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A GRAPHICAL METHOD OF DETERMINING PRESSURE DISTRIBUTION IN TWO-DIMENSIONAL FLOW

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SUMMARY

By a generalization of the Joukowski method, a procedure is developed for effecting localized modifications of airfoil shapes and for determining graphically the resultant changes in the pressure distribution. The application of the procedure to the determination of the pressure distribution over airfoils of original design is demonstrated. Formulas for the lift, the moment, and the aerodynamic center are also given.

INTRODUCTION

It is possible, by a simple geometric construction, to modify any given streamline shape in such a way that the effect of the modification on the pressure distribution, the lift, and the moment can be readily determined. The construction is essentially that used in deriving the familiar Joukowski airfoils from a circle (reference 1) although it may be applied to an airfoil shape to introduce modifications of the outline. The method is based on the concept of complex numbers but its application requires no familiarity with them.

By two or more successive applications of the construction to a circle, it is possible to derive shapes of such diversity as to permit the approximation of nearly any airfoil of current design. For the airfoil derived in this way the pressure distribution can be determined exactly, so that the only error likely to occur is in the reproduction of the exact airfoil shape. Any inaccuracy in this approximation is immediately apparent and can be made as small as is considered desirable.

MODIFICATION OF AN AIRFOIL

The method consists entirely in applications of a single construction; desired effects are obtained by the proper choice of a parameter k and of the location of the axes with respect to the figure to be transformed. This basic construction applied to an airfoil surface is demonstrated in figure 1. In order to find the point z_a on the modified airfoil to correspond to a point z on the original airfoil, the vector Oz is drawn from the origin and the vector $O\frac{k^2}{z}$ is added to it. The vector $O\frac{k^2}{z}$, which will be termed the "reciprocal" vector, is constructed with its length equal to k^2 times the reciprocal of the length of Oz and at an angle $-\phi$ with the x-axis, where ϕ is the angle made with this axis by Oz. A point of the modified airfoil, then, is located by the resultant of the vector Oz and its reciprocal vector.

The construction by which a figure is modified applies equally well to all its streamlines, altering them to conform to the distortion. The change in spacing of the streamlines near the boundary of the figure will show directly the effect of the transformation on the velocity in that region because the velocity varies inversely with the spacing. The factor by which the elements of length near a point z are changed may be shown to be the ratio of the length of the diagonal from
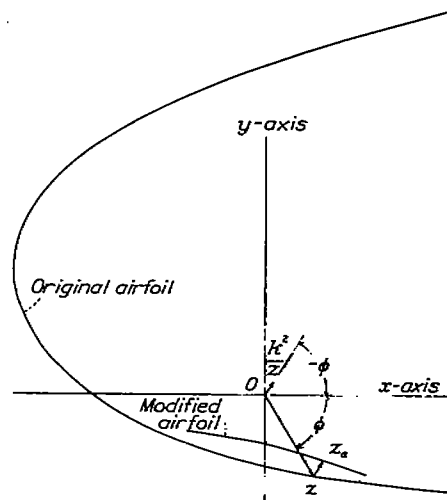

FIGURE 1.—Construction to determine a point on the modified airfoil.

k^2/z to z (the vector $z-k^2/z$) to the vector Oz. Then the ratio of the velocity at a point $z+k^2/z$ to the velocity at the corresponding point z of the original figure is the inverse of this factor.

The problem of determining the pressure distribution is reduced by Bernoulli's relation $P = P_0 - \rho V^2/2$ to that of finding the distribution of the velocity. If the velocity distribution over the original figure is known, the velocity distribution over the modified figure may then be found by applying the ratio $\dfrac{|z|}{\left|z - \dfrac{k^2}{z}\right|}$

at each point used in the construction. The vectors z and $z-k^2/z$ have already been used in transforming these points and are therefore directly measurable.

Strictly speaking, the method as outlined is applicable only to potential flows. It is reasonable to suppose, however, that the actual velocity distribution over the modified airfoil may be obtained with good accuracy

from a distribution experimentally determined for the original airfoil, provided that: (1) the experimental and the theoretical distributions do not differ too greatly and (2) the modification itself does not introduce too great a change in the distribution.

