TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOW PAST ELLIPTIC CYLINDERS

REPORT No. 624

TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOW PAST ELLIPTIC CYLINDERS

By Carl Kaplan

SUMMARY

The method of Poggi is used to calculate, for perfect fluids, the effect of compressibility upon the flow on the surface of an elliptic cylinder at zero angle of attack and with no circulation. The result is expressed in a closed form and represents a rigorous determination of the velocity of the fluid at the surface of the obstacle insofar as the second approximation is concerned.

Comparison is made with Hooker's treatment of the same problem according to the method of Janzen and Rayleigh and it is found that, for thick elliptic cylinders, the two methods agree very well. The labor of computation is, moreover, considerably reduced by the present solution.

The third approximation to the compressible flow about circular cylinders, including the terms involving the factor $(v_0/c_0)^4$, *is also obtained and compared with the result given by Poggi. It is found that the expression given by Poggi is incomplete with regard to the terms containing the factor* $(v_0/c_0)^4$.

INTRODUCTION

The purpose of this paper is to employ the method of Poggi (reference 1) to determine the effect of compressibility on the flow about elliptic cylinders. This problem has already been considered by Hooker (reference 2) who made use of the method of Janzen and Rayleigh but, owing to the necessity for expanding a certain function in the analysis, the "thickness ratio" of the ellipse to which his result applies is limited. The thickness ratio of an ellipse is defined as the ratio b/a, where a and b are the semimajor and semiminor axes, respectively. The method of Poggi, on the other hand, not only permits an unrestricted thickness ratio but also reduces the labor of computation.

Briefly, it may be said that Poggi considers compressible flow to be replaced by an incompressible flow due to a distribution of sinks and sources throughout the region of flow. The strength of the distribution in the plane of the profile is given by

$$-\frac{1}{4\pi c^2}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial v^2}{\partial\xi}+\frac{\partial\varphi}{\partial\eta}\frac{\partial v^2}{\partial\eta}\right)d\xi d\eta$$

and in the plane of the circle, into which the profile is mapped by a suitable conformal transformation, by

$$\frac{1}{4\pi c^2}\left(v_r\frac{\partial v^2}{\partial\lambda}-\frac{v_\theta}{\lambda}\frac{\partial v^2}{\partial\theta}\right)\frac{R}{\lambda}d\lambda d\theta$$

where

r, θ are the polar coordinates of a point in the plane $z(=x+iy)$ of the circle.

R, δ the radius of the circle into which the profile is mapped and the angular coordinate on this circle, respectively.

$\lambda=\frac{R}{r}$; $v_r=-\frac{\partial\phi}{\partial r}$; $v_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}$; ϕ is the velocity potential of the flow.

v, the magnitude of the velocity of the fluid in the plane of the profile.

c, the magnitude of the local velocity of sound.

Poggi then finds that the total velocity induced, at any point $P(R, \delta)$ of the circular boundary by the foregoing system of sinks and sources, is:

$$\Delta v=\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\frac{v_r\frac{\partial v^2}{\partial\lambda}-\frac{v_\theta}{\lambda}\frac{\partial v^2}{\partial\theta}}{c^2\{1-2\lambda\cos(\theta-\delta)+\lambda^2\}}\sin(\theta-\delta)\,d\lambda d\theta \quad (1)$$

Poggi's method of approximating the compressible flow of a perfect fluid is based on the assumption that the incompressible flow is a suitable first approximation and that therefore the values pertaining to that flow may be substituted for v_r, v_θ, and v^2 in equation (1). The value of Δv thus obtained then represents the effect due to compressibility and is to be added to the already known value for the velocity of the incompressible flow. That is,

$$v_{comp}=v_{incomp}+\Delta v \quad (2)$$

It is to be noted that, in equation (1), the local velocity of sound c is not a constant but is related to the velocity v of the fluid in the plane of the profile by means of Bernoulli's equation and the equation of state of the fluid. Thus, if the adiabatic equation of state is adopted,

$$c^2=c_0^2\left[1+\frac{\gamma-1}{2}\frac{v_0^2}{c_0^2}\left(1-\frac{v^2}{v_0^2}\right)\right] \quad (3)$$

where c_0, v_0 are the corresponding magnitudes in the undisturbed stream and $\gamma=1.408$ for air.

In order to facilitate the solution of equation (1), it has been the custom to replace c by c_0. This simplification of the problem may be justified by the following argument. It has been tacitly understood that nowhere in the fluid must the velocity of the fluid exceed that of the local velocity of sound since the incompressible flow has already been assumed to be a good first approximation and the effect of compressibility is merely to distort the streamlines associated with the incompressible flow. As the maximum fluid velocity occurs at the surface of the obstacle, there exists a value of v_0^2/c_0^2 for which the maximum fluid velocity equals that of the local velocity of sound. This critical velocity of the fluid is obtained from equation (3) by replacing v by c. Thus

$$c_{least}^2 = v_{crit}^2 = \frac{2c_0^2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\frac{v_0^2}{c_0^2}\right) \tag{4}$$

This value for c is a lower limit under the condition that nowhere in the fluid is the local velocity of sound exceeded. The maximum value of c occurs at the stagnation point $v=0$ and is given by

$$c_{max}^2 = c_0^2\left(1+\frac{\gamma-1}{2}\frac{v_0^2}{c_0^2}\right) \tag{5}$$

