REPORT No. 582

A THEORY FOR PRIMARY FAILURE OF STRAIGHT CENTRALLY LOADED COLUMNS

By Eugene E. Lundquist and Claude M. Fligg

SUMMARY

A theory of primary failure of straight centrally loaded columns is presented. It is assumed that the column cross section and the load are constant throughout the length.

Primary failure is defined as any type of failure in which the cross sections are translated, rotated, or translated and rotated but not distorted in their own planes. In the derivation of the general equation for the critical stress, the cross sections are assumed to rotate about any axis parallel to the column. When the location of the axis of rotation varies from zero to infinity in every direction, all combinations of translation and rotation of the column cross section are obtained.

For illustration, the theory is applied to a column of I *section. The conclusions, however, are generalized to include any column with a cross section symmetrical about its principal axes. It is shown that, for such columns, the theories for bending failure and twisting failure are special cases of this general theory and that primary failure will occur by bending about the axis of minimum moment of inertia or by twisting about the centroid, depending upon which gives the lower critical stress.*

When a column is attached to a skin, the great stiffness of the skin in its own plane causes the axis of rotation to lie in the plane of the skin. When the column cross section is symmetrical about its two principal axes, one of which is normal to the skin, the axis of rotation will be either at the point where the principal axis crosses the skin or at infinity in the plane of the skin, depending upon which location gives the smaller stress.

It is shown how the effective width of skin that may be considered to act with the column and carry the same stress as the column alters the section properties of the column and how the bending stiffness of the skin resists twisting of the column and raises the critical stress. Finally, the effective moduli that apply when the column is stressed above the proportional limit are discussed.

An illustrative problem in the first appendix (A) shows how the theory for primary failure may be used to construct the column curve for a skin-stiffener panel.

Appendix B shows how the theory may be applied to columns of closed section. For closed sections, however, the large torsional rigidity precludes anything but bending failure.

Appendix C contains a derivation of the theoretical equation for the effective modulus of elasticity when the column is stressed above the proportional limit.

INTRODUCTION

In the determination of the compressive strength of sheet and stiffener combinations as employed in stressed-skin structures for aircraft, the strength of the stiffener is a most important factor. When failure occurs by deflection normal to the skin, the accepted column curve for the material applies. (See reference 1.) When failure occurs by deflection of the outstanding portion of the stiffener in a direction parallel to the sheet, however, there is a combined action of bending and twisting in the stiffener that requires for its solution a more general theory for primary failure in columns than has been available heretofore.

Primary failure, as used in this report, is any type of column failure in which the cross sections are translated, rotated, or both translated and rotated but not distorted in their own planes (fig. 1). In keeping with this definition of primary failure, any failure in which the cross sections are distorted in their own planes but not translated or rotated is designated "secondary" or "local" failure. (See fig. 2.) Consideration is given herein only to primary failure.

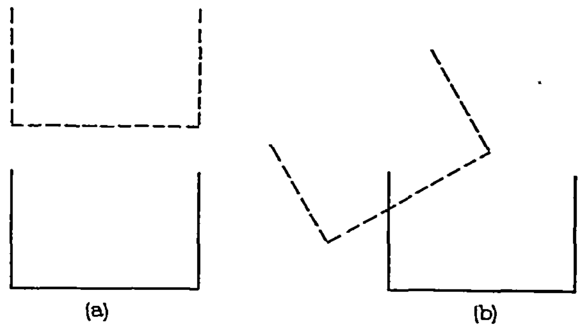

FIGURE 1.—Primary failure.
(a) Translated. (b) Translated and rotated.

FIGURE 2.—Secondary, or local, failure.

Wagner in reference 2 has presented a theory for torsion-bending failure of open-section columns formed from thin metal. A part of this theory is summarized in reference 3, which also includes the results of tests made to substantiate the theory. In his theory, Wagner considers the cross sections to rotate about an axis which is parallel to the column and which passes through the center of twist for the section. (See reference 4, p. 194, art. 41, for location of center of twist.) When the column is attached to the skin of a stressed-skin structure, the stiffness of the skin in its own plane and the anchorage of the skin at the sides of the panel are controlling factors in the location of the axis of rotation. If the stiffness of the skin in its own plane is assumed to be infinite, the axis of rotation is forced to lie in the plane of the skin. Rotation of the cross sections about any axis not lying in the plane of the skin would require a movement of the skin in its own plane. Such a movement is prevented by the stiffness of the skin in its own plane and the anchorage of the skin at the sides of the panel. Consequently, for the solution of the skin-stiffener problem the Wagner theory must be extended to include rotation of the cross sections about axes other than the one passing through the center of twist.

The purpose of this report is to present extensions of the Wagner theory, as given in reference 2, to include rotation of the cross sections about any axis parallel to the column. These extensions together with the Wagner theory constitute the general theory of primary failure of straight centrally loaded columns presented in this report. This theory is applicable to any thin-wall metal column of uniform section and contains the Euler theory for bending and the Wagner theory for twisting as special cases. The application of the general theory to columns of open section is illustrated by use of an I section column, both when the column is free and when it is restrained by the attachment of one flange to the skin of a stressed-skin structure. The application of the theory to a design problem involving an open-section column attached to a skin is given in appendix A. The application of the theory to columns of closed section is of less practical importance and is given in appendix B. Appendix C presents the derivation of the theoretical equation for the effective modulus of elasticity when the column is stressed beyond the proportional limit.

THE THEORY OF PRIMARY FAILURE

THE WAGNER EQUATION

The critical compressive load for primary failure of an open-section column that is both straight and centrally loaded when the axis of rotation passes through the shear center, in this report called "center of twist", is given by equation (9) of reference 2, which written with American notation is

$$P_{crit}=\frac{A}{I_p}\left(GJ+\frac{\pi^2}{L_0^2}E\,C_{BT}\right)$$

If both sides of this equation are divided by the cross-sectional area A, the following equation for the critical stress is obtained:

$$f_{crit}=\frac{GJ}{I_p}+\frac{C_{BT}}{I_p}\frac{\pi^2E}{L_0^2} \tag{1}$$

where

E is the tension-compression modulus of elasticity.

$G=\dfrac{E}{2(1+\mu)}$, shear modulus of elasticity.

μ, Poisson's ratio for the material.

I_p, polar moment of inertia of the cross section about the axis of rotation.

L_0, effective length of column.

J, torsion constant for the section. The product GJ in torsion problems is analogous to the product EI in bending problems. (See reference 5.)

C_{BT}, torsion-bending constant, dependent upon the location of the axis of rotation and the dimensions of the cross section. A complete discussion of how to evaluate C_{BT} is given in a later section.

In equation (1) the term $\frac{GJ}{I_p}$ is that part of the critical compressive stress caused by the resistance of the column to pure twisting. The term $\frac{C_{BT}}{I_p}\frac{\pi^2E}{L_0^2}$ is that part of the critical compressive stress caused by the resistance of the column to bending. In the derivation of equation (1) the angular displacement of the cross section about the axis of rotation was found to vary as a half sine wave along the length of the column in the same way that the lateral displacements in an Euler column vary as a half sine wave along the length. Therefore the term $\frac{C_{BT}}{I_p}$ is analogous to $\frac{I}{A}$ in the Euler column formula

$$f_{crit}=\frac{I}{A}\frac{\pi^2E}{L_0^2} \tag{2}$$

where I is the moment of inertia about a centroidal axis.

In order for a column to fail in the manner shown in figure 3 (a) the end cross sections must be free to rotate about the axis of rotation and there must be no restraint of longitudinal displacements at the ends of the column. Thus, when primary failure occurs in the manner shown in figure 3 (a), the twist per unit length is the same at all stations along the length and the column is said to be in a condition of pure twisting. In a pure twisting failure there are no longitudinal bending stresses, with the result that the second term of equation (1) is zero. The critical stress for a pure twisting failure is therefore given by $\frac{GJ}{I_p}$, which is in agreement with the value given by equation (4a) of reference 6. In order that the second term of equation (1) shall be zero the effective length of the column must be infinite ($L_0=\infty$).

In order for a column to fail in the manner shown in figure 3 (b) the end cross sections must be held against rotation about the axis of rotation but there must be no restraint of longitudinal displacements at the ends of the column. When primary failure occurs in the manner shown in figure 3 (b), the twist per unit length is variable along the length of the column with rotation about the axis of rotation and when buckling occurs, there must be complete restraint of longitudinal displacements at the ends of the column. Because the end conditions for the type of primary failure shown in figure 3 (c) correspond to built-in ends in an Euler

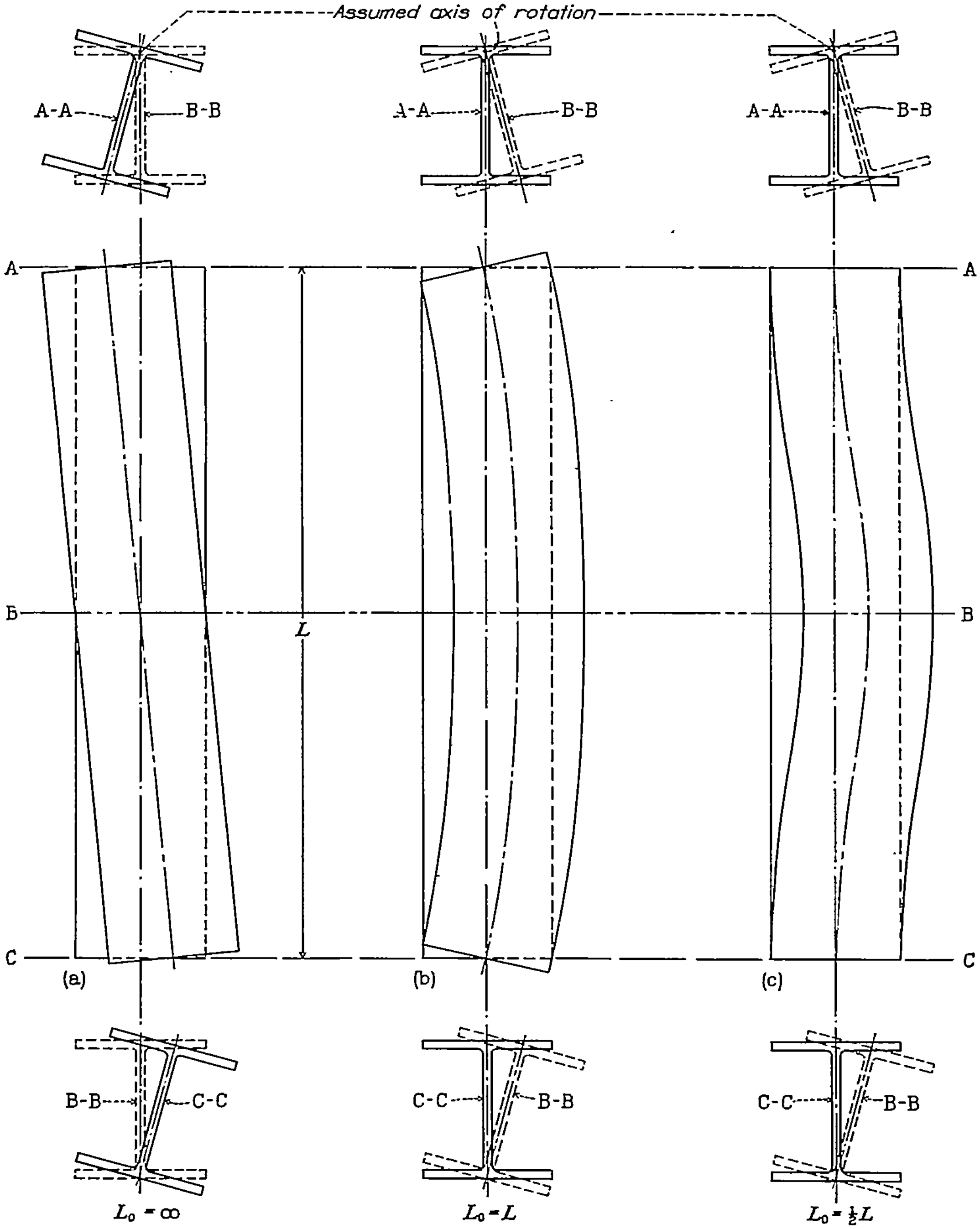

FIGURE 3.—End conditions for different effective lengths, L_0.

$$f_{crit} = \frac{GJ}{I_p} + \frac{C_{BT}}{I_p}\frac{\pi^2 E}{L_0^2}$$

the result that longitudinal bending stresses are present in addition to the shearing stresses of twisting. The end conditions for this case correspond to pin ends in an Euler column with the result that $L_0 = L$ in equation (1).

In order for a column to fail in the manner shown in figure 3 (c) the end cross sections must be held against column, $L_0 = \frac{L}{2}$ for this case. Similarly, for any degree of restraint against longitudinal displacements of the end cross sections the same effective length applies as for an Euler column with the same condition of end restraint.

GENERALIZATION OF WAGNER THEORY

In the paragraph immediately following equation (2b) on page 6 of reference 2 it is stated, "The longitudinal stresses σ_{bd} should not give a resulting bending moment (since there is no such moment acting on the member). It may easily be shown that this condition may be satisfied if and only if the magnitudes r_u and r_n refer to the shear center; that is, when the section twists about the shear axis, also in the case where longitudinal stresses arise." These statements are correct when there is no moment acting on the member. A general derivation, however, should include a moment acting on the member.

The Wagner theory is therefore based on the assumption that only torque moments are acting on the member at any station x along the column. From this assumption it follows that at failure all but the end cross sections of the column rotate about an axis parallel to the column and passing through the center of twist of the section. When it is assumed that both torque moments and bending moments are acting on the column at any station x, the combined effect is such as to cause the cross sections to rotate about some other axis parallel to the column. In this case equation (1) will give the critical stress provided that C_{BT} and I_p, which depend upon the location of the axis of rotation, are properly evaluated. The Wagner theory, together with this extension of it, of which the purpose is to include rotation of the cross sections about any axis parallel to the column, constitutes a more general theory for primary failure in columns. The development of the general theory is necessary for calculating the column strength of stiffeners attached to skin when failure occurs by deflection of the outstanding portion in a direction parallel to the skin.

EVALUATION OF C_{BT}

The torsion-bending constant C_{BT} is a section property similar to moment of inertia. Like moment of inertia it is dependent upon the axis about which the section property is calculated. Wagner has shown that, in its practical evaluation, C_{BT} may be divided into a major and a minor part, the latter of which may be neglected for most open sections formed of thin metal. In reference 3 it is shown that the major part can be expressed by a simple integral involving certain areas swept by a radius vector. In the evaluation of C_{BT} for some stiffener sections used in aircraft structures, however, the authors of the present report found it expedient to use the basic considerations of displacement from which the simple integral involving swept areas was derived. In this procedure certain concepts, not given in references 2 and 3, were introduced to clarify the method of calculating C_{BT} in the general case.

In order to evaluate C_{BT} by the general method, a portion of the column of length dx is allowed to twist about the axis of rotation an amount such that one end cross section is so displaced that it forms an angle $d\varphi$ with respect to the other end cross section. The longitudinal displacement of any point on the end cross section with respect to a reference plane, normal to the axis of rotation, is proportional to $\frac{d\varphi}{dx}$, the angle of twist per unit length hereinafter designated θ. The reference plane is then located so that the average longitudinal displacement of the elemental areas dA of the end section from this plane is zero; i. e.,

$$\frac{\int DdA}{\int dA}=\frac{\int DdA}{A}=0 \tag{3}$$

where D is the longitudinal displacement from the reference plane of the elemental area dA. Physically the reference plane establishes the neutral axis of the longitudinal bending stresses that result when the end cross section is restrained. The general expression for C_{BT}, which includes both the major and minor parts previously mentioned, is (reference 2, equation (6))

$$C_{BT}=\int u^2dA \tag{4}$$

where u is the longitudinal displacement, from the reference plane, of the elemental area dA when $\frac{d\varphi}{dx}=\theta=1$.

The general method of evaluating C_{BT} described in the preceding paragraph will now be applied to an I section column with the axis of rotation located at a distance r from the centroid in any direction. Wagner and Pretschner (reference 3) have shown how to compute C_{BT} for an I section when the axis of rotation is at the center of twist, which is at the centroid for the I section. When the axis of rotation has some other location, certain terms must be added to allow for the shift in the axis of rotation. In the derivation of C_{BT} for any location of the axis of rotation, it is convenient to resolve the displacement of the one end cross section (fig. 4 (a)) into two displacements of translation (1 and 2 of fig. 4 (b)) and one displacement of rotation about the center of twist (3 of fig. 4 (b)). The *longitudinal displacements* of the different parts of the cross section caused by the three component displacements of the cross section (fig. 4 (b)) are then added to obtain the total longitudinal displacement. In the following tabulations the longitudinal displacements at the center lines of the web and flanges are given. The algebraic sign of the displacement is positive when a point on the cross section moves in the positive direction of x and negative when it moves in the negative direction of x (figs. 5, 6, and 7). Also note in the expressions for longitudinal displacement (LD–1, 2, 3, etc.) that $\frac{d\varphi}{dx}=\theta$.

