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COMPRESSIBLE FLOW ABOUT SYMMETRICAL JOUKOWSKI PROFILES

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SUMMARY

The method of Poggi is employed for the determination of the effects of compressibility upon the flow past an obstacle. A general expression for the velocity increment due to compressibility is obtained. This general result holds whatever the shape of the obstacle; but, in order to obtain the complete solution, it is necessary to know a certain Fourier expansion of the square of the velocity of flow past the obstacle. An application is made to the case of a symmetrical Joukowski profile with a sharp trailing edge, fixed in a stream of velocity v_0 at an arbitrary angle of attack and with the circulation determined by the Kutta condition. The results are obtained in a closed form and are exact insofar as the second approximation to the compressible flow is concerned, the first approximation being the result for the corresponding incompressible flow. Formulas for the lift and moment analogous to the Blasius formulas in incompressible flow are developed and are applied to thin symmetrical Joukowski profiles for small angles of attack.

Since actual experimental data for Joukowski profiles are lacking, the theoretical results are applied to a thin and a thick profile at zero angle of attack, and the velocity and pressure distributions are calculated and compared with those for the corresponding incompressible cases. The critical values for the ratio of the stream velocity v_0 to the velocity of sound in the stream c_0, corresponding to the attainment of the local velocity of sound c by the fluid on the surface of the airfoils, are also obtained.

INTRODUCTION

When a compressible fluid streams past a fixed body with a velocity small enough so that nowhere in the fluid is the local velocity of sound exceeded, the flow may be represented by a velocity potential. The effect of compressibility is to distort the streamline picture associated with the corresponding incompressible flow. This distortion has been calculated by Janzen (reference 1) and Rayleigh (reference 2) for circular cylinders and spheres and recently for elliptical cylinders by Hooker (reference 3). The methods used by these authors, however, are not feasible for the determination of the flow about obstacles other than the simple ones mentioned. On the other hand, a method introduced by Poggi (reference 4) may be used in determining the flow about shapes resembling airfoil profiles.

The method due to Poggi is as follows: When the fluid is compressible, the equation of continuity may be written as

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=-\frac{1}{\rho}\frac{D\rho}{Dt} \tag{1}$$

where the symbol D/Dt denotes, as usual, the operator $\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}$; v_x, v_y, the fluid velocity components; and ρ, the variable density of the fluid.

This divergence will introduce extra terms in the expressions for the velocity components, the divergence at an element $dxdy$ being equivalent to a simple source of strength $\frac{-\frac{1}{\rho}\frac{D\rho}{Dt}}{2\pi}dxdy$. Poggi thus replaces the compressible flow by an incompressible flow due to a suitable distribution of sources throughout the region of flow.

If the motion of the fluid is steady, then the equation of continuity and Euler's differential equations of motion become:

$$\left.\begin{aligned}\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}&=-\frac{v_x}{\rho}\frac{\partial \rho}{\partial x}-\frac{v_y}{\rho}\frac{\partial \rho}{\partial y}\\ \text{and}\qquad v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}\\ v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\} \tag{2}$$

Assuming the pressure p to be a function of the density ρ only and introducing the local velocity of sound $c\left(=\sqrt{\frac{dp}{d\rho}}\right)$, equations (2) yield the following:

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=\frac{1}{c^2}\left[v_x^2\frac{\partial v_x}{\partial x}+v_xv_y\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+v_y^2\frac{\partial v_y}{\partial y}\right]$$

or if

$$v^2=v_x^2+v_y^2$$

then

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=\frac{1}{2c^2}\left(v_x\frac{\partial v^2}{\partial x}+v_y\frac{\partial v^2}{\partial y}\right) \tag{3}$$

If it is further assumed that the motion of the fluid is irrotational, then

$$\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}=0$$

and a velocity potential ϕ may be introduced, where

$$v_x=-\frac{\partial\phi}{\partial x},\ v_y=-\frac{\partial\phi}{\partial y}$$

The strength of the source at a point (x, y), given by the expression on the right-hand side of equation (3), then becomes

$$-\frac{1}{4\pi c^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial v^2}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial v^2}{\partial y}\right)$$

Suppose now that (ξ, η) and (x, y) are the rectangular coordinates of points in the ζ and z planes, respectively, and furthermore that these two planes are conformally related, that is

$$\zeta=f(z)$$

where $\zeta=\xi+i\eta$, $z=x+iy$. Let the ζ plane be the plane of the profile and the z plane, the plane of the circle into which the profile is mapped by the foregoing conformal transformation. It is well known that, at a pair of corresponding points at which ζ and z possess no singularities, a source at one such point corresponds to

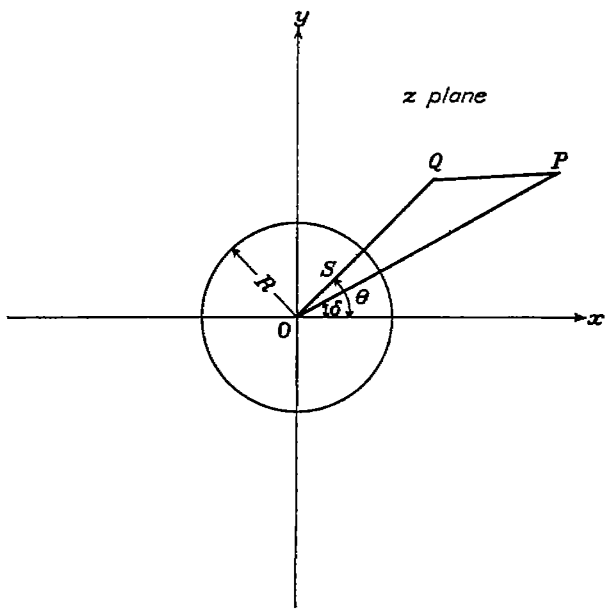

FIGURE 1.—Image of a simple source with regard to a circle.

a source of equal strength at the other. It follows then that at corresponding elements

$$-\frac{1}{4\pi c^2}\left(\frac{\partial\phi}{\partial \xi}\frac{\partial v^2}{\partial \xi}+\frac{\partial\phi}{\partial \eta}\frac{\partial v^2}{\partial \eta}\right)d\xi d\eta$$
$$=-\frac{1}{4\pi c^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial v^2}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial v^2}{\partial y}\right)dxdy \tag{4}$$

where, in the expression on the right-hand side, ϕ is the velocity potential in the z plane while v is the magnitude of the velocity in the ζ plane.

In polar coordinates (r, θ) the strength of a source at an element $dxdy$ of the z plane is

$$-\frac{1}{4\pi c^2}\left(\frac{\partial\phi}{\partial r}\frac{\partial v^2}{\partial r}+\frac{1}{r^2}\frac{\partial\phi}{\partial \theta}\frac{\partial v^2}{\partial \theta}\right)rdrd\theta$$

or, introducing a new variable $\lambda=\frac{R}{r}$ (where R is the radius of the circle into which the profile is mapped), and $v_r=-\frac{\partial\phi}{\partial r}$, $v_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}$, this expression becomes

$$\frac{1}{4\pi c^2}\left(v_r\frac{\partial v^2}{\partial\lambda}-\frac{v_\theta}{\lambda}\frac{\partial v^2}{\partial\theta}\right)\frac{R}{\lambda}d\lambda d\theta \tag{5}$$

With the source distribution known in the plane of the circle and given by equation (5), the induced tangential velocity at the circular boundary may be calculated.

Thus consider a unit source located at a point Q of the z plane. In the presence of a circular boundary of radius R, the velocity induced at any point P external to or on the boundary is given by

$$\frac{dw}{dz}=-\left(\frac{1}{z-z_Q}+\frac{1}{z-z_S}-\frac{1}{z}\right)$$

where S is the point inverse to Q in the circle. (See fig. 1.) Since the normal velocity at the boundary is zero, the velocity there is wholly tangential and is given by

$$\left(\frac{dw}{dz}\right)_{tan}=-\frac{2\lambda\sin(\theta-\delta)}{R[1-2\lambda\cos(\theta-\delta)+\lambda^2]} \tag{6}$$

where $z=Re^{i\delta}$; $z_Q=re^{i\theta}$; and $z_S=\frac{R^2}{r}e^{i\theta}$

Hence, the total velocity induced at any point of the circular boundary by the system of sources given by equation (5) is

$$\Delta v=\frac{1}{2\pi c_0^2}\int_0^1\int_0^{2\pi}\frac{v_r\frac{\partial v^2}{\partial\lambda}-\frac{v_\theta}{\lambda}\frac{\partial v^2}{\partial\theta}}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\sin(\theta-\delta)d\lambda d\theta \tag{7}$$

The justification for replacing c by c_0 in equation (7) may be shown in the following way. From the Euler equations of motion (2) and the condition for irrotational motion, it follows that

$$\frac{1}{2}dv^2+\frac{1}{\rho}dp=0$$

Then when adiabatic conditions prevail so that the relation between p and ρ is

$$p=p_0\left(\frac{\rho}{\rho_0}\right)^\gamma$$

it follows by integration that

$$c^2=c_0^2\left[1+\frac{\gamma-1}{2}\left(1-\frac{v^2}{v_0^2}\right)\frac{v_0^2}{c_0^2}\right] \tag{8}$$

where the zero subscripts denote the corresponding magnitudes in the undisturbed stream. From the foregoing equation it is seen that c has a maximum value at the stagnation point where $v=0$, that is

$$c_{max}^2=c_0^2\left(1+\frac{\gamma-1}{2}\frac{v_0^2}{c_0^2}\right)$$

Furthermore, as the streamline corresponding to the boundary of the obstacle is traversed, a point is reached where, for a definite value of the ratio v_0/c_0, the velocity of the fluid equals that of the local velocity of sound. This critical velocity is obtained from equation (8) by putting $c=v$ and solving for v. Thus

$$v_{crit}^2=\frac{2}{\gamma+1}c_0^2\left(1+\frac{\gamma-1}{2}\frac{v_0^2}{c_0^2}\right) \tag{9}$$

For example, let $v_0/c_0=0.75$. Then with $\gamma=1.408$ (for air)

$$c_{max}=1.056\ c_0$$

and

$$c_{min}=v_{crit}=0.962\ c_0$$

Away from the obstacle, v approaches v_0 and c approaches c_0. Thus it is seen that the variation of the local velocity of sound from c_0 is, in general, small enough to permit replacing c by c_0, at least to a first approximation.