The requirements of each problem will suggest the proper choice of the axes and of k. A few general observations may be helpful in this connection. It is evident that the origin must be in the neighborhood of
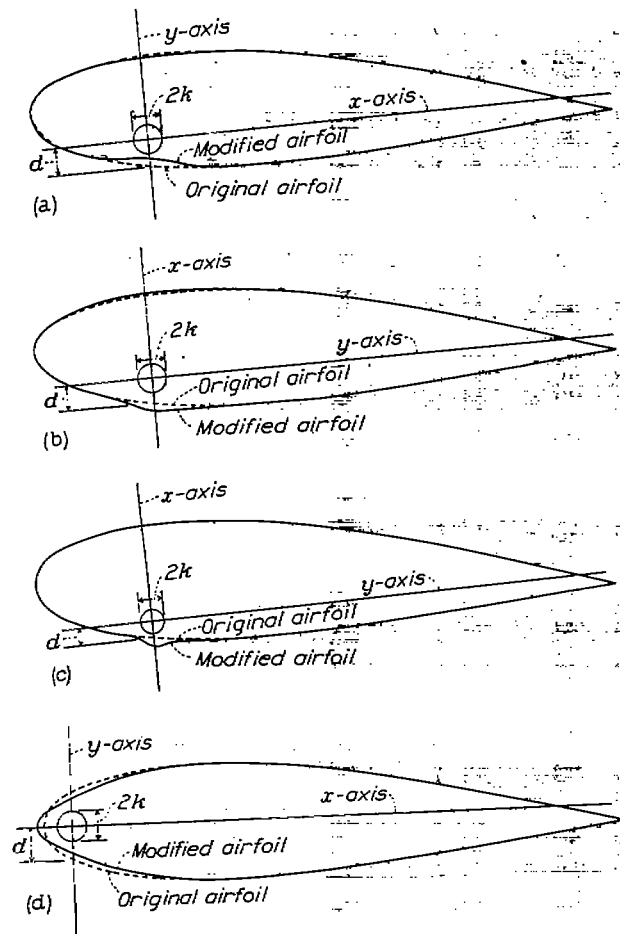

FIGURE 2.—The effect of varying the parameters of the construction.

the section to be modified since the point nearest the origin is shifted the greatest distance. The direction in which a point is shifted depends on the inclination of its vector to the axes, and the shift may have a relatively small component normal to the airfoil boundary. If one axis is roughly parallel to the section of the surface to be modified, the point of greatest deviation will usually be very near the intersection of the surface with the other axis. It will therefore be useful to note that the displacement of a point on either axis is k^2/d, where d is the distance of the origin from the point. Points on or near the x-axis are shifted outward from the origin; points near the y-axis are moved inward. The transformations of figure 2 show the nature of the modifications to be obtained by various choices of the axes and of k. Sharp modifications occur when the airfoil intersects either axis at a distance from the origin only slightly greater than k.

It is expected that the method as outlined will be useful in ascertaining the effect on the velocity distribution of a localized modification of an airfoil. It is possible, however, to approximate a more extensive modification to any desired degree of accuracy by successive applications of the same transformation, the velocity being calculated at each stage by the rule already given.
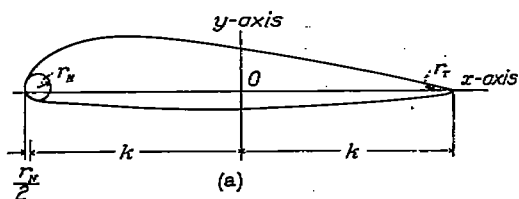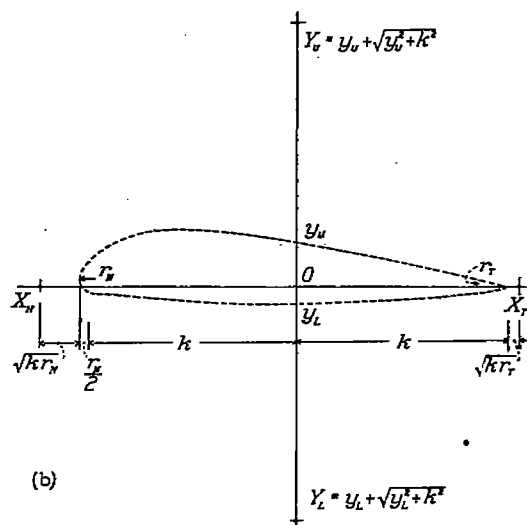
APPROXIMATION OF A GIVEN AIRFOIL