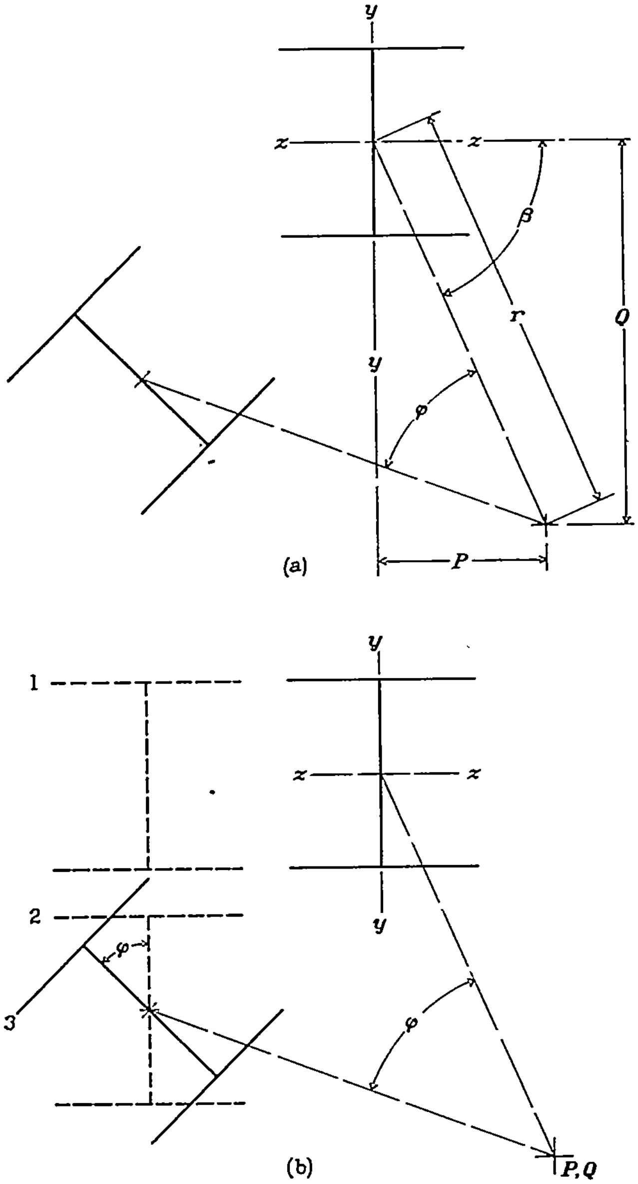

FIGURE 4.—Displacement of one end cross section with respect to the other when rotated about the point P, Q.

Displacements for rotation about the center of twist (fig. 5).—The longitudinal displacement from the original plane of the end cross section at a distance s measured from

$$\left.\begin{array}{ll}\text{B toward A is} & -\theta\frac{h}{2}s\\ \text{B toward C,} & \theta\frac{h}{2}s\\ \text{O toward B,} & 0\\ \text{O toward B}', & 0\\ \text{B}' \text{ toward C}', & -\theta\frac{h}{2}s\\ \text{B}' \text{ toward A}', & \theta\frac{h}{2}s\end{array}\right\}\text{(LD–1)}$$

Displacements for translation normal to the web (fig. 6).—The longitudinal displacement from the

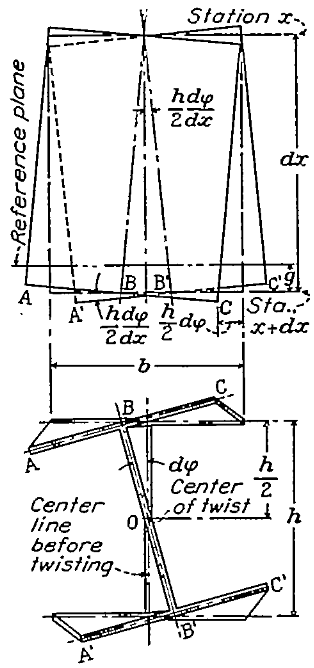

FIGURE 5.—Displacements for rotation about the center of twist.

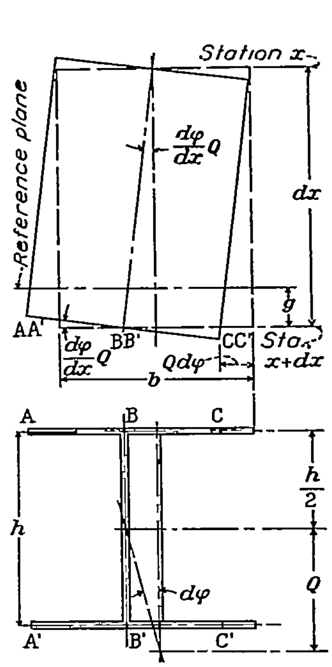

FIGURE 6.—Displacements for translation normal to the web.

original plane of the end cross section at a distance s measured from

$$\left.\begin{array}{ll}\text{B toward A is} & -\theta Qs\\ \text{B toward C,} & \theta Qs\\ \text{O toward B,} & 0\\ \text{O toward B}', & 0\\ \text{B}' \text{ toward C}', & \theta Qs\\ \text{B}' \text{ toward A}', & -\theta Qs\end{array}\right\}\text{(LD–2)}$$

Displacements for translation parallel to the web (fig. 7).—The longitudinal displacements from the original plane of the end cross section at a distance s measured from

$$\left.\begin{array}{ll}\text{B toward A is} & \theta P\frac{h}{2}\\ \text{B toward C,} & \theta P\frac{h}{2}\\ \text{O toward B,} & \theta Ps\\ \text{O toward B}', & -\theta Ps\\ \text{B}' \text{ toward C}', & -\theta P\frac{h}{2}\\ \text{B}' \text{ toward A}', & -\theta P\frac{h}{2}\end{array}\right\}\text{(LD–3)}$$

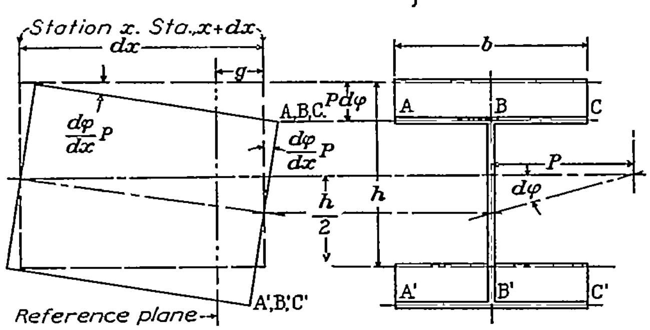

FIGURE 7.—Displacements for translation parallel to the web.

Total displacement for rotation about the point P, Q (figs. 4, 5, 6, and 7).—By addition of the displacements LD–1, LD–2, LD–3, the total longitudinal displacement from the original plane of the end cross section at a distance s measured from

$$\left.\begin{array}{ll} \text{B toward A is} & -\theta\left[s\left(\frac{h}{2}+Q\right)-P\frac{h}{2}\right] \\ \text{B toward C,} & \theta\left[s\left(\frac{h}{2}+Q\right)+P\frac{h}{2}\right] \\ \text{O toward B,} & \theta Ps \\ \text{O toward B}', & -\theta Ps \\ \text{B}' \text{ toward C}', & -\theta\left[s\left(\frac{h}{2}-Q\right)+P\frac{h}{2}\right] \\ \text{B}' \text{ toward A}', & \theta\left[s\left(\frac{h}{2}-Q\right)-P\frac{h}{2}\right] \end{array}\right\}\text{(LD–4)}$$

Therefore the longitudinal displacement of the end cross section with respect to the reference plane at a distance s measured from

$$\left.\begin{array}{ll} \text{B toward A is} & g-\theta\left[s\left(\frac{h}{2}+Q\right)-P\frac{h}{2}\right] \\ \text{B toward C,} & g+\theta\left[s\left(\frac{h}{2}+Q\right)+P\frac{h}{2}\right] \\ \text{O toward B,} & g+\theta Ps \\ \text{O toward B}', & g-\theta Ps \\ \text{B}' \text{ toward C}', & g-\theta\left[s\left(\frac{h}{2}-Q\right)+P\frac{h}{2}\right] \\ \text{B}' \text{ toward A}', & g+\theta\left[s\left(\frac{h}{2}-Q\right)-P\frac{h}{2}\right] \end{array}\right\}\text{(LD–5)}$$

Now g, the distance of the reference plane from the original plane of the end cross section, is determined by the conditions of equation (3). The term tds may be substituted for dA because the longitudinal displacements vary linearly across the thickness t_w of the web and t_b of the flanges. Then, if the longitudinal displacement of the center lines (LD–5) is substituted for D, equation (3) becomes, after multiplying by A,

$$\begin{aligned} 0=\int Dtds=&\int_0^{\frac{b}{2}}\left\{g-\theta\left[s\left(\frac{h}{2}+Q\right)-P\frac{h}{2}\right]\right\}t_b ds \\ &+\int_0^{\frac{b}{2}}\left\{g+\theta\left[s\left(\frac{h}{2}+Q\right)+P\frac{h}{2}\right]\right\}t_b ds \\ &+\int_0^{\frac{h}{2}}\left\{g+\theta Ps\right\}t_h ds \\ &+\int_0^{\frac{h}{2}}\left\{g-\theta Ps\right\}t_h ds \\ &+\int_0^{\frac{b}{2}}\left\{g-\theta\left[s\left(\frac{h}{2}-Q\right)+P\frac{h}{2}\right]\right\}t_b ds \\ &+\int_0^{\frac{b}{2}}\left\{g+\theta\left[s\left(\frac{h}{2}-Q\right)-P\frac{h}{2}\right]\right\}t_b ds \end{aligned}$$

from which

$$g=0 \tag{5}$$

From the symmetry of the I section, it might have been foreseen that $g=0$. The formal proof, however, has been presented to show the method that would be necessary for the determination of g for other sections.

Wagner has shown that for sections formed of thin metal it is convenient to divide C_{BT} into a major part C_B and a minor part C_T so that

$$C_{BT}=C_B+C_T \tag{6}$$

In the major part of C_{BT} the longitudinal displacement is assumed to be uniform across the thickness of the plate and equal to the value at its center line. For the major part, dA in equation (4) is therefore written tds. Hence

$$C_B=\int u^2tds \tag{7}$$

Substitution of the longitudinal displacements (LD–5) for u in equation (7), with $\theta=1$ and $g=0$, gives for the I section

$$\begin{aligned} C_B=&\int_0^{\frac{b}{2}}\left[s\left(\frac{h}{2}+Q\right)-P\frac{h}{2}\right]^2 t_b ds \\ &+\int_0^{\frac{b}{2}}\left[s\left(\frac{h}{2}+Q\right)+P\frac{h}{2}\right]^2 t_b ds \\ &+\int_0^{\frac{h}{2}}[Ps]^2 t_h ds \\ &+\int_0^{\frac{h}{2}}[Ps]^2 t_h ds \\ &+\int_0^{\frac{b}{2}}\left[s\left(\frac{h}{2}-Q\right)+P\frac{h}{2}\right]^2 t_b ds \\ &+\int_0^{\frac{b}{2}}\left[s\left(\frac{h}{2}-Q\right)-P\frac{h}{2}\right]^2 t_b ds \end{aligned}$$

from which

$$C_B=\frac{1}{24}b^3h^2t_b+\left(\frac{h^2bt_b}{2}+\frac{h^3t_h}{12}\right)P^2+\frac{b^3t_b}{6}Q^2 \tag{8}$$

The minor part of C_{BT} is in the nature of a correction to the major part to allow for the variation in longitudinal displacement across the thickness of the web or flange. When the thickness is constant along the web or flange, the general expression for the minor part is (reference 2, equation (6b))

$$C_T=\frac{t^3}{12}\int s^2ds \tag{9}$$

In order to evaluate $\int s^2ds$ in this equation, the origin of s must be at the point on the center line of the web

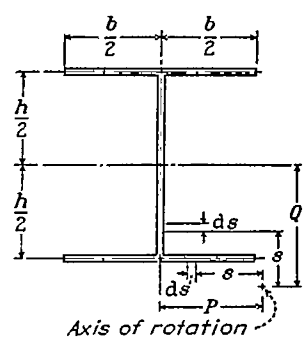

FIGURE 8.—Method of measuring s for evaluation of equation (9).

or flange, extended if necessary, from which a perpendicular may be erected to pass through the axis of rotation. (See fig. 8.) When the thickness varies with

s, t^3 should be placed under the integral sign and equation (9) evaluated by either an analytical or graphical method.

As applied to the I section, equation (9) becomes

$$C_T=2\frac{t_b^3}{12}\int_{P-\frac{b}{2}}^{P+\frac{b}{2}} s^2ds+\frac{t_h^3}{12}\int_{Q-\frac{h}{2}}^{Q+\frac{h}{2}} s^2ds$$

from which

$$C_T=\frac{b^3t_b^3}{72}+\frac{h^3t_h^3}{144}+\frac{bt_b^3}{6}P^2+\frac{ht_h^3}{12}Q^2 \tag{10}$$

When the thicknesses t_b and t_h are small as compared with b and h, respectively, C_T will be very small as compared with C_B and may be neglected in the computation of C_{BT}. Substitution in equation (6) of the values of C_B and C_T, however, as given by equations (8) and (10) gives

$$C_{BT}=\frac{b^3h^2t_b}{24}+\frac{b^3t_b^3}{72}+\frac{h^3t_h^3}{144}+\left(\frac{h^2bt_b}{2}+\frac{h^3t_h}{12}+\frac{bt_b^3}{6}\right)P^2+\left(\frac{b^3t_b}{6}+\frac{ht_h^3}{12}\right)Q^2$$

or

$$C_{BT}=(C_{BT})_{\substack{P=0\\Q=0}}+I_zP^2+I_yQ^2 \tag{11}$$

where I_y and I_z are the moments of inertia of the cross section about the principal axes y and z, respectively, (fig. 4).

CRITICAL STRESS FOR AN I SECTION COLUMN

In order to show the effect of variation in $\frac{b}{h}$ on the critical stress for the I section in a later part of this report, it is convenient to write equation (1) in the following form

$$f_{crit}=KG\frac{t_h^2}{h^2}+K_{BT}\frac{\pi^2Et_h^2}{12L_0^2} \tag{12}$$

where $G\frac{t_h^2}{h^2}$ is the critical compressive stress for a pure twisting failure of the web alone when the axis of rotation is at one edge of the web, that is, the critical compressive stress for a long outstanding flange simply supported at its base. (See reference 7, equation (91).)

$\frac{\pi^2Et_h^2}{12L_0^2}$, the critical compressive stress for the web alone acting as an Euler column.

$K=\frac{h^2}{t_h^2}\frac{J}{I_p}$, $K_{BT}=\frac{12}{I_p}\frac{C_{BT}}{t_h^2}$ — constants that vary with the dimensions of the cross section and the location of the axis of rotation.