Equation (7) is a functional equation for the fluid velocity v and may be solved by a method of successive substitutions. The procedure, due to Poggi, is to substitute for v_r, v_θ, and v^2 values pertaining to the corresponding incompressible flow and thus obtain a first approximation to the sink-source distribution in the plane of the circle. The method thus considers the incompressible flow to be the first approximation to the compressible flow. The second approximation is then obtained by superposing on the incompressible flow the effect of the sink-source distribution as given by equation (7); that is

$$v_{comp}=v_{incomp}+\Delta v \tag{10}$$

GENERAL DEVELOPMENTS

Before equation (7) is applied to any particular case, it is expedient to consider it first in a general way. Thus, suppose that v^2 can be developed in a Fourier series so that

$$\frac{v^2}{v_0^2}=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta) \tag{11}$$

where the a_n, b_n are functions of λ and also contain the parameters of the shape.

Then

$$\frac{\partial v^2}{\partial\lambda}=v_0^2\left[\frac{1}{2}a_0'+\sum_{n=1}^{\infty}(a_n'\cos n\theta+b_n'\sin n\theta)\right]$$

$$\frac{\partial v^2}{\partial\theta}=v_0^2\sum_{n=1}^{\infty}n[b_n\cos n\theta-a_n\sin n\theta]$$

where the primes denote differentiation with regard to λ. Expressions for v_r and v_θ are obtained from the complex potential of the flow past a circular cylinder of radius R, with the circulation determined by the Kutta condition. Let the stream of velocity v_0 make an angle α with the negative direction of the x axis. Then the potential is

$$w=v_0\left(ze^{i\alpha}+\frac{R^2}{ze^{i\alpha}}\right)+v_0R(e^{i\alpha}-e^{-i\alpha})\log\frac{z}{R}$$

and

$$\frac{dw}{dz}=v_0\frac{(z+R)(ze^{i\alpha}-Re^{-i\alpha})}{z^2} \tag{12}$$

Then

$$v_r=-v_0(1-\lambda^2)\cos(\theta+\alpha)=v_0(c_1\cos\theta+c_2\sin\theta)$$

and

$$\begin{aligned}v_\theta&=v_0(1+\lambda^2)\sin(\theta+\alpha)+2v_0\lambda\sin\alpha\\&=v_0\left(\frac{d_0}{2}+d_1\cos\theta+d_2\sin\theta\right)\end{aligned}$$

where

$$\begin{aligned}c_1&=-(1-\lambda^2)\cos\alpha\\c_2&=(1-\lambda^2)\sin\alpha\\\frac{d_0}{2}&=2\lambda\sin\alpha\\d_1&=(1+\lambda^2)\sin\alpha\\d_2&=(1+\lambda^2)\cos\alpha\end{aligned}$$

Therefore

$$\begin{aligned}v_r\frac{\partial v^2}{\partial\lambda}=&\frac{1}{2}v_0^3a_0'(c_1\cos\theta+c_2\sin\theta)\\&+\frac{1}{2}v_0^3\sum_{n=1}^{\infty}\{a_n'c_1[\cos(n-1)\theta+\cos(n+1)\theta]\\&+b_n'c_1[\sin(n+1)\theta+\sin(n-1)\theta]\\&+a_n'c_2[\sin(n+1)\theta-\sin(n-1)\theta]\\&+b_n'c_2[\cos(n-1)\theta-\cos(n+1)\theta]\}\end{aligned}$$

and

$$\begin{aligned}\frac{v_\theta}{\lambda}\frac{\partial v^2}{\partial\theta}=&-\frac{v_0^3}{2\lambda}\sum_{n=1}^{\infty}n\{a_nd_1[\sin(n+1)\theta+\sin(n-1)\theta]\\&-b_nd_1[\cos(n-1)\theta+\cos(n+1)\theta]\\&+a_nd_2[\cos(n-1)\theta-\cos(n+1)\theta]\\&-b_nd_2[\sin(n+1)\theta-\sin(n-1)\theta]\\&+a_nd_0\sin n\theta-b_nd_0\cos n\theta\}\end{aligned}$$

The following definite integrals will be found useful in evaluating equation (7):

$$\int_0^{2\pi}\frac{\sin(\theta-\delta)}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\sin n\theta\,d\theta=\begin{matrix}0 & \text{if } n=0\\ \pi\lambda^{n-1}\cos n\delta & \text{if } n\geqq 1\end{matrix}$$

$$\int_0^{2\pi}\frac{\sin(\theta-\delta)}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\cos n\theta\,d\theta=\begin{matrix}0 & \text{if } n=0\\ -\pi\lambda^{n-1}\sin n\delta & \text{if } n\geqq 1\end{matrix}$$

Then substituting the foregoing expressions for $v_r\frac{\partial v^2}{\partial\lambda}$ and $\frac{v_\theta}{\lambda}\frac{\partial v^2}{\partial\theta}$ into equation (7) and integrating with regard to θ, it follows, after replacing the derivatives a_n', b_n' by a_n, b_n by means of partial integrations, that

$$\Delta v=\frac{v_0^3}{4c_0^2}\Big[-(a_0)_{\lambda=0}\sin(\delta+\alpha)-(a_2)_{\lambda=0}\sin(\delta-\alpha)$$
$$+(b_2)_{\lambda=0}\cos(\delta-\alpha)-\sum_{n=2}^{\infty}2n\cos(n\delta+\alpha)\int_0^1\lambda^n b_{n-1}d\lambda$$
$$+\sum_{n=1}^{\infty}2n\cos(n\delta-\alpha)\int_0^1\lambda^{n-2}b_{n+1}d\lambda$$
$$+\sum_{n=1}^{\infty}2n\sin(n\delta+\alpha)\int_0^1\lambda^n a_{n-1}d\lambda$$
$$-\sum_{n=1}^{\infty}2n\sin(n\delta-\alpha)\int_0^1\lambda^{n-2}a_{n+1}d\lambda$$
$$+\sin\alpha\sum_{n=1}^{\infty}4n\sin n\delta\int_0^1\lambda^{n-1}b_n d\lambda$$
$$+\sin\alpha\sum_{n=1}^{\infty}4n\cos n\delta\int_0^1\lambda^{n-1}a_n\,d\lambda\Big] \qquad (13)$$

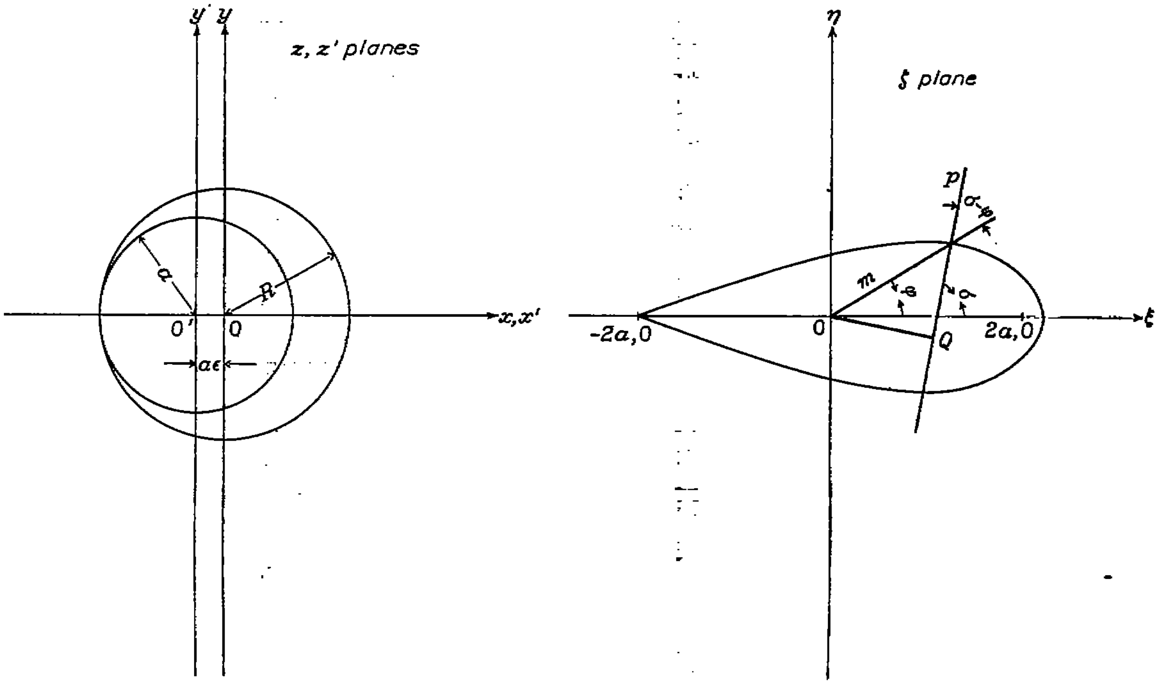

FIGURE 2.—Transformation of a symmetrical Joukowski profile into a circular contour.

It is to be especially noted that this expression for Δv is perfectly general and independent of the type of profile considered. All that is required for a complete solution of equation (13) is a knowledge of the Fourier development for v^2/v_0^2.