Thus both the maximum and the least values of c occur on the obstacle and everywhere else $c_{max} > c > c_{least}$. It follows from equations (4) and (5) that

$$\frac{c_{max}-c_{least}}{c_0} = 0.0887\left(1+0.204\frac{v_0^2}{c_0^2}\right)^{\frac{1}{2}}$$

which increases very slowly as v_0/c_0 approaches unity. In fact, it is seen that the upper limit for $\frac{c_{max}-c_{least}}{c_0}$ is 0.0973. The foregoing discussion thus shows that c/c_0 may, as a first approximation, be taken to be unity. Equation (1) then becomes

$$\Delta c = \frac{1}{2\pi c_0^2}\int_0^1\int_0^{2\pi} \frac{v_r\frac{\partial c^2}{\partial\lambda} - \frac{v_\theta}{\lambda}\frac{\partial c^2}{\partial\theta}}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\sin(\theta-\delta)\,d\lambda\, d\theta \tag{6}$$

THE FLOW OF A PERFECT COMPRESSIBLE FLUID PAST AN ELLIPTIC CYLINDER

Let the ζ plane be the plane of the ellipse and the z plane be the plane of the corresponding circle. Then it is well known that the Joukowski transformation

$$\zeta = z + \frac{a^2}{z} \tag{7}$$

maps the circle of radius a with its center at the origin of the z plane into the line segment $(-2a, 0; 2a, 0)$ in the ζ plane. Also, the circles concentric with the circle of radius a are transformed into a family of confocal ellipses with common foci at $(-2a, 0)$ and $(2a, 0)$. If $R(>a)$ denotes the radius of one of these circles, then the semimajor and semiminor axes of the ellipse into which it is transformed are, respectively, $R+\frac{a^2}{R}$ and $R-\frac{a^2}{R}$. The thickness ratio t then becomes:

$$t = \frac{R-\frac{a^2}{R}}{R+\frac{a^2}{R}} = \frac{1-\sigma^2}{1+\sigma^2}$$

or

$$\sigma^2 = \frac{1-t}{1+t}$$

where

$$\sigma = \frac{a}{R}$$

If w denotes the complex potential of the incompressible flow in the z plane when a stream of velocity v_0 impinges on a circle of radius R in the direction of the negative x axis, then

$$w = v_0\left(z+\frac{R^2}{z}\right) \tag{8}$$

The complex velocity in the ζ plane is then given by

$$\frac{dw}{d\zeta} = \frac{dw}{dz}\frac{dz}{d\zeta}$$

or

$$\frac{dw}{d\zeta} = v_0\frac{z^2-R^2}{z^2-a^2} \tag{9}$$

When $\lambda = \frac{R}{r}$ and $\sigma = \frac{a}{R}$ are introduced, it follows that

$$v^2 = \left|\frac{dw}{d\zeta}\right|^2 = v_0^2\frac{1-2\lambda^2\cos 2\theta+\lambda^4}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4} \tag{10}$$

Following Poggi's procedure, the Fourier development of v^2/v_0^2 will be obtained. Thus, by the use of the expansion

$$\frac{1}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4} = \frac{1}{1-\sigma^4\lambda^4}\left[1+2\sum_{n=1}^{\infty}(\sigma\lambda)^{2n}\cos 2n\theta\right]$$

(see appendix, sec. I),
it follows that

$$\frac{v^2}{v_0^2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_{2n}\cos 2n\theta \tag{11}$$

where

$$a_0 = 2\frac{1+(1-2\sigma^2)\lambda^4}{1-\sigma^4\lambda^4}$$

and for $n=1, 2, \ldots\ldots$

$$a_{2n} = 2\frac{(1-\sigma^2)(\sigma^2\lambda^4-1)}{\sigma^2(1-\sigma^4\lambda^4)}(\sigma\lambda)^{2n}$$

Also from equation (8)

$$\left.\begin{aligned} v_r &= -v_0(1-\lambda^2)\cos\theta \\ v_\theta &= v_0(1+\lambda^2)\sin\theta \end{aligned}\right\} \tag{12}$$

Then, inserting the expressions for r^2, r_r, and r_θ given by equations (11) and (12) into equation (6) and making use of the integrals

$$\int_0^{2\pi}\frac{\sin(\theta-\delta)}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\sin n\theta d\theta=\begin{bmatrix}0 & \text{if } n=0\\ \pi\lambda^{n-1}\cos n\delta & \text{if } n\geqq 1\end{bmatrix}$$

(see appendix, sec. II)

$$\int_0^{2\pi}\frac{\sin(\theta-\delta)}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\cos n\theta d\theta=\begin{bmatrix}0 & \text{if } n=0\\ -\pi\lambda^{n-1}\sin n\delta & \text{if } n\geqq 1\end{bmatrix}$$

it follows without difficulty that

$$\frac{\Delta v}{v_0}=\frac{\mu}{2}\Bigg[-\sin\delta+\sum_{n=0}^{\infty}(2n+1)\sin(2n+1)\delta$$

$$\int_0^1(\lambda^{2n+1}a_{-2n}-\lambda^{2n-1}a_{2n+2})d\lambda\Bigg] \tag{13}$$

where

$$\mu=\frac{v_0^2}{c_0^2}$$

Substituting for the a_{2n}'s from equation (11), equation (13) takes the form

$$\frac{\Delta v}{v_0}=\frac{\mu}{2}\frac{1-\sigma^2}{\sigma^2}\Bigg[\sin\delta-(1-\sigma^2)\sum_{n=0}^{\infty}(2n+1)\sin(2n+1)\delta$$

$$\int_0^1\frac{1-\sigma^2\lambda^4}{1-\sigma^4\lambda^4}(\sigma^2\lambda^4)^n d\lambda^2\Bigg]$$