On the assumption that the torsional stiffness GJ of the I section is equal to the sum of the torsional stiffnesses of the web and flanges (reference 4, p. 76, art. 20) the approximate equation for J is

$$J=\frac{1}{3}ht_h^3+\frac{2}{3}bt_b^3 \tag{13}$$

For any location of the axis of rotation, the value of I_p for the I section is

$$I_p=\frac{1}{12}h^3t_h+\frac{1}{2}h^2bt_b+\frac{1}{6}b^3t_b+(ht_h+2bt_b)(P^2+Q^2) \tag{14}$$

Substitution of the values of J and I_p given by equations (13) and (14) in the equation that defines K gives for the I section

$$K=\frac{4+8\frac{b}{h}\left(\frac{t_b}{t_h}\right)^3}{1+\left[2\frac{b}{h}\frac{t_b}{t_h}\right]\left[3+\left(\frac{b}{h}\right)^2\right]+12\left[1+2\frac{b}{h}\frac{t_b}{t_h}\right]\left[\left(\frac{P}{h}\right)^2+\left(\frac{Q}{h}\right)^2\right]} \tag{15}$$

For the same reason that C_{BT} has been divided into a major part C_B and a minor part C_T (see equation (6)), K_{BT} will likewise be divided into a major part K_B and a minor part K_T so that

$$K_{BT}=K_B+K_T \tag{16}$$

Substitution of the values of C_B and C_T as given by equations (8) and (10) for C_{BT} in the equation that defines K_{BT}, gives for the I section

$$K_B=\frac{6\left(\frac{b}{t_h}\right)^3\frac{t_b}{h}+12\left(\frac{P}{t_h}\right)^2\left[1+6\frac{b}{h}\frac{t_b}{t_h}\right]+24\left(\frac{b}{h}\right)^3\left(\frac{Q}{t_h}\right)^2\frac{t_b}{t_h}}{1+\left[2\frac{b}{h}\frac{t_b}{t_h}\right]\left[3+\left(\frac{b}{h}\right)^2\right]+12\left[1+2\frac{b}{h}\frac{t_b}{t_h}\right]\left[\left(\frac{P}{h}\right)^2+\left(\frac{Q}{h}\right)^2\right]} \tag{17}$$

and

$$K_T=\frac{1+2\left(\frac{b}{h}\right)^3\left(\frac{t_b}{t_h}\right)^3+12\left[2\frac{b}{h}\left(\frac{t_b}{t_h}\right)^3\left(\frac{P}{h}\right)^2+\left(\frac{Q}{h}\right)^2\right]}{1+\left[2\frac{b}{h}\frac{t_b}{t_h}\right]\left[3+\left(\frac{b}{h}\right)^2\right]+12\left[1+2\frac{b}{h}\frac{t_b}{t_h}\right]\left[\left(\frac{P}{h}\right)^2+\left(\frac{Q}{h}\right)^2\right]} \tag{18}$$

DISCUSSION

Location of the axis of rotation for a free column.—When the axis of rotation is located at a distance r from the centroid of a section, the value of $\frac{GJ}{I_p}$ in equation (1) is independent of the direction in which r is measured. Because $\frac{C_{BT}}{I_p}$ is analogous to $\frac{I}{A}$ in the Euler column formula, it seems reasonable to expect that, as the axis of rotation moves around a circle of radius r, $\frac{C_{BT}}{I_p}$ will vary from a maximum at one of the principal axes to a minimum at the other principal axis. Because I_p is independent of the direction in which r is measured, all the variation in $\frac{C_{BT}}{I_p}$ will occur in C_{BT}. It will now be shown that, for a section symmetrical about each of its two principal axes, C_{BT} is a maximum or minimum when the axis of rotation is on the principal axis about which the moment of inertia is, respectively, maximum or minimum.

It follows from the symmetry of the expressions for longitudinal displacement and the limits of integration that C_{BT} for any section symmetrical about its two principal axes will have the form given by equation (11). From figure 4

$$P = r \cos \beta$$
$$Q = r \sin \beta$$

Substitution of these values in equation (11) gives

$$C_{BT} = (C_{BT})_{\substack{P=0\\Q=0}} + I_x r^2 \cos^2\beta + I_y r^2 \sin^2\beta$$

The first and second derivatives of C_{BT} with respect to β are, respectively,

$$\frac{dC_{BT}}{d\beta} = r^2(I_y - I_x) \sin 2\beta$$

$$\frac{d^2C_{BT}}{d\beta^2} = 2r^2(I_y - I_x) \cos 2\beta$$

The first derivative set equal to zero shows that C_{BT} is either a maximum or minimum when $\beta = 0°$, $90°$, $180°$, or $270°$. When $\beta = 0°$ or $180°$, $\frac{d^2 C_{BT}}{d\beta^2}$ is negative provided that $I_y < I_x$, in which case $\beta = 0°$ or $180°$ locates the axis of rotation for $C_{BT_{max}}$. If $I_y > I_x$, then $\beta = 0°$ or $180°$ locates the axis of rotation for $C_{BT_{min}}$. Similarly, when $\beta = 90°$ or $270°$, it may be concluded that C_{BT} is a maximum or minimum when the axis of rotation is on the principal axis about which the moment of inertia is, respectively, maximum or minimum.

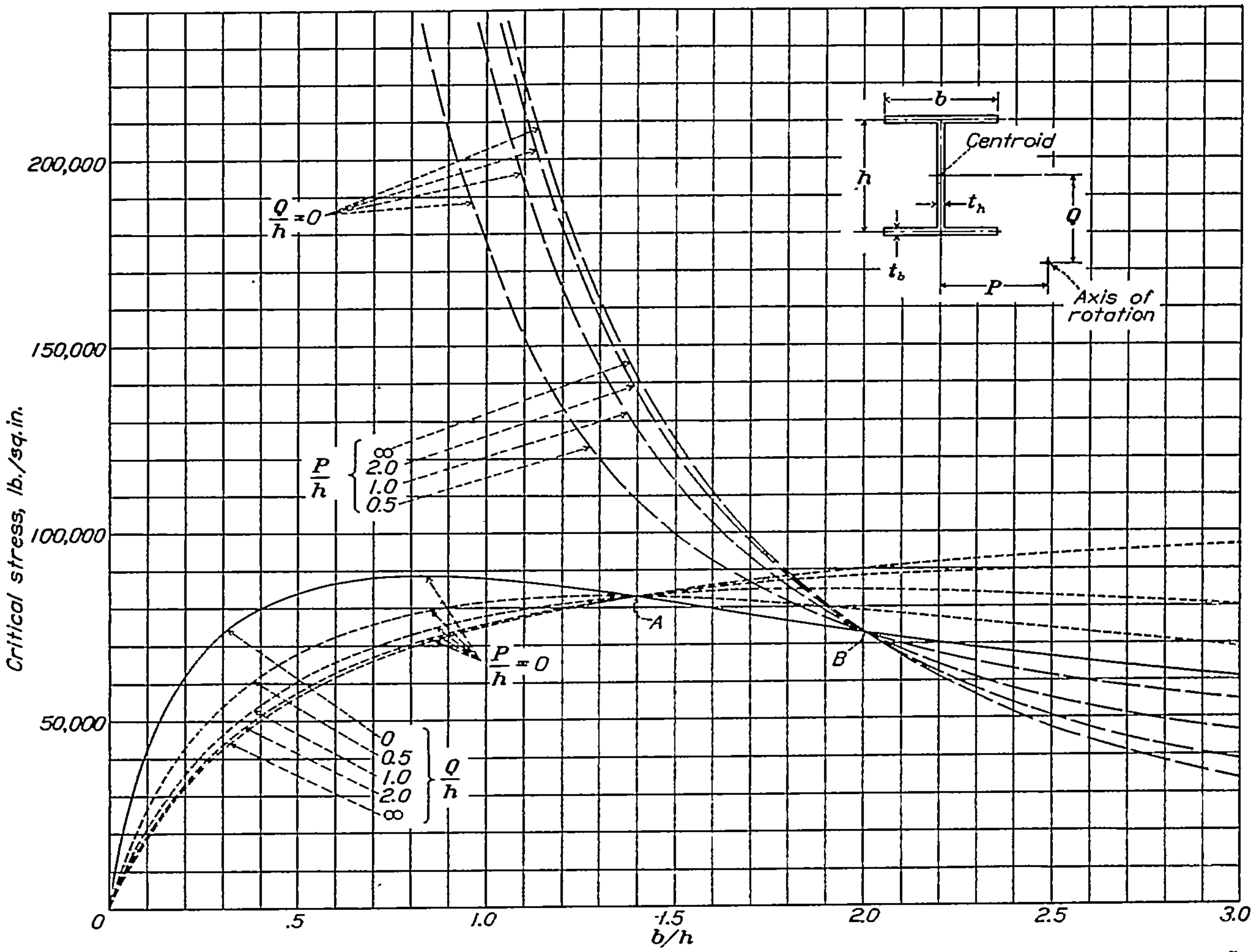

FIGURE 9.—Variation of the critical stress with b/h for different locations of the axis of rotation along the principal axes of an I section column with pin ends. Curves drawn for $b=2$ inches, $t_b = t_h = 0.1$ inch, length=17.1 inches, and $E = 10^7$ pounds per square inch.

When a free column of symmetrical section with no bending restraint at its ends (pin ends) is of such proportions that it develops a primary failure, the axis of rotation will be either at infinity on one of the principal axes or at the center of twist. Figure 9 illustrates this fact for a family of I section columns by means of curves for critical stress plotted against the ratio $\frac{b}{h}$ for different locations of the axis of ratation along each of the two principal axes. Inspection of figure 9 shows that, for values of $\frac{b}{h}$ between 0 and 1.4, the critical stress is lowest when the axis of rotation is at infinity along the principal axis parallel to the web. For values of $\frac{b}{h}$ between 1.4 and 2.0, the critical stress is lowest when the axis of rotation is at the center of twist (centroid, for the I section). For values of $\frac{b}{h}$ greater than 2.0, the critical stress is lowest when the axis of rotation is at infinity along the principal axis normal to the web. Had a different set of dimensions been selected for the family of I section columns in figure 9, the crossing points A and B would, in general, have been at different values of $\frac{b}{h}$. Regardless of the dimensions used, however, the lowest critical stress would always be given by one of the three locations of the axis of rotation previously mentioned; i. e., at the center of twist $\left(\frac{P}{h}=0; \frac{Q}{h}=0\right)$ or at infinity on either of the two principal axes $\left(\frac{P}{h}=0; \frac{Q}{h}=\infty \text{ or } \frac{P}{h}=\infty; \frac{Q}{h}=0\right)$.

In figure 9 the critical stresses are, for the most part, greater than the yield point for the present engineering materials having the same value of E as was assumed in the calculation of the curves. ($E=10^7$ pounds per square inch.) This fact does not detract from the conclusions drawn from figure 9 because, when a column is stressed above the proportional limit, equation (1) may be considered to apply with a reduced modulus of elasticity thereby giving a reduced critical stress. The reduced modulus is discussed in a later section of this report.

It will now be proved that *for a free column of* I *section* the axis of rotation will be at infinity along the principal axis parallel to the web provided that

$$\frac{t_h}{t_b}<14.7$$

and

$$\frac{b}{h}<\sqrt[3]{\frac{t_h}{t_b}}$$

Because the axis of rotation might be at the center of twist or at infinity on the principal axis normal to the web (fig. 9), the two following conditions must hold if the axis of rotation is to be at infinity on the principal axis parallel to the web:

$$(f_{crit})_{\substack{P=0\\Q=\infty}}<(f_{crit})_{\substack{P=0\\Q=0}}$$

$$(f_{crit})_{\substack{P=0\\Q=\infty}}<(f_{crit})_{\substack{P=\infty\\Q=0}}$$

The first of these conditions will be satisfied if

$$(f_{crit})_{\substack{P=0\\Q=\infty}}<\left[(f_{crit})-\left(\frac{GJ}{I_p}\right)\right]_{\substack{P=0\\Q=0}}=\left[\frac{C_{BT}}{I_p}\frac{\pi^2E}{L_0^2}\right]_{\substack{P=0\\Q=0}}$$

or if

$$\frac{I_y}{A}\frac{\pi^2E}{L_0^2}<\left[\frac{C_{BT}}{I_p}\frac{\pi^2E}{L_0^2}\right]_{\substack{P=0\\Q=0}}$$

$$\frac{I_y}{A}<\left[\frac{C_{BT}}{I_p}\right]_{\substack{P=0\\Q=0}}$$

$$\frac{\frac{1}{6}b^3t_b}{ht_h+2bt_b}<\frac{\frac{1}{24}h^2b^3t_b}{\frac{1}{12}h^3t_h+\frac{1}{2}h^2bt_b+\frac{1}{6}b^3t_b}$$

from which

$$\frac{b}{h}<\sqrt[3]{\frac{t_h}{t_b}}$$

The second condition will be satisfied if

$$I_y<I_x$$

or if

$$\frac{1}{6}b^3t_b<\frac{1}{12}h^3t_h+\frac{1}{2}h^2bt_b$$

Multiplication of both sides by $\frac{12}{h^3t_b}$ gives

$$2\left(\frac{b}{h}\right)^3<\frac{t_h}{t_b}+6\frac{b}{h}$$

from which

$$\frac{b}{h}<\sqrt[3]{\frac{1}{2}\frac{t_h}{t_b}+3\frac{b}{h}}$$

This condition holds as long as $\frac{b}{h}$ does not become too large. If $\frac{b}{h}$ is as large as $\sqrt[3]{\frac{t_h}{t_b}}$, then the following condition must be satisfied

$$\sqrt[3]{\frac{t_h}{t_b}}<\sqrt[3]{\frac{1}{2}\frac{t_h}{t_b}+3\sqrt[3]{\frac{t_h}{t_b}}}$$

This latter condition will be fulfilled provided that

$$\frac{t_h}{t_b}<14.7 \tag{19}$$

a value of $\frac{t_h}{t_b}$ much larger than will be found in any I section column of practical dimensions. It may therefore be concluded that primary failure in a free column of I section will occur by bending with the neutral axis parallel to the web when

$$\frac{b}{h}<\sqrt[3]{\frac{t_h}{t_b}} \tag{20}$$

When $\frac{b}{h}$ is greater than $\sqrt[3]{\frac{t_h}{t_b}}$ the critical stress for the axis of rotation located at the centroid should be computed and compared with the critical stress for bending about the axis of minimum moment of inertia. The smaller of these two values will be the stress at which failure occurs.

When the critical stress is to be computed for the axis of rotation at the centroid, the curves given in figures 10 and 11 may be used to determine the values of K and K_B in equation (12).

Proof that bending failure is a special case of the theory presented in this report.—When the axis of rotation is at infinity, equation (1) reduces to the Euler column formula. In this case, I_p and C_{BT} are both infinite. Hence $\frac{GJ}{I_p}=0$ and it remains to be shown that $\frac{C_{BT}}{I_p}=\frac{I}{A}$.

If $\beta=90°$ or $270°$

$$\frac{C_{BT}}{I_p}=\frac{I_y}{A}$$

Location of the axis of rotation for a column attached to a skin.—When a column with pin ends is attached to the skin of a stressed-skin structure, the stiffness of the skin in its own plane and the anchorage of the skin at the sides of the panel are controlling factors in the location of the axis of rotation. In this discussion it is assumed that the skin provides only lateral support at its point of attachment to the column. Rotation of the cross sections about any axis not lying in the plane of the skin would therefore require a movement

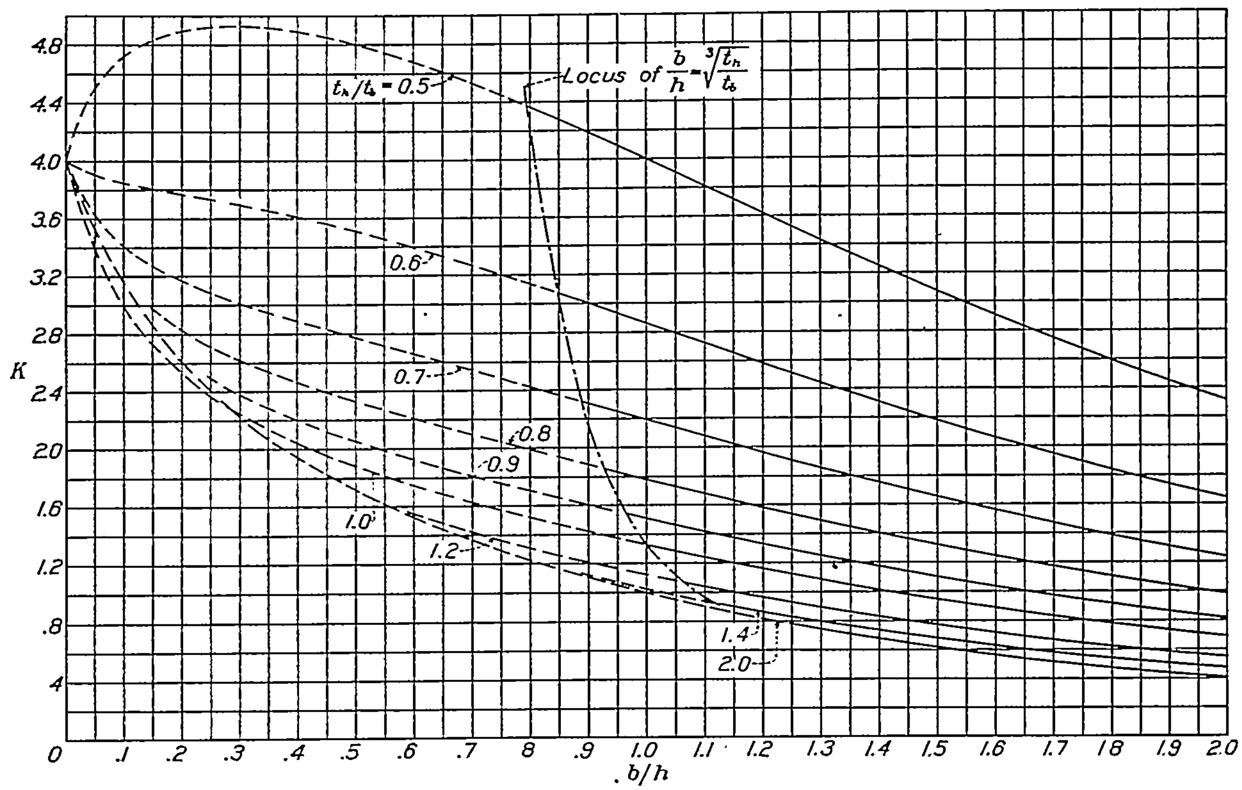

FIGURE 10.—Variation of K with b/h for different values of t_h/t_b when the axis of rotation is at the centroid of an I section column.