The foregoing discussion is concerned with the problem of effecting small modifications of existent airfoils for which the velocity distribution is known. It is sometimes required to predict the theoretical characteristics of an airfoil not derived in this way. In such a case, it is customary to use the known flow around a circle as a starting point. If the transformation discussed here is applied to a circle, the result is an oval shape with circular-arc camber, as demonstrated by Joukowski. (See reference 1.) It should be possible to proceed to modify this figure (or its special form, a Joukowski airfoil) as was done in the preceding section to an experimentally known airfoil. In general, however, it would require many steps to reproduce an arbitrary airfoil in this way because modifications of the shape already approaching an airfoil would, of necessity, be small and localized. On the other hand, if the procedure were reversed, it would be found that any airfoil may be derived by a single step from a figure closely approximating a circle, a figure which will hereinafter be called the "distorted circle" of the airfoil. The distorted circle may then be considered the result of modifying a circle, a result obtained in the same way as were the slightly modified airfoils of the preceding section.

For the first step in this process, the determination of the distorted circle corresponding to the desired airfoil, reference could be made to Theodorsen and Garrick (reference 2) or von Kármán and Burgers (reference 3), who give exact formulas for the distorted circle in terms of the airfoil coordinates. A graphical method based on the simpler transformation that is being used in this paper makes it possible, however, to obtain the distorted circle merely by trial.

The entire procedure is illustrated step by step in figure 3. In order to achieve a considerable simplification of the construction, the airfoil is considered in this figure and in the following discussion to have been drawn through the midpoints rather than through the terminals of the vectors $z+k^2/z$, so that they appear here as if to half scale.

The axes and the parameter k for the first step are found from the airfoil dimensions according to the relations given in figure 3(a). The x-axis is made to pass through the leading and the trailing edges to reduce to a minimum the distortion that will later have to be reproduced.
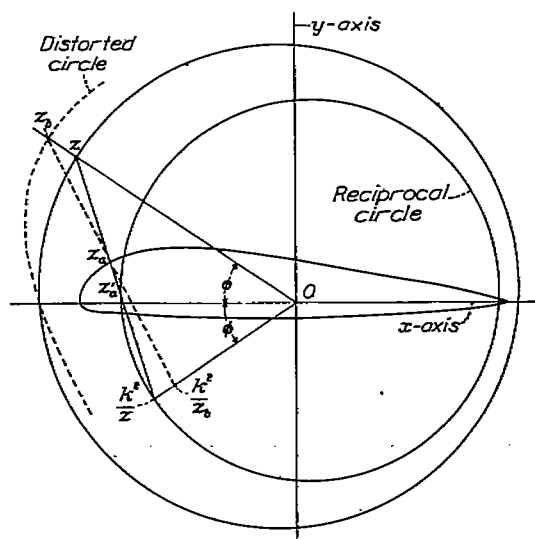
The axes and k having been chosen, the intersections of the distorted circle with the axes, X_N, X_T, Y_U, and