Replacing λ^2 by τ, for purposes of integration only, it follows that

$$I=\sum_{n=0}^{\infty}(2n+1)\sin(2n+1)\delta\int_0^1\frac{1-\sigma^2\lambda^4}{1-\sigma^4\lambda^4}(\sigma^2\lambda^4)^n d\lambda^2$$

$$=-R.P.\text{ of } i\sum_{n=0}^{\infty}(2n+1)e^{i(2n+1)\delta}\int_0^1\frac{1-\sigma^2\tau^2}{1-\sigma^4\tau^2}(\sigma\tau)^{2n}d\tau$$

or

$$I=R.P.\text{ of } ie^{i\delta}\int_0^1\frac{1+\sigma^2\tau^2e^{2i\delta}}{(1-\sigma^2\tau^2e^{2i\delta})^2}\frac{1-\sigma^2\tau^2}{1-\sigma^4\tau^2}d\tau$$

$$=\frac{1}{(1-2\sigma^2\cos 2\delta+\sigma^4)^2}\Bigg\{\frac{(1-\sigma^2)^2}{2\sigma^2}[(1+3\sigma^2+\sigma^4)\sin\delta$$

$$+\sigma^2\sin 3\delta]\log\frac{1+\sigma^2}{1-\sigma^2}$$

$$+\frac{1-\sigma^2}{\sigma}[(1+\sigma^4)\cos 2\delta-2\sigma^2]\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}$$

$$-\frac{(1+\sigma^2)(1-\sigma^2)^2}{2\sigma}\sin 2\delta\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2}$$

$$+2[(1+\sigma^2+\sigma^4)\sin\delta-\sigma^2\sin 3\delta]\Bigg\}$$

Therefore

$$\frac{\Delta v}{v_0}=\frac{\mu}{2}\frac{1-\sigma^2}{\sigma^2}\Bigg(\sin\delta$$

$$-\frac{1-\sigma^2}{(1-2\sigma^2\cos 2\delta+\sigma^4)^2}\Bigg\{\frac{(1-\sigma^2)^2}{2\sigma^2}\Big[(1+3\sigma^2+\sigma^4)\sin\delta$$

$$+\sigma^2\sin 3\delta\Big]\log\frac{1+\sigma^2}{1-\sigma^2}$$

$$-\frac{(1+\sigma^2)(1-\sigma^2)^2}{2\sigma}\sin 2\delta\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2}$$

$$+\frac{1-\sigma^2}{\sigma}\Big[(1-\sigma^4)\cos 2\delta-2\sigma^2\Big]\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}$$

$$+2\Big[(1+\sigma^2+\sigma^4)\sin\delta-\sigma^2\sin 3\delta\Big]\Bigg\}\Bigg) \tag{14}$$

For $\delta=\frac{\pi}{2}$, the position of maximum velocity on the surface of the elliptic cylinder,

$$\left(\frac{\Delta v}{v_0}\right)_{\delta=\frac{\pi}{2}}=\frac{\mu}{2}\frac{1-\sigma^2}{\sigma^2}\Bigg\{1-\frac{1-\sigma^2}{(1+\sigma^2)^2}\Bigg[\frac{(1-\sigma^2)^2}{2\sigma^2}\log\frac{1+\sigma^2}{1-\sigma^2}$$

$$-2\frac{1-\sigma^2}{\sigma}\tan^{-1}\sigma+2\Bigg]\Bigg\} \tag{15}$$

It is interesting to note that the expression for $\Delta v/v_0$ at the surface of a circular cylinder fixed in a stream of velocity v_0 impinging on it in the direction of the negative x axis may be obtained from equation (14) by allowing $\sigma\left(=\frac{a}{R}\right)$ to approach zero. Thus, making use of the expansions

$$\frac{1}{(1-2\sigma^2\cos 2\delta+\sigma^4)^2}=\frac{1}{(1-\sigma^4)^3}\Bigg\{(1+\sigma^4)+2\sum_{n=1}^{\infty}\Big[(n+1)$$

$$-(n-1)\sigma^4\Big]\sigma^{2n}\cos 2n\delta\Bigg\}\text{ (see appendix, sec. III)}$$

$$\log\frac{1+2\sigma\cos\delta+\sigma^2}{1-2\sigma\cos\delta+\sigma^2}=4\sum_{n=0}^{\infty}\frac{\sigma^{2n+1}}{2n+1}\cos(2n+1)\delta$$

$$\tan^{-1}\frac{2\sigma\sin\delta}{1-\sigma^2}=2\sum_{n=0}^{\infty}\frac{\sigma^{2n+1}}{2n+1}\sin(2n+1)\delta$$

$$\log\frac{1+\sigma^2}{1-\sigma^2}=2\sum_{n=0}^{\infty}\frac{\sigma^{2(2n+1)}}{2n+1}$$

it follows, neglecting terms containing powers of σ higher than the second, that

$$\frac{\Delta v}{v_0}=\frac{\mu}{2}(1-\sigma^2)\left(\frac{4}{3}\sin\delta-\sin 3\delta\right)$$

or

$$\lim_{\sigma\to 0}\frac{\Delta v}{v_0}=\mu\left(\frac{2}{3}\sin\delta-\frac{1}{2}\sin 3\delta\right) \tag{16}$$

This expression for $\Delta v/v_0$ agrees with that obtained by the methods of Janzen, Rayleigh, and Poggi (reference 3).

The effect of compressibility, i. e., $\Delta v/v_0$, having been found, it follows according to equation (2) that the total velocity at the circular boundary in the z plane is given by

$$\left(\frac{v}{v_0}\right)_{circle}=2\sin\delta+\frac{\Delta v}{v_0} \tag{17}$$

and on the elliptic profile in the ζ plane by

$$\left(\frac{v}{v_0}\right)_{ellipse}=\frac{1}{(1-2\sigma^2\cos 2\delta+\sigma^4)^{1/2}}\left(\frac{v}{v_0}\right)_{circle} \tag{18}$$

139778—39 —17

Table I shows the comparison between the values of $(v/v_0)_{ellipse}$ calculated according to equation (18) and those obtained by Hooker for an ellipse of thickness ratio $t=\frac{1}{2}$ or $\sigma^2=\frac{1}{3}$. The values for the corresponding incompressible flow are included. It is seen that the results of the two methods agree very well. This agreement is not unexpected since Hooker's method is particularly applicable to thick ellipses. Consider, however, a slender ellipse, say $t=\frac{1}{10}$ or $\sigma^2=\frac{9}{11}$. Table II shows the comparison between the exact calculations of the present method and the results obtained according to Hooker's method. The disagreement is more evident than that shown in table I for the thicker ellipse.