Equations (11) and (14) show that as the axis of rotation approaches infinity along a radius r the terms involving both P and Q, if P and Q both approach infinity, become very large in comparison with the remaining terms. Thus, when P and Q become infinite,

$$\frac{C_{BT}}{I_p}=\frac{I_zP^2+I_yQ^2}{A(P^2+Q^2)}$$

or

$$\frac{C_{BT}}{I_p}=\frac{I_z\cos^2\beta+I_y\sin^2\beta}{A}$$

When y and z are the principal axes of the section, $I_z\cos^2\beta+I_y\sin^2\beta$ is the moment of inertia of the cross section about a line that passes through the centroid and the axis of rotation. If $\beta=0°$ or $180°$

$$\frac{C_{BT}}{I_p}=\frac{I_z}{A}$$

of the skin in its own plane. The stiffness of the skin in its own plane and the anchorage of the skin at the sides of the panel tend to prevent such a movement and the axis of rotation is forced to lie in the plane of the skin.

For a column the cross section of which is symmetrical about its two principal axes, one of which is normal to the skin, the axis of rotation will lie in the plane of the skin and be either at infinity or at the point where the principal axis crosses the skin. This statement is illustrated in figure 12 in which values of f_{crit} for a family of I section columns having the same dimensions as those of figure 9 are plotted against $\frac{b}{h}$ for different locations of the axis of rotation in the plane of the skin. For simplicity, the skin is assumed to be at the center of one flange. Inspection of figure 12 shows

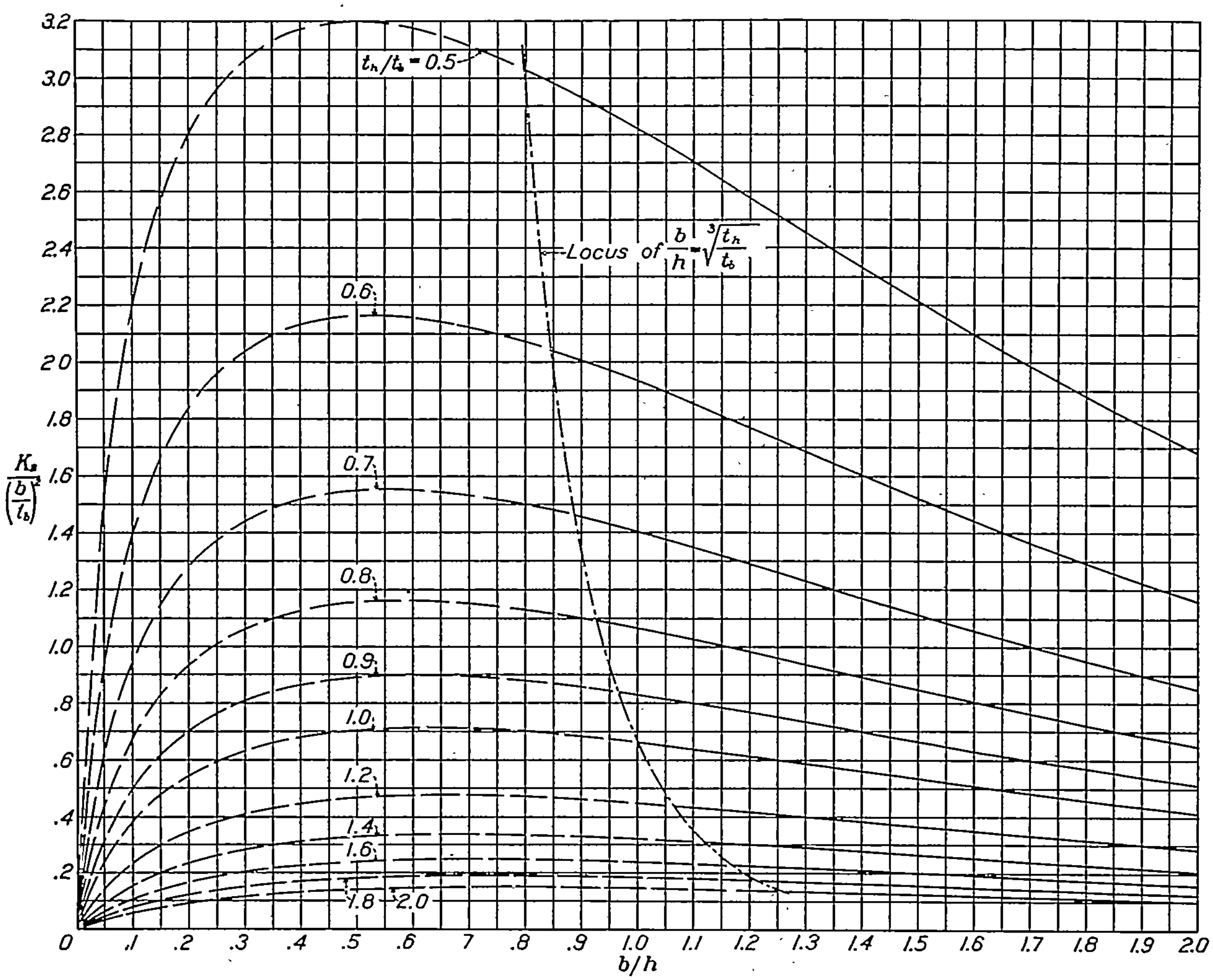

FIGURE 11.—Variation of $K_B/(b/t_b)^2$ with b/h for different values of t_h/t_b when the axis of rotation is at the centroid of an I section column.

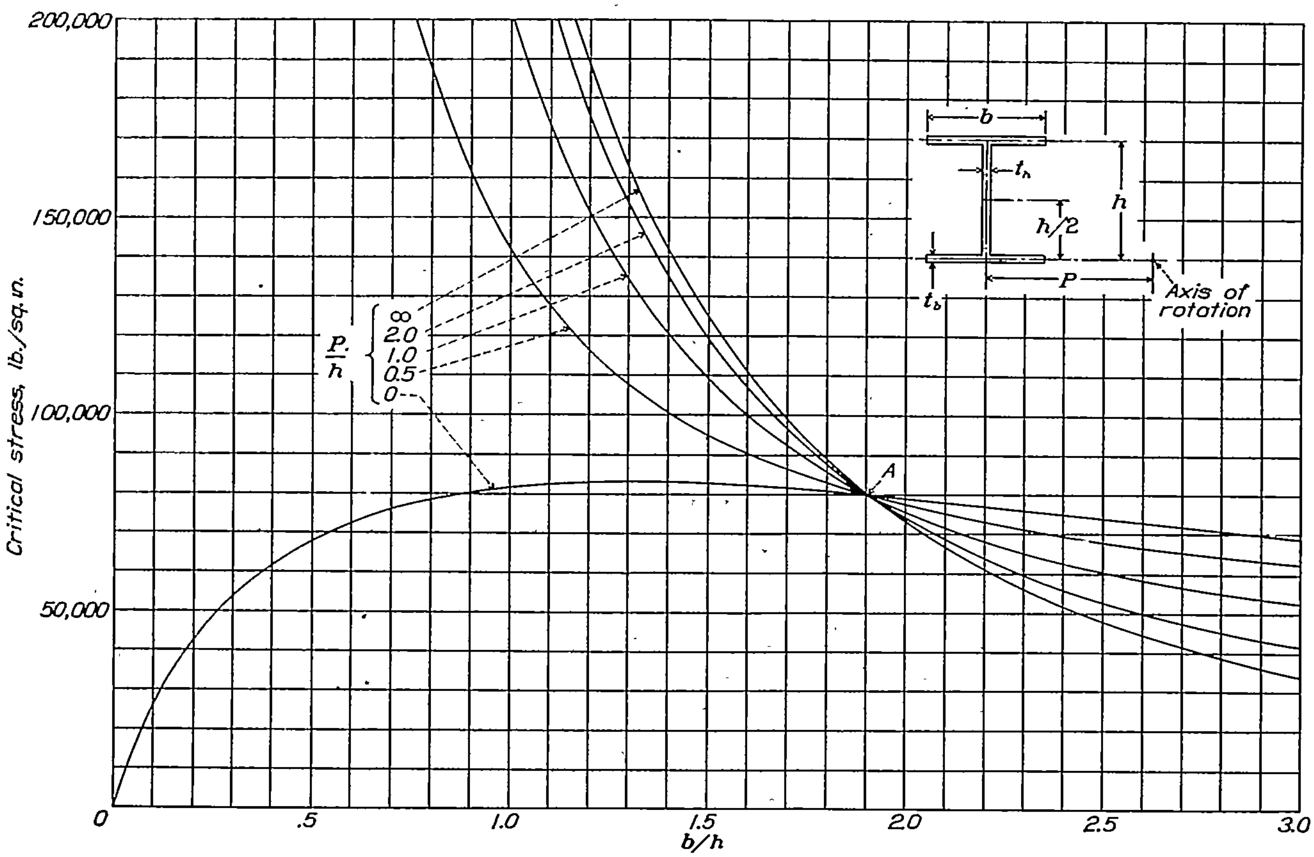

FIGURE 12.—Variation of the critical stress with b/h for different locations of the axis of rotation along the center line of one flange (extended) of an I section column with pin ends. Curves drawn for $b=2$ inches, $t_b=t_h=0.1$ inch, length=17.1 inches, and $E=10^7$ pounds per square inch.

that, for values of $\frac{b}{h}$ between 0 and 1.90, the critical stress is lowest when the axis of rotation is at the web. For values of $\frac{b}{h}$ greater than 1.90, the critical stress is lowest when the axis of rotation is at infinity in the plane of the skin.

As in the case of free columns (fig. 9), the location of the crossing point A in figure 12 will depend upon the particular dimensions selected for the family of columns. Regardless of the dimensions used, the lowest critical stress will always be given by one of the two locations of the axis of rotation previously mentioned; i. e., in the plane of the skin either at infinity $\left(\frac{P}{h}=\infty\right)$ or at the point where the principal axis crosses the skin $\left(\frac{P}{h}=0\right)$. Again, as in figure 9, the necessary use of a reduced modulus at stresses above the proportional limit does not invalidate the conclusions drawn from figure 12.

When a column of I section is attached to a skin, it is not practicable to give a simple criterion by which the location of the axis of rotation may be determined. In view of the fact that the axis of rotation will be either at infinity in the plane of the skin or at the point where the principal axis crosses the skin, the critical stress for these two locations should be computed and the lower value regarded as the failure stress. When the axis of rotation is at infinity in the plane of the skin, the critical stress is given by equation (2) with $I=I_s$. In order to facilitate the computation of f_{crtt} when the axis of rotation is at the point where the principal axis crosses the skin, figures 13 and 14 have been prepared from which the values of K and K_B may be obtained for substitution in equation (12).

Effect of the skin in changing the section properties of the column.—In the preceding section it was assumed that the only effect of the skin was to provide lateral support to the column. Inasmuch as the skin is attached to the column, however, it will also carry a part of the compression load on the column and the stress in the skin at its point of attachment will be the same as that in the column. Usually the stiffener spacing in terms of the sheet thickness is such that the skin will buckle between stiffeners and only a small width adjacent to each stiffener will be effective. In reference 1 it is shown that, when failure occurs by bending of the stiffener normal to the skin (axis of rotation at infinity in the plane of the skin), the effective width, which is dependent upon the column stress, may be considered to be a part of the column cross section and is to be included in the computation of section properties.

When the axis of rotation is at the point where the principal axis crosses the skin, twisting of the stiffener about this axis will cause a rotation of the skin near the stiffener. If it is assumed that the effective width of skin rotates with the stiffener, the following increments must be added to J, I_p, and C_{BT} as evaluated for the stiffener when the skin was assumed to provide only lateral support for the stiffener,

$$\Delta J=\frac{1}{3}Ut_s^3 \tag{21}$$

$$\Delta I_p=\frac{1}{12}U^3t_s \tag{22}$$

$$\Delta C_{BT}=\Delta C_T \tag{23}$$

where

$$\Delta C_T=\frac{1}{144}U^3t_s^3 \tag{24}$$

In these equations t_s is the thickness of the skin and U is the effective width of skin that acts with the stiffener, carries the same stress as the stiffener, and is assumed to be continuous across the stiffener and symmetrically located with respect to the web of the I section. The evaluation of U is included in the illustrative problem of appendix A.

Effect of the skin in providing restraint to twisting of the column.—When a column is attached to a skin and the axis of rotation is at a point other than infinity in the plane of the skin, the rotation of the column cross section at failure is resisted by bending of the skin provided that the skin is supported by adjacent stiffeners or other structure. A theoretical analysis of this effect has been reserved for a future report. Only a brief summary of the subject is given herein.

It may be stated that the effect of the bending stiffness of the skin in providing resistance to twisting of the column attached to the skin is such as to increase the critical stress given by equation (1) or (12) by an amount

$$\Delta f_{crtt}=\frac{K_1Et_s^3}{6(1-\mu^2)dI_p}\,\frac{L_0^2}{\pi^2} \tag{25}$$

then

$$f_{crtt}=\frac{\overline{G}J}{I_p}+\frac{C_{BT}}{I_p}\,\frac{\pi^2\overline{E}}{L_0^2}+\frac{K_1Et_s^3}{6(1-\mu^2)dI_p}\,\frac{L_0^2}{\pi^2} \tag{26}$$

where d is the stiffener spacing.

K_1, a constant depending upon the conditions of support of the skin at the adjacent stiffener or other structure.

It will be noted that in equation (26) $\overline{G}$ and $\overline{E}$ have been substituted for G and E, respectively, in equation (1). The substitution of $\overline{E}$ for E at this time was made to distinguish between the value of E associated with longitudinal stresses in the stiffener and its effective width of sheet and the value of E associated with bending of the skin between stiffeners. The desirability of distinguishing between these two values of E will be explained in a later section of this report in which the evaluation of $\overline{E}$ and $\overline{G}$ is discussed.

