REPORT No. 516

POTENTIAL FLOW ABOUT ELONGATED BODIES OF REVOLUTION

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SUMMARY

This report presents a method of solving the problem of axial and transverse potential flows around arbitrary elongated bodies of revolution. The solutions of Laplace's equation for the velocity potentials of the axial and transverse flows, the system of coordinates being an elliptic one in a meridian plane, are known to be of the following form:

$$\phi=\sum_{n=1}^{\infty} A_n Q_n(\lambda) P_n(\mu) \quad \text{(axial flow)}$$

$$\phi=\sum_{n=1}^{\infty} A_{n1} Q_n^1(\lambda) P_n^1(\mu) \cos\theta \, \text{(transverse flow)}$$

If a power-series development of λ in μ is assumed as the equation of the meridian profile in elliptic coordinates, the boundary conditions of the two types of flow yield linear equations for the determination of the coefficients A_n and A_{n1}. It is further shown that a knowledge of these coefficients leads directly to the sink-source and doublet distributions corresponding to the axial and transverse flows, respectively.

The theory is applied to a body of revolution obtained from a symmetrical Joukowsky profile, a shape resembling an airship hull. The pressure distribution and the transverse-force distribution are calculated and serve as examples of the procedure to be followed in the case of an actual airship. A section on the determination of inertia coefficients is also included in which the validity of some earlier work is questioned.

INTRODUCTION

There are two methods of handling the problem of potential flow about a body of revolution. One, the indirect method first used by Taylor (reference 1) and by G. Fuhrmann (reference 2) who computed the pressure distribution by the method of sources and sinks suggested by Rankine. Fuhrmann assumed certain sink-source distributions and calculated the pressure distribution for the streamline body resulting from the assumed sink-source system. He also constructed models of the calculated shapes and measured the pressure distributions over them when placed in a wind tunnel.

The other method, developed by von Kármán (reference 3), considered the direct problem; i. e., the calculation of the pressure distribution over a *given* streamline shape. He approximated the requisite sink-source distribution by a computed continuous system of sinks and sources arranged in stepwise constant intensity. The various strengths were determined from the condition that the airship hull is a streamline surface in the parallel flow and the flow induced by the sinks and sources. By satisfying this condition at an arbitrary number of points equal to the number of unknown sink and source segments, von Kármán obtained a system of linear equations for the determination of the unknown strengths of the sink-source distribution. He also treated the case of transverse flow (references 3 and 4) by the distribution of doublets along the axis of symmetry of the body of revolution and calculated the strengths of the various doublet segments in a manner similar to that used for the sink-source intensities.

The present paper is an attempt to treat the direct problem according to the methods of the potential theory. Thus, Laplace's equation for the velocity potential is set up in a system of elliptic-cylindrical coordinates λ, μ, θ and solved in conjunction with the appropriate boundary conditions for axial and transverse flows. It is then assumed that a power-series development of λ in μ represents the meridian profile of the elongated body of revolution. The boundary conditions for the two types of flow may then be expressed in the form of power series in μ valid for the entire range of μ. This method leads to two sets of linear equations, each set infinite in number of equations and each equation containing an infinite number of unknown coefficients which serve to determine the velocity potentials for the axial and transverse flows. Incidental to the major task of determining these coefficients, the sink-source and doublet distributions corresponding to the axial and transverse flows are also determined. Thus the results of this method are essentially the same as those obtained by the method of von Kármán but are obtained in a more rigorous and direct manner. In von Kármán's method, approximations are made prior to the analysis; whereas, in the method presented in this paper, approximations are made after the analysis has been carried through in a rigorous manner.

FUNDAMENTAL EQUATIONS

The fluid motion is assumed to be steady and irrotational. There then exists a velocity potential ϕ, which is, in general, a function of the rectangular Cartesian coordinates (x, y, z). In cases of rotational symmetry, however, it is appropriate to introduce the cylindrical coordinates (z, ρ, θ) where z denotes the distance along the axis of symmetry, $\rho(=\sqrt{x^2+y^2})$ the perpendicular distance from this axis, and θ the angle between the (z, ρ) and (z, x) planes. (See fig. 1.)

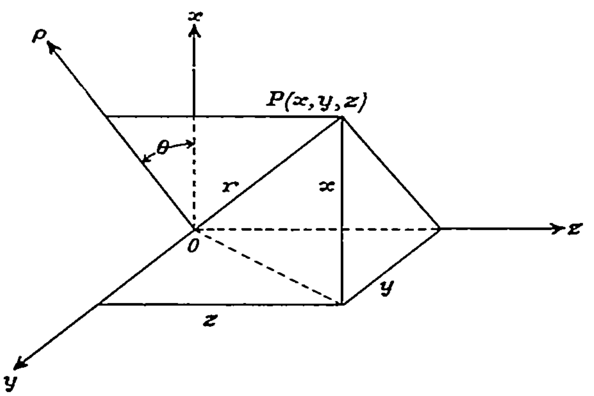

FIGURE 1.

Furthermore, since only elongated surfaces of revolution are to be considered it is natural to introduce a prolate-elliptic coordinate system in the (z, ρ) plane. The equations of transformation from the coordinates (z, ρ) to the prolate-elliptic coordinates (ζ, η) are:

$$\left.\begin{aligned} z&=2a\cosh\zeta\cos\eta \\ \rho&=2a\sinh\zeta\sin\eta \end{aligned}\right\} \tag{1}$$

where $\quad 0\leqq\zeta\leqq\infty$ and $0\leqq\eta<2\pi$

Thus $\zeta=$constant and $\eta=$constant represent confocal ellipses and hyperbolas, respectively, the distance between the foci being $4a$.

For any point in space $P(x, y, z)$ then

$$\left.\begin{aligned} x&=2a(\lambda^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\cos\theta \\ y&=2a(\lambda^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\sin\theta \\ z&=2a\lambda\mu \end{aligned}\right\} \tag{2}$$

where $\quad \lambda=\cosh\zeta$ and $\mu=\cos\eta$.

If, furthermore, the fluid is incompressible the velocity potential ϕ satisfies Laplace's equation $\Delta_2\phi=0$ and since the (λ, μ, θ) system of coordinates is an orthogonal one, takes the form:

$$\frac{\partial}{\partial\lambda}\left[(\lambda^2-1)\frac{\partial\phi}{\partial\lambda}\right]+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] +\left(\frac{1}{\lambda^2-1}+\frac{1}{1-\mu^2}\right)\frac{\partial^2\phi}{\partial\theta^2}=0 \tag{3}$$

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In this case the flow is the same for all meridian planes (z, ρ) and therefore the velocity potential ϕ is a function only of λ and μ. Equation (3) then reduces to

$$\frac{\partial}{\partial\lambda}\left[(\lambda^2-1)\frac{\partial\phi}{\partial\lambda}\right]+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right]=0 \tag{4}$$

If this equation is to be satisfied by a product

$$\phi=L(\lambda)M(\mu)$$

it follows that

$$\frac{1}{L(\lambda)}\frac{d}{d\lambda}\left[(\lambda^2-1)\frac{dL(\lambda)}{d\lambda}\right]=-\frac{1}{M(\mu)}\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM(\mu)}{d\mu}\right]$$

which separates into two ordinary differential equations

$$\left.\begin{aligned} \frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dL}{d\lambda}\right]+cL=0 \\ \frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+cM=0 \end{aligned}\right\} \tag{5}$$

where c is an arbitrary constant.

Furthermore, if $c=n(n+1)$, each of these equations is of the Legendre type and therefore the general solution of equation (4) is

$$\phi=\sum_{n=0}^{\infty}A_nP_n(\lambda)P_n(\mu) \tag{6}$$

This expression for ϕ has a singularity at infinity since $P_n(\lambda)$ is a polynomial of the nth degree in λ and is therefore infinite for $\lambda=\infty$. Since the region outside a surface is to be considered and since it must include the region at infinity, another solution for $L(\lambda)$ is required. This solution, linearly independent of $P_n(\lambda)$, is the zonal harmonic of the second kind and is denoted by $Q_n(\lambda)$ where

$$Q_n(\lambda)=P_n(\lambda)\int_{\lambda}^{\infty}\frac{d\lambda}{[P_n(\lambda)^2](\lambda^2-1)} \tag{7}$$

It vanishes for $\lambda=\infty$ but has a singular point for $\lambda=\pm1$ where it is infinite like $\log(\lambda\pm1)$.

Thus, for example, since $P_0(\lambda)=1$, $P_1(\lambda)=\lambda$, it is found that

$$Q_0(\lambda)=\int_{\lambda}^{\infty}\frac{d\lambda}{\lambda^2-1}=\frac{1}{2}\log\frac{\lambda+1}{\lambda-1}=\frac{1}{\lambda}+\frac{1}{3\lambda^3}+\frac{1}{5\lambda^5}+\ldots$$

where $|\lambda|>1$ and

$$Q_1(\lambda)=\lambda\int_{\lambda}^{\infty}\frac{d\lambda}{\lambda^2(\lambda^2-1)}=\frac{1}{2}\lambda\log\frac{\lambda+1}{\lambda-1}-1$$
$$=\frac{1}{3\lambda^2}+\frac{1}{5\lambda^4}+\frac{1}{7\lambda^6}+\ldots$$

It may also be shown that

$$Q_n(\lambda)=\frac{1}{2}P_n(\lambda)\log\frac{\lambda+1}{\lambda-1}-K_n(\lambda)$$

where $K_n(\lambda)$ is a polynomial of the $(n-1)$th degree.

Another useful expression for $Q_n(\lambda)$ is that due to F. Neumann (reference 5); namely

$$Q_n(\lambda)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(\lambda_1)d\lambda_1}{\lambda-\lambda_1} \tag{8}$$

Expanding $\frac{1}{\lambda-\lambda_1}$ in decreasing powers of λ,

$$Q_n(\lambda)=\frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{\lambda^{i+1}}\int_{-1}^{1}\lambda_1^iP_n(\lambda_1)d\lambda_1$$

Expressing λ_1^i in terms of zonal harmonics (reference 5)

$$\lambda_1^i=\sum_{k=0}^{\frac{i}{2},\frac{i-1}{2}}\frac{(2i-4k+1)i\,!}{[2i-2k+1]\,[2k]}P_{i-2k}(\lambda_1)$$

where the upper limit $\left\{\begin{matrix}\frac{i}{2}\\ \frac{i-1}{2}\end{matrix}\right\}$ used depends on whether i is $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$ and where $[2n]=2\cdot4\cdot6\ldots2n$; $[2n-1]=1\cdot3\cdot5\ldots(2n-1)$; $[0]=[1]=[-1]=1$. Also $(2n-1)\,!=[2n-1][2n-2]$.

Substituting this expression for λ_1^i in the foregoing equation for $Q_n(\lambda)$ it follows that

$$Q_n(\lambda)=\frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{\lambda^{i+1}}\sum_{k=0}^{\frac{i}{2},\frac{i-1}{2}}\frac{(2i-4k+1)i\,!}{[2i-2k+1][2k]}\int_{-1}^{1}P_n(\lambda_1)P_{i-2k}(\lambda_1)d\lambda_1$$

The zonal harmonics $P_n(\lambda_1)$ are orthogonal functions and satisfy the following relations:

$$\int_{-1}^{1}P_rP_sd\lambda_1=\begin{matrix}0 & \text{if } r\neq s\\ \frac{2}{2r+1} & \text{if } r=s\end{matrix}$$

Expanding the preceding expression for $Q_n(\lambda)$ with regard to i and writing the terms with equal indices of k in columns and adding these columns, there is obtained, using the orthogonal property of the $P_n(\lambda_1)$'s, the following equation:

$$Q_n(\lambda)=\sum_{k=0}^{\infty}\frac{(n+2k)\,!}{[2n+2k+1][2k]}\frac{1}{\lambda^{n+2k+1}}\text{ where } n=0,1,2,\cdots \tag{9}$$

This expression is convergent for $|\lambda|>1$ and divergent for $|\lambda|\leqq 1$.

Instead of being given by equation (6) the velocity potential is now given by the following expression:

$$\phi=\sum_{n=0}^{\infty}A_nQ_n(\lambda)P_n(\mu) \tag{10}$$

which gives the general solution of equation (4) for regions outside a surface of revolution and extending to infinity.

In cases of rotational symmetry where the lines of flow are in meridian planes, it is convenient to introduce Stokes' stream function ψ. This function arises from the statement that the fluid is incompressible (equation of continuity) and is related to the velocity potential ϕ according to the following equations:

$$\frac{\partial\psi}{\partial\rho}=\rho\frac{\partial\phi}{\partial z}\text{ and }\frac{\partial\psi}{\partial z}=-\rho\frac{\partial\phi}{\partial\rho} \tag{11}$$

The lines $\psi=$ constant represent the streamlines. It may be remarked that, unlike the two-dimensional case where both the stream function and the velocity potential satisfy Laplace's equation, Stokes' stream function does not satisfy it.