APPLICATION TO SYMMETRICAL JOUKOWSKI PROFILES

Suppose now that the boundary in the ζ plane is a symmetrical Joukowski profile with a sharp trailing edge. The Joukowski transformation given by

$$\zeta=z'+\frac{a^2}{z'} \qquad (14)$$

maps the circle of radius a with center at the origin 0′ of the z' plane into the line segment $(-2a,\ 0;\ 2a,\ 0)$ in the ζ plane, and the circle of radius R with center at the origin 0 of the z plane into a Joukowski profile with a sharp trailing edge in the ζ plane. (See fig. 2.) The distance 00′ is denoted by $a\epsilon$ and ϵ is therefore a measure of the thickness of the profile. Since the profile has a sharp trailing edge, the two circles touch at the corresponding point $(-a,\ 0)$. The relation between the z, z' planes is

$$z'=\epsilon a+z$$

If w denotes the complex potential of the incompressible flow in the ζ plane, then the complex velocity is given by

$$\frac{dw}{d\zeta}=\frac{dw}{dz}\frac{dz}{dz'}\frac{dz'}{d\zeta}$$

where

$$\frac{dz}{dz'}=1 \text{ and } \frac{dz'}{d\zeta}=\frac{z'^2}{(z'+a)(z'-a)}$$

According to equation (12)

$$\frac{dw}{dz}=v_0\frac{(z+R)(ze^{i\alpha}-Re^{-i\alpha})}{z^2}$$

Hence

$$\frac{dw}{d\zeta}=v_0\frac{(z+R)(ze^{i\alpha}-Re^{-i\alpha})z'^2}{z^2(z'+a)(z'-a)}$$

But

$$z'+a=z+a(1+\epsilon)=z+R$$
$$z'-a=z-a(1-\epsilon)$$

Therefore

$$\frac{dw}{d\zeta}=v_0\frac{(z+\epsilon a)^2(ze^{i\alpha}-Re^{-i\alpha})}{z^2[z-a(1-\epsilon)]}$$

Introducing

$$\lambda=\frac{R}{r},\ \frac{\epsilon a}{R}=h, \text{ and } \frac{a(1-\epsilon)}{R}=k$$

it follows that

$$v^2=\left|\frac{dw}{d\zeta}\right|^2=v_0^2\frac{(1+2h\lambda\cos\theta+h^2\lambda^2)^2[1-2\lambda\cos(\theta+2\alpha)+\lambda^2]}{1-2k\lambda\cos\theta+k^2\lambda^2}$$

It is now required to obtain the Fourier series for v^2/v_0^2. Thus, making use of the following developments:

$$\frac{\sin\theta}{1-2k\lambda\cos\theta+k^2\lambda^2}=\sum_{n=1}^{\infty}(k\lambda)^{n-1}\sin n\theta$$

$$\frac{\cos\theta-k\lambda}{1-2k\lambda\cos\theta+k^2\lambda^2}=\sum_{n=1}^{\infty}(k\lambda)^{n-1}\cos n\theta$$

$$\frac{1}{1-2k\lambda\cos\theta+k^2\lambda^2}=\frac{1}{1-k^2\lambda^2}\left[1+2\sum_{n=1}^{\infty}(k\lambda)^n\cos n\theta\right]$$

it follows that

$$\begin{aligned}\frac{v^2}{v_0^2}=(1+2h\lambda\cos\theta+h^2\lambda^2)^2\Bigg[&\frac{1+\lambda^2-2k\lambda^2\cos 2\alpha}{1-k^2\lambda^2}\\ &+2\lambda\sin 2\alpha\sum_{n=1}^{\infty}(k\lambda)^{n-1}\sin n\theta\\ &+2\frac{k(1+\lambda^2)-(1+k^2\lambda^2)\cos 2\alpha}{k(1-k^2\lambda^2)}\sum_{n=1}^{\infty}(k\lambda)^n\cos n\theta\Bigg]\end{aligned}$$

Reducing this expression to the form of a Fourier series, it turns out that, for $n>2$

$$\left.\begin{aligned}a_n&=2k^{n-3}(h+k)^2(1+hk\lambda^2)^2\lambda^n\\ &\quad\left[\cos 2\alpha+\frac{k(1+\lambda^2)-2\cos 2\alpha}{1-k^2\lambda^2}\right]\\ b_n&=2k^{n-3}(h+k)^2(1+hk\lambda^2)^2\lambda^n\sin 2\alpha\end{aligned}\right\}\qquad(15)$$

For later use it will be convenient to introduce the following notation for $n\geqq 0$:

$$\left.\begin{aligned}a_n^1&=2k^{n-3}(h+k)^2(1+hk\lambda^2)^2\lambda^n\cos 2\alpha\\ a_n^2&=2k^{n-3}(h+k)^2(1+hk\lambda^2)^2\lambda^n\frac{k(1+\lambda^2)-2\cos 2\alpha}{1-k^2\lambda^2}\\ \overline{b}_n&=2k^{n-3}(h+k)^2(1+hk\lambda^2)^2\lambda^n\sin 2\alpha\\ \overline{a}_n&=a_n^1+a_n^2\end{aligned}\right\}\qquad(16)$$

Also

$$\left.\begin{aligned}\tfrac{1}{2}a_0&=AD+(B+Ck\lambda)Fk\lambda\\ a_1&=2BD+(A+Bk\lambda+Ck^2\lambda^2)Fk\lambda\\ a_2&=2CD+(B+Ak\lambda+Bk^2\lambda^2+Ck^3\lambda^3)Fk\lambda\\ b_1&=E(A+Bk\lambda+Ck^2\lambda^2)\\ b_2&=E(B+Ak\lambda+Bk^2\lambda^2+Ck^3\lambda^3)\end{aligned}\right\}\qquad(17)$$

where

$$A=1+4h^2\lambda^2+h^4\lambda^4$$

$$B=2h\lambda(1+h^2\lambda^2)$$

$$C=h^2\lambda^2$$

$$D=\frac{1+(1-2k\cos 2\alpha)\lambda^2}{1-k^2\lambda^2}$$

$$E=2\lambda\sin 2\alpha$$

$$F=\frac{2(k-\cos 2\alpha)+2k(1-k\cos 2\alpha)\lambda^2}{k(1-k^2\lambda^2)}$$

It is seen that

$$(a_0)_{\lambda=0}=2 \text{ and } (a_2)_{\lambda=0}=(b_2)_{\lambda=0}=0$$

Equation (13) then becomes

$$\left.\begin{aligned}\Delta v=\frac{v_0^3}{4c_0^2}\Bigg[&-2\sin(\delta+\alpha)+2\cos(\delta-\alpha)\int_0^1(b_1-\overline{b}_1)d\lambda\\ &+4\cos(2\delta-\alpha)\int_0^1\lambda(b_2-\overline{b}_2)d\lambda\\ &-2\cos(\delta+\alpha)\int_0^1(b_1-\overline{b}_1)d\lambda\\ &-4\cos(2\delta+\alpha)\int_0^1\lambda(b_2-\overline{b}_2)d\lambda\\ &-4\cos(2\delta+\alpha)\int_0^1\lambda^2(b_1-\overline{b}_1)d\lambda\\ &-6\cos(3\delta+\alpha)\int_0^1\lambda^3(b_2-\overline{b}_2)d\lambda\\ &+2\cos(\delta+\alpha)\int_0^1\lambda\overline{b}_0d\lambda\\ &+2\sin(\delta+\alpha)\int_0^1(a_1-\overline{a}_1)d\lambda\\ &+4\sin(2\delta+\alpha)\int_0^1\lambda(a_2-\overline{a}_2)d\lambda\\ &-2\sin(\delta-\alpha)\int_0^1(a_1-\overline{a}_1)d\lambda\\ &-4\sin(2\delta-\alpha)\int_0^1\lambda(a_2-\overline{a}_2)d\lambda\\ &+2\sin(\delta+\alpha)\int_0^1\lambda(a_0-\overline{a}_0)d\lambda\\ &+4\sin(2\delta+\alpha)\int_0^1\lambda^2(a_1-\overline{a}_1)d\lambda\\ &+6\sin(3\delta+\alpha)\int_0^1\lambda^3(a_2-\overline{a}_2)d\lambda\\ &+2\cos(\delta-\alpha)\int_0^1\frac{1}{\lambda}(b_2-\overline{b}_2)d\lambda\\ &-2\sin(\delta-\alpha)\int_0^1\frac{1}{\lambda}(a_2-\overline{a}_2)d\lambda\\ &-\sum_{n=1}^{\infty}2n\cos(n\delta+\alpha)\int_0^1(\lambda^n\overline{b}_{n-1}+\lambda^{n-1}\overline{b}_n)d\lambda\\ &+\sum_{n=1}^{\infty}2n\cos(n\delta-\alpha)\int_0^1(\lambda^{n-2}\overline{b}_{n+1}+\lambda^{n-1}\overline{b}_n)d\lambda\\ &+\sum_{n=1}^{\infty}2n\sin(n\delta+\alpha)\int_0^1(\lambda^n\overline{a}_{n-1}+\lambda^{n-1}\overline{a}_n)d\lambda\\ &-\sum_{n=1}^{\infty}2n\sin(n\delta-\alpha)\int_0^1(\lambda^{n-2}\overline{a}_{n+1}+\lambda^{n-1}\overline{a}_n)d\lambda\Bigg]\end{aligned}\right\}\qquad(18)$$

where

$$\left.\begin{aligned}
a_0-\overline{a}_0&=\frac{2}{k^3}\{(h+k)^2(\cos 2\alpha-k)+k^3+[k(h^4+k^2)\\
&\quad -k(1+h^2)(h+k)^2+2hk(h+k)^2\cos 2\alpha\\
&\quad +2h^2k^2\cos 2\alpha]\lambda^2+[-h^2k(h+k)^2\\
&\quad +h^4k+h^2k^2(h+k)^2\cos 2\alpha]\lambda^4\}\\
a_1-\overline{a}_1&=\frac{2\lambda\cos 2\alpha}{k^2}[h(h+2k)+2h^3k\lambda^2]\\
&\qquad -\frac{2h^2\lambda}{k}\,\frac{1+\lambda^2(1-2k\cos 2\alpha)}{1-k^2\lambda^2}\\
a_2-\overline{a}_2&=\frac{2h^2}{k}\lambda^2\cos 2\alpha\\
b_1-\overline{b}_1&=-\frac{2\lambda\sin 2\alpha}{k^2}[h(h+2k)+2h^3k\lambda^2]\\
b_2-\overline{b}_2&=-\frac{2h^2}{k}\lambda^2\sin 2\alpha\\
b_0&=\frac{2}{k^3}(h+k)^2(1+hk\lambda^2)^2\sin 2\alpha
\end{aligned}\right\}\quad(19)$$