If the two ends of the stiffener are held against rotation about the axis of rotation and the end cross sec-

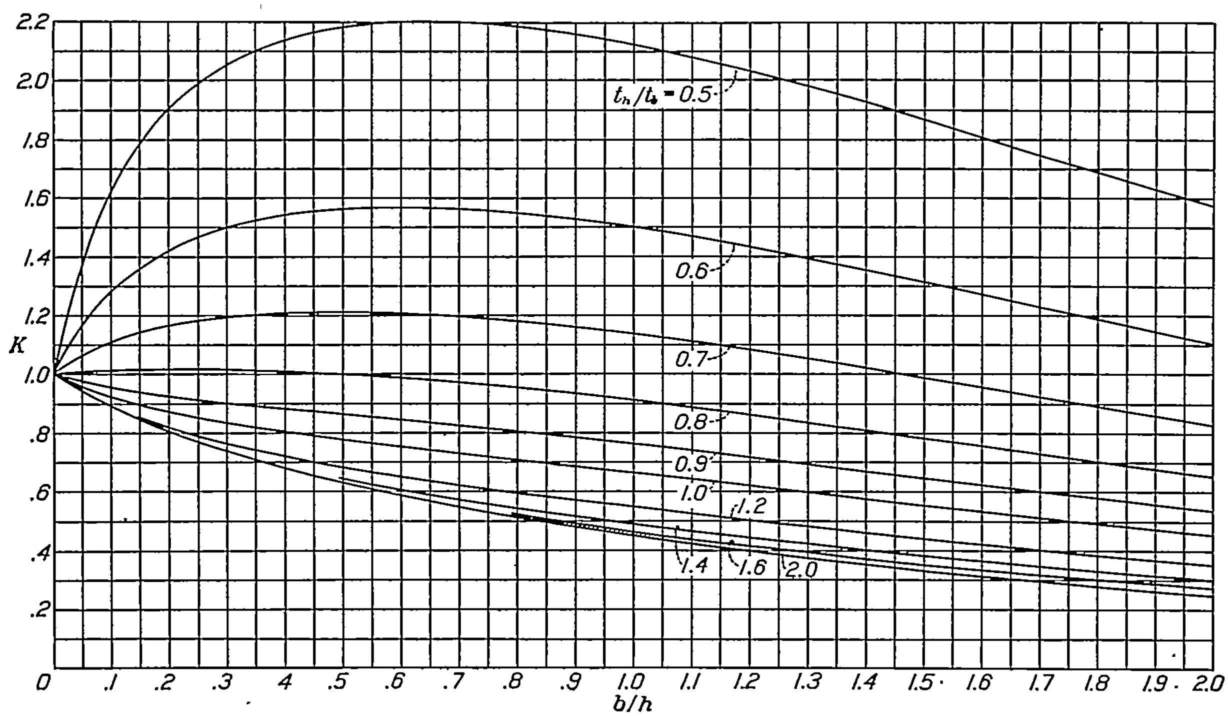

FIGURE 13.—Variation of K with b/h for different values of t_h/t_b when the axis of rotation is at the intersection of the center lines of the web and flange of an I section column

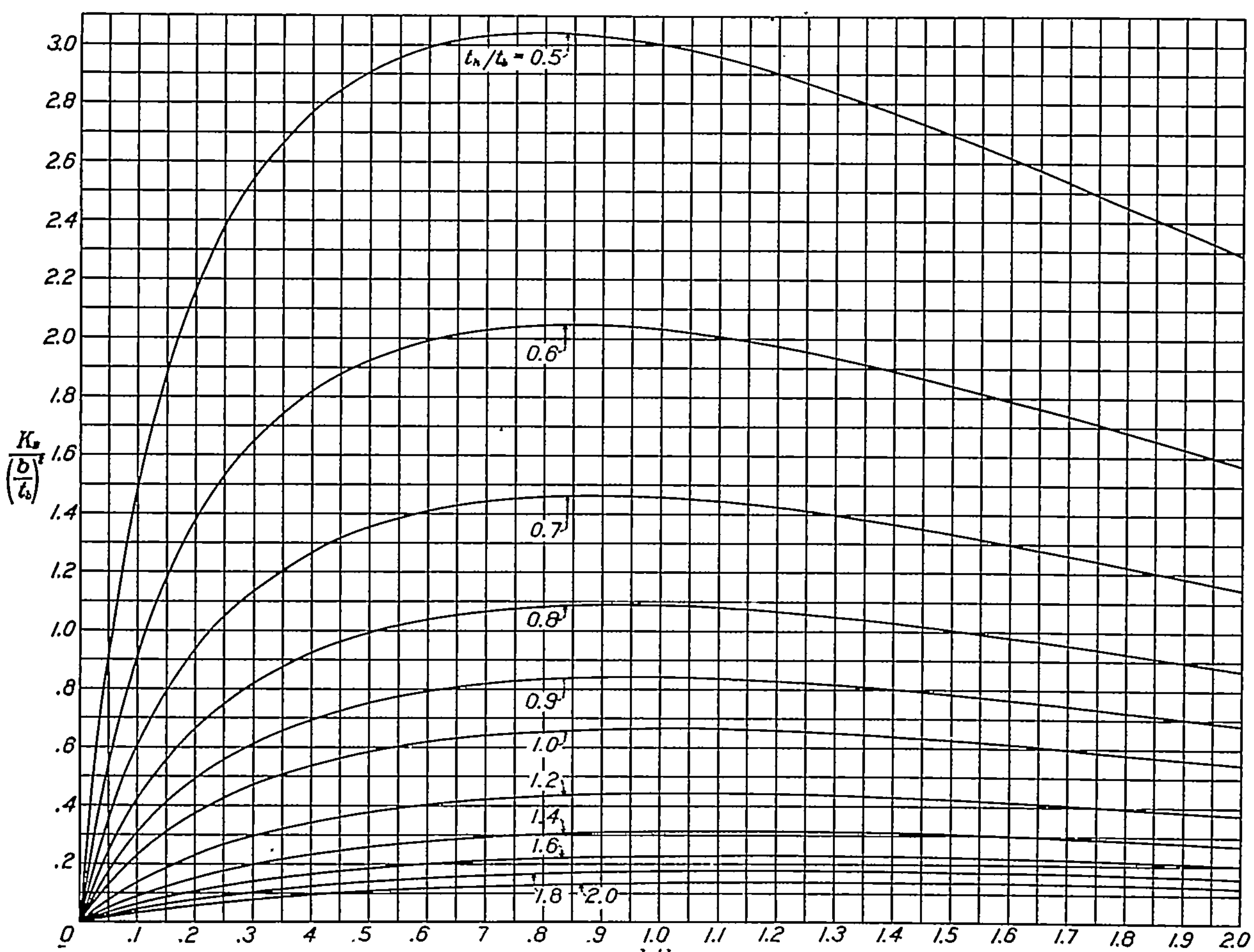

FIGURE 14.—Variation of $K_B/(b/t_b)^2$ with b/h for different values of t_h/t_b when the axis of rotation is at the intersection of the center lines of the web and flange of an I section column.

tions are free to have longitudinal displacements, L_0 cannot exceed the length L. For a skin approaching zero thickness L_0 will be equal to L. (See fig. 3 (b).) In general, however, $L_0=\frac{L}{n}$ where n has integral values ($n=1, 2, 3, 4$, etc.). Thus, when $L_0=\frac{L}{n}$ there will be a particular value of n for each skin-stiffener combination that will cause $f_{cr_{tt}}$ to be a minimum. A trial calculation should be made with $n=1, 2, 3, 4$, etc. to determine which value of n gives the lowest critical stress. This critical stress should then be compared with that for bending in a plane normal to the skin (reference 1) and the lower of these two stresses regarded as the stress at failure for the stiffener and its effective width of skin.

No information has thus far been given regarding the value of K_1 to be used in equation (26). For a stiffener that has one principal axis normal to the skin and that is also symmetrical about this principal axis, the value of K_1 may be taken from the curve given in figure 15 provided that the total compression load is equally divided among several stiffeners of the same dimensions spaced at equal intervals along the skin. This curve for K_1 was calculated by the energy method (reference 8, p. 584, art. 39) on the following assumptions:

(a) The full width of skin between stiffeners provides resistance to twisting of the stiffener.

(b) The skin is not under edge compression and is therefore flat until twisting of the stiffener occurs.

(c) When the stiffener twists, the skin takes the shape of a circular arc between stiffeners and a sine curve of half wave length L_0 parallel to the stiffeners.

Because the width of the effective skin that acts with the stiffener is small, any error that may result from assumption (a) is likely to be small. Of the three assumptions, (b) is probably the most questionable. Under load the skin is always subjected to edge compression and usually buckling of the skin occurs prior to twisting of the stiffeners. Because L_0 is usually several times the half wave length that forms when the skin alone buckles, any buckling of the skin prior to twisting of the stiffener tends to increase the effective thickness of the skin and hence the resistance of the skin to twisting of the stiffener. The increase in strength caused by the increase in effective thickness of the skin tends to offset any reduction in strength caused by the edge compression. The assumptions made under (c) are the most reasonable that could be made following (a) and (b) without greatly complicating the mathematics of the problem.

Until the curve for K_1 given in figure 15 has been checked by tests, it should be used only as a guide to design. As such, it will point the direction toward a ore efficient proportioning of material between skin and stiffeners. (See appendix A.) In the skin-stiffener combinations that are likely to be used in practice $\frac{L_0}{d}$ will usually be greater than 3. For these cases it will be satisfactory to use $K_1=2$, the asymptote for the curve of figure 15.

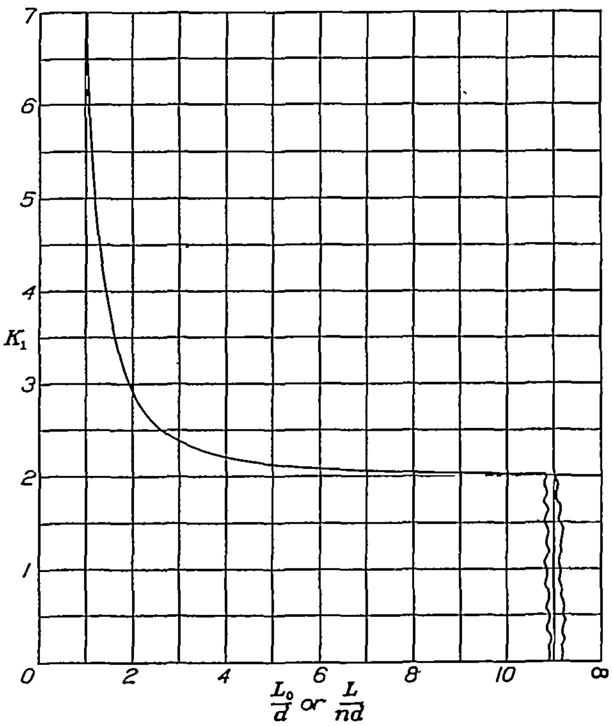

FIGURE 15.—Values of K_1 for use in equations (25) and (26).

$$K_1=\left[2+\frac{\pi^2}{3}\left(\frac{d}{L_0}\right)^2+\frac{\pi^4}{60}\left(\frac{d}{L_0}\right)^4\right]$$

Effective modulus of elasticity.—For columns that fail by bending, the critical stresses depart from the theoretical values given by the Euler formula at low values of the slenderness ratio. Consequently, an empirical straight line or parabolic curve is frequently drawn on the column chart to give the critical stress in this range. Likewise, for the general theory there will be a similar departure of the critical stress from the theoretical values given in this report and empirical curves must be found to give the strength for short lengths.

For a column that fails by bending, the reduced strength at short lengths is explained by the double-modulus theory of column action (reference 8, p. 572, art. 37, and references 9 and 10). This theory follows briefly: When a straight, centrally loaded column is stressed above the proportional limit for the material and deflected, the stress on the concave side increases according to the tangent modulus E' for the material (the slope of the stress-strain curve at the stress concerned) while the stress on the convex side decreases according to Young's modulus E for the material. The critical stress is then given by the Euler formula when an effective modulus $\overline{E}$ is substituted for E. The effective modulus is dependent upon the shape of the column cross section as well as upon E' and E and is given by the following general expression (references 9 and 10):

$$\overline{E}=\frac{E'I_1+EI_2}{I} \tag{27}$$

where, according to Osgood (reference 9), "I_1 is the moment of inertia about the axis of average stress [zero bending stress, see fig. 16] of the part of the cross-sectional area which suffers an increase of stress at the instant of failure of the column, I_2 is the moment of inertia about the axis of average stress of the part of the cross-sectional area which suffers a decrease of stress at the instant of failure of the column, and I is the moment of inertia of the total cross-sectional area of the column about the centroidal axis normal to the plane of bending. The position of the axis of average stress is defined by the relation $E'S_1=ES_2$ where S_1 and S_2 are the statical moments about the axis of average stress, respectively, of the two parts of the cross-sectional area just mentioned in connection with I_1 and I_2."

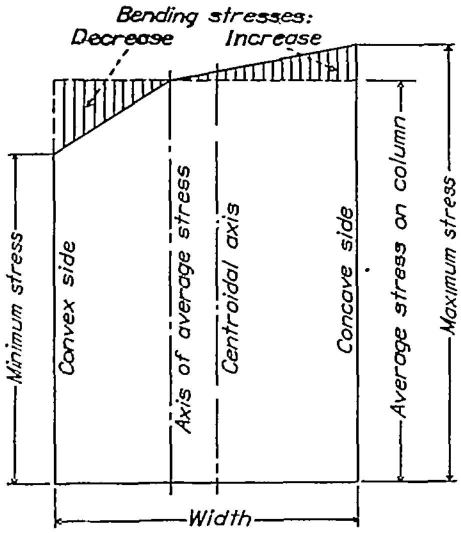

FIGURE 16.—Stress distribution for double-modulus theory.

The effective modulus has been evaluated for a number of cross sections. For a rectangular section (reference 4, p. 242, equation (161))

$$\overline{E}=\frac{4EE'}{(\sqrt{E}+\sqrt{E'})^2} \tag{28}$$

from which

$$\frac{\overline{E}}{E}=\frac{4\left(\frac{E'}{E}\right)}{\left(1+\sqrt{\frac{E'}{E}}\right)^2} \tag{29}$$

For an I section with a web of negligible thickness and with bending in the plane of the web (reference 9, equation (4))

$$\overline{E}=\frac{2EE'}{E+E'} \tag{30}$$

from which

$$\frac{\overline{E}}{E}=\frac{2\left(\frac{E'}{E}\right)}{1+\left(\frac{E'}{E}\right)} \tag{31}$$

In the theory for primary failure as herein presented there is a double-modulus action, similar to the double-modulus action in bending, when the column is stressed above the proportional limit for the material. In view of the fact that this double-modulus action is concerned only with longitudinal bending stresses, an effective modulus $\overline{E}$ will be substituted for E in the second term of equations (1) and (12). It is shown theoretically in appendix C that this value of $\overline{E}$ is

$$\overline{E}=\frac{E'C_{BT_1}+EC_{BT_2}}{C_{BT}} \tag{32}$$

where C_{BT_1} is the value obtained from equation (4) when the integration is made over the part of the cross section that suffers an increase of stress at the instant of failure of the column, C_{BT_2} is the value obtained from equation (4) when the integration is made over the part of the cross section that suffers a decrease of stress at the instant of failure of the column, and C_{BT} is the value obtained from equation (4) when the integration is made over the entire cross section as previously outlined. In order to locate the points of average stress (zero bending stress), which define the limits of integration for C_{BT_1} and C_{BT_2}, the reference plane must be so located that

$$E'\int D_1dA+E\int D_2dA=0 \tag{33}$$

where D_1 and D_2 are the longitudinal displacements used in the evaluation of C_{BT_1} and C_{BT_2}, respectively. Physically, equation (33) means that the summation of the forces on the cross section that result from the longitudinal displacements is zero.

When the column is stressed above the proportional limit for the material, the shear modulus G, which is related to E, must be corrected to correspond to the reduced modulus $\overline{E}$ for the column. A theoretical treatment of this problem does not appear to have been published. Bleich (reference 11) used for the effective shear modulus

$$\overline{G}=\sqrt{\tau}G \tag{34}$$

where

$$\tau=\frac{\overline{E}}{E} \tag{35}$$

It was reasoned that the percentage reduction in G was not so great as in E. Because τ is always equal to or less than unity, Bleich selected $\sqrt{\tau}\ G$ as a convenient expression for the effective shear modulus.

After analyzing the results of some 500 tests on angle columns where failure occurred by twisting, Kollbrunner (reference 12) concluded that the effective shear modulus was best given by the equation

$$\overline{G}=\frac{\tau+\sqrt{\tau}}{2}G \qquad (36)$$

As this value of $\overline{G}$ is based upon test data, it is recommended that it be used in preference to the value given by equation (34) to express the reduced shear modulus. Thus, when the column is stressed above the proportional limit, the value of $\overline{G}$ given by equation (36) should be substituted for G in the first term of equations (1) and (12).

When the axis of rotation is at infinity on either of the principal axes, equation (32) reduces to equation (27). It can be shown that, when the axis of rotation is at the centroid of an I section, the value of $\overline{E}$ is the same as when the axis of rotation is at infinity on the principal axis parallel to the web. For these two locations of the axis of rotation the value of $\overline{E}$ can conservatively be assumed to be the same as that given by equation (28) for the bending of a rectangular cross section. This close agreement is shown in figure 17 where values of $\frac{\overline{E}}{E}$ are plotted against $\frac{E'}{E}$.

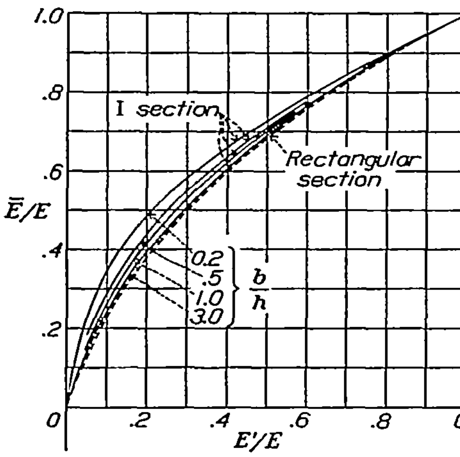

FIGURE 17.—Variation of $\overline{E}/E$ with E'/E for an I section column when the axis of rotation is at the centroid or at infinity on the principal axis parallel to the web.

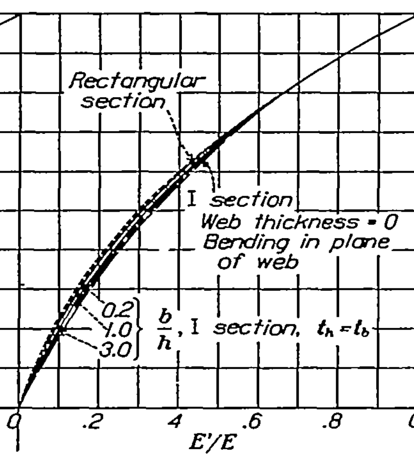

FIGURE 18.—Variation of $\overline{E}/E$ with E'/E for an I section column when the axis of rotation is at infinity on the principal axis normal to the web.