(a) Location of the axes and determination of the parameter k from the airfoil.
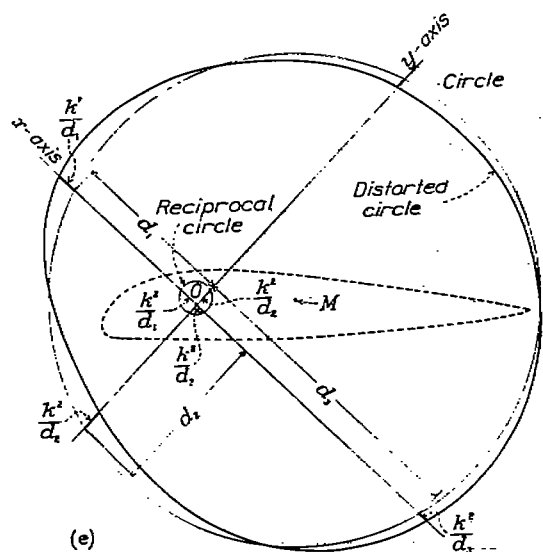
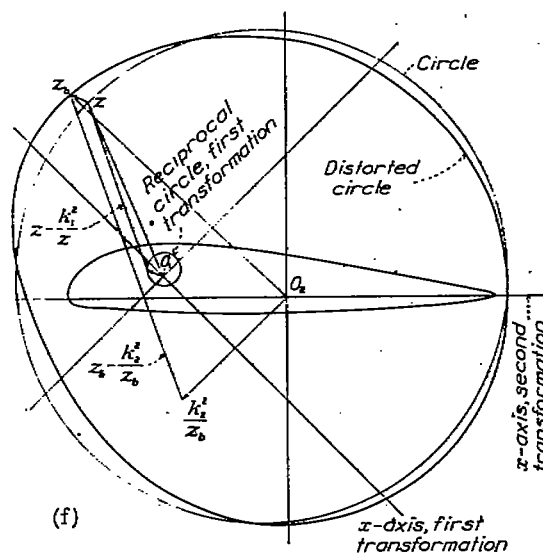(b) Location of the intersections of the distorted circle with the axes.
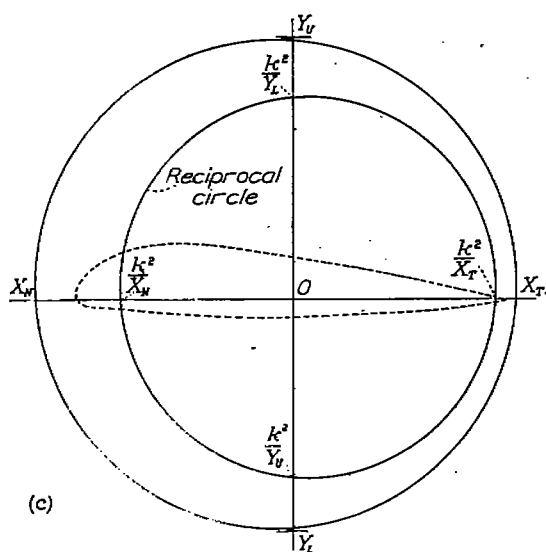
(c) Construction of the circle to serve as a first approximation of the distorted circle; the reciprocal circle.

(d) Construction of the distorted circle of the airfoil.

(e) Approximation of the distorted circle by a small modification of a circle.

(f) Reconstruction of the airfoil from the circle, showing the lines needed in computing the velocity distribution.

FIGURE 3.—Steps in the approximation of a given airfoil by the transformation of a circle.

Γ_L, are easily calculated (fig. 3(b)). The construction then proceeds as follows:

In order to serve as a first approximation of the distorted circle, a circle is drawn to fit the four intersections with the axes as closely as possible. It should be remembered that, because this step is a first approximation, great accuracy is not a consideration. When the vectors z follow this circle, the vectors k^2/z, constructed as previously described, are found to terminate in a smaller circle, which is termed for convenience the "reciprocal" circle. This reciprocal circle may be easily located from its intersections with the axes (fig. 3 (c)).

Vectors Oz and $O\frac{k^2}{z}$ are then drawn from the origin to the circle and to the reciprocal circle at equal angles below and above the x-axis (fig. 3(d)). The midpoint z_a' of the diagonal $z-k^2/z$ connecting the ends of these vectors will not, in general, fall exactly on the airfoil. A point of the distorted circle z_b is located by adjusting with dividers the length of Oz_b along Oz so that the diagonal to its reciprocal is bisected by the airfoil. The point of the airfoil z_a at which this bisection occurs will actually be indistinguishable from the intersection of the first diagonal with the airfoil; this fact is useful in making the correction. Because the corrections to the vectors Oz and $O\frac{k^2}{z}$ are in approximately the same ratio as the vectors, the decrease in k^2/z that corresponds to an increase in the length of Oz may be estimated and the reciprocal relation be maintained by eye. Any desired degree of accuracy may be obtained by checking numerically the lengths of the vectors.

The distorted circle obtained in this way must now be approximated by applying a transformation of the form $z+k^2/z$ to the circle most nearly approximating it. For this purpose the circle already drawn for the preceding step will usually be satisfactory, although it may have to be shifted slightly to obtain a more convenient distribution of the distortion. Depressions should occur as nearly as possible on opposite sides of the circle, with distended portions between them. (See fig. 3(e).) The x-axis should then be passed through the distended portions and the y-axis, through the depressions. Some adjustment will be needed to satisfy the condition that the segments of the axes cut off by the circle and the corresponding distortions be approximately in inverse proportion, that is, that their product at each intersection be constant. The parameter k for the transformation is then the square root of this product.