It is a great simplification to replace $\overline{a}_n$ by $a_n{}^1+a_n{}^2$ in the last two integrals of equation (18) before performing the integrations. Then

$$\left.\begin{aligned}
&-\sum_{n=1}^{\infty}2n\cos(n\delta+\alpha)\int_0^1(\lambda^n\overline{b}_{n-1}+\lambda^{n-1}\overline{b}_n)d\lambda\\
&+\sum_{n=1}^{\infty}2n\sin(n\delta+\alpha)\int_0^1(\lambda^n a_{n+1}{}^1+\lambda^{n-1}a_n{}^1)d\lambda\\
&=(1+k)(h+k)^2\sum_{n=1}^{\infty}4nk^{n-4}\sin(n\delta-\alpha)\\
&\int_0^1(1+hk\lambda^2)^2\lambda^{2n-1}d\lambda\\
\text{and}\quad&\\
&\sum_{n=1}^{\infty}2n\cos(n\delta-\alpha)\int_0^1(\lambda^{n-2}\overline{b}_{n+1}+\lambda^{n-1}\overline{b}_n)d\lambda\\
&-\sum_{n=1}^{\infty}2n\sin(n\delta-\alpha)\int_0^1(\lambda^{n-2}a_{n+1}{}^1+\lambda^{n-1}a_n{}^1)d\lambda\\
&=-(1+k)(h+k)^2\sum_{n=1}^{\infty}4nk^{n-8}\sin(n\delta-3\alpha)\\
&\int_0^1(1+hk\lambda^2)^2\lambda^{2n-1}d\lambda
\end{aligned}\right\}\quad(20)$$

Also

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty}2n\sin(n\delta+\alpha)\int_0^1(\lambda^n a_{n-1}{}^2+\lambda^{n-1}a_n{}^2)d\lambda\\
&=-(1+k)(h+k)^2\sum_{n=1}^{\infty}4nk^{n-5}\sin(n\delta+\alpha)\\
&\int_0^1(1+hk\lambda^2)^2\lambda^{2n-1}d\lambda\\
&+(1-2k\cos 2\alpha+k^2)(1+k)(h+k)^2\\
&\sum_{n=1}^{\infty}4nk^{n-5}\sin(n\delta+\alpha)\int_0^1\frac{(1+hk\lambda^2)^2\lambda^{2n+1}}{1-k^2\lambda^2}d\lambda
\end{aligned}\right\}\quad(21a)$$

and

$$\left.\begin{aligned}
&\sum_{n=1}^{\infty}2n\sin(n\delta-\alpha)\int_0^1(\lambda^{n-2}a_{n+1}{}^2+\lambda^{n-1}a_n{}^2)d\lambda\\
&=-(1+k)(h+k)^2\sum_{n=1}^{\infty}4nk^{n-4}\sin(n\delta-\alpha)\\
&\int_0^1(1+hk\lambda^2)^2\lambda^{2n-1}d\lambda\\
&+(1-2k\cos 2\alpha+k^2)(1+k)(h+k)^2\\
&\sum_{n=1}^{\infty}4nk^{n-4}\sin(n\delta-\alpha)\int_0^1\frac{(1+hk\lambda^2)^2\lambda^{2n-1}}{1-k^2\lambda^2}d\lambda
\end{aligned}\right\}\quad(21b)$$

Consider now the integrals

$$J_2=\sum_{n=1}^{\infty}nk^n\sin(n\delta+\beta)\int_0^1(1+hk\lambda^2)^2\lambda^{2n-1}d\lambda$$

and

$$J_1=\sum_{n=1}^{\infty}nk^n\cos(n\delta+\beta)\int_0^1(1+hk\lambda^2)^2\lambda^{2n-1}d\lambda$$

Then

$$J_1+iJ_2=e^{i\beta}\sum_{n=1}^{\infty}\int_0^1(1+hk\lambda^2)^2nk^n\lambda^{2n-1}e^{in\delta}d\lambda$$

Let $s=k\lambda^2$. Then

$$J_1+iJ_2=\frac{e^{i\beta}}{2}\int_0^k\frac{e^{i\delta}(1+hs)^2}{(1-se^{i\delta})^2}ds=\frac{e^{-i(\delta-\beta)}}{2}\Big[h^2k+\frac{(1+he^{-i\delta})^2}{e^{-i\delta}-k}$$
$$-e^{i\delta}(1+he^{-i\delta})^2+2h(1+he^{-i\delta})\log(1-ke^{i\delta})\Big]$$

Hence

$$\begin{aligned}
J_2&=\frac{k(1+hk)^2}{2}\,\frac{\sin(\delta+\beta)-k\sin\beta}{1-2k\cos\delta+k^2}\\
&\quad-h\Big[\cos(\delta-\beta)\tan^{-1}\frac{k\sin\delta}{1-k\cos\delta}\\
&\quad+\frac{1}{2}\sin(\delta-\beta)\log(1-2k\cos\delta+k^2)-k\sin\beta\Big]\\
&\quad-h^2\Big[k\sin(\delta-\beta)+\cos(2\delta-\beta)\tan^{-1}\frac{k\sin\delta}{1-k\cos\delta}\\
&\quad+\frac{1}{2}\sin(2\delta-\beta)\log(1-2k\cos\delta+k^2)-\frac{1}{2}k^2\sin\beta\Big]
\end{aligned}$$

Replacing β by α, $-\alpha$, or -3α, the corresponding integrals in equations (20) and (21) are obtained.

Consider finally the integrals

$$I_1=\sum_{n=1}^{\infty}nk^n\cos n\delta\int_0^1\frac{(1+hk\lambda^2)^2\lambda^{2n-1}}{1-k^2\lambda^2}d\lambda$$

and

$$I_2=\sum_{n=1}^{\infty}nk^n\sin n\delta\int_0^1\frac{(1+hk\lambda^2)^2\lambda^{2n-1}}{1-k^2\lambda^2}d\lambda$$

Then

$$I_1+iI_2=\int_0^1\frac{(1+hk\lambda^2)^2}{\lambda(1-k^2\lambda^2)}\sum_{n=1}^{\infty}n(k\lambda^2)^n e^{in\delta}d\lambda$$

Again let $s=k\lambda^2$. Then

$$I_1+iI_2=\frac{e^{i\delta}}{2}\int_0^k\frac{(1+hs)^2}{(1-ks)(1-se^{i\delta})^2}ds=$$
$$-\frac{(h+k)^2e^{i\delta}}{2k(k-e^{i\delta})^2}\log(1-k^2)$$
$$-\frac{h(h+e^{i\delta})}{2e^{i\delta}(k-e^{i\delta})}\log(1-ke^{i\delta})$$
$$+\frac{(h+k)(h+e^{i\delta})}{2(k-e^{i\delta})^2}\log(1-ke^{i\delta})$$
$$-\frac{k(h+e^{i\delta})^2}{2(1-ke^{i\delta})(k-e^{i\delta})}$$

Therefore

$$I_1=\frac{(h+k)^2}{2k}\frac{2k-(1+k^2)\cos\delta}{(1-2k\cos\delta+k^2)^2}\log(1-k^2)$$
$$-\frac{h}{4}\frac{(hk-1)\cos\delta+k-h\cos 2\delta}{1-2k\cos\delta+k^2}\log(1-2k\cos\delta+k^2)$$
$$+\frac{h}{2}\frac{(hk-1)\sin\delta-h\sin 2\delta}{1-2k\cos\delta+k^2}\tan^{-1}\frac{k\sin\delta}{1-k\cos\delta}$$
$$+\frac{h+k}{4}\frac{[hk^2-2k+(1-2kh+k^2)\cos\delta+h\cos 2\delta]}{(1-2k\cos\delta+k^2)^2}$$
$$\log(1-2k\cos\delta+k^2)$$
$$-\frac{h+k}{2}\frac{(1-2kh-k^2)\sin\delta+h\sin 2\delta}{(1-2k\cos\delta+k^2)^2}\tan^{-1}\frac{k\sin\delta}{1-k\cos\delta}$$
$$+\frac{k}{2}\frac{(1+h^2)\cos\delta+2h}{1-2k\cos\delta+k^2}$$

and

$$I_2=\frac{(h+k)^2}{2k}\frac{(1-k^2)\sin\delta}{(1-2k\cos\delta+k^2)^2}\log(1-k^2)$$
$$+\frac{h}{4}\frac{(hk-1)\sin\delta-h\sin 2\delta}{1-2k\cos\delta+k^2}\log(1-2k\cos\delta+k^2)$$
$$+\frac{h}{2}\frac{(hk-1)\cos\delta+k-h\cos 2\delta}{1-2k\cos\delta+k^2}\tan^{-1}\frac{k\sin\delta}{1-k\cos\delta}$$
$$-\frac{h+k}{2}\frac{hk^2-2k+(1-2hk+k^2)\cos\delta+h\cos 2\delta}{(1-2k\cos\delta+k^2)^2}$$
$$\tan^{-1}\frac{k\sin\delta}{1-k\cos\delta}$$
$$-\frac{h+k}{4}\frac{(1-2hk-k^2)\sin\delta+h\sin 2\delta}{(1-2k\cos\delta+k^2)^2}$$
$$\log(1-2k\cos\delta+k^2)$$
$$+\frac{k}{2}\frac{(1-h^2)\sin\delta}{1-2k\cos\delta+k^2}$$