The introduction of the variables λ, μ into equations (11) by means of equations (1) leads to the following relations:

$$\frac{\partial\psi}{\partial\lambda}=2a(1-\mu^2)\frac{\partial\phi}{\partial\mu}\text{ and }\frac{\partial\psi}{\partial\mu}=-2a(\lambda^2-1)\frac{\partial\phi}{\partial\lambda} \tag{12}$$

If a substitution is made for ϕ from equation (10) and $P_n(\mu)$ is replaced by its value obtained from Legendre's differential equation, that is:

$$P_n(\mu)=-\frac{1}{n(n+1)}\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_n}{d\mu}\right]$$

it is found that

$$\psi=2a(1-\mu^2)(\lambda^2-1)\sum_{n=1}^{\infty}\frac{A_n}{n(n+1)}\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}+\text{constant} \tag{13}$$

Furthermore, if the body of revolution is moving with a velocity U in the direction of the axis of symmetry z, it may be conveniently supposed to be at rest and the fluid to have a translation $-U$ superposed on its actual motion. This consideration adds a term $2aU\lambda\mu$ to the velocity potential and $2a^2U(1-\mu^2)(\lambda^2-1)$ to the stream function. Therefore

$$\psi=2a^2(1-\mu^2)(\lambda^2-1)U\left[\sum_{n=1}^{\infty}\frac{A_n}{aUn(n+1)}\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}+1\right] \tag{14}$$

At the surface of the fixed body of revolution the normal velocity of the fluid must be zero and therefore the boundary must coincide with a streamline $\psi=$ constant, say 0. Hence the boundary condition at the surface is given by

$$\sum_{n=1}^{\infty}\frac{A_n}{n(n+1)}\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}+aU=0 \tag{15}$$

In order to find the velocity components u_λ, u_μ, in the directions of the coordinate lines λ, μ, respectively, it is to be noted that since the system of coordinates is an orthogonal one,

$$u_\lambda=-\frac{\partial\phi}{\partial s_\lambda}\text{ and } u_\mu=-\frac{\partial\phi}{\partial s_\mu}$$

where $\quad ds^2=dz^2+d\rho^2=ds_\lambda^2+ds_\mu^2$

By means of the equations of transformation (1), it is found that

$$ds_\lambda = 2a\left(\frac{\lambda^2-\mu^2}{\lambda^2-1}\right)^{\frac{1}{2}} d\lambda.$$

and

$$ds_\mu = 2a\left(\frac{\lambda^2-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}} d\mu$$

Therefore:

$$u_\lambda = -\frac{1}{2a}\left(\frac{\lambda^2-1}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\frac{\partial\phi}{\partial\lambda}$$

and

$$u_\mu = -\frac{1}{2a}\left(\frac{1-\mu^2}{\lambda^2-\mu^2}\right)^{\frac{1}{2}}\frac{\partial\phi}{\partial\mu}$$

Hence:

$$u^2 = \frac{1}{4a^2(\lambda^2-\mu^2)}\left[(\lambda^2-1)\left(\frac{\partial\phi}{\partial\lambda}\right)^2 + (1-\mu^2)\left(\frac{\partial\phi}{\partial\mu}\right)^2\right] \quad (16)$$

SINK-SOURCE DISTRIBUTION

The distribution of sinks and sources is assumed to lie along the segment of the axis of symmetry z, $-2a \leqq z_1 \leqq 2a$, and to be of intensity $I(z_1)$ per unit length. At any point (z, ρ) in any meridian plane the velocity potential due to this distribution is given (reference 6, p. 60) by

$$\phi = \frac{1}{4\pi}\int_{-2a}^{2a}\frac{I(z_1)dz_1}{[(z-z_1)^2+\rho^2]^{\frac{1}{2}}} \quad (17)$$

For points lying on the z axis but outside the distribution, the velocity potential is given by the simplified expression

$$\phi = \frac{1}{4\pi}\int_{-2a}^{2a}\frac{I(z_1)dz_1}{z-z_1}$$

Substituting for z and z_1, $2a\lambda$ and $2a\lambda_1$, respectively, the preceding equation takes the form

$$\phi = \frac{1}{4\pi}\int_{-1}^{1}\frac{I(2a\lambda_1)d\lambda_1}{\lambda-\lambda_1}$$

Finally, substituting for ϕ from equation (10) and noting that $P_n(1)=1$ for all values of n,

$$\sum_{n=1}^{\infty} A_n Q_n(\lambda) = \frac{1}{4\pi}\int_{-1}^{1}\frac{I(2a\lambda_1)d\lambda_1}{\lambda-\lambda_1} \quad (18)$$

This is an integral equation for the unknown function $I(2a\lambda_1)$. It may be solved in the following manner:

From F. Neumann's expression for $Q_n(\lambda)$ given by equation (8) the following development is suggested for the distribution function:

$$I(2a\lambda_1) = \sum_{n=1}^{\infty} a_n P_n(\lambda_1)$$

where

$$-1 \leqq \lambda_1 \leqq 1$$

It then follows directly from equation (8) that

$$\sum_{n=1}^{\infty}\left(A_n - \frac{1}{2\pi}a_n\right)Q_n(\lambda) = 0$$

for all values of λ.

Hence

$$a_n = 2\pi A_n$$

and

$$I(2a\lambda_1) = 2\pi\sum_{n=1}^{\infty} A_n P_n(\lambda_1) \quad (19)$$

Thus, given the potential function ϕ, that is the A_n's, this expression determines the equivalent sink-source distribution.

FLOW NORMAL TO THE AXIS OF SYMMETRY

The differential equation for the velocity potential in the case of transverse flow is given by equation (3). Recalling that this expression is Laplace's equation in the coordinates λ, μ, θ, it may be solved by supposing ϕ to be a product $N(\lambda, \mu)\ R(\theta)$. Replacing ϕ in equation (3) by this product, the following pair of differential equations is obtained:

$$\left.\begin{aligned} &\frac{d^2R}{d\theta^2} + k^2R = 0 \\ &\frac{\partial}{\partial\lambda}\left[(\lambda^2-1)\frac{\partial N}{\partial\lambda}\right] + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial N}{\partial\mu}\right] \\ &\qquad -k^2\frac{\lambda^2-\mu^2}{(1-\mu^2)(\lambda^2-1)}N = 0 \end{aligned}\right\} \quad (20)$$

The general solution of the first equation is given by

$$R = A\cos k\,\theta + B\sin k\,\theta$$

where A and B are arbitrary constants.

Putting $N(\lambda, \mu) = L(\lambda)\ M(\mu)$ in the second equation leads to the following pair of ordinary differential equations:

$$\left.\begin{aligned} &\frac{d}{d\lambda}\left[(1-\lambda^2)\frac{dL}{d\lambda}\right] + \left(c - \frac{k^2}{1-\lambda^2}\right)L = 0 \\ &\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right] + \left(c - \frac{k^2}{1-\mu^2}\right)M = 0 \end{aligned}\right\} \quad (21)$$

where c is an arbitrary constant.

Both of the latter equations are of the form of the differential equation for the associated Legendre functions provided that $c = n(n+1)$. Accordingly,

$$M(\mu) = P_n^k(\mu) \text{ and } L(\lambda) = P_n^k(\lambda)$$

where, for example,

$$P_n^k(\mu) = (1-\mu^2)^{\frac{k}{2}}\frac{d^kP_n(\mu)}{d\mu^k}$$

The general solution of equation (3) may then be written as

$$\phi_1 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_n^k(\mu)P_n^k(\lambda)[A_{nk}\cos k\theta + B_{nk}\sin k\theta]$$

This expression, however, has a singularity at infinity and since only the region outside a given surface of revolution is of interest, the infinite region, or the neighborhood of $\lambda = \infty$, must be considered. Therefore $P_n^k(\lambda)$ is replaced by the associated Legendre function of the second kind $Q_n^k(\lambda)$, where by definition,

$$Q_n^k(\lambda) = (\lambda^2-1)^{\frac{k}{2}}\frac{d^kQ_n(\lambda)}{d\lambda^k}$$

Then

$$\phi_1 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_n^k(\mu)Q_n^k(\lambda)[A_{nk}\cos k\theta + B_{nk}\sin k\theta] \quad (22)$$

If the body of revolution moves with a uniform velocity V in the direction of the x axis, it may be supposed to be at rest and the fluid to have a translation $-V$ superposed on its actual motion. Then

$$\phi=\phi_1+xV \tag{23}$$

Consider the body profile in any one of the meridian planes θ. At any arbitrary point of it the normal derivative of ϕ is given by

$$-\frac{\partial \phi}{\partial n}ds=\frac{\partial \phi}{\partial s_\mu}ds_\lambda-\frac{\partial \phi}{\partial s_\lambda}ds_\mu \tag{24}$$

Since the normal velocity along the meridian curve is zero, it follows from equations (23) and (24) that

$$\left(\frac{\partial \phi_1}{\partial s_\lambda}+V\frac{\partial x}{\partial s_\lambda}\right)ds_\mu-\left(\frac{\partial \phi_1}{\partial s_\mu}+V\frac{\partial x}{\partial s_\mu}\right)ds_\lambda=0$$

Also, $x=\rho\cos\theta$, so that

$$\frac{\partial x}{\partial s_\lambda}=\cos\theta\frac{\partial \rho}{\partial s_\lambda} \text{ and } \frac{\partial x}{\partial s_\mu}=\cos\theta\frac{\partial \rho}{\partial s_\mu}$$

Therefore

$$\left(\frac{\partial \phi_1}{\partial s_\lambda}+V\cos\theta\frac{\partial \rho}{\partial s_\lambda}\right)ds_\mu=\left(\frac{\partial \phi_1}{\partial s_\mu}+V\cos\theta\frac{\partial \rho}{\partial s_\mu}\right)ds_\lambda \tag{25}$$

In order that the condition of no flow normal to the body of revolution be valid for all values of θ, there must be chosen from among all the solutions given by equation (22) that one which has $\cos\theta$ as a factor; namely:

$$\phi_1=\sum_{n=1}^{\infty}A_{n1}P_n^1(\mu)Q_n^1(\lambda)\cos\theta$$

or

$$\phi_1=2aV\cos\theta(\lambda^2-1)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}\sum_{n=1}^{\infty}C_n\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda} \tag{26}$$

where $C_n=\dfrac{A_{n1}}{2aV}$

Furthermore

$$ds_\lambda=2a\left(\frac{\lambda^2-\mu^2}{\lambda^2-1}\right)^{\frac{1}{2}}d\lambda \text{ and } ds_\mu=2a\left(\frac{\lambda^2-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}}d\mu$$

so that equation (25) becomes

$$\frac{\dfrac{\partial}{\partial \lambda}(\phi_1+\rho V\cos\theta)}{\dfrac{\partial}{\partial \mu}(\phi_1+\rho V\cos\theta)}=\frac{1-\mu^2}{\lambda^2-1}\frac{d\lambda}{d\mu}$$

Finally, by means of equation (26) and the differential equation for the Legendre polynomials the foregoing boundary condition takes the following form:

$$\sum_{n=1}^{\infty}C_n\left[\frac{d(\lambda\mu)}{d\mu}\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}-n(n+1)\frac{d}{d\mu}(P_nQ_n)\right]=\frac{d(\lambda\mu)}{d\mu} \tag{27}$$

DISTRIBUTION OF DOUBLETS

The doublets are assumed to have their axes in the x direction and to lie along the segment $-2a\leqq z_1\leqq 2a$ of the axis of symmetry z. The velocity potential at any point (z, ρ) of some meridian plane θ then takes the form (see reference 6):

$$\phi_1=\frac{\rho\cos\theta}{4\pi}\int_{-2a}^{2a}\frac{J(z_1)dz_1}{[(z-z_1)^2+\rho^2]^{\frac{3}{2}}}$$

where $J(z_1)$ is the intensity of the doublets per unit length.

Substituting for ϕ_1 from equation (26) it follows that

$$V\sum_{n=1}^{\infty}C_n\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}=\frac{1}{4\pi}\int_{-2a}^{2a}\frac{J(z_1)dz_1}{[(z-z_1)^2+\rho^2]^{\frac{3}{2}}}$$

For points lying on the z axis but outside the distribution this equation takes the following simplified form:

$$V\sum_{n=1}^{\infty}\frac{n(n+1)}{2}C_n\frac{dQ_n}{d\lambda}=\frac{1}{16\pi a^2}\int_{-1}^{1}\frac{J(2a\lambda_1)d\lambda_1}{(\lambda-\lambda_1)^3} \tag{28}$$

where z_1 is replaced by $2a\lambda_1$, z by $2a\lambda$, and $\left(\dfrac{dP_n}{d\mu}\right)_{\mu=1}$ by $\dfrac{n(n+1)}{2}$. This is an integral equation for the unknown function $J(2a\lambda_1)$. In the solution of this integral equation it is necessary that a development of $\dfrac{1}{(\lambda-\lambda_1)^3}$ as a series of Legendre ploynomials in λ_1 be obtained. The form of this development is suggested by Neumann's equation (8). Thus assume that

$$\frac{1}{\lambda-\lambda_1}=\sum_{n=1}^{\infty}b_nP_n(\lambda_1)Q_n(\lambda)$$

Then substituting this expression for $\dfrac{1}{\lambda-\lambda_1}$ in Neumann's equation and making use of the orthogonality relations satisfied by the Legendre polynomials, it is found that

$$b_n=2n+1$$

Therefore

$$\frac{1}{\lambda-\lambda_1}=\sum_{n=1}^{\infty}(2n+1)P_n(\lambda_1)Q_n(\lambda)$$

Differentiating this last expression once with regard to λ and once with regard to λ_1, it follows that

$$\frac{1}{(\lambda-\lambda_1)^3}=-\sum_{n=1}^{\infty}\frac{2n+1}{2}\frac{dP_n(\lambda_1)}{d\lambda_1}\frac{dQ_n(\lambda)}{d\lambda}$$

Equation (28) then becomes

$$V\sum_{n=1}^{\infty}C_n\frac{n(n+1)}{2}\frac{dQ_n}{d\lambda}=$$

$$-\frac{1}{32\pi a^2}\sum_{n=1}^{\infty}(2n+1)\frac{dQ_n}{d\lambda}\int_{-1}^{1}J(2a\lambda_1)\frac{dP_n}{d\lambda_1}d\lambda \tag{29}$$

It is now obvious that the following assumption must be made:

$$J(2a\lambda_1)=-8\pi a^2V(1-\lambda_1^2)\sum_{m=1}^{\infty}c_m\frac{dP_m}{d\lambda_1}$$

and substituting this expression in equation (29) it follows that

$$\sum_{n=1}^{\infty}(C_n-c_n)\frac{dQ_n}{d\lambda}=0$$

In order that this equation be valid for arbitrary values of λ,

$$c_n=C_n$$

and therefore

$$J(2a\lambda_1)=-8\pi a^2V(1-\lambda_1^2)\sum_{n=1}^{\infty}C_n\frac{dP_n}{d\lambda_1} \qquad (30)$$

Thus, given the velocity potential ϕ_1, that is, the C_n's, this expression determines the equivalent doublet distribution.