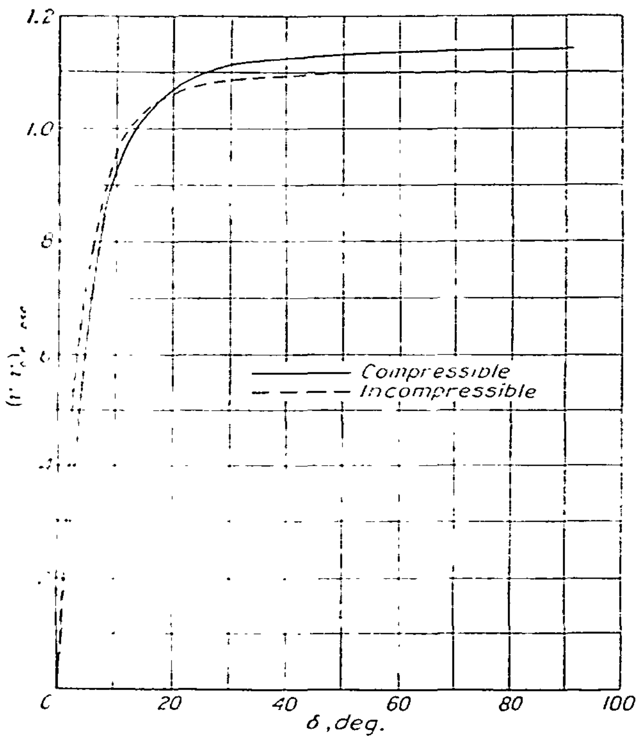

FIGURE 1.—The velocity of the fluid on the surface of an elliptic cylinder of thickness ratio 1/10 for compressible and incompressible flows with $v_0/c_0=0.857$.

Figure 1 shows the graph $(v/v_0)_{ellipse}$ calculated according to Poggi's method for both the compressible and the incompressible flows past the ellipse of thickness ratio 1/10 with $v_0/c_0=0.857$.

TABLE I

$\frac{v_0}{c_0}=0.5$; thickness ratio $=\frac{1}{2}$

δ (deg.)	$\left(\frac{v}{v_0}\right)_{ellipse}$		
	Hooker's method	Poggi's method	Incompressible
0	0	0	0
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	1.[illegible]10	1.[illegible]59
[illegible]	1.[illegible]	1.[illegible]	1.3417
[illegible]	1.[illegible]	1.[illegible]	1.4[illegible]2
[illegible]	1.[illegible]	1.[illegible]	1.48[illegible]7
[illegible]	1.[illegible]67	1.[illegible]	1.[illegible]

TABLE II

$\frac{v_0}{c_0}=0.857$; thickness ratio $=\frac{1}{10}$

δ (deg.)	$\left(\frac{v}{v_0}\right)_{ellipse}$		
	Hooker's method	Poggi's method	Incompressible
0	0	0	0
5	.6793	.6342	.7244
10	.9005	.9395	.9569
15	.9904	1.0056	1.0307
20	1.0433	1.0749	1.0608
30	1.1080	1.1146	1.0839
40	1.1332	1.1271	1.0924
50	1.1320	1.1338	1.0962
60	1.1273	1.1374	1.0983
70	1.1346	1.1394	1.0994
80	1.1501	1.1403	1.0999
90	1.1582	1.1406	1.1000

THE PRESSURE DISTRIBUTION

According to Bernoulli's theorem and the adiabatic equation of state, if p and ρ are the pressure and density of the fluid, then

$$\frac{p}{p_0}=\left(\frac{\rho}{\rho_0}\right)^{\gamma}=\left[1+\frac{\gamma-1}{2}\mu\left(1-\frac{v^2}{v_0^{\,2}}\right)\right]^{\frac{\gamma}{\gamma-1}}$$

where p_0 and ρ_0 are the pressure and density, respectively, in the undisturbed stream. Expanding the right-hand side of the foregoing equation and neglecting