From equations (18), (19), and (20) and the integrals J_2, I_1, and I_2, it follows that

$$\frac{\Delta v}{v_0}=\frac{\mu}{4}\left\{\left[1-\frac{(9+5h^2)(h+k)^2}{3k^2}-\frac{3h^2}{k}+\frac{5}{3}\frac{h^4}{k^2}-\frac{2h(h+2k)}{k^2}\right]\right.$$
$$\sin(\delta+\alpha)+\left[\frac{2}{3}\frac{(3+3hk+h^2k^2)(h+k)^2}{k^3}+\frac{2h^2}{k}\right.$$
$$\left.+\frac{2h(h+2k)}{k^2}\right]\sin(\delta+3\alpha)-\frac{16}{3}\frac{h^2}{k}\sin(2\delta+\alpha)$$
$$+\left[\frac{10}{3}\frac{h^2}{k}+\frac{2h(h+2k)}{k^2}\right]\sin(2\delta+3\alpha)+\frac{2h^2}{k}\sin(3\delta+3\alpha)$$
$$+\frac{4h^2}{k}\sin(\delta-\alpha)+\frac{8(1+k)(h+k)^2}{k^4}J_2(-\alpha)$$
$$-\frac{4(1+k)(h+k)^2}{k^3}J_2(-3\alpha)-\frac{4(1+k)(h+k)^2}{k^5}J_2(\alpha)$$
$$+\frac{4(1+k)^2(h+k)^2}{k^5}(1-2k\cos 2\alpha+k^2)\,I_1\sin\alpha$$
$$\left.+\frac{4(1-k^2)(h+k)^2}{k^5}(1-2k\cos 2\alpha+k^2)\,I_2\cos\alpha\right\}\qquad(22)$$

where $J_2(-\alpha)$, for example, means that in the expression for $J_2(\beta)$, $-\alpha$ has been substituted for β; and $\mu=\left(\frac{v_0}{c_0}\right)^2$. There is no difficulty in evaluating $\Delta v/v_0$ for any value of h except $h=0$. For $h=0$, $k=1$, however, the Joukowski profile degenerates into a line segment and

$$\frac{\Delta v}{v_0}=\mu\sin\alpha\left[\cos(\delta+2\alpha)+2\sin\alpha\frac{\sin(\delta+\alpha)+\sin\alpha}{1-\cos\delta}\right.$$
$$\left.-\frac{2\sin^2\alpha}{1-\cos\delta}\log 2(1-\cos\delta)-\frac{4\sin^2\alpha}{1-\cos\delta}\sum_{n=1}^{\infty}\frac{1}{n}\right]\qquad(23)$$

The last term in this expression contains the divergent series $\sum_{n=1}^{\infty}\frac{1}{n}$, which approaches infinity like $\lim_{n\to\infty}\log n$. This infinite term shows that streamline flow cannot be maintained about a straight-line profile except for the trivial case of zero angle of attack.

CORRECTION FOR THE CIRCULATION

It is noted that the expression for the complex velocity about the circular profile given by equation (12) was obtained with the circulation fixed by the Kutta condition. When Δv, representing the effect of compressibility, is added to the incompressible velocity obtained from equation (12) to yield the compressible velocity, the Kutta condition no longer holds. In order to restore the Kutta condition, an additional circulation $\Delta\Gamma$ is added to the incompressible one. Thus, the velocity at the boundary of the circular profile is given by

$$\frac{v_c}{v_0}=2\ \sin\ (\delta+\alpha)+\frac{\Gamma_0+\Delta\Gamma}{2\pi R v_0}+\frac{\Delta v}{v_0} \tag{24}$$

where $\Gamma_0+\Delta\Gamma=\Gamma$ and $\Gamma_0=4\pi R v_0 \sin \alpha$, the circulation in the incompressible flow. The Kutta condition, i. e., $\left(\frac{v_c}{v_0}\right)_{\delta=\pi}=0$, thus serves to evaluate $\Delta\Gamma$. For $\delta=\pi$ the expressions for $J_2(\beta)$, I_1, and I_2 simplify considerably. Thus

$$(J_2(\beta))_{\delta=\pi}=-\left\{\frac{k(1+hk)^2}{2(1+k)}+h[\log\ (1+k)-k]\right.$$
$$\left.+h^2[k-\frac{1}{2}k^2-\log\ (1+k)]\right\}\sin\ \beta$$
$$(I_1)_{\delta=\pi}=\frac{h^2}{2k}\log\ (1+k)+\frac{1}{8k}\log\ (1-k)-\frac{k}{8}$$
$$(I_2)_{\delta=\pi}=0$$

The relation between the velocities in the plane of the circle and the plane of the profile is given by

$$\frac{dw}{d\zeta}=\frac{dw}{dz}\frac{dz}{dz'}\frac{dz'}{d\zeta}$$

where $z'=z+a\epsilon$ and $\zeta=z'+\frac{a^2}{z'}$.

Then

$$\left|\frac{dz}{dz'}\right|=1 \text{ and}$$
$$\left|\frac{dz'}{d\zeta}\right|=\frac{1+2h\lambda\ \cos\ \delta+h^2\lambda^2}{\sqrt{1+2\lambda\ \cos\ \delta+\lambda^2}\sqrt{1-2k\lambda\ \cos\ \delta+k^2\lambda^2}}$$

It follows that on the profile where $\lambda=1$

$$\frac{v_P}{v_0}=\frac{1+2h\ \cos\ \delta+h^2}{\sqrt{2(1+\cos\ \delta)}\sqrt{1-2k\ \cos\ \delta+k^2}}\frac{v_c}{v_0} \tag{25}$$

where v_P is the velocity on the profile corresponding to v_c on the circle.

When the profile is assumed to be thin so that only the first power of h is retained and the angle of attack is small so that $\cos\ \alpha\cong 1$ and $\sin\ \alpha\cong\alpha$, then the Kutta condition leads to the following expression for $\frac{\Delta\Gamma}{2\pi R v_0}$

$$\frac{\Delta\Gamma}{2\pi R v_0}=\mu(1+h)\alpha \tag{26}$$

It then follows that

$$\frac{\Gamma}{\Gamma_0}=1+\frac{1+h}{2}\mu \tag{27}$$

This value for the ratio Γ/Γ_0 corroborates Glauert's result (reference 5)

$$\frac{\Gamma}{\Gamma_0}=\frac{1}{\sqrt{1-\mu}}=1+\frac{1}{2}\mu+\ \cdot\ \cdot\ \cdot$$

when the profile is very thin, i. e., when h is negligible in comparison with unity.

Since the rigorous expressions are available, it may be interesting to compare the approximate result given by equation (26) with the exact result obtained from equation (22). Thus, for a very thin profile defined by $\epsilon=0.01\left(h=\frac{1}{101}, k=\frac{99}{101}\right)$ and for the more conventional profiles defined by $\epsilon=0.05\left(h=\frac{1}{21}, k=\frac{19}{21}\right)$ and $\epsilon=0.10\left(h=\frac{1}{11}, k=\frac{9}{11}\right)$, at angles of attack $\alpha=10°$ and $\alpha=5°$, the following table presents the results:

α (deg.)	$\epsilon=0.01$		$\epsilon=0.05$		$\epsilon=0.10$	
	$\frac{\Delta\Gamma}{2\pi R v_0}$ (approximate)	$\frac{\Delta\Gamma}{2\pi R v_0}$ (exact)	$\frac{\Delta\Gamma}{2\pi R v_0}$ (approximate)	$\frac{\Delta\Gamma}{2\pi R v_0}$ (exact)	$\frac{\Delta\Gamma}{2\pi R v_0}$ (approximate)	$\frac{\Delta\Gamma}{2\pi R v_0}$ (exact)
10	0.17625μ	0.20796μ	0.18281μ	0.20281μ	0.19036μ	0.20850μ
5	$.08816\mu$	$.09218\mu$	$.09146\mu$	$.09146\mu$	$.09524\mu$	$.09597\mu$

It is to be noticed that for $\alpha=10°$ the exact evaluation of $\Delta\Gamma/2\pi R v_0$ yields a greater value for $\epsilon=0.01$ than for $\epsilon=0.05$, a fact not given by the approximate equation (26). This reversal appears, in general, for larger values of ϵ as the angle of attack increases; e. g., for $\epsilon=0.05$ at $\alpha=20°$. This feature of the exact expression for the additional lift has no practical significance, however, insofar as the lift is concerned, since the appropriate combination of ϵ and α showing this reversal is outside the practical range.

In the calculation of the local velocities and pressures on the surface of the airfoil, the rigorous expressions for $\Delta v/v_0$ are to be used. The rigorous derivation, however, of the total integrated lift and moment on the airfoil involves great mathematical difficulties. A simplified form for $\Delta v/v_0$ may, however, be obtained for a thin Joukowski profile at small angles of attack. Its use in integrating for the lift yields, as will be shown later, the expected result that

$$\text{Lift}=\rho_0 v_0 \Gamma$$

where $\Gamma=\Gamma_0\left(1+\frac{1+h}{2}\mu\right)$

This result justifies the use of the simplified form of $\Delta v/v_0$ in calculating the lift, but its use in integrating for the moment, although reasonable, is somewhat uncertain.

If, then, only the first power of h is retained and the angle of attack taken small enough so that $\cos\ \alpha\cong 1$ and $\sin\ \alpha\cong\alpha$, it follows from equation (22) that

$$\frac{\Delta v}{v_0}=\mu\{[\cos\delta+h(4+8\cos\delta+3\cos 2\delta)]\alpha+h(\sin\delta+\sin 2\delta)\}$$

The expression for v_c/v_0, replacing $\Delta\Gamma/2\pi R v_0$ by the value given by equation (26), then becomes:

$$\frac{v_c}{v_0}=2\ \sin\ \delta+2\alpha(1+\cos\ \delta)$$
$$+\mu[(1+\cos\ \delta)\alpha+h(5+8\ \cos\ \delta+3\ \cos\ 2\delta)\alpha$$
$$+h(\sin\ \delta+\sin\ 2\delta)] \tag{28}$$

CALCULATION OF THE PRESSURE AND LIFT ON THE AIRFOIL

According to equation (8)

$$c^2=c_0^2\left[1+\frac{\gamma-1}{2}\left(1-\frac{v^2}{v_0^2}\right)\frac{v_0^2}{c_0^2}\right]$$

Then from the adiabatic equation of state

$$p=p_0\left(\frac{\rho}{\rho_0}\right)^\gamma$$

and the definition of the local velocity of sound c

$$c=\sqrt{\frac{dp}{d\rho}}$$

it follows that

$$\frac{c^2}{c_0^2}=\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}=\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$$

Therefore

$$\frac{p}{p_0}=\left[1+\frac{\gamma-1}{2}\left(1-\frac{v^2}{v_0^2}\right)\frac{v_0^2}{c_0^2}\right]^{\frac{\gamma}{\gamma-1}} \tag{29}$$

Expanding the right-hand side of the foregoing equation according to powers of $\frac{v_0^2}{c_0^2}$ $(=\mu)$ it follows that

$$p=p_0+\frac{1}{2}\rho_0v_0^2\left(1-\frac{v^2}{v_0^2}\right)+\frac{1}{8}\rho_0v_0^2\mu\left(1-\frac{v^2}{v_0^2}\right)^2+\cdots \tag{30}$$

The pressure distribution may be calculated by means of equation (29) together with the values for v/v_0 obtained from equations (22), (24), and (25). Equation (30) will be used in obtaining the total lift and moment on the airfoil.