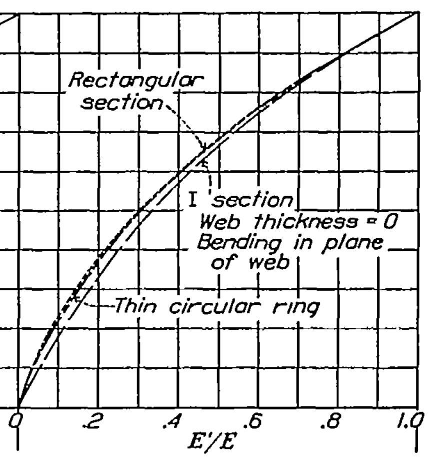

FIGURE 19.—Variation of $\overline{E}/E$ with E'/E for a rectangular section, a thin circular ring, and an I section in bending.

When the axis of rotation is at infinity on the principal axis normal to the web of an I section, the value of $\overline{E}$ will in all cases lie between that given by equations (28) and (30), as shown in figure 18. It will therefore be conservative to assume that $\overline{E}$ is given by equation (30) for this case.

When the axis of rotation is at the point where the principal axis crosses the skin, the considerations of the double-modulus action result in a lack of symmetry for the I section. This lack of symmetry may cause the critical stress to be a minimum when the axis of rotation is slightly shifted in the plane of the skin. Although a study of this condition has not been completed, it appears that the shift of the axis of rotation in the plane of the skin is small, for columns of practical dimension, and that the values of $\overline{E}$ are near those given by equations (28) and (30).

In figure 19 it is shown that the values of $\overline{E}$ as given by equations (28) and (30) are very nearly the same as the values for a thin circular ring or a tube. In view of this fact it appears justifiable for practical use to assume that $\overline{E}$ for the I section is the same as $\overline{E}$ for the thin-wall tube in bending. Dr. W. R. Osgood of the National Bureau of Standards suggested that the column curves constructed by the theory of this report be made consistent with the curves now used for tubes, which are determined from column tests, by evaluating $\overline{E}$ according to the following procedure:

1. Assume a series of values for the slenderness ratio $\frac{L_0}{\rho}$.
2. By means of the accepted column curve for tubes of the material under consideration, determine the critical stress f_{crit}.
3. Substitute the assumed values of $\frac{L_0}{\rho}$ and the corresponding values of f_{crit} in the following equation to obtain $\overline{E}$ and plot a curve of f_{crit} against $\overline{E}$:

$$\overline{E}=f_{crit}\frac{1}{\pi^2}\left(\frac{L_0}{\rho}\right)^2 \qquad (37)$$

4. Correct this value of $\overline{E}$ for the cross-sectional shape being used (figs. 17 to 19), if desired.

In the construction of a column curve for a particular I section, the following procedure should be used:

1. Select the location of the axis of rotation for which the column curve is to be drawn.
2. Assume a series of values of f_{crit}.
3. From the curve of $\overline{E}$ against f_{crit} previously derived, tabulate the values of $\overline{E}$ and $\overline{G}$ that correspond to the assumed values of f_{crit}.
4. Evaluate J, I_p, and C_{BT}.

5. Substitute J, I_p, C_{BT}, the assumed values of f_{crit}, and the corresponding values of $\overline{E}$ and $\overline{G}$ in equation (1) or (12) and solve for the length L_0.

6. The column curve is obtained by plotting the assumed values of f_{crit} against the computed lengths L_0.

If the column is attached to a skin, the values of J, I_p, and C_{BT} calculated under 4 should be increased by the amounts ΔJ, ΔI_p, and ΔC_{BT}, respectively. These values together with the assumed values of f_{crit} and the corresponding values of $\overline{E}$ and $\overline{G}$ are then substituted in equation (26), which is solved for the length L_0. A curve is then drawn by plotting the assumed values of f_{crit} against the computed values of L_0. This curve will be found to have a minimum point at some particular value of L_0. Because $L_0=\frac{L}{n}$, where n is an integral value ($n=1, 2, 3, 4$, etc.), the strength for any particular length L is obtained by choosing such a value of n as will cause the critical stress to be a minimum. (See appendix A.)

CONCLUSIONS

The following conclusions apply when primary column failure is defined as any type of failure in which the cross sections are translated, rotated, or both translated and rotated but not distorted.

1. When primary failure occurs in a pin-end column that is straight and centrally loaded, the general equation for the critical stress is

$$f_{crit}=\frac{\overline{G}J}{I_p}+\frac{C_{BT}}{I_p}\frac{\pi^2\overline{E}}{L_0^2}$$

In the derivation of this equation it is assumed that the cross sections rotate about an axis parallel to the column. The factors I_p and C_{BT} depend upon the location of this axis, which is called the "axis of rotation." The first term $\frac{\overline{G}J}{I_p}$ gives the critical stress for a pure twisting failure about the axis of rotation. The second term $\frac{C_{BT}}{I_p}\frac{\pi^2\overline{E}}{L_0^2}$ is in the nature of a correction for the effect of length caused by longitudinal bending stresses when the end cross sections are held against rotation. All possible combinations of translation and rotation of the column cross section are obtained by letting the location of the axis of rotation vary from zero to infinity in every direction.

2. The theory for primary failure shows that, for a free column with a cross section symmetrical about its two principal axes, the axis of rotation will be at either of the two following locations depending upon which location gives the lower stress:

(a) The center of twist, which is at the centroid of the section.

(b) Infinity on the principal axis about which the moment of inertia is the smaller.

Location (a) gives the condition for twisting failure; location (b), the condition for bending failure.

3. For a pin-end free column of I section symmetrical about its two principal axes the critical stress will be a minimum when the axis of rotation is at infinity on the principal axis parallel to the web, provided that the two following conditions are met:

$$\frac{t_h}{t_b}<14.7$$

and

$$\frac{b}{h}<\sqrt[3]{\frac{t_h}{t_b}}$$

When these conditions are not satisfied, the critical stress should be computed for the axis of rotation located at the centroid and compared with the critical stress for bending about the axis of minimum moment of inertia. The smaller of these two values will then be the stress at which failure occurs.

4. When a column is attached to a skin, the great stiffness of the skin in its own plane causes the axis of rotation to lie in the plane of the skin. When the column cross section is symmetrical about its two principal axes, one of which is normal to the skin, the axis of rotation will be at either of the two following locations depending upon which location gives the smaller stress:

(a) The point where the principal axis crosses the skin.

(b) Infinity in the plane of the skin.

Location (a) gives the condition for twisting failure when the column is attached to a skin; location (b), the condition for bending normal to the skin.

5. When a column is attached to a skin and the axis of rotation is at a point other than infinity in the plane of the skin, the rotation of the cross sections about the axis of rotation is resisted by the bending stiffness of the skin. The effect of this restraint is to increase the critical stress by an amount

$$\Delta f_{crit}=\frac{K_1Et_s^3}{6(1-\mu^2)dI_p}\frac{L^2}{n^2\pi^2}$$

and the critical stress becomes

$$f_{crit}=\frac{\overline{G}J}{I_p}+\frac{C_{BT}}{I_p}\frac{n^2\pi^2\overline{E}}{L^2}+\frac{K_1Et_s^3}{6(1-\mu^2)dI_p}\frac{L^2}{n^2\pi^2}$$

In this equation $n=1, 2, 3, 4$, etc., the number of half waves that develop in the stiffener in the length L. A trial calculation is necessary to determine which value of n gives the lowest critical stress. This critical stress should then be compared with that for bending in a plane normal to the skin and the lower of these two stresses regarded as the stress at failure for the stiffener and its effective width of skin.

6. When the column length is small, there will be a departure of the critical stresses from the theoretical values given by this theory that is similar to the departure from the Euler values in standard column curves. It is because of this fact that the effective moduli $\overline{E}$ and $\overline{G}$ have been substituted for E and G, respectively,

in certain terms of the equations for the critical stress. So long as the column is not stressed above the proportional limit, $\overline{E}$ and $\overline{G}$ are equal to E and G, respectively. Above the proportional limit the substitution of $\overline{E}$ for E follows from the double-modulus theory of bending where

$$\overline{E}=\frac{E'C_{BT_1}+EC_{BT_2}}{C_{BT}}$$

For the evaluation of G, the following empirical expression is recommended:

$$\overline{G}=\frac{\tau+\sqrt{\tau}}{2}G$$

where

$$\tau=\frac{\overline{E}}{E}$$

7. When the axis of rotation of a symmetrical I section column is at the center of twist (centroid) or at infinity on one of the principal axes, the value of $\overline{E}$ is very nearly the same as that for a thin-wall tube of the same material in bending. When the axis of rotation is at the point where the principal axis crosses the skin, the considerations of the double-modulus action result in a lack of symmetry for the I section. This lack of symmetry may cause the critical stress to be a minimum when the axis of rotation is slightly shifted in the plane of the skin. Although a study of this condition has not been completed, it appears that the shift of the axis of rotation in the plane of the skin is small for columns of practical dimensions and that the values of $\overline{E}$ are also near those for a thin-wall tube in bending.

8. The value of $\overline{E}$ varies with the critical stress and should be computed from the accepted column curve for the material by use of the following equation:

$$\overline{E}=f_{crit}\frac{1}{\pi^2}\left(\frac{L_0}{\rho}\right)^2$$

If desired, this value of $\overline{E}$ may be corrected for different cross-sectional shapes.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *August 17, 1936.*

APPENDIX A

ILLUSTRATIVE PROBLEM

Problem: To construct the column curve for an I section column of 24S-T aluminum-alloy material (E=10,537,000 pounds per square inch), with the dimensions shown in figure 20, used as a stiffener on skin

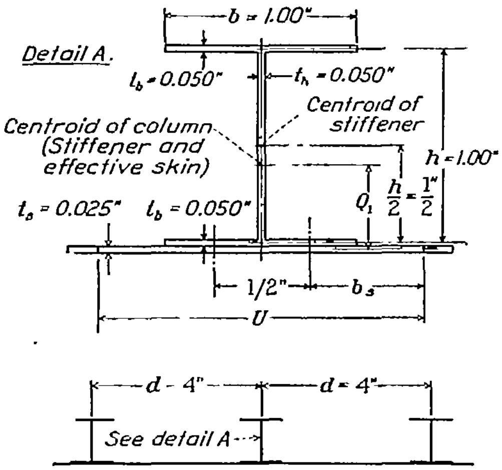

FIGURE 20.—A skin-stiffener combination.

0.025 inch thick. It is assumed that the stiffeners are spaced at 4-inch intervals along the skin and that all stiffeners are equally loaded in compression.

Effective moduli $\overline{E}$ and $\overline{G}$ for 24S-T aluminum alloy.— It is assumed that the pin-end column strength of 24S-T tubes is given by the straight-line equation

$$f_{crit}=58{,}000-527\frac{L_0}{\rho} \tag{38}$$

for values of the slenderness ratio $\frac{L_0}{\rho}$ between 9.5 and 73. Below $\frac{L_0}{\rho}$=9.5 it is assumed that the critical stress is 53,000 pounds per square inch. Above $\frac{L_0}{\rho}$=73 the stress is assumed to be given by the Euler formula

$$f_{crit}=\frac{\pi^2 E}{\left(\frac{L_0}{\rho}\right)^2} \tag{39}$$

The calculations for the effective moduli $\overline{E}$ and $\overline{G}$ are made as follows, the results of which are given in table I:

1. Assume a series of values of $\frac{L_0}{\rho}$

2. Compute f_{crit} from

$$f_{crit}=58{,}000-527\frac{L_0}{\rho} \text{ for } 9.5<\frac{L_0}{\rho}<73$$

$$f_{crit}=\frac{\pi^2 E}{\left(\frac{L_0}{\rho}\right)^2} \qquad \text{for } \frac{L_0}{\rho}>73$$

3. Using the computed values of f_{crit}, compute $\overline{E}$, from

$$\overline{E}=\frac{f_{crit}\left(\frac{L_0}{\rho}\right)^2}{\pi^2} \tag{37}$$

4. Compute τ from

$$\tau=\frac{\overline{E}}{E},\ E=10{,}537{,}000$$

5. Compute $\overline{G}$ from

$$\overline{G}=\left[\frac{\tau+\sqrt{\tau}}{2}\right]G,\ G=0.385E=4{,}057{,}000$$

Effective width of skin that acts with the column.— It is assumed that the column is attached to the skin with two lines of rivets one-half inch apart. The width of the skin between the rivet lines is therefore $20t_s$. The effective width outside the rivet lines is assumed to be given by the von Kármán equation for the effective width with the coefficient of 1.70, established in reference 1,

$$2b_s=1.70t_s\sqrt{\frac{E}{f_{crit}}} \tag{40}$$

Professor Joseph S. Newell and Mr. Walter H. Gale in an unpublished report of aircraft materials research at the Massachusetts Institute of Technology for 1931–32 recommend the value of 1.73 for the coefficient in the von Kármán equation.

As the width $20t_s$ between the two rivet lines is less than the smallest value of $2b_s$ given by equation (40) when f_{crit}= 53,000 pounds per square inch, all the material between the two rivet lines must be considered as effective and the total effective width of skin that acts with the column and carries the same stress as the column is

$$U=0.5+2b_s \tag{41}$$

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The effective width of skin is calculated as follows, the results of which are given in table II:

1. Assume a series of values of f_{crit}. (For convenience, use the same values as given in table I.)
2. Compute $2b_s$ by equation (40).
3. Compute U by equation (41).

Axis of rotation at infinity in the plane of the skin for bending failure.—In the report proper it has been shown that, when an I section column is attached to a skin, the axis of rotation will be either at infinity in the plane of the skin or at the point where the principal axis crosses the skin. The column curve must therefore be drawn for each location to determine which location gives the lower critical stress.

When the axis of rotation is at infinity in the plane of the skin, the critical stress is given by the Euler formula, equation (2) or (39), with $\overline{E}$ substituted for E. For this case $\frac{I}{A}$, equation (2), is calculated about a centroidal axis parallel to the skin considering the effective area of the skin Ut_s as a part of the column cross section. The calculations for the construction of the column curve are made as follows, the results of which are given in table III:

1. Assume a series of values of f_{crit}. (For convenience use the same values as in table I.)
2. Compute area of effective skin, Ut_s. (For U see table II.) $t_s=0.025$.
3. Compute total area of column cross section, from $A=A_{stiff}+A_U$
   where A_{stiff}=area of stiffener=0.15 sq. in.
   A_U=area of effective skin=0.025 U
4. Compute the centroid of the column cross section (including the effective skin) and tabulate the distance Q_1 from the center line of the skin to the centroid,

$$Q_1=\frac{A_{stiff}\left(\frac{h}{2}+\frac{t_b+t_s}{2}\right)}{A}=\frac{A_{stiff}\,(0.5375)}{A}\ \text{(See fig. 20.)}$$

5. Compute the moment of inertia, of the complete column cross section (area A), about the centroidal axis parallel to the skin

$$I=\frac{1}{12}t_h h^3+2bt_b\frac{h^2}{4}+[2bt_b+ht_h]\left[\frac{h}{2}+\frac{t_b+t_s}{2}-Q_1\right]^2+Ut_sQ_1^2$$

$$=0.004167+0.025+0.15\ (0.5375-Q_1)^2+Ut_sQ_1^2$$

6. From table I obtain the values of $\overline{E}$ that correspond to the assumed values of f_{crit}.
7. Compute the lengths L_0 that correspond to the assumed critical stresses by use of the Euler formula where $\overline{E}$ has replaced E,

$$L_0=\pi\sqrt{\frac{I}{A}\frac{\overline{E}}{f_{crit}}}$$

In figure 21 the assumed values of f_{crit} are plotted against the computed values of L_0. For a column with pin ends, $L_0=L$. Hence figure 21 is the column curve for the axis of rotation at infinity in the plane of the skin (bending failure). This direct calculation for obtaining the column curve when failure occurs by bending normal to the skin is preferable to the trial and error procedure recommended in reference 1.

Axis of rotation at the intersection of the center lines of the web and skin—twisting failure.—The calculation for the construction of the column curve when the axis of rotation is at the intersection of the center lines of the web and skin are similar to those for the axis of rotation at infinity in the plane of the skin. The calculations are made as follows; the results are given in table IV.