Carrying out the transformation of the circle at this point serves at once to check the estimated parameters or to suggest an adjustment and to provide the construction lines that will be needed to determine the velocity distribution. Similarly, the distorted circle thus obtained from the circle should be reduced to the desired airfoil.

In figure 3(f) the complete double transformation and the lines necessary for the computation of the veloc-ity distribution are shown applied to a point of the circle of figure 3(e).

Occasionally an airfoil will give rise to a distorted circle that cannot be obtained by a single modification of a circle. In such a case an additional transformation may be applied either to achieve the desired distortion of the circle or to modify locally an airfoil that can be derived by two transformations.

VELOCITY DISTRIBUTION OVER AN AIRFOIL DERIVED FROM A CIRCLE

It is apparent that a large variety of useful airfoil shapes can be obtained by two or more successive applications to a circle of the transformation $'z+k^2/z$. (Two typical examples are shown in fig. 4.) The first transformation primarily determines the general outline of the airfoil. The second transformation reduces the figure to the dimensions of an airfoil and determines the nose radius, the thickness, and the camber. It may be of interest to note that in the first transformation k is small relative to the radius of the circle; in the second transformation the ratio of k to the radius of the circle is only slightly less than 1.

The method of finding the velocity at a point on an airfoil derived in this manner is deduced from the following considerations:

The velocity at a point z on the circle is given by the formula

$$V_z = 2V_0 \sin\theta + \frac{\Gamma}{2\pi r}$$

where r is the radius of the circle, V_0 is the wind velocity, and θ is the angle that the radius to the point z makes with the direction of the air stream. The circulation Γ is determined by the Kutta condition. Then

$$V_z = 2V_0(\sin\theta - \sin\theta_0)$$

where θ_0 equals the angle between the air stream and the radius to $z_{T.E.}$, the point of the circle that transforms to the trailing edge of the airfoil. This expression for V_z lends itself readily to graphical evaluation as a part of the construction; the velocity factor $(\sin\theta - \sin\theta_0)$ is proportional to the ordinate of the circle measured from a line drawn through $z_{T.E.}$ and parallel to the wind velocity V_0.

The velocity at the corresponding point of the transformed figure has been shown in a preceding section to be $\frac{|z|}{\left|z-\frac{k^2}{z}\right|}V_z$. The "stretching factor" $\frac{|z|}{\left|k-\frac{k^2}{z}\right|}$ will have to be applied again to the velocities over the distorted circle to transform the flow to that over the airfoil. A simple procedure is to plot the stretching factor alone for the first transformation, as a function of the angular position of the transformed point, with the lengths $|z|$ and $\left|z-\frac{k^2}{z}\right|$ measured from the construction. The stretching factor for the second transformation could then be applied at convenient points to values of the
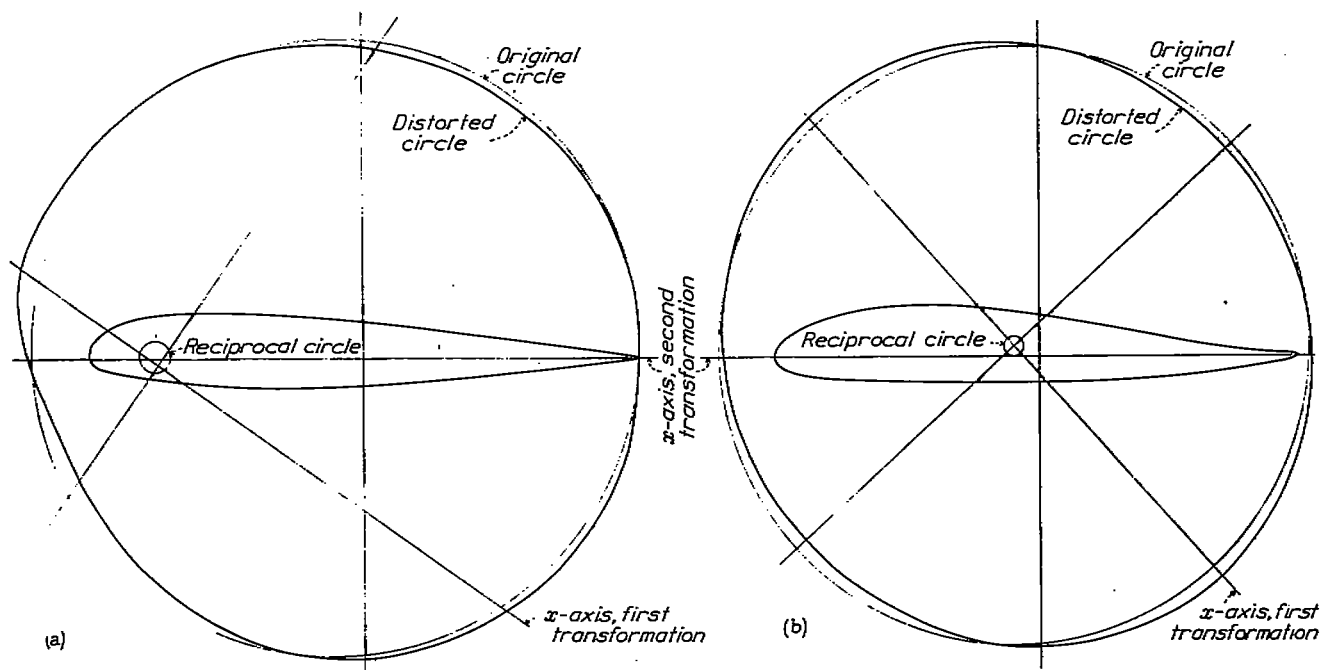
FIGURE 4.—Airfoils derived by two applications of transformation to circle. Airfoils are shown half scale with respect to circles.