DETERMINATION OF THE COEFFICIENTS A_n AND C_n

Any symmetrical profile may be represented by a power series in $\mu(=\cos\eta)$. That is

$$\lambda=\sum_{r=0}^{\infty}a_r\mu^r \qquad (31)$$

The rapidity of convergence of this series depends, however, on the choice of origin with respect to the profile. Since $\lambda=a_0$ defines an ellipse, the rapidity of convergence of the foregoing series may be looked upon as a measure of the resemblance of the profile to the ellipse $\lambda=a_0$. The proper choice of origin may be attained in the following manner. The radius of curvature R of an ellipse at the end of its major axis is given by

$$R=\frac{B^2}{A}$$

where A and B are its semimajor and semiminor axes, respectively.

Eliminating η from equations (1), the following equation of a system of confocal ellipses results: $\frac{z^2}{(2a\cosh\zeta)^2}+\frac{\rho^2}{(2a\sinh\zeta)^2}=1$. (The distance between foci is $4a$.)

In terms of elliptic coordinates then

$$R=2a\frac{\sinh^2\zeta}{\cosh\zeta}$$

Furthermore, for an elongated ellipse the semimajor axis $2a\cosh\zeta$ is large compared to the semiminor axis $2a\sinh\zeta$. This limitation means that ζ is small. Neglecting powers of ζ higher than the second it follows that (see reference 7)

$$R=2a\zeta^2 \text{ (approximately)}$$

The ends of the ellipse are at

$$\pm 2a\cosh\zeta=\pm 2a\left(1+\frac{\zeta^2}{2}+\cdots\right)=\pm\left(2a+\frac{R}{2}\right)$$
(approximately)

and therefore the focus of an elongated ellipse very nearly bisects the line joining the end of the semimajor axis and the center of curvature. Thus the proper choice of origin is the point bisecting the line of length $4a$ extending from the point midway between the leading edge and the center of curvature of that edge to a point midway between the trailing edge and the center of curvature of that edge. Having thus chosen a reference frame (z, ρ) in which to present the profile, the next step is to obtain the series equation (31). This equation may be obtained with the help of the following expressions. From equation (1) it can be found that

$$\left.\begin{aligned}\lambda&=\frac{1}{2}\left[\sqrt{\left(\frac{z}{2a}+1\right)^2+\left(\frac{\rho}{2a}\right)^2}+\sqrt{\left(\frac{z}{2a}-1\right)^2+\left(\frac{\rho}{2a}\right)^2}\right]\\ \mu&=\frac{1}{2}\left[\sqrt{\left(\frac{z}{2a}+1\right)^2+\left(\frac{\rho}{2a}\right)^2}-\sqrt{\left(\frac{z}{2a}-1\right)^2+\left(\frac{\rho}{2a}\right)^2}\right]\end{aligned}\right\} \qquad (32)$$

where $-1\leqq\mu\leqq 1$.

A series of corresponding values of λ and μ are thus obtained. In order to express λ as a polynomial in μ of, say, degree n, it is most convenient to employ the method of least squares for determining the $(n+1)$ constants a_r (reference 8).

FLOW PARALLEL TO THE AXIS OF SYMMETRY

The boundary condition for this type of flow is given by equation (15). In that expression functions of the type $\frac{dQ_n}{d\lambda}$ appear and these are to be expressed as power series in μ.

Suppose the meridian profile to be given by the following analytic expression:

$$\lambda=a_0+\mu\sum_{n=0}^{\infty}a_{1,n}\mu^n \qquad (33)$$

Then on the profile, $\frac{dQ_n}{d\lambda}$ may be looked upon as a function of μ and can be developed in a Taylor series in the neighborhood of $\mu=0$ or $\lambda=a_0$. That is,

$$\frac{dQ_n}{d\lambda}=\sum_{p=0}^{\infty}\frac{(\lambda-a_0)^p}{p!}\frac{d^{p+1}Q_n}{da_0^{p+1}}$$

$$\text{where } \frac{d^{p+1}Q_n}{da_0^{p+1}}=\left(\frac{d^{p+1}Q_n}{d\lambda^{p+1}}\right)_{\lambda=a_0}$$

Substituting for $\lambda-a_0$ from equation (33), it follows that

$$\frac{dQ_n}{d\lambda}=\sum_{p=0}^{\infty}\frac{\mu^p}{p!}\left(\sum_{q=0}^{\infty}a_{1,q}\mu^q\right)^p\frac{d^{p+1}Q_n}{da_0^{p+1}}$$

In the following the expansion of S^p in powers of μ is to be determined (reference 9, p. 122), p being any positive integer and where

$$S=\sum_{q=0}^{\infty}a_{1,q}\mu^q$$

Thus

$$S^2=\sum_{q=0}^{\infty}a_{1,r}\mu^r\sum_{q=0}^{\infty}a_{1,q}\ \mu^q$$

or

$$S^2=\sum_{r=0}^{\infty}a_{1,r}\sum_{p=r}^{\infty}a_{1,p-r}\ \mu^p$$

Expanding S^2 with respect to r and writing the terms with equal indices of p in columns and adding these columns,

$$S^2=\sum_{p=0}^{\infty}\mu^p\sum_{r=0}^{p}a_{1,r}\ a_{1,p-r}=\sum_{p=0}^{\infty}a_{2,p}\ \mu^p$$

where

$$a_{2,p}=\sum_{r=0}^{p}a_{1,p-r}\,a_{1,r}$$

In a similar manner,

$$S^3=\sum_{p=0}^{\infty}a_{3,p}\ \mu^p$$

where

$$a_{3,p}=\sum_{r=0}^{p}a_{1,p-r}\ a_{2,r}$$

and, in general, it is permissible to write

$$S^t=\sum_{p=0}^{\infty}a_{t,p}\ \mu^p$$

where

$$a_{t,p}=\sum_{r=0}^{p}a_{1,p-r}\ a_{t-1,r}\text{ and }a_{0,r}=\begin{matrix}0\text{ if }r\neq0\\1\text{ if }r=0\end{matrix}$$

Hence

$$\frac{dQ_n}{d\lambda}=\sum_{p=0}^{\infty}\frac{\mu^p}{p!}\frac{d^{p+1}Q_n}{da_0^{p+1}}\sum_{q=0}^{\infty}a_{p,q}\ \mu^q$$

or

$$\frac{dQ_n}{d\lambda}=\sum_{p=0}^{\infty}\sum_{q=p}^{\infty}\frac{\mu^q}{p!}\frac{d^{p+1}Q_n}{da_0^{p+1}}a_{p,q-p}$$

Expanding according to p and writing the terms with equal indices of q in columns and adding these columns,

$$\frac{dQ_n}{d\lambda}=\sum_{q=0}^{\infty}\mu^q\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_n}{da_0^{p+1}}\tag{34}$$

where

$$\left.\begin{aligned}
&a_{p,0}=a^p_{1,0}\\
&a_{p,1}=p\,a^{p-1}_{1,0}a_{1,1}\\
&a_{p,2}=p\,a^{p-1}_{1,0}a_{1,2}+\frac{p(p-1)}{2!}a^{p-2}_{1,0}a^2_{1,1}\\
&a_{p,3}=p\,a^{p-1}_{1,0}a_{1,3}+p(p-1)a^{p-2}_{1,0}a_{1,1}a_{1,2}+\frac{p(p-1)(p-2)}{3!}a^{p-3}_{1,0}a^3_{1,1}
\end{aligned}\right\}\tag{35}$$

and so on.

The boundary condition also contains terms of the type $\frac{dP_n}{d\mu}$ where $P_n(\mu)$ is the Legendre polynomial in μ of degree n and is given by

$$P_n(\mu)=\sum_{j=0}^{\frac{n}{2},\frac{n-1}{2}}(-1)^j\frac{[2n-2j-1]!}{(n-2j)!\,[2j]}\mu^{n-2j}\tag{36}$$

where the upper limit for j is $\left\{\begin{matrix}\frac{n}{2}\\ \frac{n-1}{2}\end{matrix}\right\}$ according as n is $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$

Then,
$$\frac{dP_n}{d\mu}=\sum_{j=0}^{\frac{n-2}{2},\frac{n-1}{2}}(-1)^j\frac{[2n-2j-1]}{(n-2j-1)!\,[2j]}\mu^{n-2j-1}\tag{37}$$

Substituting for $\frac{dQ_n}{d\lambda}$ and $\frac{dP_n}{d\mu}$ the expressions given by equations (34) and (37) into the boundary condition equation (15),

$$\sum_{n=1}^{\infty}\frac{A_n}{n(n+1)}\sum_{j=0}^{\frac{n-2}{2},\frac{n-1}{2}}(-1)^j\frac{[2n-2j-1]}{(n-2j-1)!\,[2j]}\mu^{n-2j-1}\sum_{q=0}^{\infty}\mu^q\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_n}{da_0^{p+1}}+aU=0$$

Expanding according to powers of n and writing the terms with equal indices of j in columns and then adding these columns,

$$\sum_{j=0}^{\infty}\sum_{n=2j+1}^{\infty}\frac{A_n}{n(n+1)}(-1)^j\frac{[2n-2j-1]}{(n-2j-1)!\,[2j]}\mu^{n-2j-1}\sum_{q=0}^{\infty}\mu^q\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_n}{da_0^{p+1}}+aU=0$$

Putting $n-2j-1+q=m$ this becomes

$$\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=q}^{\infty}\mu^m(-1)^j\frac{A_{2j+1+m-q}}{(2j+1+m-q)(2j+2+m-q)}\frac{[2j+1+2m-2q]}{(m-q)!\,[2j]}\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_{2j+1+m-q}}{da_0^{p+1}}+aU=0$$

Expanding with respect to q and writing the terms with equal indices of m in columns and adding these columns,

$$\sum_{m=0}^{\infty}\mu^m\sum_{q=0}^{m}\sum_{j=0}^{\infty}(-1)^j\frac{A_{2j+1+m-q}}{(2j+1+m-q)(2j+2+m-q)}\frac{[2j+1+2m-2q]}{(m-q)!\,[2j]}\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_{2j+1+m-q}}{da_0^{p+1}}+aU=0 \tag{38}$$

If this expression is to be valid everywhere on the boundary surface, it must hold for the entire range of μ. It follows that the coefficients of the various powers of μ are identically zero. Finally, the introduction of k and n by means of the substitutions $q=m-k$ and $p=m-n$, respectively, leads to the following expression of the boundary condition:

$$\sum_{n=0}^{m}\sum_{k=0}^{n}\frac{a_{m-n,n-k}}{k!(m-n)!}\sum_{j=0}^{\infty}(-1)^j\frac{[2j+1+2k]A_{2j+1+k}}{(2j+1+k)(2j+2+k)[2j]}\frac{d^{m-n+1}Q_{2j+1+k}}{da_0^{m-n+1}}=-aU\delta_0^m \tag{39}$$

where

$$\delta_0^m=\begin{matrix}1 \text{ if } m=0\\ 0 \text{ if } m\neq 0\end{matrix} \text{ and } m=0,\ 1,\ 2,\ \ldots\ \infty.$$

Equation (39) represents a set of linear equations infinite in number and containing an infinite number of unknowns A_n. It provides a formal and rigorous solution of the problem of potential flow about a body of revolution, parallel to its axis of symmetry.

In the foregoing equations the only unknowns are the A_n's. The $a_{p,q}$'s are related to the coefficients of the power series of λ in μ (giving the meridian profile equation (33)) and are evaluated by means of equations (35). Finally, the Q_n's and their derivatives are well-defined Legendre functions.

For example, if the meridian profile is an ellipse $\lambda=a_0$, then equation (39) becomes

$$\sum_{j=0}^{\infty}(-1)^j\frac{A_{2j+1+m}}{(2j+1+m)(2j+2+m)}\frac{[2j+2m+1]}{m!\,[2j]}\frac{dQ_{2j+1+m}}{da_0}=-\delta_0^m aU$$

For $m=1,\ 2,\ 3,\ \ldots$ this is an infinite set of linear homogeneous equations for the unknowns A_2, A_3, ... and, since the determinant of the coefficients is different from zero, the only solution is that A_2, A_3, ... are zero. From the first equation, i. e., $m=0$, it is then found that

$$A_1=-\frac{2aU}{\frac{dQ_1}{da_0}}=-\frac{2aU}{\frac{1}{2}\log\frac{a_0+1}{a_0-1}-\frac{a_0}{a_0^2-1}}$$

Hence

$$\phi=-\frac{2aU}{\frac{1}{2}\log\frac{a_0+1}{a_0-1}-\frac{a_0}{a_0^2-1}}\left(\frac{1}{2}\lambda\log\frac{\lambda+1}{\lambda-1}-1\right)\mu$$

If A and B are the semimajor and semiminor axes of the meridian ellipse and e its eccentricity, then

$$2a=Ae,\ a_0=\frac{1}{e},\ 2a(a_0^2-1)^{\frac{1}{2}}=B$$

so that

$$\phi=-\frac{UA}{\frac{1}{2e}\log\frac{1+e}{1-e}-\frac{1}{1-e^2}}\left(\frac{1}{2}\lambda\log\frac{\lambda+1}{\lambda-1}-1\right)\mu$$

This result agrees with the well-known expression for the velocity potential of a prolate ellipsoid of revolution (reference 10, p. 132).