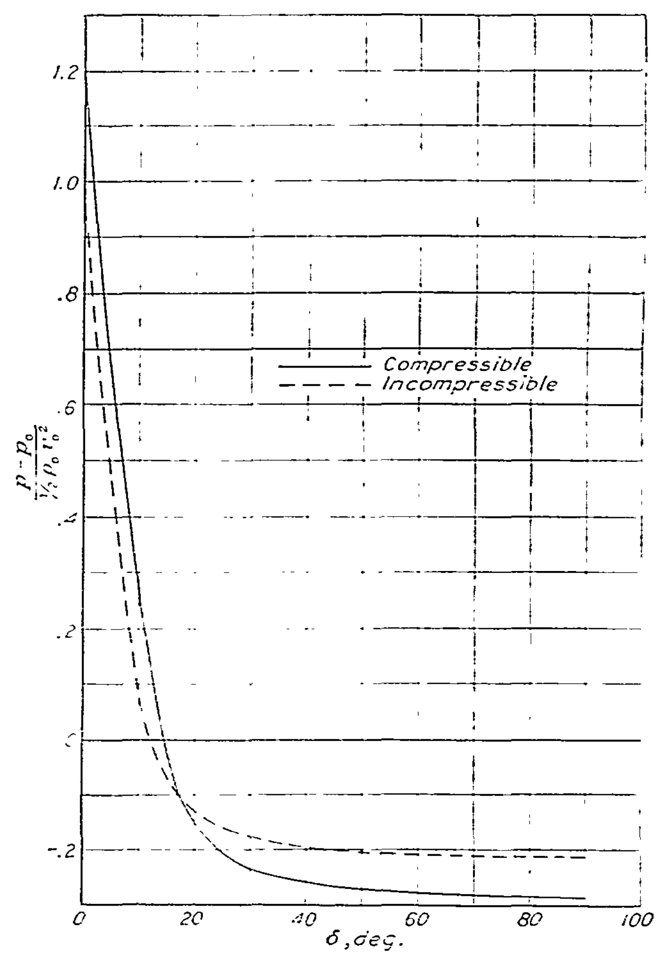

FIGURE 2.—The pressure of the fluid on the surface of an elliptic cylinder of thickness ratio 1/10 for compressible and incompressible flows with $v_0/c_0=0.857$

terms involving powers of μ higher than the first yields

$$\frac{p-p_0}{\frac{1}{2}\rho_0 v_0^2}=\left(1-\frac{v^2}{v_0^2}\right)+\frac{\mu}{4}\left(1-\frac{v^2}{v_0^2}\right)^2+\ldots \tag{19}$$

The pressure distribution is then obtained by substituting for v/v_0 from equation (18). Table III shows the pressure distribution over the surface of an ellipse of thickness ratio 1/10 with $v_0/c_0=0.857$, and figure 2 shows the graph of this distribution together with the one due to the corresponding incompressible flow.

TABLE III

$\frac{v_0}{c_0}=0.857$; thickness ratio = 1/10

δ (deg.)	$\frac{p-p_0}{\frac{1}{2}\rho v_0^2}$	
	Compressible	Incompressible
0	1.1836	1.0000
5	.6635	.4753
10	.1198	.0843
15	−.0113	−.0623
20	−.1511	−.1253
30	−.2315	−.1718
40	−.2570	−.1933
50	−.2706	−.2017
60	−.2779	−.2063
70	−.2818	−.2087
80	−.2838	−.2098
90	−.2843	−.2100

THE ATTAINMENT OF THE LOCAL VELOCITY OF SOUND AT THE SURFACE OF AN ELLIPTIC CYLINDER

According to equation (4) the critical velocity of the fluid is given by

$$\left(\frac{v_{crit}}{v_0}\right)^2=\frac{2}{\gamma+1}\,\frac{1}{\mu}+\frac{\gamma-1}{\gamma+1} \tag{20}$$

For an elliptic cylinder, at zero angle of attack, the critical velocity occurs at $\delta=\frac{\pi}{2}$, the position of maximum velocity on the cylinder and also in the region of flow. Hence substituting from equation (18) for $(v/v_0)_{ellipse}$ at $\delta=\frac{\pi}{2}$ yields a cubic equation in the variable μ.

Thus, from equation (15), if

$$f(\sigma)=\frac{1-\sigma^2}{2\sigma^2}\left\{1-\frac{1-\sigma^2}{(1+\sigma^2)^2}\left[\frac{(1-\sigma^2)^2}{2\sigma^2}\log\frac{1+\sigma^2}{1-\sigma^2}\right.\right.$$
$$\left.\left.-2\frac{1-\sigma^2}{\sigma}\tan^{-1}\sigma+2\right]\right\}$$

then

$$[f(\sigma)]^2\mu^3+4f(\sigma)\mu^2+\left[4-\frac{\gamma-1}{\gamma+1}(1+\sigma^2)^2\right]\mu$$
$$-\frac{2(1+\sigma^2)^2}{\gamma+1}=0 \tag{21}$$

where $\gamma=1.408$ for air.

Table IV gives the critical values of v_0/c_0 for the entire range of thickness ratios including the limiting cases of the straight-line segment and the circular profile. Figure 3 shows the critical values of $v_0/c_0(=\sqrt{\mu})$ plotted against the thickness ratio.

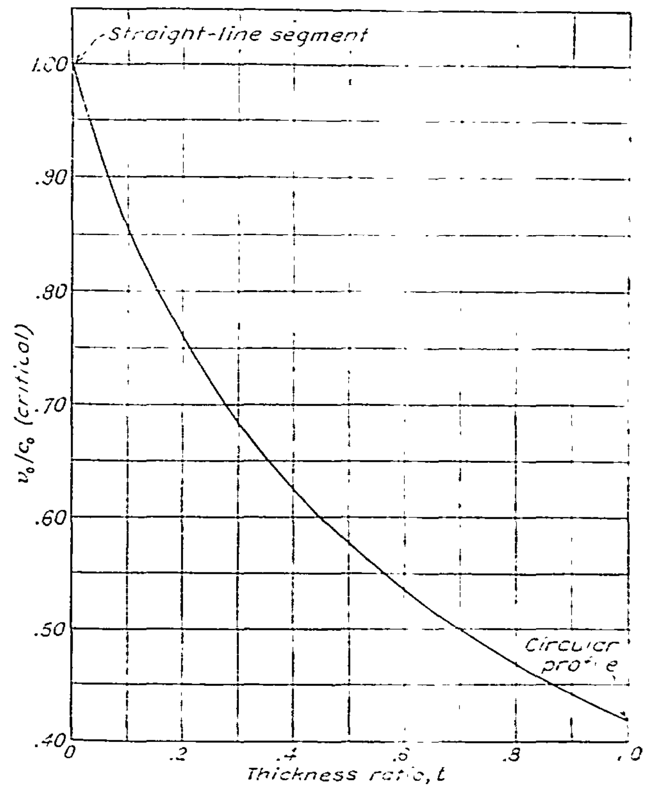

FIGURE 3.—The critical ratio v_0/c_0 as a function of the thickness ratio t.