stretching factor for the first transformation read from the plot. The resultant stretching factor obtained in this way can be directly applied to the velocity at points of the circle to give the velocity distribution over the airfoil at any angle of attack.

Figure 5 shows a sample pressure distribution ovei an airfoil (the NACA 4412 airfoil) approximated by two transformations of a circle. Comparison is made with the distribution derived by the theory of reference 2 and with experimental results taken from reference 4. The theoretical distributions were computed at an angle of attack of 6.4°, which corresponds to a geometric angle of attack of 8.5° for the finite-span airfoil of reference 4.

CONCLUSION

The method of this paper has been found useful in determining the effect of small modifications of airfoil shapes on the pressure distribution. It is apparent that a family of related shapes can be derived in this way with greater simplicity than by standard methods because the effect of the modification alone can be calculated for each shape. It is also possible to foresee the manner in which a shape must be modified to produce a desired change in the pressure distribution. The method is, in fact, reversible and by it an airfoil may be designed to have a predetermined pressure
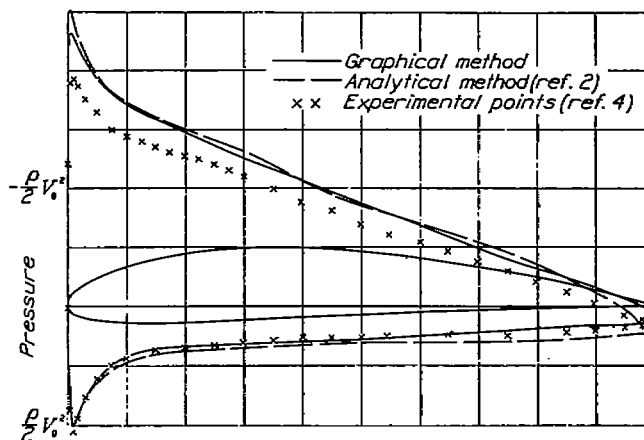

FIGURE 5.—Theoretical and experimental pressure distributions over NACA 4412 airfoil. $\alpha_a=6.4°$.

distribution, provided that a somewhat similar airfoil is already known. The modified pressure distributions obtained in this way have closely checked with experiment.

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    LANGLEY FIELD, VA., *November 12, 1940.*

APPENDIX

FORMULAS FOR LIFT AND MOMENT

In the case of an airfoil derived from a circle, the method of derivation, interpreted in terms of complex numbers, makes it possible to give concise formulas for the lift and the moment in terms of the parameters used in the construction and to locate by simple geometric methods the theoretical aerodynamic center. The general theory for a transformation of the form

$$z_a = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \qquad (1)$$

has been developed by von Mises (reference 5, ch. VII). In order to apply the formulas of von Mises, it is necessary to express the resultant of two successive transformations in a series of the foregoing form.

Let $z_b = z + k_1^2/z$ express the first transformation, z being a point on the circle or one of its streamlines and z_b, the corresponding point associated with the distorted circle. The axes for the second transformation are described by the complex parameter z_1, which locates the origin, and β, the angle between the two sets of axes.