FLOW NORMAL TO THE AXIS OF SYMMETRY

The case of flow normal to the axis of symmetry will now be treated in a manner similar to the case of parallel flow. The boundary condition is given by equation (27):

$$\sum_{n=1}^{\infty}C_n\left[\frac{d(\lambda\mu)}{d\mu}\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}-n(n+1)\frac{d}{d\mu}(P_nQ_n)\right]=\frac{d(\lambda\mu)}{d\mu}$$

From equation (33),

$$\frac{d(\lambda\mu)}{d\mu}=a_0+\sum_{n=0}^{\infty}(n+2)a_{1,n}\mu^{n+1} \tag{40}$$

Referring to equation (38) it follows that

$$\sum_{n=1}^{\infty} C_n \frac{d(\lambda\mu)}{d\mu}\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda} = a_0 \sum_{m=0}^{\infty}\mu^m \sum_{q=0}^{m}\sum_{j=0}^{\infty}(-1)^j \frac{[2j+1+2m-2q]}{(m-q)![2j]} C_{2j+1+m-q}\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_{2j+1+m-q}}{da_0^{p+1}}$$
$$+\sum_{n=0}^{\infty}(n+2)a_{1,n}\mu^{n+1}\sum_{m=0}^{\infty}\mu^m \sum_{q=0}^{m}\sum_{j=0}^{\infty}(-1)^j \frac{[2j+1+2m-2q]}{(m-q)![2j]} C_{2j+1+m-q}\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_{2j+1+m-q}}{da_0^{p+1}}$$

or

$$\sum_{n=1}^{\infty} C_n \frac{d(\lambda\mu)}{d\mu}\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda} = a_0 \sum_{m=0}^{\infty}\mu^m \sum_{q=0}^{m}\sum_{j=0}^{\infty}(-1)^j \frac{[2j+1+2m-2q]}{(m-q)![2j]} C_{2j+1+m-q}\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_{2j+1+m-q}}{da_0^{p+1}}$$
$$+\sum_{n=0}^{\infty}\mu^{n+1}\sum_{m=0}^{n}(n-m+2)a_{1,n-m}\sum_{q=0}^{m}\sum_{j=0}^{\infty}(-1)^j \frac{[2j+1+2m-2q]}{(m-q)![2j]} C_{2j+1+m-q}\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^{p+1}Q_{2j+1+m-q}}{da_0^{p+1}} \tag{41}$$

Analogous to equation (34)

$$Q_n=\sum_{q=0}^{\infty}\mu^q\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^pQ_n}{da_0^p}$$

Hence, in a manner similar to the derivation of equation (38)

$$\sum_{n=1}^{\infty} n(n+1)C_n\frac{d}{d\mu}(P_nQ_n)=$$
$$\sum_{m=1}^{\infty} m\mu^{m-1}\sum_{q=0}^{m}\sum_{j=0}^{\infty}(-1)^j\frac{(2j+m-q)(2j+1+m-q)}{(m-q)![2j]}[2j-1+2m-2q]\; C_{2j+m-q}\sum_{p=0}^{q}\frac{a_{p,q-p}}{p!}\frac{d^pQ_{2j+m-q}}{da_0^p} \tag{42}$$

Substituting equations (40), (41), and (42) in the boundary condition and equating the coefficients of the various powers of μ equal to zero, the following set of equations is obtained:

$$\sum_{n=0}^{\infty}(-1)^n\frac{[2n+1]}{[2n]}\left\{C_{2n+1}\left[a_0\frac{dQ_{2n+1}}{da_0}-(2n+1)(2n+2)Q_{2n+1}\right]-2na_{1,0}C_{2n}\frac{dQ_{2n}}{da_0}\right\}=a_0 \text{ for } h=0$$

and (after rearranging as for equation (39))

$$\left.\begin{aligned}&\sum_{k=0}^{h-1}\sum_{n=0}^{k}\sum_{m=0}^{n}\frac{(h-k+1)}{m!(k-n)!}a_{1,h-k-1}a_{k-n,n-m}\sum_{j=0}^{\infty}(-1)^j\frac{[2j+1+2m]}{[2j]}C_{2j+1+m}\frac{d^{k-n+1}Q_{2j+1+m}}{da_0^{k-n+1}}\\ &+a_0\sum_{n=0}^{h}\sum_{m=0}^{n}\frac{a_{h-n,n-m}}{m!(h-n)!}\sum_{j=0}^{\infty}(-1)^j\frac{[2j+1+2m]}{[2j]}C_{2j+1+m}\frac{d^{h-n+1}Q_{2j+1+m}}{da_0^{h-n+1}}\\ &-(h+1)\sum_{n=0}^{h+1}\sum_{m=0}^{n}\frac{a_{h+1-n,n-m}}{m!(h+1-n)!}\sum_{j=0}^{\infty}(-1)^j\frac{(2j+m)(2j+1+m)[2j-1+2m]}{[2j]}C_{2j+m}\frac{d^{h-n+1}Q_{2j+m}}{da_0^{h-n+1}}=(h+1)a_{1,h-1}\end{aligned}\right\} \tag{43}$$

where $h=1, 2, 3, \ldots\ldots \infty$.

This set of equations represents a formal and rigorous solution of the problem of potential flow about a body of revolution with flow normal to the axis of symmetry. The only unknown quantities are the infinite number of C_n's. The other quantities appearing in the equations are determined as in equations (39).

If the meridian profile is an ellipse $\lambda=a_0$, the $a_{1,n}$'s are all zero and, from the second expression of equation (43),

$$\sum_{n=0}^{\infty}(-1)^n\frac{[2n+2h+1]}{h![2n]}\left\{a_0\frac{dQ_{2n+1+h}}{da_0}-\frac{(2n-h-2)!}{(2n+h)!}Q_{2n+1+h}\right\}C_{2n+1+h}=0$$

where $h=1, 2, \ldots\ldots \infty$.

This expression represents an infinite set of linear homogenous equations in the unknowns $C_2, C_3, \ldots$ and, since the determinant of the coefficients is different from zero, it immediately follows that the only solution is $C_2=C_3=C_4=\ldots=0$. From the first expression of equation (43), it follows that

$$C_1=\frac{a_0}{a_0\frac{dQ_1}{da_0}-2Q_1(a_0)}$$

This result could have been easily obtained from the general expression for the velocity potential given by equation (26). Thus, assume that

$$\phi_1=2aV\cos\theta(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2}C_1\frac{dP_1}{d\mu}\frac{dQ_1}{d\lambda}$$

If the body moves in the positive direction of the y axis with constant velocity V, it may be supposed to be at rest and the fluid to have a translation $-V$ superposed on its actual motion. Accordingly

$$\phi=2aV\cos\theta(\lambda^2-1)^{1/2}(1-\mu^2)^{1/2}\left(C_1\frac{dQ_1}{d\lambda}+1\right)$$

At the surface of the ellipsoid of revolution generated by the ellipse $\lambda=a_0$, the normal velocity of the fluid must be zero. Therefore

$$\frac{\partial\phi}{\partial n}=2aV\cos\theta\left(\frac{1-\mu^2}{a_0^2-1}\right)^{1/2}\left[C_1(a_0^2-1)\frac{d^2Q_1}{da_0^2}+C_1\,a_0\frac{dQ_1}{da_0}+a_0\right]\frac{d\lambda}{dn}=0$$

From Legendre's equation

$$(a_0^2-1)\frac{d^2Q_1}{da_0^2}=-2a_0\frac{dQ_1}{da_0}+2Q_1(a_0)$$

Hence (see reference 10, p. 133)

$$C_1=\frac{a_0}{a_0\frac{dQ_1}{da_0}-2Q_1(a_0)}=\frac{-a_0}{\frac{a_0}{2}\log\frac{a_0+1}{a_0-1}+\frac{a_0^2}{a_0^2-1}-2}$$

In the appendix an application of the boundary condition (equations (39) and (43)) for axial and transverse flows, respectively, is made to a body of revolution obtained from a symmetrical Joukowsky profile.

INERTIA COEFFICIENTS OF BODIES OF REVOLUTION

It is of some interest to obtain the coefficients of inertia for axial and transverse flows and also to compare them with those of an ellipsoid of revolution of equal fineness ratio (references 11 and 12).

When a body moves in a fluid at rest at infinity the total kinetic energy of the fluid is given by

$$2T=-\sigma\int\int\phi\frac{\partial\phi}{\partial n}dS \tag{44}$$

where ϕ is the velocity potential of the fluid motion, $\frac{\partial\phi}{\partial n}$ the normal derivative of ϕ where the positive direction of the normal to the surface of the body is into the fluid and the integration is performed over the surface of the body; σ denotes the density of the fluid.

FLOW PARALLEL TO THE AXIS OF SYMMETRY

Since the velocity potential of this type of flow is independent of the angular coordinate θ, the following equation may be written for the element of surface:

$$dS=2\pi\rho ds$$

where ds denotes the element of length along a meridian profile. Hence,

$$2T=-2\pi\sigma\oint\rho\phi\frac{\partial\phi}{\partial n}ds$$

If the body moves in the direction of its axis of symmetry with a uniform velocity U the boundary condition is

$$\frac{\partial\phi}{\partial n}ds=-U\frac{\partial z}{\partial n}ds$$

Also, according to equation (24)

$$-\frac{\partial *}{\partial n}ds=\left[\left(\frac{1-\mu^2}{\lambda^2-1}\right)^{\frac{1}{2}}\frac{\partial *}{\partial\mu}d\lambda-\left(\frac{\lambda^2-1}{1-\mu^2}\right)^{\frac{1}{2}}\frac{\partial *}{\partial\lambda}d\mu\right] \tag{45}$$

Therefore,

$$2T=-8\pi a^2\sigma U\int_{-1}^{1}\phi\left[(1-\mu^2)\lambda\frac{d\lambda}{d\mu}-(\lambda^2-1)\mu\right]d\mu$$

In general then

$$2T=-8\pi a^2\sigma U\int_{-1}^{1}\left[(1-\mu^2)\lambda\frac{d\lambda}{d\mu}-(\lambda^2-1)\mu\right]\sum_{n=1}^{\infty}A_nP_n(\mu)Q_n(\lambda)d\mu \tag{46}$$

If M is the mass of fluid displaced by the body, then the coefficient of inertia k_a is the quantity multiplying MU^2 in the expression for $2T$.

If the body is a prolate spheroid $\lambda=a_0$ the foregoing expression for $2T$ becomes:

$$2T=\frac{4}{3}\pi\sigma(2a)^3U^2(a_0^2-1)\;\frac{\frac{1}{2}a_0\log\frac{a_0+1}{a_0-1}-1}{\frac{a_0}{a_0^2-1}-\frac{1}{2}\log\frac{a_0+1}{a_0-1}}$$

But $2a=Ae$, $a_0=\frac{1}{e}$ and $2a(a_0^2-1)^{\frac{1}{2}}=B$ where A, B are the semimajor and semiminor axes, respectively, and e the eccentricity of the elliptical meridian section. Therefore

$$2T=\frac{\frac{1}{2e}\log\frac{1+e}{1-e}-1}{\frac{1}{1-e^2}-\frac{1}{2e}\log\frac{1+e}{1-e}}\;\frac{4}{3}\pi AB^2\sigma U^2=k_aMU^2$$

The coefficient of inertia for a prolate ellipsoid in uniform axial motion is then given by

$$k_a=\frac{\frac{1}{2e}\log\frac{1+e}{1-e}-1}{\frac{1}{1-e^2}-\frac{1}{2e}\log\frac{1+e}{1-e}} \tag{47}$$

(See reference 10, p. 144)

Equation (46) is now evaluated for the case of a body of revolution obtained from the Joukowsky profile $\epsilon_1=0.15$, $\epsilon_2=0.10$. (See appendix.) The volume of this body is found to be

$$Q=\frac{4}{3}\pi(2a)^3\times 0.05342$$

so that the expression for $2T$ may be written:

$$2T=\frac{2\pi(2a)^3\times 0.003139}{\frac{4}{3}\pi(2a)^3\times 0.05342}\sigma Q U^2$$

or $k_a=0.0881$. (See table I.)
Compare this value of k_a with that of a prolate spheroid whose fineness ratio is the same as for the above-mentioned body of revolution. The fineness ratio f is defined as the ratio of the length to the maximum diameter of the body. The maximum diameter is obtained from equation (53) by means of the condition $\frac{d\rho}{d\nu}=0$ and the length of the body is given by $l=2a(\lambda_{\mu=1}+\lambda_{\mu=-1})$. By means of these expressions it is found that $f=4.208$. The fineness ratio for an ellipse is given by

$$f=\frac{A}{B}=\frac{1}{\sqrt{1-e^2}}$$

or

$$e=\sqrt{1-\frac{1}{f^2}}=0.971$$

where e is the eccentricity of the ellipse.

Substituting this value of e into equation (47), the following value of k_a is obtained,

$$k_a=0.0757$$

A theorem enunciated by Munk (reference 13) states that when the disturbance caused by a body moving in an infinite fluid is replaced by fictitious sinks and sources, the total mass is the sum of the products obtained by multiplying the intensity of each source or sink by the potential of the parallel flow. This theorem will now be shown to be only a first approximation and to hold exactly only for ellipsoids of revolution. Thus from equation (19),

$$I(z_1)=2\pi\sum_{n=1}^{\infty}A_n\,P_n(\lambda_1)$$

where $z_1=2a\lambda_1$

The strength per length dz_1 is then

$$4\pi a\sigma\sum_{n=1}^{\infty}A_n\,P_n(\lambda_1)\,d\lambda_1$$

The velocity potential at $(z_1, 0)$ of the parallel flow is given by

$$\phi=2a\,U\lambda_1$$

Hence, according to Munk's theorem

$$2T_{total}=8\pi\sigma a^2 U\sum_{n=1}^{\infty}A_n\int_{-1}^{1}\lambda_1\,P_n(\lambda_1)\,d\lambda_1$$

or

$$2T_{total}=\frac{4}{3}\pi(2a)^3\sigma U^2\frac{A_1}{2aU}$$

Therefore

$$2T_{fluid}=\left(\frac{4}{3}\frac{\pi(2a)^3}{Q}\frac{A_1}{2aU}-1\right)MU^2$$

where Q is the volume of the body and M is the mass of the displaced fluid.

The coefficient of inertia for axial flow is therefore

$$k_a=\frac{4}{3}\frac{\pi(2a)^3}{Q}\frac{A_1}{2aU}-1$$

This expression for k_a is valid for a prolate ellipsoid but is not valid for a more general shape.