TABLE IV

σ^2	Thickness ratio	$\frac{v_0}{c_0}$ (critical)
1	0	1.000
19/21	1/20	.919
9/11	1/10	.857
4/5	1/9	.845
7/9	1/8	.830
3/4	1/7	.811
5/7	1/6	.788
2/3	1/5	.779
3/5	1/4	.719
1/2	1/3	.663
1/3	1/2	.577
1/5	2/3	.512
1/7	3/4	.485
1/19	9/10	.444
0	1	.420

THE THIRD APPROXIMATION TO THE COMPRESSIBLE FLOW ABOUT CIRCULAR CYLINDERS

In reference 2, the opinion is expressed by Hooker that the terms involving $(v_0/c_0)^4$, thus far neglected, may become of considerable importance as the local velocity of sound is approached on the ellipse. Hooker, however, did not investigate the matter any further. In reference 4, Poggi calculated these terms for the compressible flow about a circular cylinder, but a close examination of his work shows that not all such terms were taken into account. In what follows the terms neglected by Poggi will be obtained and compared with the already existing ones.

The fundamental integral equation (1) may be written as follows:

$$\frac{\Delta v}{v_0}=\frac{\mu}{2\pi}\int_0^1\int_0^{2\pi}\frac{\frac{v_r}{v_0}\frac{\partial\frac{v^2}{v_0^2}}{\partial\lambda}-\frac{1}{\lambda}\frac{v_\theta}{v_0}\frac{\partial\frac{v^2}{v_0^2}}{\partial\theta}}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\left[1-\frac{\gamma-1}{2}\mu\left(1-\frac{v^2}{v_0^2}\right)+\ldots\right]\sin(\theta-\delta)d\lambda d\theta \quad (22)$$

where $1/c^2$ has been replaced by a power series in $\mu(=v_0^2/c_0^2)$ obtained from equation (3); i. e.:

$$\frac{1}{c^2}=\frac{1}{c_0^2}\left[1-\frac{\gamma-1}{2}\mu\left(1-\frac{v^2}{v_0^2}\right)+\ .\ .\ .\ .\ \right]$$

The method followed by Poggi was to substitute for v_r, v_θ, and v^2 expressions pertaining to the incompressible flow and thus obtain the following result: [1]

$$\frac{\Delta v}{v_0}=\left(\frac{2}{3}\sin\delta-\frac{1}{2}\sin 3\delta\right)\mu+(\gamma-1)\left(\frac{23}{120}\sin\delta-\frac{11}{40}\sin 3\delta+\frac{1}{8}\sin 5\delta\right)\mu^2+\ .\ . \quad (23)$$

The velocity for the compressible flow at the surface of the circular cylinder then becomes:

$$\frac{v_{comp}}{v_0}=\frac{v_{incomp}}{v_0}+\frac{\Delta v}{v_0} \quad (24)$$

where $\Delta v/v_0$ is given by equation (23).

Equation (24) thus represents the second approximation to the compressible flow, the first approximation being the purely incompressible flow given by v_{incomp}/v_0.

The third approximation may be obtained, at least in principle, by substituting for v_r, v_θ, and v^2 in equation (22) expressions based on the second approximation. Such expressions, as far as the terms involving μ are concerned, are given in reference 3 and are as follows:

$$\frac{v_r}{v_0}=-(1-\lambda^2)\cos\theta-\mu\left[\left(-\frac{13}{12}\lambda^2+\frac{3}{2}\lambda^4-\frac{5}{12}\lambda^6\right)\cos\theta+\left(\frac{1}{4}\lambda^2-\frac{1}{4}\lambda^4\right)\cos 3\theta\right]+\ .\ .\ .$$

$$[illegible]\ (1+\lambda^2)\sin\theta+\mu\left[\left(\frac{13}{12}\lambda^2-\frac{1}{2}\lambda^4+\frac{1}{12}\lambda^6\right)\sin\theta+\left(-\frac{3}{4}\lambda^2+\frac{1}{4}\lambda^4\right)\sin 3\theta\right]+\ .\ .$$

$$[illegible]\ 1\ [illegible]\ \lambda\ [illegible]\cos 2\ [illegible]\ \mu\left[\left(\frac{19}{6}\lambda^4-\frac{7}{3}\lambda^6-\frac{1}{2}\lambda^8\right)+\left([illegible]\ \frac{1}{3}\lambda^8\right)\cos 2\theta+\lambda^2\cos 4\theta\right]+\ .\ .$$

[1] The [illegible] were obtained for elliptic cylinders [illegible] that they reduce to those given in equation (23) for the circle. However, [illegible] reference 3 the coefficients of sin δ and sin 5δ are given, respectively, as [illegible] and [illegible] owing to a slight error in the calculations. Pistolesi (reference 3) gives for the coefficient of sin [illegible] the value [illegible] which, in view of the independent check of Poggi's result [illegible] the authors, is believed to be a misprint.

where it is recalled that $\lambda=\frac{R}{r}$.

When the foregoing expressions are substituted into equation (22) and only the terms involving μ and μ^2 are evaluated, it is found that, besides the terms given by equation (23), the following ones involving μ^2 must be included:

$$\mu^2\left(\frac{37}{40}\sin\delta-\frac{25}{24}\sin 3\delta+\frac{3}{8}\sin 5\delta\right) \quad (25)$$

These terms seem to have been overlooked by both Poggi and Pistolesi (reference 3).