Then, if primes denote the vectors drawn to the new origin,

$$z_b' = (z_b - z_1)e^{-i\beta} \qquad (2)$$

Substitute

$$z_b = z + \frac{k_1^2}{z}$$

Then

$$z_b' = \left(z + \frac{k_1^2}{z} - z_1\right)e^{-i\beta} \qquad (3)$$

which is the point on the distorted circle located with respect to the axes for the second transformation. Apply the second transformation to z_b'; then

$$z_a' = z_b' + \frac{k_2^2}{z_b'} \qquad (4)$$

gives the corresponding point associated with the airfoil. Substitute for z_b' from equation (3). Equation (4) then becomes

$$z_a' = \left(z + \frac{k_1^2}{z} - z_1\right)e^{-i\beta} + \frac{k_2^2 e^{i\beta} z}{z^2 - z_1 z + k_1^2} \qquad (5)$$

In order to restore the wind velocity V_0 to its original magnitude and direction, it is necessary to return to the original axes.

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$$z_a' = (z_a - z_1)e^{-i\beta} \qquad (6)$$

and the inverse transformation is

$$z_a = z_a' e^{i\beta} + z_1 \qquad (7)$$

Then

$$z_a = z + \frac{k_1^2}{z} + \frac{k_2^2 e^{2i\beta} z}{z^2 - z_1 z + k_1^2} \qquad (8)$$

This expression for the complete transformation can be expanded, by carrying out the division of the last term, into the series

$$z_a = z + \frac{k_1^2 + k_2^2 e^{2i\beta}}{z} + \frac{k_2^2 e^{2i\beta} z_1}{z^2} + \cdots \qquad (9)$$

which is in the form of equation (1), where

$$\left. \begin{array}{l} a_1 = k_1^2 + k_2^2 e^{2i\beta} \\ \\ a_2 = k_2^2 e^{2i\beta} z_1 \end{array} \right\} \qquad (10)$$

The formulas for the lift and the moment as given by Glauert (reference 5, pp. 84 and 85) may now be applied. The circulation, and consequently the lift, is unchanged by the transformation. Thus the lift depends only on the radius of the original circle.

$$L = \rho V_0 \Gamma = 8\pi r \frac{\rho}{2} V_0^2 \sin \theta_0 \qquad (11)$$

or $4\pi r \frac{\rho}{2} V_0^2 \sin \theta_0$ for the half-scale airfoil. The value of M_0, the moment about O (fig. 6), is given by the imaginary part of the expression (reference 5, p. 84)

$$\pi\rho \left[2a_1 V_0^2 e^{2i\alpha} - 2r^2 V_0^2 - \frac{m V_0 \Gamma}{\pi} i e^{i(\alpha-\delta)} - \frac{\Gamma^2}{4\pi^2} \right] \qquad (12)$$

where the quantities not already defined are as defined by the figure. Substitution for a_1 from equation (10) gives

$$\pi\rho \left[2(k_1^2 + k_2^2 e^{2i\beta})e^{2i\alpha} - \frac{m V_0 \Gamma}{\pi} i e^{i(\alpha-\delta)} \right] \qquad (13)$$

for expression (12), or

$$M_0 = 2\pi\rho V_0^2 [k_1^2 \sin 2\alpha + k_2^2 \sin 2(\alpha+\beta)] \\ - m\rho V_0 \Gamma \cos (\alpha-\delta) \qquad (14)$$

Since

$$\Gamma = 4\pi r V_0 \sin (\alpha - \alpha_0) \qquad (15)$$

$$M_0 = 2\pi\rho V_0^2 [k_1^2 \sin 2\alpha + k_2^2 \sin 2(\alpha+\beta) \\ - 2rm \cos (\alpha-\delta) \sin (\alpha-\alpha_0)] \qquad (16)$$
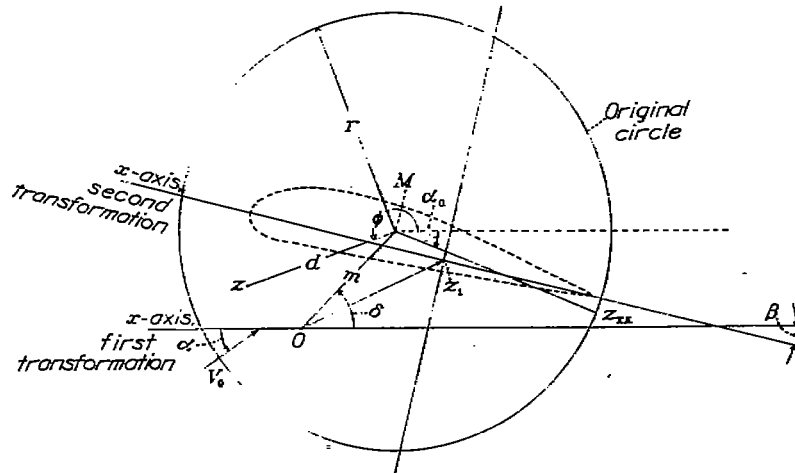
FIGURE 6.—Symbols used in development of lift and moment formulas.