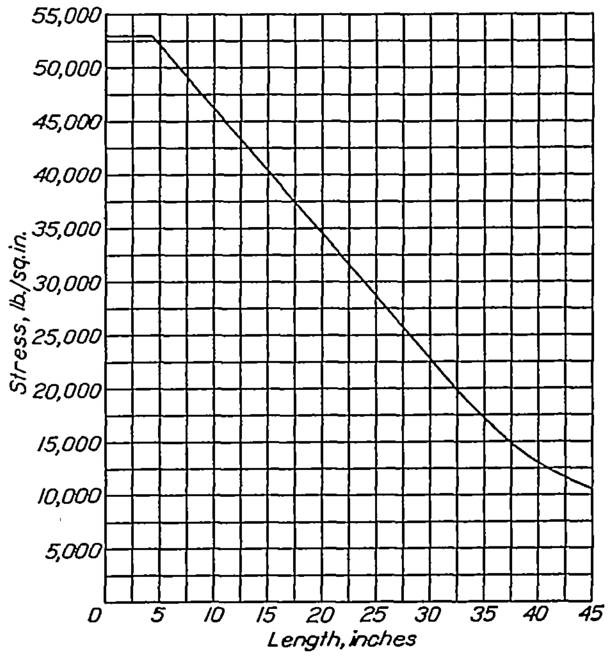

FIGURE 21.—The column curve for bending failure of the skin-stiffener combination shown in figure 20. The axis of rotation is at infinity in the plane of the skin.

1. Assume a series of values for f_{crit}. (For convenience use the same values as in table I.)
2. Compute ΔJ from

$$\Delta J=\frac{1}{3}Ut_s^3 \tag{21}$$

3. Compute J from

$$J=J_{stiff}+\Delta J$$
$$\text{where } J_{stiff}=\frac{1}{3}ht_h^3+\frac{2}{3}bt_b^3 \tag{13}$$

4. Compute ΔI_p from

$$\Delta I_p=\frac{1}{12}U^3t_s \tag{22}$$

5. Compute I_p from

$$I_p=I_{p_{stiff}}+\Delta I_p$$

where

$$I_{p_{stiff}}=\frac{1}{12}h^3t_h+\frac{1}{2}h^2bt_b+\frac{1}{6}b^3t_b+[ht_h+2bt_b]\,[P^2+Q^2] \tag{14}$$

(In the evaluation of equation (14), note that

$$P=0 \text{ and } Q=\frac{h}{2}+\frac{t_b+t_s}{2}=0.5375.)$$

6. Compute ΔC_{BT} from

$$\Delta C_{BT}=\Delta C_T=\frac{1}{144}U^3t_s^3 \tag{24}$$

7. Compute C_{BT} from

$$C_{BT}=C_{BT_{stiff}}+\Delta C_{BT}$$

where

$$C_{BT_{stiff}}=C_B+C_T \tag{6}$$

$$C_B=\frac{1}{24}b^3h^2t_b+\left[\frac{h^2bt_b}{2}+\frac{h^3t_h}{12}\right]P^2+\frac{b^3t_b}{6}Q^2 \tag{8}$$

$$C_T=\frac{b^3t_b^3}{72}+\frac{h^3t_h^3}{144}+\frac{bt_b^3}{6}P^2+\frac{ht_h^3}{12}Q^2 \tag{10}$$

(In the evaluation of equation (6), note that $P=0$ and $Q=0.5375$.)

8. From table I obtain the values of $\overline{E}$ and $\overline{G}$ that correspond to the assumed values of f_{crit}.

9. Solve equation (26) for L_0.

$$L_0=\sqrt{\frac{-\left[\frac{\overline{G}J}{I_p}-f_{crit}\right]\pm\sqrt{\left[\frac{\overline{G}J}{I_p}-f_{crit}\right]^2-4\left[\frac{C_{BT}\pi^2\overline{E}}{I_p}\right]\left[\frac{K_1Et_s^3}{6(1-\mu^2)d\pi^2I_p}\right]}}{\frac{2K_1Et_s^3}{6(1-\mu^2)d\pi^2I_p}}} \tag{42}$$

Evaluate equation (42) using values of J, I_p, C_{BT}, $\overline{G}$, and $\overline{E}$ that correspond to the assumed values of f_{crit}; and

$\mu=0.3$

$E=10,537,000$ lb. per sq. in.

$d=4$ in.

$t_s=0.025$ in.

$K_1=2$

In figure 22 the assumed values of f_{crit} are plotted against the computed values of L_0. From this figure the column curve for twisting failure is derived in the following manner. Put L_0 equal to $\frac{L}{n}$ and then plot curves of f_{crit} against L for $n=1, 2, 3, 4$, etc. The column curve is then given by the lowest portions of the several curves and is shown by full lines in figure 23.

Column curve for primary failure.—It has been previously shown that primary failure will occur either by bending or by twisting, depending upon which type of failure gives the lower critical stress. The column curves of figures 21 and 23 are therefore combined as shown in figure 24 to obtain the column curve for

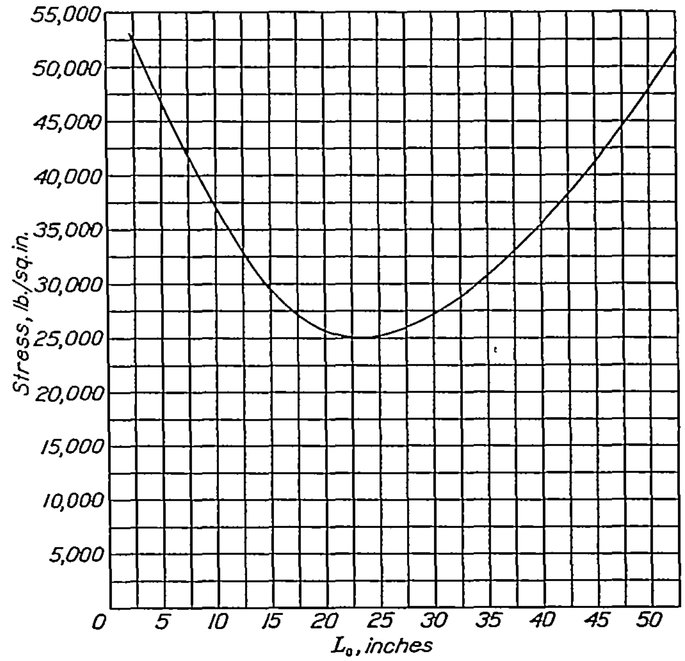

FIGURE 22.—Critical stress plotted against L_0 for twisting failure of the skin-stiffener combination shown in figure 20. The axis of rotation is at the intersection of the center lines of the web and the skin.

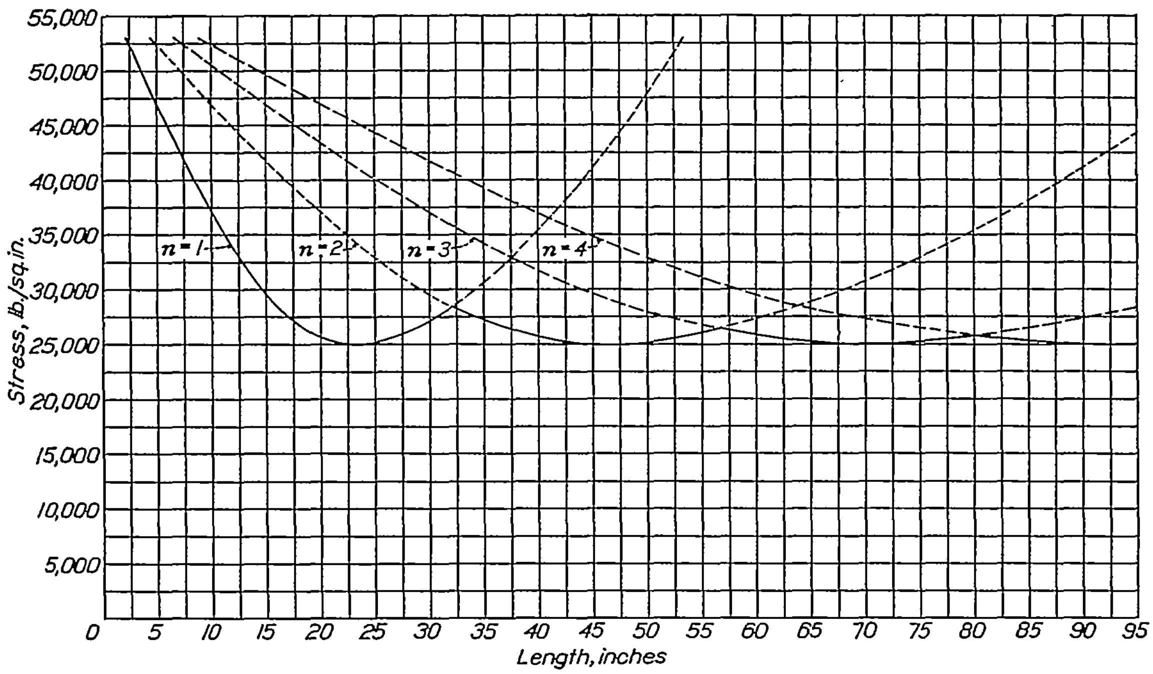

FIGURE 23.—The column curve for twisting failure of the skin-stiffener combination shown in figure 20. The axis of rotation is at the intersection of the web and the skin.

primary failure. It will be noted that, at lengths less than 27.4 inches, failure occurs by twisting; whereas, at lengths greater than 27.4 inches, failure occurs by bending.

Discussion.—In the computed tables for this illustrative problem it will be noted that some of the factors are small and might have been neglected. All of the factors, however, have been included to show their relative numerical values and the method of evaluation. The designer may therefore shorten the calculations here outlined by neglecting the unimportant factors, if desired.

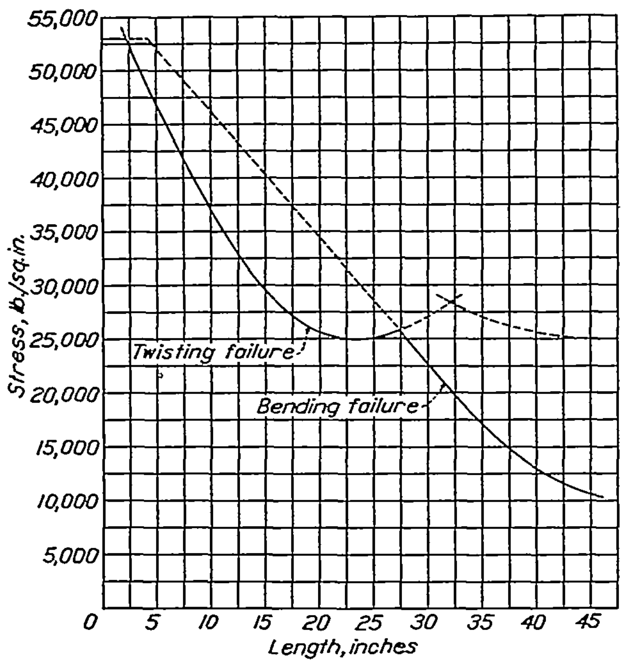

FIGURE 24.—The column curve for primary failure of the skin-stiffener combination shown in figure 20.

In the foregoing calculations for twisting failure it was assumed that $K_1=2$ regardless of the value of $\frac{L_0}{d}$. This value of K_1 was selected because of the possible uncertainty in establishing a more definite value, as discussed in this report. If it had been desired to use the values of K_1 given by the curve of figure 15 rather than the asymptotic value $K_1=2$, the calculation of L_0 would of necessity have been by trial and error because K_1 varies with $\frac{L_0}{d}$.

When a skin-stiffener combination is loaded in compression, buckling will first occur in the skin provided that the stiffener spacing divided by the skin thickness $\frac{d}{t_s}$ is sufficiently large. Because the skin is attached to the stiffeners, the buckling of the skin will twist the stiffeners and form small waves in them, the lengths of which are the same as those in the skin. In this condition the stiffeners are not ready to buckle of themselves but are forced to buckle by the skin. The stiffeners therefore resist buckling of the skin.

Now, if the load on the skin-stiffener combination is increased, the waves in the skin and the corresponding waves in the stiffeners grow larger. Finally a load is reached at which the stiffeners buckle of themselves. The type of buckling that occurs in the stiffeners will be that associated with the lowest critical stress. On the assumption that local buckling does not occur, the stiffeners will either buckle by deflection perpendicular to the skin in the manner of an ordinary column or will twist about an axis in the plane of the skin. If twisting occurs, the skin will resist twisting of the stiffeners. The column curves derived by the methods of this report give the critical stress at which the stiffeners begin to buckle (bend or twist) of themselves. Because the stiffeners are the main strength element in a skin-stiffener combination, it seems quite proper that the strength of the combination should be based on the strength of the stiffeners.

When the stiffeners fail by twisting, it is quite possible that tests will show the ultimate load for a skin-stiffener panel in compression to be greater than the critical load at which twisting begins. The reason for this belief is that when the stiffener twists, the material adjacent to the axis of rotation is not laterally displaced and is therefore capable of further compression. The amount by which the ultimate load will exceed the critical load at which buckling begins is dependent upon a number of factors the consideration of which is beyond the scope of this report.

Until the results of extensive tests made especially to check the theoretical behavior of skin-stiffener combinations in compression become available, the designer should conservatively assume that failure occurs when the buckling load is reached. The methods outlined in this report and illustrated in this appendix may therefore be used to derive column curves for different skin-stiffener combinations. By comparison of the strength-weight ratios the most efficient combination of skin and stiffeners can be selected.

APPENDIX B

APPLICATION OF THE THEORY FOR PRIMARY FAILURE TO A COLUMN OF CLOSED SECTION

Equation (1), which has heretofore been applied to columns of open section, can also be applied to columns of closed section provided that all the factors appearing on the right-hand side of the equality sign can be evaluated. It will be shown how these factors can be evaluated for a thin-wall column of closed rectangular section, symmetrical about its two principal axes. (See fig. 25.)

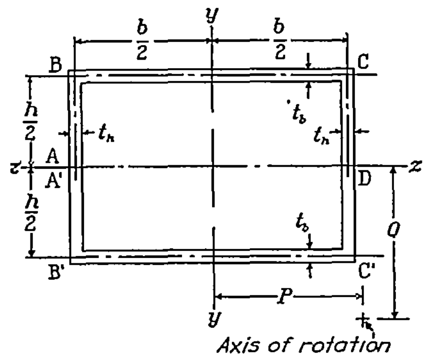

FIGURE 25.—A thin-wall rectangular tube.

Evaluation of GJ/I_p.—Except for J and C_{BT} all of the factors that enter into equation (1) are readily evaluated by standard methods. For the closed section

$$J=\frac{4A^2}{\int\frac{ds}{t}} \tag{43}$$

where A is the area enclosed by the center lines of the wall of the rectangular tube.

ds, differential element of the perimeter.

t, wall thickness of ds.

For a square tube of constant thickness equation (43) becomes

$$J=b^3t$$

Because the square tube is symmetrical about its two principal axes, the critical stress will be a minimum when the axis of rotation for the free column is either at the centroid (center of twist $P=0$, $Q=0$) or at infinity on one of the principal axes. The critical stress when the axis of rotation is at the centroid will be greater than that given by the first term of equation (1) or

$$(f_{crit})_{\substack{P=0\\Q=0}}>\frac{GJ}{I_p}=\frac{0.385Eb^3t}{\frac{4}{3}b^3t}=\frac{3}{4}(0.385E)$$

or, if $E=10^7$ pounds per square inch,

$$(f_{crit})_{\substack{P=0\\Q=0}}>2{,}885{,}000 \text{ pounds per square inch}$$

As this value of the critical stress is much greater than the yield-point stress for any engineering material with $E=10^7$ pounds per square inch, it may be concluded that the large torsional rigidity of a closed section precludes any type of primary failure except bending failure; i. e., axis of rotation at infinity on one of the principal axes.

Evaluation of C_{BT}.—In order to show that C_{BT} can be evaluated for a closed section, the expressions for the longitudinal displacement at the center lines of the wall of the tube will be derived. In view of the conclusion in the preceding paragraph, the value of this work will be more in the possibilities offered in the calculation of the stresses in monocoque shells, such as airplane wings, fuselages, floats, and hulls than in the solution of the column problem.