It is obvious from this expression that Munk's theorem applies exactly only to ellipsoids of revolution since only the coefficient A_1 appears.[1] In order to provide a numerical comparison between Munk's theorem and the exact method, the foregoing equation is evaluated for the body of revolution whose meridian curve is the Joukowsky profile $\epsilon_1=0.15$, $\epsilon_2=0.10$. It yields a value $k_a=0.0717$ as compared with the more exact value $k_a=0.0867$ already obtained by means of the fundamental equation (46).

FLOW NORMAL TO THE AXIS OF SYMMETRY

For flow normal to the axis of symmetry the velocity potential depends not only on the elliptic coordinates λ, μ, but also on the cylindrical coordinate θ. Hence, the equation for the element of surface dS is

$$dS=\rho d\theta ds$$

and equation (44) becomes

$$2T=-\sigma\int\int\rho\phi\frac{\partial\phi}{\partial n}ds\,d\theta$$

If the body moves in the direction of the transverse axis Ox with a constant velocity V the boundary condition is

$$\frac{\partial\phi}{\partial n}ds=-V\cos\theta\frac{\partial\rho}{\partial n}ds$$

[1] The above criticism of Munk's theorem has been found to be incorrect. (It is to be noted that the volume of the body contains all of the coefficients A_n implicitly.) This theorem may be readily verified by applying Green's second theorem to the space internal to the shape and enclosing the appropriate distribution of sinks and sources γ. Then if $\Phi_1=Ux$ and $\Phi_2=\Phi$ (where $\Delta_2\Phi+\gamma=0$), Green's second theorem together with equation (44) immediately yields the following expression on Munk's theorem:

$$\frac{2T_{fluid}}{\sigma}+U^2.\sigma\text{ (volume of body)}=-\int\int\int_\tau Ux.\Delta_2\Phi\,d\tau=\int\int\int_\tau Ux.\gamma\,d\tau$$

Also, in general

$$\phi=2a\ V\cos\theta(\lambda^2-1)^{\frac{1}{2}}\ (1-\mu^2)^{\frac{1}{2}}\sum_{n=1}^{\infty}C_n\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}$$

Hence

From equation (45), it follows that

$$\frac{\partial\rho^2}{\partial n}ds=-8a^2(1-\mu^2)^{\frac{1}{2}}\ (\lambda^2-1)^{\frac{1}{2}}\ (\mu d\lambda+\lambda d\mu)$$

$$2T=(2a)^3\pi\sigma V^2\int_{-1}^{1}(1-\mu^2)(\lambda^2-1)\left(\mu\frac{d\lambda}{d\mu}+\lambda\right)\sum_{n=1}^{\infty}C_n\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}d\mu \tag{48}$$

For a prolate spheroid, $\lambda=a_0$ and

$$\phi=2aV\cos\theta(\lambda^2-1)^{\frac{1}{2}}\ (1-\mu^2)^{\frac{1}{2}}\ \frac{C_1dQ_1}{d\lambda}$$

where

$$C_1=\frac{a_0}{a_0\frac{dQ_1}{da_0}-2Q_1(a_0)}$$

Therefore

$$2T=\frac{\frac{a_0}{a_0^2-1}-\frac{a_0}{2}\log\frac{a_0+1}{a_0-1}}{\frac{a_0}{2}\log\frac{a_0+1}{a_0-1}+\frac{a_0^2}{a_0^2-1}-2}\frac{4}{3}\pi AB^2\sigma V^2=k_TMV^2$$

or

$$k_T=\frac{\frac{1}{e^2}-\frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}}{2-\frac{1}{e^2}+\frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}} \tag{49}$$

(See reference 10, p. 145.)

For the body of revolution whose meridian curve is the Joukowsky profile $\epsilon_1=0.15$, $\epsilon_2=0.10$ (see appendix) it is seen from equation (48) that

$$2T=\frac{\pi\sigma V^2(2a)^3\times0.059587}{\frac{4}{3}\quad\pi(2a)^3\times0.05342}Q=0.8366\times MV^2$$

Therefore $k_T=0.8366$. (See table II.)

According to equation (49) for the prolate ellipsoid of equal fineness ratio $f=4.208$ and $k_T=0.8689$.

According to Munk's theorem the inertia coefficient k_T of transverse flow may be obtained from the doublet distribution along the axis of symmetry. Again, as in the case of axial flow, this theorem is a first approximation and holds exactly only for ellipsoids of revolution since an expression for k_T is obtained that contains only the coefficient C_1. Thus from equation (30) it follows that

$$J(z_1)=-8\pi a^2V(1-\lambda_1^2)\sum_{n=1}^{\infty}C_n\frac{dP_n}{d\lambda_1}$$

Then according to Munk

$$2T_{total}=\sigma V\int_{-1}^{1}J(2a\lambda_1)2ad\lambda_1$$

or

$$2T_{total}=-16\pi a^3\sigma V^2\sum_{n=1}^{\infty}C_n\int_{-1}^{1}(1-\lambda_1^2)\ \frac{dP_n}{d\lambda_1}\frac{dP_1}{d\lambda_1}d\lambda_1$$

since

$$\frac{dP_1}{d\lambda_1}=1$$

Hence

$$2T_{total}=-\frac{8}{3}\pi\sigma(2a)^3V^2C_1$$

and

$$2T_{fluid}=-\frac{\frac{8}{3}\pi(2a)^3C_1+Q}{Q}MV^2$$

or

$$k_T=-\frac{\frac{8}{3}\pi(2a)^3C_1+Q}{Q}$$

In order to give a numerical comparison between Munk's theorem and the exact method, the foregoing equation is evaluated for the body of revolution whose meridian curve is the symmetrical Joukowsky profile $\epsilon_1=0.15$, $\epsilon_2=0.10$. It yields a value $k_T=0.8210$ as compared with the more exact value $k_T=0.8366$ obtained from the fundamental equation (48).

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *November 12, 1934.*

APPENDIX

APPLICATION OF THE ANALYSIS TO SURFACES OF REVOLUTION OBTAINED FROM SYMMETRICAL JOUKOWSKY PROFILES

By means of the mapping function

$$\zeta=\zeta'+\frac{a^2}{\zeta'} \tag{50}$$

the circle k_1 of radius a in the ζ' plane is transformed into the line segment $(-2a, 0; 2a, 0)$ in the ζ plane and the circle k_2 of radius $(1+\epsilon_1+\epsilon_2)a$ with center at $(\epsilon_1\, a, O)$ is transformed into a symmetrical Joukowsky profile J in the ζ plane. (See fig. 2.)

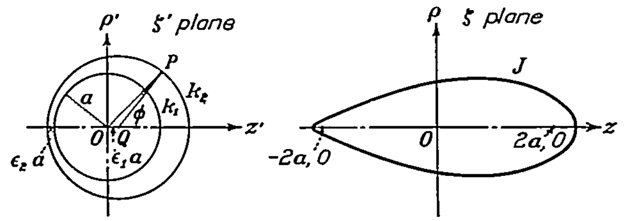

FIGURE 2.—Mapping of a circle into a symmetrical Joukowsky profile.

If in the ζ' plane $PQ=ae^{\xi_0}$, $PO=ae^{\xi}$, angle $POQ=\eta$, and angle $PQz'=\phi$ then, according to the law of cosines,

$$e^{2(\xi-\xi_0)}=1+2\delta\cos\phi+\delta^2 \tag{51}$$

where $\qquad \delta=\dfrac{\epsilon_1}{1+\epsilon_1+\epsilon_2}$

Again, by the law of sines

$$\tan\eta=\frac{\sin\phi}{\delta+\cos\phi} \tag{52}$$

Putting $\zeta'=ae^{\xi+i\eta}$ into equation (48),

$$\zeta=2a\cosh(\xi+i\eta)$$

or $z=2a\cosh\xi\cos\eta$, $\rho=2a\sinh\xi\sin\eta$

The latter two equations are, in fact, the equations of transformation from the rectangular coordinates (z, ρ) to the elliptic coordinates (ξ, η). Since (z, ρ) refer to points of the Joukowsky profile J, using equations (51) and (52), the following parametric equations of the system of symmetrical Joukowsky profiles may be obtained

$$\left.\begin{aligned}\lambda&=\frac{\epsilon_1(1+2\delta\nu+\delta^2)^{1/2}}{2\delta}+\frac{\delta}{2\epsilon_1(1+2\delta\nu+\delta^2)^{1/2}}\\ \mu&=\frac{\delta+\nu}{(1+2\delta\nu+\delta^2)^{1/2}}\end{aligned}\right\} \tag{53}$$

where $\lambda=\cosh\xi$, $\mu=\cos\eta$, and $\nu=\cos\phi$ (the independent parameter).

From these equations λ can be expressed as a power series in μ by means of a Maclaurin's expansion in the neighborhood of $\mu=0$ (i. e. $\nu=-\delta$). Thus,[1]

$$\lambda=a_0\left(1+\alpha\gamma+\frac{1}{2}\gamma^2-\frac{1}{8}\gamma^4+\frac{1}{16}\gamma^6\cdots\right) \tag{54}$$

where

$$a_0=\frac{\epsilon_1^2(1-\delta^2)+\delta^2}{2\epsilon_1\delta(1-\delta^2)^{1/2}},\ \alpha=\frac{\epsilon_1^2(1-\delta^2)-\delta^2}{\epsilon_1^2(1-\delta^2)+\delta^2}\text{ and }\gamma=\frac{\delta}{\sqrt{1-\delta^2}}\mu$$

FLOW PARALLEL TO THE AXIS OF SYMMETRY

Equation (39) is a set of linear equations for the infinite number of unknown coefficients A_n and provides a solution of the problem of axial potential flow about a body of revolution. In practice it is necessary to evaluate only the first few coefficients A_n. From equation (39), neglecting the A_n's after A_5, the following equations are obtained:

[1] This power series suggests the form

$$\lambda=a_0(\alpha\gamma+\sqrt{1+\gamma^2})$$

In fact, if the expression for μ from equation (53) is herein substituted the equation for λ is obtained.

$$
\left.\begin{aligned}
&-8\frac{dQ_1}{da_0}A_1+2\frac{dQ_3}{da_0}A_3-\frac{dQ_5}{da_0}A_5=16aU\\
&8a_{1,0}\frac{d^2Q_1}{da_0^2}A_1+8\frac{dQ_2}{da_0}A_2-2a_{1,0}\frac{d^2Q_3}{da_0^2}A_3-6\frac{dQ_4}{da_0}A_4+a_{1,0}\frac{d^2Q_5}{da_0^2}A_5=0\\
&8\left(2a_{1,1}\frac{d^2Q_1}{da_0^2}+a_{1,0}^2\frac{d^3Q_1}{da_0^3}\right)A_1+16a_{1,0}\frac{d^2Q_2}{da_0^2}A_2+2\left(10\frac{dQ_3}{da_0}-2a_{1,1}\frac{d^2Q_3}{da_0^2}-a_{1,0}^2\frac{d^3Q_3}{da_0^3}\right)A_3-12a_{1,0}\frac{d^2Q_4}{da_0^2}A_4\\
&\qquad-\left(28\frac{dQ_5}{da_0}-2a_{1,1}\frac{d^2Q_5}{da_0^2}-a_{1,0}^2\frac{d^3Q_5}{da_0^3}\right)A_5=0\\
&8\left(6a_{1,2}\frac{d^2Q_1}{da_0^2}+6a_{1,0}a_{1,1}\frac{d^3Q_1}{da_0^3}+a_{1,0}^3\frac{d^4Q_1}{da_0^4}\right)A_1+24\left(2a_{1,1}\frac{d^2Q_2}{da_0^2}+a_{1,0}^2\frac{d^3Q_2}{da_0^3}\right)A_2\\
&\qquad+2\left[6(5a_{1,0}-a_{1,2})\frac{d^2Q_3}{da_0^2}-6a_{1,0}a_{1,1}\frac{d^3Q_3}{da_0^3}-a_{1,0}^3\frac{d^4Q_3}{da_0^4}\right]A_3\\
&\quad+6\left(14\frac{dQ_4}{da_0}-6a_{1,1}\frac{d^2Q_4}{da_0^2}-3a_{1,0}^2\frac{d^3Q_4}{da_0^3}\right)A_4+\left[6(a_{1,2}-14a_{1,0})\frac{d^2Q_5}{da_0^2}+6a_{1,0}a_{1,1}\frac{d^3Q_5}{da_0^3}+a_{1,0}^3\frac{d^4Q_5}{da_0^4}\right]A_5=0\\
&8\left[24a_{1,3}\frac{d^2Q_1}{da_0^2}+12(2a_{1,0}a_{1,2}+a_{1,1}^2)\frac{d^3Q_1}{da_0^3}+12a_{1,1}a_{1,0}^2\frac{d^4Q_1}{da_0^4}+a_{1,0}^4\frac{d^5Q_1}{da_0^5}\right]A_1\\
&\qquad+32\left(6a_{1,2}\frac{d^2Q_2}{da_0^2}+6a_{1,0}a_{1,1}\frac{d^3Q_2}{da_0^3}+a_{1,0}^3\frac{d^4Q_2}{da_0^4}\right)A_2\\
&\quad+2\left[24(5a_{1,1}-a_{1,3})\frac{d^2Q_3}{da_0^2}+12(5a_{1,0}^2-2a_{1,0}a_{1,2}-a_{1,1}^2)\frac{d^3Q_3}{da_0^3}-12a_{1,1}a_{1,0}^2\frac{d^4Q_3}{da_0^4}-a_{1,0}^4\frac{d^5Q_3}{da_0^5}\right]A_3\\
&\quad+24\left[2(7a_{1,0}-3a_{1,2})\frac{d^2Q_4}{da_0^2}-6a_{1,0}a_{1,1}\frac{d^3Q_4}{da_0^3}-a_{1,0}^3\frac{d^4Q_4}{da_0^4}\right]A_4\\
&\quad+\left[504\frac{dQ_5}{da_0}-24(14a_{1,1}-a_{1,3})\frac{d^2Q_5}{da_0^2}+12(2a_{1,0}a_{1,2}+a_{1,1}^2-14a_{1,0}^2)\frac{d^3Q_5}{da_0^3}+12a_{1,1}a_{1,0}^2\frac{d^4Q_5}{da_0^4}+a_{1,0}^4\frac{d^5Q_5}{da_0^5}\right]A_5=0
\end{aligned}\right\}\quad(55)
$$

The coefficients of the unknown A_n's can be calculated simply by a knowledge of power-series development of λ in μ (i. e., the quantities $a_0, a_{1,0}, a_{1,1}, \ldots$ are obtained from equation (33)). The zonal harmonics $Q_n(a_0)$ are given by means of the recursion formula:

$$(n+1)Q_{n+1}(a_0)-(2n+1)a_0Q_n(a_0)+nQ_{n-1}(a_0)=0$$

In the application of this recursion formula it is necessary to calculate $Q_0(a_0)$ and $Q_1(a_0)$ independently where

$$Q_0(a_0)=\frac{1}{2}\log\frac{a_0+1}{a_0-1}\text{ and }Q_1(a_0)=a_0Q_0(a_0)-1$$

The first derivatives $\frac{dQ_n}{da_0}$ are then obtained from the following relation:

$$\frac{dQ_{n+1}}{da_0}+\frac{dQ_n}{da_0}=(1+a_0)\frac{dQ_0}{da_0}+Q_0(a_0)+3Q_1(a_0)+5Q_2(a_0)+\ldots+(2n+1)Q_n(a_0)$$

where it is necessary to determine $\frac{dQ_0}{da_0}$ independently.