Since the profile is a streamline, the normal velocity $-\partial\phi/\partial n=0$ and, accordingly, if s denotes the length along it, then Bernoulli's equation may be written

$$p=\text{constant}-\frac{1}{2}\rho_0\left(1+\frac{1}{2}\mu\right)\left(\frac{\partial\phi}{\partial s}\right)^2+\frac{1}{8}\frac{\rho_0\mu}{v_0^2}\left(\frac{\partial\phi}{\partial s}\right)^4+\cdots$$

Let n denote the inward-drawn normal to the contour. Then from figure 2, it is seen that $p\cos(\xi,n)$ and $p\cos(\eta,n)$ are the components of the pressure along the ξ and η axes, respectively. Accordingly, the force on the airfoil is given by

$$\overline{P}=P_\xi-iP_\eta=-\frac{\rho_0}{2}\left(1+\frac{1}{2}\mu\right)\oint_C\left(\frac{\partial\phi}{\partial s}\right)^2[\cos(\xi,n)-i\cos(\eta,n)]ds+\frac{\rho_0\mu}{8v_0^2}\oint_C\left(\frac{\partial\phi}{\partial s}\right)^4[\cos(\xi,n)-i\cos(\eta,n)]ds$$

where the profile C is traversed in the counterclockwise positive sense. On the other hand

$$d\xi=ds\cos(\eta,n),\ d\eta=ds\cos[\pi-(\xi,n)]=-ds\cos(\xi,n)$$

and therefore

$$\overline{P}=\frac{i\rho_0}{2}\left(1+\frac{1}{2}\mu\right)\oint_C\left(\frac{\partial\phi}{\partial s}\right)^2(d\xi-id\eta)-\frac{i\rho_0\mu}{8v_0^2}\oint_C\left(\frac{\partial\phi}{\partial s}\right)^4(d\xi-id\eta)$$

Now, by definition,

$$\frac{dw}{d\zeta}=-v_\xi+iv_\eta=\frac{\partial\phi}{\partial\xi}-i\frac{\partial\phi}{\partial\eta}$$

and, since the velocity normal to the profile equals zero, it follows that

$$\frac{\partial\phi}{\partial\xi}=\frac{\partial\phi}{\partial s}\sin(\xi,n),\ \frac{\partial\phi}{\partial\eta}=\frac{\partial\phi}{\partial s}\sin(\eta,n)$$

Therefore

$$\frac{dw}{d\zeta}=\frac{\partial\phi}{\partial s}\frac{d\xi-id\eta}{ds}$$

But

$$ds^2=d\xi^2+d\eta^2=(d\xi+id\eta)(d\xi-id\eta)$$

or

$$d\xi-id\eta=\frac{ds^2}{d\zeta}$$

Hence

$$\frac{\partial\phi}{\partial s}=\frac{dw}{d\zeta}\frac{\partial\zeta}{\partial s}$$

and

$$\overline{P}=\frac{i\rho_0}{2}\left(1+\frac{1}{2}\mu\right)\oint_C\left(\frac{dw}{d\zeta}\right)^2d\zeta-\frac{i\rho_0\mu}{8v_0^2}\oint_C\left(\frac{dw}{d\zeta}\right)^4\left(\frac{\partial\zeta}{\partial s}\right)^2d\zeta$$

Now

$$\left(\frac{\partial\zeta}{\partial s}\right)^2=\frac{d\zeta}{d\bar{\zeta}}\text{ where }\bar{\zeta}\text{ is the conjugate of }\zeta$$

Therefore

$$\overline{P}=\frac{i\rho_0}{2}\left(1+\frac{1}{2}\mu\right)\oint_C\left(\frac{dw}{d\zeta}\right)^2d\zeta-\frac{i\rho_0\mu}{8v_0^2}\oint_C\left(\frac{dw}{d\zeta}\right)^4\frac{d\zeta}{d\bar{\zeta}}d\zeta \tag{31}$$

Referring to the plane of the circle of radius R

$$\overline{P}=\frac{i\rho_0}{2}\left(1+\frac{1}{2}\mu\right)\oint_{circle}\left(\frac{dw}{dz}\right)^2\frac{dz}{d\zeta}dz-\frac{i\rho_0\mu}{8v_0^2}\oint_{circle}\left(\frac{dw}{dz}\right)^4\left(\frac{dz}{d\zeta}\right)^2\frac{dz}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}dz \tag{32}$$

where $z=e^{i\delta}$ and δ is the polar angle of the circle of radius R.

Since by definition

$$\frac{dw}{dz}=R(-v_x+iv_y)$$

it follows from equation (28) that

$$\frac{dw}{dz}=iRv_ce^{-i\delta}=Rv_c\left(a_1z+a_0+\frac{a_{-1}}{z}+\frac{a_{-2}}{z^2}+\frac{a_{-3}}{z^3}\right) \tag{33}$$

where

$$a_0=\left(1+\frac{h\mu}{2}\right)+i\alpha\left(1+\frac{\mu}{2}+4h\mu\right)$$

$$a_1=\frac{h\mu}{2}+\frac{3}{2}ih\mu\alpha$$

$$a_{-1}=i\alpha(2+\mu+5h\mu)$$

$$a_{-2}=-\left(1+\frac{h\mu}{2}\right)+i\alpha\left(1+\frac{\mu}{2}+4h\mu\right)$$

$$a_{-3}=-\frac{h\mu}{2}+\frac{3}{2}ih\mu\alpha$$

Also from the Joukowski transformation

$$\zeta = z' + \frac{a^2}{z'}$$

and the relation

$$z' = R(z+h)$$

it follows that

$$\frac{dz}{d\zeta} = \frac{1}{R\left[1-\frac{(1-h)^2}{(z+h)^2}\right]}$$

$$= \frac{1}{R}\left(1+\frac{1-2h}{z^2}-\frac{2h}{z^3}+\frac{1-4h}{z^4}-\frac{4h}{z^5}+\cdots\right)$$

$$\frac{d\bar{z}}{d\bar{\zeta}} = \frac{1}{R\left[1-\frac{(1-h)^2\bar{z}^2}{(1+h\bar{z})^2}\right]}$$

$$\frac{d\bar{z}}{dz} = -\bar{z}^2$$

Then making use of the well-known relations

$$\oint z^m dz = 0; \text{ if } m \neq -1$$

and

$$\oint_C \frac{dz}{z} = 2\pi i; \text{ if } m=-1$$

it turns out, neglecting as usual terms containing powers of μ, h, and α higher than the first, that

$$\bar{P} = P_\xi - iP_\eta = -i\rho_0 v_0 \Gamma_0\left[1+\left(1+\frac{7}{2}h\right)\mu\right] + i\rho_0 v_0 \Gamma_0 \frac{\mu}{2}(1+6h)$$

$$= -i\rho_0 v_0 \Gamma_0\left(1+\frac{1+h}{2}\mu\right)$$

or

$$P_\eta = \rho_0 v_0 \Gamma_0\left(1+\frac{1+h}{2}\mu\right)$$

This last expression shows that, when the angle of attack is assumed small enough so that only the first power of α is retained, the component P_ξ of the lift vanishes in comparison with the component P_η.

Thus

$$\text{Lift} = P_\eta = \rho_0 v_0 \Gamma \tag{34}$$

This expression agrees in form with the corresponding one in incompressible flow with the auxiliary definition $\Gamma = \Gamma_0\left(1+\frac{1+h}{2}\mu\right)$, e. g., equation (27).

CALCULATION OF THE MOMENT

The moment arm $0Q = m \sin(\sigma-\varphi)$ and the force per unit length along the airfoil is pds (fig. 2). Hence the total moment about the origin 0 is given by

$$M = \oint_C pm \sin(\sigma-\varphi)ds = \oint_C p(m \sin\sigma\cos\varphi - m\cos\sigma\sin\varphi)ds$$

But

$$d\xi = ds\cos\sigma \text{ and } d\eta = -ds\sin\sigma$$

Hence

$$M = \oint_C p(m\cos\varphi d\xi + m\sin\varphi d\eta)$$

$$= \oint_C p(\xi d\xi + \eta d\eta) = \frac{1}{2}\oint_C pdm^2$$

Now

$$\xi d\xi + \eta d\eta = R.P.\zeta d\bar{\zeta}$$

and since $d\bar{\zeta} = \frac{ds^2}{d\zeta}$

$$\xi d\xi + \eta d\eta = R.P.\zeta\frac{ds^2}{d\zeta}$$

Substituting for the pressure p the expression

$$\text{Const.} - \frac{1}{2}\rho_0\left(1+\frac{1}{2}\mu\right)\left(\frac{\partial\phi}{\partial s}\right)^2 + \frac{1}{8}\frac{\rho_0\mu}{v_0^2}\left(\frac{\partial\phi}{\partial s}\right)^4 + \ldots$$

it follows that

$$M = -\frac{1}{2}\rho_0\left(1+\frac{1}{2}\mu\right)R.P.\oint_C\left(\frac{dw}{d\zeta}\right)^2\zeta d\zeta + \frac{1}{8}\frac{\rho_0\mu}{v_0^2}R.P.\oint_C\left(\frac{dw}{d\zeta}\right)^4\frac{d\zeta}{d\bar{\zeta}}\zeta d\zeta \tag{35}$$

Referring to the plane of the circle of radius R

$$M = -\frac{1}{2}\rho_0\left(1+\frac{1}{2}\mu\right)R.P.\oint_{circle}\left(\frac{dw}{dz}\right)^2\zeta\frac{dz}{d\zeta}dz + \frac{1}{8}\frac{\rho_0\mu}{v_0^2}R.P.\oint_{circle}\left(\frac{dw}{dz}\right)^4\left(\frac{dz}{d\zeta}\right)^2\frac{dz}{d\bar{z}}\frac{d\bar{z}}{d\bar{\zeta}}\zeta dz \tag{36}$$

Performing the integrations in a manner analogous to that for the lift, it turns out that

$$M = 4\pi\rho_0 R^2 v_0^2\alpha\left[1-h+\left(1+\frac{9}{2}h\right)\mu\right] - \pi\rho_0 R^2 v_0^2\mu\alpha(2+6h)$$

or

$$M = 4\pi\rho_0 R^2 v_0^2\alpha\left(1-h+\frac{1+6h}{2}\mu\right) \tag{37}$$

This expression represents the moment about the origin of coordinates, and the moment about the center of the circle of radius R (into which the profile is mapped) can be obtained at once as

$$M_c = M - L a\epsilon = 4\pi\rho_0 v_0^2 a^2\alpha + 4\pi\rho_0 v_0^2 a^2\alpha\frac{1+7h}{2}\mu$$

or

$$M_c = M_0\left(1+\frac{1+7h}{2}\mu\right) \tag{38}$$

where $M_0 = 4\pi\rho_0 v_0^2 a^2\alpha$ is the moment about the center of the circle of radius R for the corresponding incompressible flow.

