kg/m/s = 550 lb./ft./sec.
- 1 kg/m/s = 0.01315 hp
- 1 mi./hr. = 0.44704 m/s
- 1 m/s = 2.23693 mi./hr.
- 1 lb. = 0.4535924277 kg.
- 1 kg = 2.2046224 lb.
- 1 mi. = 1609.35 m = 5280 ft.
- 1 m = 3.2808333 ft.