In order to calculate the higher derivatives the preceding recursion formula may be repeatedly differentiated with regard to a_0. The higher derivatives of Q_0 and Q_1 are obtained independently by means of the $(r-1)$th derivative of Legendre's differential equation:

$$(a_0^2-1)\frac{d^{r+1}Q_n}{da_0^{r+1}}+2ra_0\frac{d^rQ_n}{da_0^r}-(n+r)(n-r+1)\frac{d^{r-1}Q_n}{da_0^{r-1}}=0$$

If the constants a_0, $a_{1,1}$, $a_{1,2}$, . . . and the various derivatives of Q_i are known, the coefficients of the unknowns A_i in equation (54) are easily calculated. The resulting system of linear equations can then be solved for the A_i's which in turn determine the potential function ϕ given by equation (10). A knowledge of the potential function yields directly the velocity u given by equation (16). Finally, according to Bernoulli's equation, the pressure p on the surface is given by

$$p+\frac{1}{2}\rho u^2=p_0+\frac{1}{2}\rho U^2$$

or

$$C_p=\frac{p-p_0}{p_s}=1-\left(\frac{u}{U}\right)^2\text{ where }p_s=\frac{\rho}{2}U^2$$

The numerical work is straightforward but somewhat tedious owing largely to a lack of tables of the zonal harmonics of the second kind.

As an illustration of the procedure here outlined, consider the body of revolution whose meridian curve is the Joukowsky profile (reference 14) defined by $\epsilon_1=0.15$ and $\epsilon_2=0.10$. (See fig. 2.) From equation (53) there can be written then for the power-series development of λ:

$$\lambda=1.02340+0.02630\mu+0.007476\mu^2 -0.0000273\mu^4+\ldots, \tag{56}$$

a very rapidly convergent series.

Here

$$a_0=1.02340,\ a_{1,0}=0.0263,\ a_{1,1}=0.007476,\ a_{1,2}=0,\ a_{1,3}=-0.0000273,\ \ldots$$

The zonal harmonics of the second kind and their derivatives are given by table III.

Substituting these numerical values into equations (55) the following set of five linear equations is obtained for A_1, A_2, A_3, A_4, and A_5:

$$\left.\begin{array}{l}
19.385\,A_1 - 3.780\,A_3 + 1.400\,A_5=2aU \\
10.372\,A_1 - 7.635\,A_2 - 2.342\,A_3 + 4.333\,A_4 + A_5=0 \\
-1.152\,A_1 + 1.294\,A_2 - 0.810\,A_3 - 0.850\,A_4 + A_5=0 \\
-0.500\,A_1 + 0.685\,A_2 - 0.763\,A_3 + 0.300\,A_4 + A_5=0 \\
-27.957\,A_1 + 39.785\,A_2 - 57.432\,A_3 + 63.549\,A_4 + A_5=0
\end{array}\right\} \tag{57}$$

The solution is given by [2]

$$A_1=0.0573\times 2aU,\ A_2=0.0726\times 2aU,\ A_3=0.0296\times 2aU,\ A_4=0.0065\times 2aU,\ A_5=0.0015\times 2aU$$

The sink-source distribution obtained from equation (19) is:

$$I(2a\lambda_1)=4\pi aU(-0.0339+0.0157\lambda_1+0.0846\lambda_1^2+0.0606\lambda_1^3+0.0283\lambda_1^4+0.0119\lambda_1^5) \tag{58}$$

Figure 3 shows a graph of this function with $I(2a\lambda_1)/4\pi aU$ as ordinate and λ_1 as abscissa.

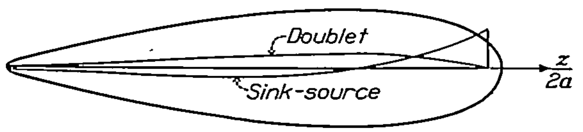

FIGURE 3.—Sink-source and doublet distributions.

In order to obtain the pressure distribution, the following expressions can be evaluated at a sufficient number of points of the boundary:

$$\frac{\partial\phi}{\partial\lambda}=\sum_{n=1}^{5}A_n\frac{dQ_n}{d\lambda}P_n(\mu)$$

$$\frac{\partial\phi}{\partial\mu}=\sum_{n=1}^{5}A_nQ_n(\lambda)\frac{dP_n}{d\mu}$$

and then substituted into equation (16) for u^2 where u is now the velocity at the surface of the body with the body considered to be at rest with regard to the fluid. The velocity u is calculated by means of the following expression:

$$\left(\frac{u}{U}\right)^2=\frac{1}{\lambda^2-\mu^2}\left[(\lambda^2-1)\left(\frac{1}{2aU}\sum_{n=1}^{5}A_n\frac{dQ_n}{d\lambda}P_n(\mu)+\mu\right)^2+(1-\mu^2)\left(\frac{1}{2aU}\sum_{n=1}^{5}A_nQ_n(\lambda)\frac{dP_n}{d\mu}+\lambda\right)^2\right] \tag{59}$$

Note here that the velocity potential

$$\phi=2aU[0.0573\ P_1(\mu)Q_1(\mu)+0.0726\ P_2(\mu)Q_2(\mu)+\ \ldots\ +0.0015\ P_5(\mu)Q_5(\mu)]$$

is exact for the body of revolution obtained by superposing a uniform velocity U on the flow from the sink-source distribution given by equation (58). This body is a very good approximation to the actual body obtained by revolving the Joukowsky profile about the axis of symmetry, so that in calculating the pressure distribution it is permissible to use the (λ, μ) values as given by equation (55).

Table IV shows the sequence of operations to be followed in obtaining the pressure distribution and figure 4 presents graphically the pressure distribution.

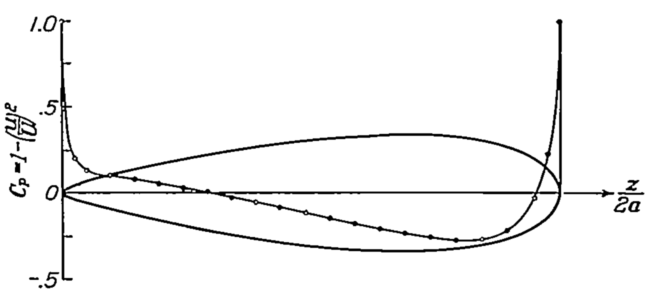

FIGURE 4.—Theoretical pressure distribution (axial flow).

[2] In its exact form the system contains an infinite number of equations with an infinite number of unknowns A_1, A_2, ... For practical purposes, however, the following method of solution is suggested. Suppose the system of equations to have been solved to an arbitrary degree of approximation, say three. Then to this solution there corresponds a definite sink-source (or doublet, as the case may be) distribution from which can be obtained the corresponding profile and hence a (λ, μ) curve. This (λ, μ) curve can then be compared to the (λ, μ) curve of the actual profile. In order to improve the approximation, the true (λ, μ) curve can be shifted in such a manner that a repetition of the process of solution, to the same degree of approximation, yields a new system of (λ, μ) values closer to the actual set of (λ, μ) values than the first approximation. In this manner the process can be carried on until the *desired* degree of accuracy is obtained.

FLOW NORMAL TO THE AXIS OF SYMMETRY

From equations (43), for the first five coefficients C_1, C_2, C_3, C_4, C_5 the following set of linear equations can be obtained:

$$A_1^1C_1+A_1^2C_2+A_1^3C_3+A_1^4C_4+A_1^5C_5=a_0$$
$$A_i^1C_1+A_i^2C_2+A_i^3C_3+A_i^4C_4+A_i^5C_5=i\,a_{1,\,i-2} \tag{60}$$

where $i=2, 3, 4, 5$
and

$$A_1^1=a_0\frac{dQ_1}{da_0}-2Q_1$$

$$A_1^2=3a_{1.0}\frac{dQ_2}{da_0}$$

$$A_1^3=-3\left(\frac{1}{2}a_0\frac{dQ_3}{da_0}-6Q_3\right)$$

$$A_1^4=-\frac{15}{2}a_{1.0}\frac{dQ_4}{da_0}$$

$$A_1^5=\frac{15}{4}\left(\frac{1}{2}a_0\frac{dQ_5}{da_0}-15Q_5\right)$$

$$A_2^1=a_{1,0}\left(a_0\frac{d^2Q_1}{da_0^2}-2\frac{dQ_1}{da_0}\right)$$

$$A_2^2=3\left[a_{1,0}^2\frac{d^2Q_2}{da_0^2}+(a_0+2a_{1,1})\frac{dQ_2}{da_0}-6Q_2\right]$$

$$A_2^3=-3a_{1,0}\left(\frac{a_0}{2}\frac{d^2Q_3}{da_0^2}-11\frac{dQ_3}{da_0}\right)$$

$$A_2^4=-\frac{15}{2}\left[a_{1,0}^2\frac{d^2Q_4}{da_0^2}+(a_0+2a_{1,1})\frac{dQ_4}{da_0}-20Q_4\right]$$

$$A_2^5=\frac{15}{4}a_{1,0}\left(\frac{a_0}{2}\frac{d^2Q_5}{da_0^2}-29\frac{dQ_5}{da_0}\right)$$

$$A_3^1=\frac{a_0a_{1,0}^2}{2}\frac{d^3Q_1}{da_0^3}+(a_0a_{1,1}-a_{1,0}^2)\frac{d^2Q_1}{da_0^2}-3a_{1,1}\frac{dQ_1}{da_0}$$

$$A_3^2=3\left[\frac{a_{1,0}^3}{2}\frac{d^3Q_2}{da_0^3}+a_{1,0}(3a_{1,1}+a_0)\frac{d^2Q_2}{da_0^2}+(3a_{1,2}-7a_{1,0})\frac{dQ_2}{da_0}\right]$$

$$A_3^3=-3\left[\frac{a_0a_{1,0}^2}{4}\frac{d^3Q_3}{da_0^3}+\frac{1}{2}(a_0a_{1,1}-16a_{1,0}^2)\frac{d^2Q_3}{da_0^2}-\frac{1}{2}(5a_0+33a_{1,1})\frac{dQ_3}{da_0}+30Q_3\right]$$

$$A_3^4=-\frac{15}{2}\left[\frac{a_{1,0}^3}{2}\frac{d^3Q_4}{da_0^3}+a_{1,0}(3a_{1,1}+a_0)\frac{d^2Q_4}{da_0^2}+(3a_{1,2}-28a_{1,0})\frac{dQ_4}{da_0}\right]$$

$$A_3^5=\frac{15}{8}\left[\frac{a_0a_{1,0}^2}{2}\frac{d^3Q_5}{da_0^3}+(a_0a_{1,1}-43a_{1,0}^2)\frac{d^2Q_5}{da_0^2}-(14a_0+87a_{1,1})\frac{dQ_5}{da_0}+420Q_5\right]$$

$$A_4^1=\frac{a_0a_{1,0}^3}{6}\frac{d^4Q_1}{da_0^4}+a_{1,0}\left(a_0a_{1,1}-\frac{a_{1,0}^2}{3}\right)\frac{d^3Q_1}{da_4^3}+(a_0a_{1,2}-3a_{1,0}a_{1,1})\frac{d^2Q_1}{da_0^2}-4a_{1,2}\frac{dQ_1}{da_0}$$

$$A_4^2=3\left[\frac{a_{1,0}^4}{3!}\frac{d^4Q_2}{da_0^4}+a_{1,0}^2\frac{4a_{1,1}+a_0}{2}\frac{d^3Q_2}{da_0^3}+(4a_{1,0}a_{1,2}+2a_{1,1}^2+a_0a_{1,1}-4a_{1,0}^2)\frac{d^2Q_2}{da_0^2}+(4a_{1,3}-9a_{1,1})\frac{dQ_2}{da_0}\right]$$

$$A_4^3=-\left[\frac{a_0a_{1,0}^3}{4}\frac{d^4Q_3}{da_0^4}+\frac{3}{2}a_{1,0}(a_0a_{1,1}-7a_{1,0}^2)\frac{d^3Q_3}{da_0^3}+\frac{3}{2}(a_0a_{1,2}-43a_{1,0}a_{1,1}-5a_0a_{1,0})\frac{d^2Q_3}{da_0^2}+3(35a_{1,0}-22a_{1,2})\frac{dQ_3}{da_0}\right]$$