In general, the moment about any point z (see fig. 6) is given by

$$M_z = 2\pi\rho V_0^2[k_1^2 \sin 2\alpha + k_2^2 \sin 2(\alpha+\beta) + 2rd \cos (\alpha-\phi) \sin (\alpha-\alpha_0)] \tag{17}$$

or

$$\frac{M_z}{2\pi\rho V_0^2} = [k_1^2 + k_2^2 \cos 2\beta + rd \cos (\phi+\alpha_0)] \sin 2\alpha$$
$$+ [k_2^2 \sin 2\beta - rd \sin (\phi+\alpha_0)] \cos 2\alpha$$
$$+ rd \sin (\phi-\alpha_0) \tag{18}$$

The location of the aerodynamic center (or focus) is determined by the condition that the moment about that point be independent of the angle of attack. In order to satisfy this condition, the coefficients of $\cos 2\alpha$ and of $\sin 2\alpha$ in equation (18) must vanish simultaneously.

Thus

$$\frac{1}{r}(k_1^2 + k_2^2 \cos 2\beta) = -d \cos (\phi+\alpha_0) \tag{19}$$

$$1\frac{k_2^2}{r} \sin 2\beta = d \sin (\phi+\alpha_0) \tag{20}$$

and

$$(\phi+\alpha_0) = \tan^{-1}\frac{-k_2^2 \sin 2\beta}{k_1^2 + k_2^2 \cos 2\beta}$$
$$d^2 = \frac{1}{r^2}(k_1^4 + 2k_1^2k_2^2 \cos 2\beta + k_2^4) \tag{21}$$

Equations (21) are found to have a simple geometric representation. If k_1^2/r and k_2^2/r are two sides of a triangle and $(180°-2\beta)$ is the angle between them, then d is the length of the third side and $(\phi+\alpha_0)$ is the supplement of the angle opposite k_1^2/r. This triangle may be used to locate the aerodynamic center directly on the construction, as shown in figure 7. As in the preceding illustrations, this airfoil has been drawn to half scale with respect to the circle and the distance d has therefore been bisected. The moment about the aerodynamic center
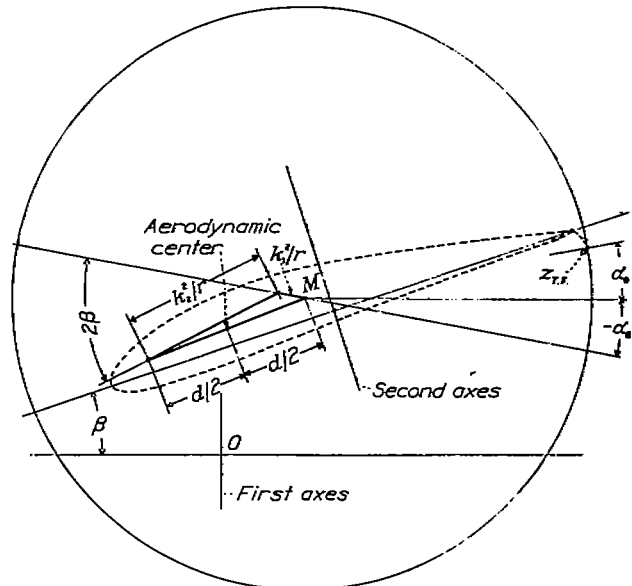

FIGURE 7.—Construction to find the aerodynamic center.

$$M_0 = 2\pi\rho V_0^2 rd \sin (\phi-\alpha_0) \tag{22}$$

is obtained directly from equation (18). The moment for the half-scale airfoil is one-quarter of this value.

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