If now d represents the distance of the center of pressure from the origin of coordinates, then

$$M = Ld$$

or

$$d = \frac{4\pi\rho_0 R^2 v_0^2\alpha\left(1-h+\frac{1+6h}{2}\mu\right)}{4\pi\rho_0 R v_0^2\alpha\left(1+\frac{1+h}{2}\mu\right)} \cong a(1+3h\mu) \tag{39}$$

This expression shows that the airfoil has a constant center of pressure at a distance equal to $1/4\ (1-3h\mu)$ of the chord from the leading edge. For a thin airfoil, say $\epsilon=0.05$ and for a stream velocity $v_0=0.835\ c_0$, the center of pressure, as compared with the corresponding incompressible case, is nearer to the leading edge by about 2.5 percent of the chord.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *November 19, 1937.*

APPENDIX A

APPLICATION OF THE THEORETICAL RESULTS

As an example of the application of the theory to any particular case, the flows past a thin and a fairly thick symmetrical Joukowski profile for zero angle of attack will be calculated. Since no experimental results are available for purposes of comparison, it was considered hardly worth while to perform the rather lengthy and tedious calculations associated with angle of attack or circulation.

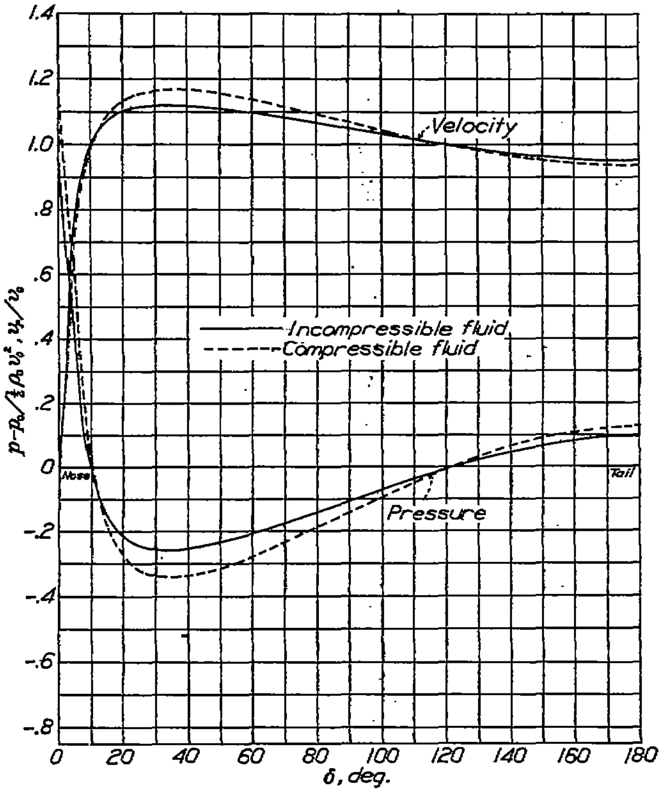

FIGURE 3.—Pressure and velocity distribution for the symmetrical Joukowski airfoil section $\epsilon=0.05$. $\mu=0.70$, $\alpha=0°$.

Equation (23) for $\Delta v/v_0$ simplifies considerably when the angle of attack α is taken to be zero. Thus

$$\frac{\Delta v}{v_0}=\frac{\mu}{4}\left[\frac{h(12-21h-4h^2+30h^3-24h^4-8h^5)}{3k^3}\sin\delta + \frac{4h(1-h)^2}{k^2}\sin 2\delta+\frac{2h^2}{k}\sin 3\delta - \frac{32h^2(1-h)^3}{k^5}J_2(0)+\frac{64h^3(1-h)^3}{k^5}I_2\right] \quad (40)$$

where

$$J_2(0)=\frac{(1+2h)(1-4h^2)(1-h)^2}{2}\frac{\sin\delta}{1-2k\cos\delta+k^2} - h^2k\sin\delta-h(\cos\delta+h\cos 2\delta)\tan^{-1}\frac{k\sin\delta}{1-k\cos\delta} - \frac{h}{2}(\sin\delta+h\sin 2\delta)\log(1-2k\cos\delta+k^2)$$

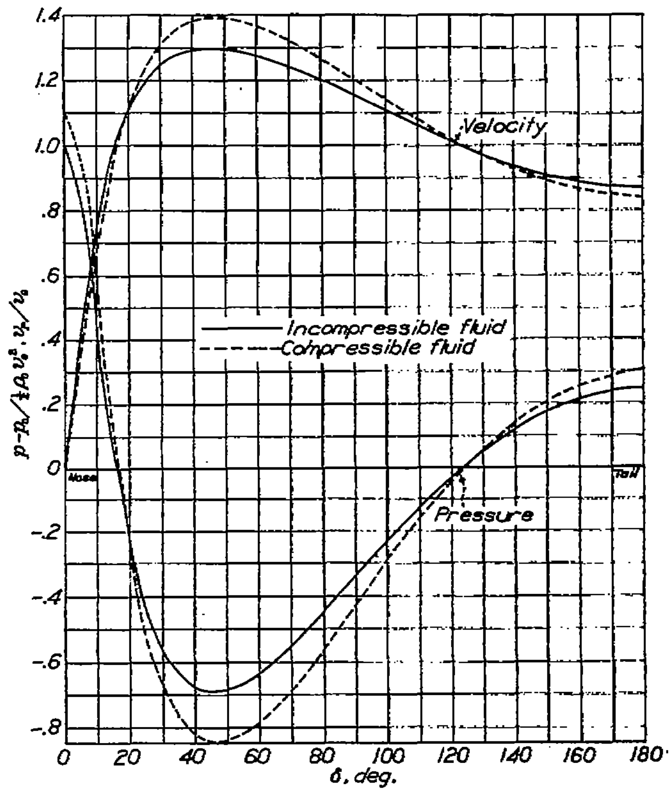

FIGURE 4.—Pressure and velocity distributions for the symmetrical Joukowski airfoil section $\epsilon=0.15$. $\mu=0.47$, $\alpha=0°$.

The profiles chosen are defined by $\epsilon=0.05$ and $\epsilon=0.15$ or $h=1/21$, $k=19/21$; and $h=3/23$, $k=17/23$, respectively. Tables I and II present the calculations and figures 3 and 4 show the velocity and pressure distributions for both incompressible and compressible flow. The values of μ chosen were the critical values obtained by plotting $(v_P/v_0)_{max}$ against v_0/c_0 and then noting the intersection of this graph with that of $(v/v_0)_{crit}$ against v_0/c_0 as given by equation (9). Table III presents the data and figure 5 shows the corresponding graphs.

The expressions for $\Delta v/v_0$ are given by:

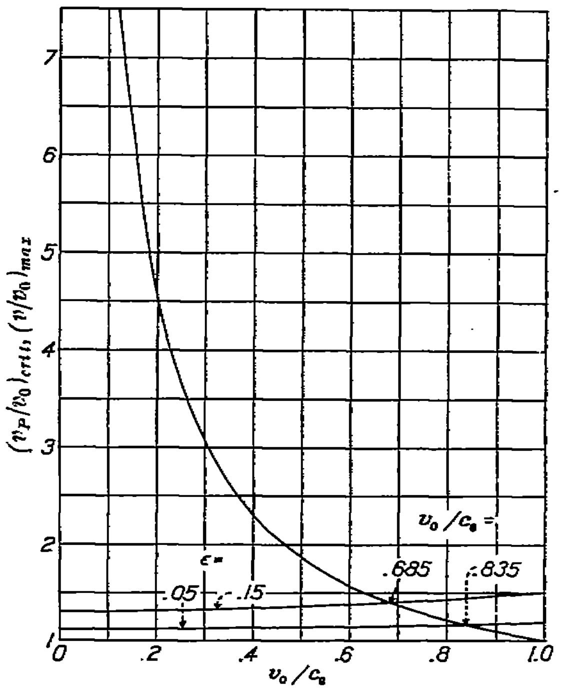

FIGURE 5.—Limiting values of v_0/c_s at zero angle of attack for ε=0.05 and ε=0.15

for =0.05

$$\frac{\Delta v}{v_0}=\mu[0.05890 \sin\delta+0.05276 \sin 2\delta+0.00125 \sin 3\delta -0.02585J_2(0)+0.00246I_2]$$

and

for ε=0.15

$$\frac{\Delta v}{v_0}=\mu[0.24905 \sin\delta+0.18053 \sin 2\delta+0.01151 \sin 3\delta -0.40568J_2(0)+0.10583I_2]$$

It is to be noted in tables I, II, and III that the maximum velocity v_P/v_0 for the incompressible flow occurs at about $\delta=35°$ and $\delta=45°$ for ε=0.05 and ε=0.15,

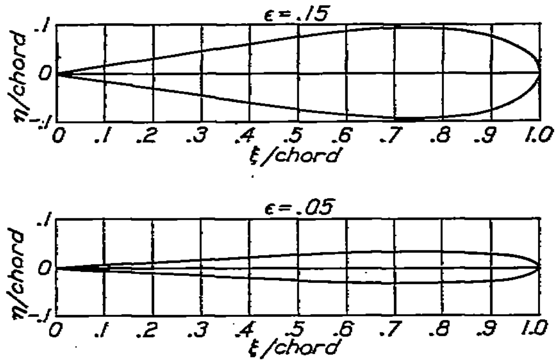

FIGURE 6.—Symmetrical Joukowski profiles ε=0.05 and ε=0.15.

respectively. It is then assumed that the position of maximum velocity is independent of μ and maximum values for v_P/v_0 are calculated for various values of μ. These values of $(v_P/v_0)_{max}$ are given in table III and are used in obtaining the critical values of μ as shown in figure 5. The coordinates of the airfoils ε=0.05, ε=0.15 are given in table IV and the corresponding contours in figure 6.