$$A_4^4=-\frac{5}{2}\left[\frac{a_{1,0}^4}{2}\frac{d^4Q_4}{da_0^4}+3a_{1,0}^2\frac{4a_{1,1}+a_0}{2}\frac{d^3Q_4}{da_0^3}+3(4a_{1,0}a_{1,2}+2a_{1,1}^2+a_0a_{1,1}-18a_{1,0}^2)\frac{d^2Q_4}{da_0^2}+(12a_{1,3}-111a_{1,1}-7a_0)\frac{dQ_4}{da_0}+140Q_4\right]$$

$$A_4^5=\frac{5}{2}\left[\frac{a_0a_{1,0}^3}{8}\frac{d^4Q_5}{da_0^4}+\frac{3}{4}a_{1.0}(a_0a_{1,1}-19a_{1,0}^2)\frac{d^3Q_5}{da_0^3}+\frac{3}{4}(a_0a_{1,2}-115a_{1,0}a_{1,1}-14a_0a_{1,0})\frac{d^2Q_5}{da_0^2}+3(133a_{1,0}-29a_{1,2})\frac{dQ_5}{da_0}\right]$$

$$A_5^1=\frac{a_0a_{1,0}^4}{24}\frac{d^5Q_1}{da_0^5}+\frac{a_{1,0}^2}{2}\left(a_0a_{1,1}-\frac{a_{1,0}^2}{6}\right)\frac{d^4Q_1}{da_0^4}+\left(a_0a_{1,0}a_{1,2}-a_{1,1}\frac{a_0a_{1,1}-3a_{1,0}^2}{2}\right)\frac{d^3Q_1}{da_0^3}+(a_0a_{1,3}-2a_{1,1}^2-4a_{1,0}a_{1,2})\frac{d^2Q_1}{da_0^2}-5a_{1,3}\frac{dQ_1}{da_0}$$

$$A_5^2=3\left[\frac{a_{1,0}^5}{4!}\frac{d^5Q_2}{da_0^5}+a_{1,0}^3\frac{5a_{1,1}+a_0}{3!}\frac{d^4Q_2}{da_0^4}+a_{1,0}\left(5\frac{a_{1,0}a_{1,2}+a_{1,1}^2}{2!}+a_0a_{1,1}-\frac{3}{2}a_{1,0}^2\right)\frac{d^3Q_2}{da_0^3}\right.$$

$$\left.+(5a_{1,3}a_{1,0}+5a_{1,2}a_{1,1}+a_0a_{1,2}-10a_{1,0}a_{1,1})\frac{d^2Q_2}{da_0^2}+(5a_{1,4}-11a_{1,2})\frac{dQ_2}{da_0}\right]$$

$$A_5^3=-\frac{1}{2}\left\{\frac{a_0a_{1,0}^4}{8}\frac{d^5Q_3}{da_0^5}+\frac{a_{1,0}^2}{3}(3a_0a_{1,1}-13a_{1,0}^2)\frac{d^4Q_3}{da_0^4}+\frac{3}{2}\left[a_0(a_{1,1}^2+2a_{1,0}a_{1,2})-a_{1,0}^2(5a_0+53a_{1,1})\right]\frac{d^3Q_3}{da_0^3}\right.$$

$$\left.+3\left[a_0(a_{1,3}-5a_{1,1})+2a_{1,0}(20a_{1,0}-27a_{1,2})-27a_{1,1}^2\right]\frac{d^2Q_3}{da_0^2}+15(17a_{1,1}-11a_{1,3})\frac{dQ_3}{da_0}\right\}$$

$$A_5^4=-\frac{5}{2}\left[\frac{3a_{1,0}^5}{4!}\frac{d^5Q_4}{da_0^5}+a_{1,0}^3\frac{5a_{1,1}+a_0}{2}\frac{d^4Q_4}{da_0^4}+3a_{1,0}\left(5\frac{a_{1,0}a_{1,2}+a_{1,1}^2}{2}+a_0a_{1,1}-\frac{22}{3}a_{1,0}^2\right)\frac{d^3Q_4}{da_0^3}\right.$$

$$\left.+(15a_{1,3}a_{1,0}+15a_{1,2}a_{1,1}+3a_0a_{1,2}-135a_{1,0}a_{1,1}-7a_0a_{1,0})\frac{d^2Q_4}{da_0^2}+(15a_{1,4}-138a_{1,2}+161a_{1,0})\frac{dQ_4}{da_0}\right]$$

$$A_5^5=5\left\{\frac{a_0a_{1,0}^4}{64}\frac{d^5Q_5}{da_0^5}+\frac{a_{1,0}^2}{32}(6a_0a_{1,1}-71a_{1,0}^2)\frac{d^4Q_5}{da_0^4}+\frac{3}{16}\left[a_0(a_{1,1}^2+2a_{1,0}a_{1,2})-a_{1,0}^2(14a_0+143a_{1,1})\right]\frac{d^3Q_5}{da_0^3}\right.$$

$$\left.+3\left[a_0\frac{a_{1,3}-14a_{1,1}}{8}-9(a_{1,1}^2+2a_{1,0}a_{1,2})+\frac{161}{4}a_{1,0}^2\right]\frac{d^2Q_5}{da_0^2}+\frac{3}{8}(21a_0-145a_{1,3}+658a_{1,1})\frac{dQ_5}{da_0}-\frac{945}{4}Q_5\right\}$$

Again, substituting the numerical values given by equation (53) and table III into the foregoing expressions, there result the following equations:

$$\begin{aligned}
-22.402\,C_1-\;1.363\,C_2+\;34.018\,C_3+\;2.578\,C_4-\;39.602\,C_5&=1.02340\\
25.036\,C_1-67.392\,C_2-\;45.658\,C_3+163.240\,C_4+\;66.762\,C_5&=0.05260\\
-20.665\,C_1+78.034\,C_2-132.860\,C_3-221.940\,C_4+497.330\,C_5&=0.02243\\
-15.235\,C_1+64.050\,C_2-178.260\,C_3+202.310\,C_4+391.330\,C_5&=0\\
10.437\,C_1-46.495\,C_2+150.380\,C_3-361.150\,C_4+197.350\,C_5&=0.00014
\end{aligned}\tag{61}$$

The solution is given by

$$C_1=-0.0486$$
$$C_2=-0.0178$$
$$C_3=-0.0027$$
$$C_4=-0.00028$$
$$C_5=-0.00005$$

The doublet distribution function $J(2a\lambda_1)$ then becomes:

$$J(2a\lambda_1)=8\pi a^2V(1-\lambda_1^2)(0.0447+0.0512\lambda_1+0.0187\lambda_1^2+0.0049\lambda_1^3+0.0022\lambda_1^4)\tag{62}$$

The graph of this function with $J/8\pi a^2V$ as ordinate and λ_1 as abscissa is shown in figure 3.

DETERMINATION OF THE TRANSVERSE-FORCE DISTRIBUTION

When the axial flow is combined with the transverse flow some information regarding the distribution of forces over the surface of the body can be obtained by introducing the notion of the transverse-force coefficient. For the pressure difference at the surface, according to Bernoulli's equation:

$$p-p_0=\frac{\sigma}{2}(U^2+V^2-q^2)$$

where p_0 is the pressure at an infinite distance from the body and q is the velocity of the fluid at the surface, supposing the body to be at rest and the fluid to strike it at an angle α where $\tan\alpha=\frac{U}{V}$.

Now q has three components—in the directions ds_λ, ds_μ, and $\rho d\theta$. They may be denoted by q_λ, q_μ, and q_θ, respectively. Also denote by u_λ and u_μ the velocity components of the axial flow and by v_λ, v_μ the velocity components of the transverse flow taken in the plane $\theta=0$. Then,

$$q_\lambda=u_\lambda+v_\lambda\cos\theta$$
$$q_\mu=u_\mu+v_\mu\cos\theta$$
$$q_\theta=v_\theta\sin\theta$$

Introducing these values into Bernoulli's equation,

$$p-p_0=\frac{\sigma}{2}\left[U^2+V^2-u_\lambda^2-u_\mu^2-v_\lambda^2\cos^2\theta-v_\mu^2\cos^2\theta-v_\theta^2\sin^2\theta-2(u_\lambda v_\lambda+u_\mu v_\mu)\cos\theta\right]$$

It is only the term

$$\sigma(u_\lambda v_\lambda + u_\mu v_\mu)\cos\theta = \sigma(\bar{u}\,\bar{v})\cos\theta$$

that need be considered, for the other terms have equal values for θ and $\pi-\theta$ and accordingly vanish when integrated over the entire cross section. The resulting transverse force, relative to an annular element of width unity, is therefore

$$\frac{dQ}{dz} = \sigma\int_0^{2\pi}(\bar{u}\,\bar{v})\cos^2\theta\,\rho\,d\theta = \sigma\,\pi\,\rho\,(\bar{u}\,\bar{v})$$

Klemperer defines the transverse-force coefficient by

$$\beta = \frac{\dfrac{dQ}{dz}}{\dfrac{\sigma}{2}(U^2+V^2)} = 2\pi\rho\frac{(\bar{u}\,\bar{v})}{U^2+V^2}$$

or

$$\beta = \pi\rho\left(\frac{\bar{u}}{U}\frac{\bar{v}}{V}\right)\sin 2\alpha \qquad (63)$$

By means of the velocity potentials of the axial and transverse flows this last expression takes the following form:[3]

$$\frac{\beta}{2a\pi\sin 2\alpha} = \frac{(\lambda^2-1)(1-\mu^2)}{\lambda^2-\mu^2}\Bigg[\left(\sum_{n=1}^{\infty}\frac{A_n}{2aU}P_n(\mu)\frac{dQ_n}{d\lambda}+\mu\right)\left\{\sum_{n=1}^{\infty}C_n\frac{dP_n}{d\mu}\left[n(n+1)Q_n(\lambda)-\lambda\frac{dQ_n}{d\lambda}\right]+\lambda\right\}$$
$$-\left(\sum_{n=1}^{\infty}\frac{A_n}{2aU}\frac{dP_n}{d\mu}Q_n(\lambda)+\lambda\right)\left\{\sum_{n=1}^{\infty}C_n\frac{dQ_n}{d\lambda}\left[n(n+1)P_n(\mu)-\mu\frac{dP_n}{d\mu}\right]+\mu\right\}\Bigg] \qquad (64)$$

Tables IV and V give the numerical data for the evaluation of the right-hand side of equation (61) and figure 5 represents graphically these numerical results

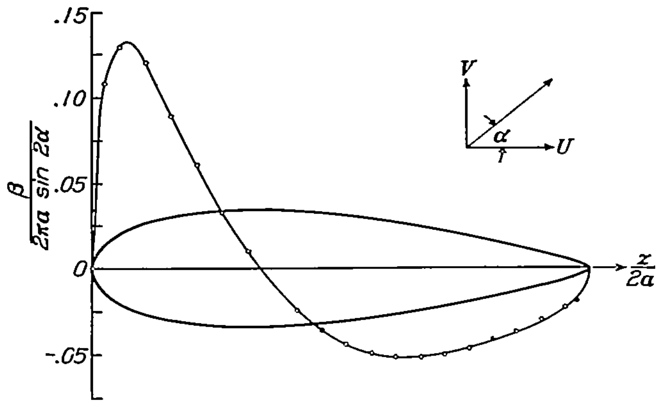

FIGURE 5.—Transverse-force distribution.

with $\dfrac{\beta}{2a\pi\sin 2\alpha}$ as ordinate and $\dfrac{z}{2a}$ $(=\lambda\mu)$ as abscissa.

According to theory the positive and negative areas included by the β curve and the x axis are equal; that is, there is no resultant lift force but only a simple couple.

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[3] Mr. Upson has kindly pointed out to the author the very close check of his approximation formula (equation (10), N. A. C. A. Report 405) for the transverse force with the more exact equation (63).

TABLE I

μ	$(\lambda^2-1)\mu$	$\frac{d\lambda}{d\mu}$	$(1-\mu^2)\lambda$	$(1-\mu^2)\lambda \frac{d\lambda}{d\mu}$	$(\lambda^2-1)\mu - (1-\mu^2)\lambda \frac{d\lambda}{d\mu}$	$\sum_{n=1}^{5} A_n P_n Q_n(\lambda)$	$\left[(\lambda^2-1)\mu - (1-\mu^2)\lambda\frac{d\lambda}{d\mu}\right] \sum_{n=1}^{5} A_n P_n(\mu) Q_n(\lambda)$
1	0.11757	0.04114	0	0	0.11757	0.10025	0.01179
.9	.09814	.03968	.20000	.00794	.09120	.08114	.00740
.8	.08067	.03821	.37772	.01443	.06624	.06337	.00420
.7	.06511	.03673	.53319	.01958	.04553	.04695	.00214
.6	.05129	.03525	.66680	.02351	.02778	.03191	.00089
.5	.03916	.03376	.77882	.02629	.01287	.01818	.00023
.4	.02859	.03227	.86950	.02806	.00053	.00575	.00000
.3	.01950	.03078	.93908	.02891	−.00941	−.00503	.00001
.2	.01175	.02929	.98780	.02893	−.01718	−.01464	.00025
.1	.00529	.02780	1.01585	.02824	−.02295	−.02294	.00053
0	0	.02630	1.02340	.02692	−.02692	−.02991	.00081
−.1	−.00421	.02481	1.01064	.02507	−.02928	−.03560	.00104
−.2	−.00744	.02331	.97770	.02279	−.03023	−.04004	.00121
−.3	−.00979	.02182	.92472	.02018	−.02997	−.04324	.00130
−.4	−.01136	.02033	.85183	.01732	−.02868	−.04522	.00130
−.5	−.01219	.01884	.75908	.01430	−.02649	−.04604	.00122
−.6	−.01244	.01735	.64660	.01122	−.02366	−.04132	.00107
−.7	−.01217	.01587	.51441	.00816	−.02033	−.04433	.00090
−.8	−.01145	.01439	.36257	.00522	−.01667	−.04203	.00070
−.9	−.01041	.01292	.19110	.00247	−.01288	−.03895	.00050
−1	−.00912	.01146	0	0	−.00912	−.03542	.00032