The third approximation to the compressible flow at the surface of the circular cylinder then becomes:

$$\frac{v_{comp}}{v_0}=2\sin\delta+\left(\frac{2}{3}\sin\delta-\frac{1}{2}\sin 3\delta\right)\mu+\left[\left(\frac{37}{40}\sin\delta-\frac{25}{24}\sin 3\delta+\frac{3}{8}\sin 5\delta\right)+(\gamma-1)\left(\frac{23}{120}\sin\delta-\frac{11}{40}\sin 3\delta+\frac{1}{8}\sin 5\delta\right)\right]\mu^2+\ .\ . \quad (26)$$

It is interesting to compare the magnitudes of the various terms in equation (26) at the position of maximum velocity $\delta=\pi/2$ and for the critical value $\mu=0.1670$ (obtained by means of equations (20) and (26)). Thus

$$(2\sin\delta)_{\delta=\frac{\pi}{2}}=2$$

$$\mu\left(\frac{2}{3}\sin\delta-\frac{1}{2}\sin 3\delta\right)_{\delta=\frac{\pi}{2}}=0.1948$$

$$\mu^2\left(\frac{37}{40}\sin\delta-\frac{25}{24}\sin 3\delta+\frac{3}{8}\sin 5\delta\right)_{\delta=\frac{\pi}{2}}=0.0653$$

$$\mu^2(\gamma-1)\left(\frac{23}{120}\sin\delta-\frac{11}{40}\sin 3\delta+\frac{1}{8}\sin 5\delta\right)_{\delta=\frac{\pi}{2}}=0.0067$$

Thus, it is seen that the terms involving μ^2 do become of importance with regard to the μ terms as the local velocity of sound is approached on the circle and that the main contribution is made by expression (25).

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *February 11, 1938.*

APPENDIX

I. The Fourier Expansion of $\frac{1}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4}$

If $2\cos 2\theta$ is replaced by $e^{2i\theta}+e^{-2i\theta}$, then

$$H=\frac{1}{1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4}=\frac{1}{(1-\sigma^2\lambda^2e^{2i\theta})}\cdot\frac{1}{(1-\sigma^2\lambda^2e^{-2i\theta})}$$

Since, by the binomial theorem,

$$(1-\sigma^2\lambda^2e^{2i\theta})^{-1}=\sum_{j=0}^{\infty}(\sigma^2\lambda^2)^j e^{2ij\theta}$$

and

$$(1-\sigma^2\lambda^2e^{-2i\theta})^{-1}=\sum_{k=0}^{\infty}(\sigma^2\lambda^2)^k e^{-2ik\theta}$$

it follows that

$$H=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(\sigma^2\lambda^2)^{j+k}e^{2i(j-k)\theta}$$

Let

$$j+k=n$$

and therefore

$$j-k=n-2k,\quad j=n-k$$

The double series then becomes

$$H=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(\sigma^2\lambda^2)^n e^{2i(n-2k)\theta}$$

The terms of this series can be grouped in pairs such that

$$H=2\sum_{0}^{\infty}\sum_{k=0}^{\frac{n}{2},\frac{n-1}{2}}(\sigma^2\lambda^2)^n\cos(n-2k)2\theta \qquad (1)$$

where $\frac{n}{2}$ or $\frac{n-1}{2}$ is the upper limit according as n is even or odd and where the factor 2 is omitted from the term for which n is even and $k=\frac{n}{2}$. This term is independent of θ and there is only one such term, not two.

The series (1) may be written as

$$H=\sum_{n=0}^{\infty}\sum_{k=0}^{\frac{n}{2},\frac{n-1}{2}}A_n\cos(n-2k)2\theta$$

where

$$A_n=2(\sigma^2\lambda^2)^n$$

Expanding this series and rearranging the terms in the form of a Fourier series,

$$H=\frac{1}{2}\sum_{n=0}^{\infty}A_{2n}+\sum_{n=1}^{\infty}\cos 2n\theta\sum_{k=0}^{\infty}A_{n+2k}$$

But

$$\frac{1}{2}\sum_{n=0}^{\infty}A_{2n}=\sum_{n=0}^{\infty}(\sigma^4\lambda^4)^n=\frac{1}{1-\sigma^4\lambda^4}$$

and

$$\sum_{k=0}^{\infty}A_{n+2k}=2(\sigma^2\lambda^2)^n\sum_{k=0}^{\infty}(\sigma^4\lambda^4)^k=\frac{2(\sigma^2\lambda^2)^n}{1-\sigma^4\lambda^4}$$

Therefore

$$H=\frac{1}{1-\sigma^4\lambda^4}\left[1+2\sum_{n=1}^{\infty}(\sigma\lambda)^{2n}\cos 2n\theta\right] \qquad (2)$$

II. The Integrals

$$J_1=\int_0^{2\pi}\frac{\sin(\theta-\delta)}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\cos n\theta d\theta$$

and

$$J_2=\int_0^{2\pi}\frac{\sin(\theta-\delta)}{1-2\lambda\cos(\theta-\delta)+\lambda^2}\sin n\theta d\theta$$

If $2\cos(\theta-\delta)$ is replaced by $e^{i(\theta-\delta)}+e^{-i(\theta-\delta)}$, then

$$\frac{1}{1-2\lambda\cos(\theta-\delta)+\lambda^2}=\frac{1}{(1-\lambda e^{i(\theta-\delta)})(1-\lambda e^{-i(\theta-\delta)})}$$

$$=\frac{1}{e^{i(\theta-\delta)}-e^{-i(\theta-\delta)}}\left\{\frac{e^{i(\theta-\delta)}}{1-\lambda e^{i(\theta-\delta)}}-\frac{e^{-i(\theta-\delta)}}{1-\lambda e^{-i(\theta-\delta)}}\right\}$$

$$=\frac{1}{2i\sin(\theta-\delta)}\left[\sum_{m=0}^{\infty}\lambda^m e^{i(m+1)(\theta-\delta)}-\sum_{m=0}^{\infty}\lambda^m e^{-i(m+1)(\theta-\delta)}\right]$$