APPENDIX B

NOTATION

$x, y,$ rectangular coordinates in the plane of the circle.

$\xi, \eta,$ rectangular coordinates in the plane of the profile.

$z = x + iy,\ \zeta = \xi + i\eta$

$r, \theta,$ plane polar coordinates in the z plane.

$v_x, v_y,$ fluid velocity components along the x and y axes, respectively.

$v_c,$ tangential velocity on the circle.

$v_P,$ tangential velocity on the profile corresponding to v_c.

$v = \sqrt{v_x^2 + v_y^2}$, magnitude of the fluid velocity.

$c,$ local velocity of sound in the fluid.

$\rho,$ density of the fluid.

$p,$ static pressure in the fluid.

$v_0, c_0, \rho_0, p_0,$ corresponding magnitudes in the undisturbed stream.

$\mu = \left(\frac{v_0}{c_0}\right)^2$

$\Delta v,$ correction term to the velocity in incompressible flow due to compressibility.

$\phi,$ velocity potential of the incompressible flow.

$v_r = -\frac{\partial \phi}{\partial r}$, component of velocity along the radius vector.

$v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$, component of velocity perpendicular to the radius vector in the sense of θ increasing.

$\lambda = \frac{R}{r}$

$R,$ radius of circle into which the profile is mapped.

$\gamma,$ ratio of specific heats.

$w,$ complex potential of the incompressible flow in the ζ plane.

$\alpha,$ angle of attack.

$\epsilon,$ thickness coefficient of Joukowski profile (see fig. 2).

$h = \frac{\epsilon}{1+\epsilon}$

$k = \frac{1-\epsilon}{1+\epsilon} = 1 - 2h$

$\Gamma,$ circulation about profile in the compressible fluid.

$\Gamma_0,$ circulation about profile in the incompressible fluid.

$\Delta\Gamma = \Gamma - \Gamma_0,$ contribution to the circulation due to compressibility.

$\overline{P},$ force vector on the airfoil.

$P_\xi, P_\eta,$ components of $\overline{P}$ along the ξ and η axes, respectively.

$M,$ moment about origin of coordinates in the plane of compressible flow.

$M_c,$ moment about center of circle of radius R in the plane of compressible flow.

$M_0,$ moment corresponding to M_c in the plane of incompressible flow.

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TABLE I

Case 1: $\epsilon=0.05, h=\frac{1}{21}, k=\frac{19}{21}, \alpha=0$

δ (deg.)	$J_1(0)$	I_2	$\frac{1}{\mu}\frac{\Delta v}{v_0}$	Jacobian[1]	$\frac{v_P}{v_0}$ (Incompressible)	$\frac{v_P}{v_0}$ ($\mu=0.70$)	$\frac{p-p_0}{\frac{1}{2}\rho_0 v_0^2}$ Compressible	$\frac{p-p_0}{\frac{1}{2}\rho_0 v_0^2}$ Incompressible
0	0.0000	0.0000	0.0000	5.7619	0.0000	0.0000	1.1743	1.0000
5	2.6629	10.8700	−.0275	4.3458	.7876	.6743	.5971	.4261
10	2.3054	8.2845	−.0104	2.8772	.9993	.9784	.0431	.0015
15	1.7670	5.5612	.0105	2.0752	1.0742	1.0894	−.1808	−.1539
20	1.3904	3.9260	.0289	1.6122	1.1028	1.1353	−.2744	−.2162
30	.9477	2.2576	.0575	1.1197	1.1197	1.1645	−.3340	−.2537
40	.7041	1.4504	.0764	.8700	1.1185	1.1648	−.3346	−.2510
50	.5508	1.0536	.0861	.7247	1.1103	1.1538	−.3120	−.2327
60	.4448	.7907	.0872	.6341	1.0953	1.1368	−.2774	−.2062
70	.3665	.6148	.0807	.5767	1.0838	1.1163	−.2356	−.1747
80	.3055	.4893	.0683	.5422	1.0679	1.0937	−.1894	−.1404
90	.2560	.3951	.0520	.5255	1.0511	1.0701	−.1415	−.1047
100	.2146	.3214	.0341	.5250	1.0341	1.0466	−.0938	−.0694
110	.1789	.2583	.0168	.5414	1.0176	1.0239	−.0480	−.0354
120	.1474	.2112	.0020	.5785	1.0020	1.0028	−.0055	−.0039
130	.1190	.1779	−.0089	.6443	.9878	.9839	.0322	.0242
140	.0929	.1294	−.0151	.7588	.9755	.9675	.0647	.0486
150	.0683	.0944	−.0165	.9657	.9657	.9546	.0902	.0674
160	.0540	.0617	−.0139	1.4010	.9594	.9448	.1094	.0876
170	.0223	.0305	−.0077	2.7465	.9539	.9391	.1205	.0002
180	.0000	.0000	.0000	.0000	.9524	.9358	.1270	.0930

[1] Jacobian $=\frac{221+21\cos\delta}{21\cos(\delta/2)\sqrt{802-798\cos\delta}}$

TABLE II

Case 2: $\epsilon=0.15, h=\frac{3}{23}, k=\frac{17}{23}, \alpha=0$

δ (deg.)	$J_2(0)$	I_2	$\frac{1}{\mu}\frac{\Delta v}{v_0}$	Jacobian[1]	$\frac{v_P}{v_0}$ (Incompressible)	$\frac{v_P}{v_0}$ ($\mu=0.47$)	$\frac{p-p_0}{\frac{1}{2}\rho_0 v_0^2}$ Compressible	$\frac{p-p_0}{\frac{1}{2}\rho_0 v_0^2}$ Incompressible
0	0.0000	0.0000	0.0000	2.4493	0.0000	0.0000	1.1175	1.0000
5	.5079	.9412	−.0504	2.3542	.4104	.3546	.9640	.8316
10	.8207	1.4940	−.0541	2.1253	.7381	.6741	.5805	.4559
15	.9299	1.6509	−.0397	1.8598	.9627	.9250	.1410	.0732
20	.9199	1.5576	.0060	1.6162	1.1055	1.1101	−.2260	−.2222
30	.7844	1.2790	.1095	1.2473	1.2473	1.3115	−.6590	−.5556
40	.6468	.9927	.1930	1.0064	1.2938	1.3851	−.8193	−.6739
50	.5249	.7756	.2438	.8465	1.2970	1.3940	−.8386	−.6821
60	.4345	.6215	.2615	.7876	1.2776	1.3681	−.7826	−.6320
70	.3630	.5038	.2504	.6619	1.2433	1.3218	−.6816	−.5474
80	.3052	.4130	.2170	.6106	1.2026	1.2649	−.5576	−.4463
90	.2571	.3409	.1693	.5783	1.1566	1.2026	−.4229	−.3378
100	.2163	.2818	.1156	.5629	1.1087	1.1393	−.2576	−.2293
110	.1507	.2321	.0635	.5646	1.0611	1.0780	−.1589	−.1260
120	.1491	.1893	.0189	.5865	1.0158	1.0210	−.0422	−.0318
130	.1205	.1516	−.0141	.6360	.9743	.9701	.0592	.0507
140	.0940	.1175	−.0335	.7299	.9334	.9269	.1432	.1195
150	.0692	.0860	−.0393	.9090	.9090	.8922	.2068	.1737
160	.0456	.0564	−.0334	1.2972	.8873	.8670	.2556	.2126
170	.0226	.0279	−.0190	2.5166	.8740	.8516	.2837	.2361
180	.0000	.0000	.0000	.0000	.8636	.8389	.3067	.2439

[1] Jacobian $=\frac{269+69\cos\delta}{23\cos(\delta/2)\sqrt{818-782\cos\delta}}$

TABLE III

$\frac{v_0}{c_0}$	$\left(\frac{v_P}{v_0}\right)_{max}$ $\epsilon=0.05$	$\left(\frac{v_P}{v_0}\right)_{max}$ $\epsilon=0.15$	[1] $\left(\frac{v}{v_0}\right)_{crit}$
0.0	1.120	1.300	0.000
.2	1.122	1.307	4.575
.4	1.130	1.333	2.815
.5	1.136	1.350	1.870
.6	1.143	1.372	1.574
.7	1.151	1.400	1.366
.8	1.161	1.430	1.211
1.0	1.195	1.503	1.000

[1] $\left(\frac{v}{v_0}\right)^2_{crit}=\frac{2}{\gamma+1}\frac{1}{\mu}+\frac{\gamma-1}{\gamma+1}$, with $\gamma=1.408$

or $\left(\frac{v}{v_0}\right)^2_{crit}=\frac{0.83056}{\mu}+0.16944$

TABLE IV

COORDINATES OF THE AIRFOILS

δ (deg.)	$\epsilon=0.05$ $\frac{\xi}{\text{chord}}$	$\epsilon=0.05$ $\frac{\eta}{\text{chord}}$	$\epsilon=0.15$ $\frac{\xi}{\text{chord}}$	$\epsilon=0.15$ $\frac{\eta}{\text{chord}}$
0	[1] 1.0000	0.0000	1.0000	0.0000
10	.9930	.0080	.9938	.0213
20	.9723	.0155	.9751	.0414
30	.9474	.0220	.9444	.0592
40	.8917	.0270	.9023	.0739
50	.8339	.0304	.8495	.0845
60	.7662	.0318	.7871	.0907
70	.6904	.0315	.7163	.0922
80	.6084	.0295	.6386	.0891
90	.5226	.0261	.5558	.0820
100	.4355	.0218	.4695	.0715
110	.3496	.0171	.3830	.0589
120	.2678	.0124	.2980	.0454
130	.1928	.0082	.2178	.0324
140	.1271	.0049	.1457	.0211
150	.0732	.0025	.0849	.0123
160	.0331	.0010	.0358	.0062
170	.0084	.0008	.0098	.0025
180	[2] .0000	.0000	.0000	.0000

[1] Nose. [2] Tail.