According to the trapezoidal rule

$$\int_{-1}^{1}\left[(\lambda^2-1)\mu-(1-\mu^2)\lambda\frac{d\lambda}{d\mu}\right]\sum_{n=1}^{5}A_nP_n(\mu)\,Q_n(\lambda)\,d\mu=0.003139$$

TABLE II

μ	$(\lambda^2-1)(1-\mu^2)$	$\mu\frac{d\lambda}{d\mu}+\lambda$	$(\lambda^2-1)(1-\mu^2)\left(\mu\frac{d\lambda}{d\mu}+\lambda\right)$	$\sum_{n=1}^{5} C_n \frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}$	$(\lambda^2-1)(1-\mu^2)\left(\mu\frac{d\lambda}{d\mu}+\lambda\right)\sum_{1}^{5} C_n \frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}$
1	0	1.09839	0	0.66858	0
.9	.02072	1.08882	.02256	.75868	.01712
.8	.03630	1.07978	.03920	.77059	.03022
.7	.04744	1.07118	.05081	.78269	.03977
.6	.05471	1.06302	.05816	.79534	.04626
.5	.05874	1.05530	.06199	.80838	.05011
.4	.06004	1.04803	.06292	.82192	.05172
.3	.05914	1.04120	.06158	.83623	.05149
.2	.05641	1.03482	.05837	.85079	.01966
.1	.05237	1.02889	.05388	.86590	.04666
0	.04735	1.02340	.04846	.88209	.04363
−.1	.04172	1.01837	.04249	.89411	.03799
−.2	.03573	1.01378	.03622	.91522	.03315
−.3	.02968	1.00964	.02997	.93289	.02796
−.4	.02382	1.00595	.02396	.95040	.02278
−.5	.01828	1.00269	.01833	.96979	.01777
−.6	.01327	.99990	.01327	.98811	.01311
−.7	.00886	.99754	.00884	1.00804	.00891
−.8	.00515	.99562	.00513	1.03084	.00529
−.9	.00219	.99414	.00218	1.05385	.00230
−1	0	.99309	0	1.27840	0

According to the trapezoidal rule

$$\int_{-1}^{1}(\lambda^2-1)(1-\mu^2)\left(\mu\frac{d\lambda}{d\mu}+\lambda\right)\sum_{1}^{5}C_n\frac{dP_n}{d\mu}\frac{dQ_n}{d\lambda}d\mu=0.050557$$

TABLE III.—($a_0=1.0234$)

n	Q_n	$\frac{dQ_n}{da_0}$	$\frac{d^2Q_n}{da_0^2}$	$\frac{d^3Q_n}{da_0^3}$	$\frac{d^4Q_n}{da_0^4}$	$\frac{d^5Q_n}{da_0^5}$
0	2.2299	−21.1202	913.16	−78071	100107×10^2	-17115×10^5
1	1.2821	−19.3845	892.28	−77157	99326	−17015
2	.8532	−17.2740	855.00	−75394	97792	−16817
3	.6005	−15.1187	805.90	−72882	95556	−16526
4	.4356	−13.0705	749.00	−69752	92690	−16148
5	.3220	−11.1983	638.24	−66141	89278	−15692

TABLE IV

μ	λ	$1-\mu^2$	λ^2-1	$\lambda^2-\mu^2$	P_1	P_2	P_3	P_4	P_5	Q_1	Q_2	Q_3	Q_4	Q_5
1	1.0572	0.000	0.11757	0.11757	1	1	1	1	1	0.8941	0.5220	0.3236	0.2072	0.1353
.95	1.0551	.0975	.1133	.2108	.95	.8538	.7184	.5541	.3727	.9091	.5341	.3332	.2146	.1411
.90	1.0531	.19	.1090	.2990	.90	.7150	.473	.2079	−.0411	.9244	.5466	.3431	.2223	.1470
.80	1.0492	.36	.1008	.4608	.80	.46	.08	−.233	−.3995	.9563	.5728	.3641	.2389	.1599
.70	1.0455	.51	.0930	.6030	.70	.2350	−.193	−.412	−.3652	.9697	.6005	.3855	.2568	.1740
.60	1.0419	.64	.0855	.7255	.60	.04	−.36	−.408	−.1523	1.0249	.6300	.4106	.2763	.1895
.50	1.0384	.75	.0783	.8283	.50	−.1250	−.4375	−.289	.0898	1.0620	.6613	.4366	.2974	.2066
.40	1.0351	.84	.0715	.9115	.40	−.2600	−.44	−.113	.2706	1.1011	.6947	.4645	.3203	.2253
.30	1.0320	.91	.0650	.9749	.30	−.3650	−.3825	.0729	.3454	1.1425	.7304	.4947	.3455	.2460
.20	1.0290	.96	.0588	1.0188	.20	−.44	−.28	.232	.3075	1.1862	.7685	.5271	.3728	.2688
.10	1.0261	.99	.0529	1.0429	.10	−.4850	−.1475	.3379	.1788	1.2331	.8098	.5629	.4034	.2947
0	1.0234	1	.0474	1.0474	0	−.5	0	.375	0	1.2821	.8532	.6005	.4356	.3220
−.10	1.0209	.99	.0421	1.0321	−.10	−.4850	.1475	.3379	−.1788	1.3346	.9002	.6419	.4716	.3538
−.20	1.0184	.96	.0372	.9972	−.20	−.44	.28	.232	−.3075	1.3911	.9512	.6872	.5113	.3876
−.30	1.0162	.91	.0326	.9426	−.30	−.3650	.3825	.0729	−.3454	1.4516	1.0064	.7367	.5553	.4263
−.40	1.0141	.84	.0284	.8684	−.40	−.26	.44	−.113	−.2706	1.5165	1.0660	.7907	.6037	.4694
−.50	1.0121	.75	.0244	.7744	−.50	−.1250	.4375	−.289	−.0898	1.5874	1.1317	.8508	.6581	.5183
−.60	1.0103	.64	.0207	.6607	−.60	.04	.36	−.408	.1523	1.6637	1.2030	.9165	.7182	.5729
−.70	1.0087	.51	.0174	.5274	−.70	.2350	.1925	−.412	.3652	1.7474	1.2819	.9900	.7860	.6351
−.80	1.0071	.36	.0143	.3743	−.80	.46	−.08	−.233	.3995	1.8402	1.3699	1.0727	.8632	.7066
−.90	1.0058	.19	.0116	.2016	−.90	.7150	−.4725	.2079	.0411	1.9424	1.4677	1.1653	.9503	.7881
−.95	1.0052	.0975	.0103	.1078	−.95	.8538	−.7184	.5541	−.3727	1.9986	1.5216	1.2167	.9988	.6337
−1.00	1.0016	0	.0091	.0091	−1.00	1	−1	1	−1	2.0579	1.5789	1.2715	1.0511	.8333

TABLE IV—Continued

$\frac{dP_1}{d\mu}$	$\frac{dP_2}{d\mu}$	$\frac{dP_3}{d\mu}$	$\frac{dP_4}{d\mu}$	$\frac{dP_5}{d\mu}$	$\frac{dQ_1}{d\lambda}$	$\frac{dQ_2}{d\lambda}$	$\frac{dQ_3}{d\lambda}$	$\frac{dQ_4}{d\lambda}$	$\frac{dQ_5}{d\lambda}$	$\frac{1}{2aU}\frac{\partial\phi}{\partial\lambda}+\mu$	$\frac{1}{2aU}\frac{\partial\phi}{\partial\mu}+\lambda$	$C_p=1-\left(\frac{u}{U}\right)^2$
1	3	6	10	15	−7.2003	−5.8236	−4.5906	−3.5586	−2.7262	0.0015	1.2961	1
1	2.85	5.2688	9.9328	10.256	−7.5067	−6.1021	−4.8360	−3.7697	−2.9043	.0451	1.2857	.2341
1	2.70	4.575	6.0075	5.9464	−7.8306	−6.3970	−5.0976	−3.9961	−3.0966	.0876	1.2697	−.0270
1	2.40	3.3	2.96	1.203	−8.5399	−7.0476	−5.6760	−4.4991	−3.5260	.1687	1.2443	−.2157
1	2.10	2.175	.7525	−1.5336	−9.3375	−7.7827	−6.3351	−5.0772	−4.0242	.2446	1.2195	−.2670
1	1.80	1.2	−.72	−2.472	−10.2431	−8.6222	−7.0933	−5.7472	−4.6072	.3147	1.1955	−.2725
1	1.50	.375	−1.5625	−2.2266	−11.2737	−9.5829	−7.9671	−6.5269	−5.2909	.3787	1.1724	−.2581
1	1.20	−.3	−1.88	−1.317	−12.4528	−10.6880	−8.9793	−7.4367	−6.0962	.4363	1.1502	−.2342
1	.90	−.825	−1.7775	−.1686	−13.8145	−11.9710	−10.1628	−8.5085	−7.0531	.4871	1.1290	−.2056
1	.60	−1.2	−1.36	.888	−15.3869	−13.4600	−11.5443	−9.7701	−8.1890	.5308	1.1088	−.1750
1	.30	−1.425	−.7325	1.6164	−17.2203	−15.2037	−13.1712	−11.2637	−9.5409	.5779	1.0895	−.1437
1	0	−1.5	0	1.875	−19.3834	−17.2778	−15.1225	−13.0743	−11.2022	.5953	1.0711	−.1124
1	−.30	−1.425	.7325	1.6164	−21.9413	−19.7295	−18.4402	−14.2363	−14.1961	.6268	1.0538	−.0811
1	−.60	−1.2	1.36	.888	−25.0150	−22.694	−20.259	−17.8839	−15.6569	.6240	1.0374	−.0505
1	−.90	−.825	1.7775	−.1686	−28.7377	−26.2994	−23.7058	−21.1426	−18.7083	.6224	1.0217	−.0211
1	−1.20	−.3	1.88	−1.317	−33.2780	−30.7135	−27.9478	−25.1785	−22.5146	.6061	1.0074	.0063
1	−1.50	.375	1.5625	−2.2266	−38.9802	−36.2774	−33.3215	−30.321	−27.3984	.5752	.9942	.0322
1	−1.80	1.2	.72	−2.472	−46.1090	−43.2571	−40.0941	−36.8414	−33.6301	.5224	.9822	.0569
1	−2.10	2.175	−.7525	−1.5336	−55.3285	−52.3123	−48.9193	−45.3826	−41.8450	.4466	.9718	.0802
1	−2.40	3.3	−2.96	1.203	−67.5552	−64.3565	−60.7056	−56.8476	−52.9372	.3456	.9633	.1030
1	−2.70	4.575	−6.0075	5.9464	−83.9788	−80.5785	−76.6404	−72.4214	−68.0879	.2178	.9571	.1338
1	−2.85	5.2688	−9.9328	10.256	−94.5425	−91.0314	−86.9344	−82.5145	−77.9454	.1492	.9429	.1998
1	−3	6	−10	15	−107.0955	−103.4671	−99.2013	−94.5668	−89.7418	.0812	.9560	.9934

TABLE V

μ	$n(n+1)Q_n-\lambda\frac{dQ_n}{d\lambda}$					$n(n+1)P_n-\mu\frac{dP_n}{d\mu}$					$\frac{\beta}{2\pi a\sin 2a}$
	$n=1$	2	3	4	5	$n=1$	2	3	4	5	
0.95	9.7386	9.6432	9.1009	8.2703	7.2967	0.95	2.415	3.6160	1.6456	1.439	−0.1084
.9	10.095	10.017	9.4853	8.6549	7.6705	.9	1.86	1.5585	−1.2480	−6.5860	−.1299
.8	10.873	10.831	10.324	9.4983	8.4968	.8	.84	−1.68	−7.028	−12.948	−.1207
.7	11.742	11.739	11.261	10.443	9.4272	.7	−.06	−3.8385	−13.5075	−9.8325	−.0891
.6	12.722	12.763	12.318	11.513	10.486	.6	−.84	−5.04	−3.84	−3.093	−.0600
.5	13.831	13.919	13.512	12.725	11.691	.5	−1.50	−5.4375	−4.9988	3.8085	−.0331
.4	15.092	15.232	14.868	14.105	12.876	.4	−2.04	−5.16	−1.508	8.646	−.0102
.3	16.541	16.736	16.423	15.690	14.659	.3	−2.46	−4.3425	1.9921	10.4123	−.0093
.2	18.205	18.461	18.204	17.509	16.489	.2	−2.76	−3.12	4.912	9.048	.0244
.1	20.136	20.460	20.270	19.625	18.632	.1	−2.94	−1.6275	6.8321	5.2033	.0359
0	22.406	22.801	22.682	22.092	21.125	0	−3	0	7.5	0	.0441
−.1	25.068	25.542	26.527	23.964	25.206	−.1	−2.94	1.6275	6.8321	−5.2033	.0492
−.2	28.258	28.820	28.879	28.440	27.575	−.2	−2.76	3.12	4.912	−9.048	.0515
−.3	32.106	32.763	32.930	32.590	31.801	−.3	−2.46	4.3425	1.9921	−10.4123	.0519
−.4	36.780	37.542	37.830	37.607	36.914	−.4	−2.04	5.16	−1.508	−8.646	.0503
−.5	42.627	43.507	43.935	43.852	43.290	−.5	−1.50	5.4375	−4.9988	−3.8085	.0469
−.6	49.912	50.921	51.506	51.586	51.164	−.6	−.84	5.04	−3.84	3.093	.0415
−.7	59.297	60.456	61.221	61.498	61.261	−.7	−.06	3.8385	−13.5075	9.8825	.0376
−.8	71.717	73.035	74.011	74.516	74.513	−.8	.84	1.68	−7.028	12.948	.0302
−.9	88.348	89.850	91.066	91.845	92.124	−.9	1.86	−1.5585	−1.2480	6.5960	.0233
−.95	99.026	100.63	101.98	102.91	103.36	−.95	2.415	−3.6160	1.6456	−1.439	.0196