Therefore

$$J_1+iJ_2=\frac{1}{2i}\int_0^{2\pi}\left\{\sum_{m=0}^{\infty}\lambda^m e^{i(m+1)(\theta-\delta)}-\sum_{m=0}^{\infty}\lambda^m e^{-i(m+1)(\theta-\delta)}\right\}e^{in\theta}d\theta$$

Replacing $e^{i\theta}$ by z,

$$J_1+iJ_2=-\frac{1}{2}\oint\left\{\sum_{m=0}^{\infty}\lambda^m e^{-i(m+1)\delta}z^{m+n}\right.$$

$$\left.-\sum_{m=0}^{\infty}\lambda^m e^{i(m+1)\delta}z^{-(m-n+2)}\right\}dz$$

Since

$$\oint z^p dz=\begin{cases}0, \text{ in general}\\ 2\pi i, \text{ when } p=-1\end{cases}$$

it follows that $m=n-1$ and therefore

$$J_1+iJ_2=\pi i\lambda^{n-1}e^{in\delta}$$

Hence, for $n\geqq 1$,

$$J_1=-\pi\lambda^{n-1}\sin n\delta \text{ and } J_2=\pi\lambda^{n-1}\cos n\delta \qquad (3)$$

III. The Fourier Expansion of

$$\frac{1}{(1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4)^2}$$

In analogy to section I, replace $2\cos 2\theta$ by $e^{2i\theta}+e^{-2i\theta}$. Then

$$H^2=\frac{1}{(1-2\sigma^2\lambda^2\cos 2\theta+\sigma^4\lambda^4)^2}$$

$$=\frac{1}{(1-\sigma^2\lambda^2e^{2i\theta})^2(1-\sigma^2\lambda^2e^{-2i\theta})^2}$$

According to the binomial theorem

$$(1-\sigma^2\lambda^2e^{2i\theta})^{-2}=\sum_{j=0}^{\infty}(j+1)(\sigma^2\lambda^2)^j e^{2ij\theta}$$

and

$$(1-\sigma^2\lambda^2e^{-2i\theta})^{-2}=\sum_{k=0}^{\infty}(k+1)(\sigma^2\lambda^2)^k e^{-2ik\theta}$$

Therefore

$$H^2=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(j+1)(k+1)(\sigma^2\lambda^2)^{j+k}e^{2i(j-k)\theta}$$

Let

$$j+k=n$$

and therefore

$$j-k=n-2k,\ j=n-k$$

Then

$$H^2=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n-k+1)(k+1)(\sigma^2\lambda^2)^n e^{2i(n-2k)\theta}$$

The exponent of e is $2i[(n-k)-k]\theta$. If k and $n-k$ are interchanged, the exponent of e changes sign but the coefficient of e remains unaltered. The terms can therefore be grouped in pairs so that:

$$H^2=2\sum_{n=0}^{\infty}\sum_{k=0}^{\frac{n}{2},\frac{n-1}{2}}(n-k+1)(k+1)(\sigma^2\lambda^2)^n\cos(n-2k)2\theta \quad (4)$$

where the factor 2 is omitted from the term for which n is even and $k=\frac{n}{2}$.

The series (4) may be written as

$$H^2=\sum_{n=0}^{\infty}\sum_{k=0}^{\frac{n}{2},\frac{n-1}{2}}A_{n,k}\cos(n-2k)2\theta$$

where

$$A_{n,k}=2(n-k+1)(k+1)(\sigma^2\lambda^2)^n$$

Expanding this series and rearranging the terms in the form of a Fourier series,

$$H^2=\frac{1}{2}\sum_{k=0}^{\infty}A_{2k,k}+\sum_{n=1}^{\infty}\cos 2n\theta\sum_{k=0}^{\infty}A_{n+2k,k}$$

But

$$\frac{1}{2}\sum_{k=0}^{\infty}A_{2k,k}=\sum_{k=0}^{\infty}(k+1)^2(\sigma^4\lambda^4)^k=\frac{1+\sigma^4\lambda^4}{(1-\sigma^4\lambda^4)^3}$$

and

$$\sum_{k=0}^{\infty}A_{n+2k,k}=2(\sigma^2\lambda^2)^n\sum_{k=0}^{\infty}(n+k+1)(k+1)(\sigma^4\lambda^4)^k$$

$$=2(\sigma^2\lambda^2)^n\frac{(n+1)-(n-1)\sigma^4\lambda^4}{(1-\sigma^4\lambda^4)^3}$$

Therefore

$$H^2=\frac{1}{(1-\sigma^4\lambda^4)^3}\Big\{(1+\sigma^4\lambda^4)+2\sum_{n=1}^{\infty}[(n+1)$$

$$-(n-1)\sigma^4\lambda^4](\sigma^2\lambda^2)^n\cos 2n\theta\Big\} \quad (5)$$

REFERENCES

1. Poggi, Lorenzo: Campo di velocitá in una corrente piana di fluido compressibile. Parte II.—Caso dei profili ottenuti con rappresentazione conforme dal cerchio ed in particolare dei profili Joukowski. L'Aerotecnica, vol. XIV, fasc. 5, May 1934, pp. 532–550.
2. Hooker, S. G.: The Two-Dimensional Flow of Compressible Fluids at Sub-Sonic Speeds Past Elliptic Cylinders. R. & M. No. 1684, British A. R. C., 1936.
3. Pistolesi, Enrico: La portanza alle alte velocitá inferiori a quella del suono. Reale Accademia d'Italia, Classe delle Scienze Fisiche, Matematiche e Naturali, vol. XIII, 1935.
4. Poggi, Lorenzo: Campo di velocitá in una correnta piana di fluido compressibile. L'Aerotecnica, vol. XII, fasc. 12, Dec. 1932, pp. 1579–1593.