First, the longitudinal displacements caused by the twisting of the section about its centroid will be determined ($P=0$, $Q=0$ in fig. 25). If the tube is assumed to be slit longitudinally on the z axis at A—A′, the closed section becomes an open section. Now imagine a portion of length dx to be twisted an amount $d\varphi$ about the centroid (center of twist for the closed section). The longitudinal displacements of the points on the end cross section caused by such twisting can then be determined in the same manner as for an open section. These displacements with respect to the original plane of the end cross section are, at a distance s measured from

$$\left.\begin{array}{ll}
\text{B toward A,} & -\theta\left[\frac{3hb}{4}+\frac{bs}{2}\right]\\
\text{C toward B,} & -\theta\left[\frac{hb}{4}+\frac{hs}{2}\right]\\
\text{D toward C,} & -\theta\left[\frac{bs}{2}\right]\\
\text{D toward C}', & \theta\left[\frac{bs}{2}\right]\\
\text{C}' \text{ toward B}', & \theta\left[\frac{hb}{4}+\frac{hs}{2}\right]\\
\text{B}' \text{ toward A}', & \theta\left[\frac{3hb}{4}+\frac{bs}{2}\right]
\end{array}\right\} \tag{LD—6}$$

The longitudinal displacement of A (just above the slit) is

$$-\theta[hb]$$

and of A′ (just below the slit) is

$$\theta[hb]$$

The longitudinal displacement of A′ with respect to A is therefore

$$\theta[2hb]$$

In order to transform the open section, slit at A—A′, into a closed section, equal and opposite shearing forces F are introduced in the slit to draw A and A′ together. The magnitude of these shearing forces is determined by equating the integral of the shear strain in the section between A and A′ to the longitudinal displacement of A′ with respect to A when the section is slit

$$\int_A^{A'}\frac{F}{tG}\frac{ds}{dx}=\theta[2hb]$$

which becomes for the section shown in figure 25

$$\frac{2F}{G}\frac{1}{dx}\left[\frac{h}{t_h}+\frac{b}{t_b}\right]=\theta[2hb]$$

from which

$$\frac{F}{dx}=\theta\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}G \qquad (44)$$

The longitudinal displacement with respect to the original plane of the end cross section caused by the shearing force F in the slit is at a distance s measured from

$$\left.\begin{aligned}
&\text{B toward A,} && \frac{F}{dx}\frac{1}{G}\left[\frac{1}{2}\frac{h}{t_h}+\frac{b}{t_b}+\frac{s}{t_h}\right]\\
&\text{C toward B,} && \frac{F}{dx}\frac{1}{G}\left[\frac{1}{2}\frac{h}{t_h}+\frac{s}{t_b}\right]\\
&\text{D toward C,} && \frac{F}{dx}\frac{1}{G}\left[\frac{s}{t_h}\right]\\
&\text{D toward C}', && -\frac{F}{dx}\frac{1}{G}\left[\frac{s}{t_h}\right]\\
&\text{C}'\text{ toward B}', && -\frac{F}{dx}\frac{1}{G}\left[\frac{1}{2}\frac{h}{t_h}+\frac{s}{t_b}\right]\\
&\text{B}'\text{ toward A}', && -\frac{F}{dx}\frac{1}{G}\left[\frac{1}{2}\frac{h}{t_h}+\frac{b}{t_b}+\frac{s}{t_h}\right]
\end{aligned}\right\}\text{(LD}-7)$$

Adding of these longitudinal displacements to those of (LD−6) and substituting the value of F/dx from equation (44) gives at a distance s measured from

$$\left.\begin{aligned}
&\text{B toward A,} && -\theta\left[\frac{3hb}{4}+\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{b}{t_b}+\frac{s}{t_h}\right)\right]\\
&\text{C toward B,} && -\theta\left[\frac{hb}{4}+\frac{hs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{s}{t_b}\right)\right]\\
&\text{D toward C,} && -\theta\left[\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{s}{t_h}\right)\right]\\
&\text{D toward C}', && \theta\left[\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{s}{t_h}\right)\right]\\
&\text{C}'\text{ toward B}', && \theta\left[\frac{hb}{4}+\frac{hs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{s}{t_b}\right)\right]\\
&\text{B}'\text{ toward A}', && \theta\left[\frac{3hb}{4}+\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{b}{t_b}+\frac{s}{t_h}\right)\right]
\end{aligned}\right\}\text{(LD-8)}$$

The longitudinal displacements of (LD-8) apply to the closed section of figure 25 when the portion of length dx is twisted an amount $d\varphi$ about the centroid. If the axis of rotation is now shifted from the centroid to the location defined by P and Q, in figure 25, certain terms must be added to (LD-8) that are analogous to the longitudinal displacements of (LD-2) and (LD-3) for the I section. These longitudinal displacements caused by translation are, at a distance s measured from

$$\left.\begin{aligned}
&\text{B toward A,} && \theta\left[P\left(\frac{h}{2}-s\right)-\frac{Qb}{2}\right]\\
&\text{C toward B,} && \theta\left[P\frac{h}{2}+Q\left(\frac{b}{2}-s\right)\right]\\
&\text{D toward C,} && \theta\left[Ps+Q\frac{b}{2}\right]\\
&\text{D toward C}', && \theta\left[-Ps+Q\frac{b}{2}\right]\\
&\text{C}'\text{ toward B}', && \theta\left[-P\frac{h}{2}+Q\left(\frac{b}{2}-s\right)\right]\\
&\text{B}'\text{ toward A}', && \theta\left[-P\left(\frac{h}{2}-s\right)-Q\frac{b}{2}\right]
\end{aligned}\right\}\text{(LD-9)}$$

Addition of the longitudinal displacements given by equations (LD–8) and (LD–9) give at a distance s measured from

$$\left.\begin{array}{ll}
\text{B toward A,} & -\theta\left[\frac{3hb}{4}+\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{b}{t_b}+\frac{s}{t_h}\right)-P\left(\frac{h}{2}-s\right)+\frac{Qb}{2}\right] \\
\text{C toward B,} & -\theta\left[\frac{hb}{4}+\frac{hs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{s}{t_b}\right)-\frac{Ph}{2}-Q\left(\frac{b}{2}-s\right)\right] \\
\text{D toward C,} & -\theta\left[\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{s}{t_h}\right)-Ps-\frac{Qb}{2}\right] \\
\text{D toward C}', & \theta\left[\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{s}{t_h}\right)-Ps+\frac{Qb}{2}\right] \\
\text{C}'\text{ toward B}', & \theta\left[\frac{hb}{4}+\frac{hs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{s}{t_b}\right)-\frac{Ph}{2}+Q\left(\frac{b}{2}-s\right)\right] \\
\text{B}'\text{ toward A}', & \theta\left[\frac{3hb}{4}+\frac{bs}{2}-\frac{hb}{\frac{h}{t_h}+\frac{b}{t_b}}\left(\frac{h}{2t_h}+\frac{b}{t_b}+\frac{s}{t_h}\right)-P\left(\frac{h}{2}-s\right)-\frac{Qb}{2}\right]
\end{array}\right\}\text{(LD–10)}$$

Because the rectangular tube of figure 25 is symmetrical about its two principal axes the reference plane coincides with the original plane of the end cross section. (See derivation of C_{BT} for the I section.) Hence, (LD–10) gives the longitudinal displacements with respect to the reference plane. These longitudinal displacements when substituted for u in equation (7) with $\theta=1$ give the major part of C_{BT}. The minor part of C_{BT} is calculated in the same manner as for an open section.

APPENDIX C

DERIVATION OF THE THEORETICAL VALUE OF THE EFFECTIVE MODULUS $\overline{E}$

If C_{BT_1} is the value obtained from equation (4) when the integration is made over the part of the cross section that suffers an increase of stress at the instant of failure of the column, and E' is the modulus of elasticity for increasing stress, the work done by the increase in compressive stresses is (see equation (3) of reference 2)

$$\frac{1}{2}E'C_{BT_1}\int_0^L(\varphi'')^2dx$$

If C_{BT_2} is the value obtained from equation (4) when the integration is made over the part of the cross section that suffers a decrease of stress at the instant of failure of the column, and E is the modulus of elasticity for decreasing stress, the work done by the decrease in compressive stresses is

$$\frac{1}{2}EC_{BT_2}\int_0^L(\varphi'')^2dx$$

The total work done by the longitudinal bending stresses is therefore

$$\frac{1}{2}(E'C_{BT_1}+EC_{BT_2})\int_0^L(\varphi'')^2dx \qquad \text{(a)}$$

When the modulus of elasticity is the same for increasing stress as for decreasing stress, as it is in the elastic range, the total work done by the longitudinal bending stresses is

$$\frac{1}{2}EC_{BT}\int_0^L(\varphi'')^2dx \qquad \text{(b)}$$

If a modulus $\overline{E}$ is substituted for E in this expression, the total work given by expression (b) can be made to have any desired value depending upon the value assigned to $\overline{E}$. If $\overline{E}$ is allowed to have only such values as will cause the total work given by (b) to equal that given by (a), it is found that

$$\overline{E}=\frac{E'C_{BT_1}+EC_{BT_2}}{C_{BT}}$$

This value of $\overline{E}$ is called the "effective" modulus when the column is loaded above the proportional limit.

The total work done by the longitudinal bending stresses when the column is loaded above the proportional limit is therefore given by the expression

$$\frac{1}{2}\overline{E}C_{BT}\int_0^L(\varphi'')^2dx$$

Thus when the column is loaded above the proportional limit, $\overline{E}$ should be substituted for E in Wagner's equation for the critical stress, i. e., equation (1) of this report.

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TABLE I

EFFECTIVE MODULI $\overline{E}$ AND $\overline{G}$ FOR 24ST ALUMINUM ALLOY

$\frac{L_0}{\rho}$	f_{crit} lb./sq. in.	$\overline{E}$ lb./sq. in.	τ	$\frac{\tau+\sqrt{\tau}}{2}$	$\overline{G}$ lb./sq. in.
9.49	53,000	483,600	0.0459	0.1301	527,600
13.28	51,000	911,300	.0865	.1903	771,000
17.08	49,000	1,453,000	.1379	.2382	966,400
20.87	47,000	2,074,000	.1968	.3203	1,299,000
24.67	45,000	2,775,000	.2633	.3883	1,575,000
28.46	43,000	3,529,000	.3349	.4568	1,853,000
32.26	41,000	4,323,000	.4103	.5254	2,132,000
36.05	39,000	5,135,000	.4874	.5891	2,390,000
39.85	37,000	5,953,000	.5650	.6583	2,671,000
43.64	35,000	6,754,000	.6409	.7208	2,924,000
47.44	33,000	7,525,000	.7141	.7796	3,163,000
51.23	31,000	8,244,000	.7823	.8334	3,381,000
55.03	29,000	8,898,000	.8444	.8817	3,577,000
58.82	27,000	9,465,000	.8982	.9230	3,744,000
60.72	26,000	9,713,000	.9215	.9407	3,817,000
62.62	25,000	9,933,000	.9426	.9503	3,881,000
66.41	23,000	10,278,000	.9754	.9815	3,982,000
70.21	21,000	10,489,000	.9954	.9965	4,043,000
73.00	19,520	10,537,000	1.0000	1.0000	4,057,000
75.00	18,490	10,537,000	1.0000	1.0000	4,057,000
80.00	16,250	10,537,000	1.0000	1.0000	4,057,000
85.00	14,360	10,537,000	1.0000	1.0000	4,057,000
90.00	12,840	10,537,000	1.0000	1.0000	4,057,000
95.00	11,520	10,537,000	1.0000	1.0000	4,057,000
100.00	10,400	10,537,000	1.0000	1.0000	4,057,000

TABLE II

EFFECTIVE WIDTH OF SKIN THAT ACTS WITH THE COLUMN

f_{crit} lb./sq. in.	$2b_e$ inches	U inches
53,000	0.599	1.099
51,000	.611	1.111
49,000	.623	1.123
47,000	.636	1.136
45,000	.650	1.150
43,000	.665	1.165
41,000	.681	1.181
39,000	.699	1.199
37,000	.717	1.217
35,000	.737	1.237
33,000	.759	1.259
31,000	.784	1.284
29,000	.810	1.310
27,000	.840	1.340
26,000	.856	1.356
25,000	.873	1.373
23,000	.910	1.410
21,000	.952	1.452
19,520	.988	1.488
18,490	1.015	1.515
16,250	1.082	1.582
14,390	1.150	1.650
12,840	1.218	1.718
11,520	1.285	1.785
10,400	1.353	1.853

TABLE III

CRITICAL STRESS FOR BENDING FAILURE

f_{crit} lb./sq. in.	Ut_e sq. in.	A sq. in.	Q_1 inch	I in.^{4}	$\overline{E}$ lb./sq. in.	L_0 inches
53,000	0.0275	0.1775	0.4542	0.0359	483,600	4.27
51,000	.0278	.1778	.4535	.0359	911,300	5.97
49,000	.0281	.1781	.4527	.0360	1,453,000	7.60
47,000	.0284	.1784	.4519	.0361	2,074,000	9.38
45,000	.0288	.1788	.4509	.0362	2,775,000	11.09
43,000	.0291	.1791	.4502	.0362	3,529,000	12.80
41,000	.0295	.1795	.4492	.0363	4,323,000	14.51
39,000	.0300	.1800	.4479	.0364	5,135,000	16.21
37,000	.0304	.1804	.4469	.0365	5,953,000	17.92
35,000	.0309	.1809	.4457	.0366	6,754,000	19.62
33,000	.0315	.1815	.4442	.0367	7,525,000	21.33
31,000	.0321	.1821	.4428	.0368	8,244,000	23.03
29,000	.0328	.1828	.4411	.0369	8,898,000	24.73
27,000	.0335	.1835	.4394	.0371	9,465,000	26.44
26,000	.0339	.1839	.4384	.0372	9,713,000	27.29
25,000	.0343	.1843	.4375	.0372	9,933,000	28.15
23,000	.0353	.1853	.4351	.0374	10,278,000	29.84
21,000	.0363	.1863	.4328	.0376	10,489,000	31.55
19,520	.0372	.1872	.4307	.0378	10,537,000	32.79
18,490	.0379	.1879	.4291	.0379	10,537,000	33.68
16,250	.0396	.1896	.4252	.0382	10,537,000	35.91
14,390	.0413	.1913	.4215	.0385	10,537,000	38.15
12,840	.0430	.1930	.4177	.0388	10,537,000	40.36
11,520	.0446	.1946	.4143	.0391	10,537,000	42.60
10,400	.0463	.1963	.4107	.0394	10,537,000	45.80

TABLE IV.—CRITICAL STRESS FOR TWISTING FAILURE

f_{crit} lb./sq. in.	ΔJ in.^{4}	J in.^{4}	ΔI_p in.^{4}	I_p in.^{4}	ΔC_{BT} in.^{6}	$^*C_{BT}$ in.^{6}	$\overline{G}$ lb./sq. in.	$\overline{E}$ lb./sq. in.	L_0 in.	L_0 in.
53,000	0.0000057	0.0001307	0.00277	0.08360	0.0000001	0.00450	527,600	483,600	53.4	2.2
51,000	.0000058	.0001308	.00286	.08369	.0000001	.00450	771,900	911,300	52.1	3.1
49,000	.0000059	.0001309	.00295	.08379	.0000003	.00450	996,400	1,453,000	50.9	4.0
47,000	.0000059	.0001309	.00306	.08389	.0000002	.00450	1,209,000	2,074,000	49.4	5.0
45,000	.0000060	.0001310	.00317	.08401	.0000002	.00450	1,575,000	2,775,000	48.0	5.9
43,000	.0000061	.0001311	.00330	.08413	.0000002	.00450	1,853,000	3,529,000	46.5	6.9
41,000	.0000062	.0001312	.00343	.08427	.0000002	.00450	2,132,000	4,328,000	44.9	7.9
39,000	.0000062	.0001312	.00359	.08442	.0000002	.00450	2,390,000	5,135,000	43.2	8.9
37,000	.0000063	.0001313	.00376	.08459	.0000002	.00450	2,671,000	5,953,000	41.5	10.0
35,000	.0000064	.0001314	.00395	.08478	.0000002	.00450	2,924,000	6,754,000	39.6	11.2
33,000	.0000066	.0001316	.00416	.08500	.0000002	.00450	3,163,000	7,525,000	37.5	12.5
31,000	.0000067	.0001317	.00441	.08524	.0000002	.00450	3,381,000	8,244,000	35.3	13.9
29,000	.0000068	.0001318	.00468	.08552	.0000002	.00450	3,577,000	8,898,000	32.8	15.5
27,000	.0000070	.0001320	.00501	.08584	.0000003	.00450	3,744,000	9,465,000	29.7	17.7
26,000	.0000071	.0001321	.00519	.08603	.0000003	.00450	3,817,000	9,713,000	27.7	19.2
25,000	.0000072	.0001322	.00539	.08622	.0000003	.00450	3,881,000	9,933,000	23.6	22.8
23,000	.0000073	.0001323	.00584	.08667	.0000003	.00450	3,982,000	10,278,000	Imaginary	

$^*C_B=0.00149$; $C_T=0.000006$.