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COMPRESSIBLE POTENTIAL FLOW WITH CIRCULATION ABOUT A CIRCULAR CYLINDER

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SUMMARY

The potential function for flow, with circulation, of a compressible fluid about a circular cylinder is obtained in series form including terms of the orders of M^4 where M is the Mach number of the free stream. The resulting equations are used to obtain pressure coefficient as a function of Mach number at a point on the surface of the cylinder for different values of circulation. The coefficients derived are compared with the Glauert-Prandtl and Kármán-Tsien approximations which are functions of the pressure coefficients of an incompressible fluid. For the cases considered, the values of the pressure coefficients computed from the theory were found to be somewhere between the two approximations, the first underestimating and the second overestimating it.

INTRODUCTION

In the two-dimensional irrotational flow of a compressible fluid, where the expansion is assumed to be adiabatic, the velocity potential is known to satisfy a nonlinear partial differential equation of the second order. For subsonic velocities, at least three methods are known for the approximate solution of this equation. They are usually denoted as the method of small perturbations, the Rayleigh-Janzen method, and the hodograph method.

The method of small perturbations (references 1 and 2) assumes that velocity changes which are brought about by the airfoil in the uniform parallel air stream are small in comparison with the velocity of the undisturbed stream. Under this assumption it is possible to introduce new variables which reduce the differential equation to a Laplace equation, and, as a consequence, the problem becomes one concerning flow in an incompressible fluid, provided the body is assumed distorted to correspond to the change of variables. The assumed distortion consists in expansion of the dimensions of the airfoil perpendicular to the direction of the free stream in the ratio $1/\sqrt{1-M^2}$, where M is the Mach number of the undisturbed stream.

The Rayleigh-Janzen method (references 3 and 4) assumes that the general expression for velocity potential may be written as a series in rising powers of M and with variable coefficients. These coefficients can be shown to satisfy certain Poisson differential equations and, if the equations are integrable, the solution becomes a matter of determining these coefficients. Successive steps, however, become increasingly laborious and the convergence of the series may be slow, even at relatively small Mach numbers, if the shape of the body is such that the speed of sound is approached locally. Solutions, using this method of attack, have been carried out by C. Kaplan (references 5 and 6), S. G. Hooker (reference 7), I. Imai (reference 8), K. Tamada and Y. Saito (reference 9), and L. Poggi (reference 10). Poggi introduced certain refinements and some of the preceding references employ this process. It is tantamount to using the so-called Neumann function in solving given Poisson equations and will be discussed in the appendix.

The hodograph method is ascribed by writers on that subject to P. Molenbrock and A. Tschaplygin. Instead of expressing the velocity potential as a function of coordinates in the Cartesian or polar plane, the magnitude of velocity V and its inclination θ to an assumed axis are chosen as independent variables. The resulting differential equation is linear and can be further simplified by replacing the pressure-volume relationship for adiabatic expansion by the equation of a line tangent at a point corresponding to the state of the fluid in the ambient stream. This artifice was suggested by T. von Kármán (references 2 and 11) and used successfully by Hsue-Shen Tsien (reference 12). K. Tamada (reference 13) has also applied Tsien's more general results on elliptic cylinders to compressible flow past a circular cylinder.

One noteworthy result of the hodograph method has been the Kármán-Tsien expression for pressure coefficient P in terms of Mach number M and $P_{M=0}$, the pressure coefficient for $M=0$. This expression may be written

$$P=P_{M=0}\frac{1}{\sqrt{1-M^2}+\dfrac{M^2}{1+\sqrt{1-M^2}}\dfrac{P_{M=0}}{2}} \tag{1}$$

It always gives, for negative pressure coefficients, a result greater in absolute value than the Glauert-Prandtl formula which is based on the method of small perturbations,

$$P=P_{M=0}\frac{1}{\sqrt{1-M^2}} \tag{2}$$

and is currently accepted as the more accurate of the two.

From equations (1) and (2) it is possible to compute the critical Mach number M_c, the value of M at which the local speed of sound is attained, in terms of $P_{M=0}$. The relations involving M_c and $P_{M=0}$, corresponding respectively to formulas (1) and (2), are

$$\frac{2}{\gamma M_c^2}\left\{\left(\frac{2}{\gamma+1}+\frac{\gamma-1}{\gamma+1}M_c^2\right)^{\frac{\gamma}{\gamma-1}}-1\right\}= P_{M=0}\frac{1}{\sqrt{1-M_c^2}+\frac{M_c^2}{1+\sqrt{1-M_c^2}}\frac{P_{M=0}}{2}} \tag{3}$$

and

$$P_{M=0}=\frac{2\sqrt{1-M_c^2}}{\gamma M_c^2}\left[\left(\frac{2}{\gamma+1}+\frac{\gamma-1}{\gamma+1}M_c^2\right)^{\frac{\gamma}{\gamma-1}}-1\right] \tag{4}$$

The difficulties inherent in the two latter procedures are quite as distinctive as their respective approaches to the problem. As stated before, the Rayleigh-Janzen method employs classical mathematics, the required terms being solutions of Poisson equations with given boundary conditions, but the work involved is arduous. In the hodograph method the principal difficulty is to determine proper boundary conditions in the V, θ plane. In available calculations the solution is given with a slight distortion in the given boundary. It is possible to correct this distortion, in some cases, so that the final results are not too seriously affected. When the flow around the body involves circulation, however, the change in the boundary is more serious, for nonperiodic terms appear and the boundary is no longer a closed curve. Added circulation does not involve any essential variations in the Rayleigh-Janzen method, however, and in this report the velocity potential for such compressible flow about a circular cylinder has been derived. Since no theoretical study has been presented, as far as is known, to determine the error in the Kármán-Tsien pressure coefficient, the results obtained in this report furnish a means of approaching this problem. The results of such calculations, for various values of circulation, are therefore included.

ANALYSIS

Consider a gas obeying the adiabatic law and flowing irrotationally in two dimensions. Its equation of motion may be written in polar coordinates in the form

$$\left[1-\frac{\gamma-1}{2}M^2\left(\frac{V^2}{U^2}-1\right)\right]\nabla^2\Phi=\frac{1}{2}\frac{M^2}{U^2}\left(\frac{\partial\Phi}{\partial r}\frac{\partial V^2}{\partial r}+\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}\frac{\partial V^2}{\partial\theta}\right) \tag{5}$$

where

Φ	velocity potential
$\nabla^2\Phi$	$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}$
γ	ratio of specific heats of gas
c_0	velocity of sound in undisturbed flow
U	velocity of free stream
M	Mach number of free stream $\left(\frac{U}{c_0}\right)$
V^2	local velocity squared $\left[\left(\frac{\partial\Phi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\Phi}{\partial\theta}\right)^2\right]$

By the introduction of the variables ϕ and v so that

$$\phi=\frac{\Phi}{U} \text{ and } v=\frac{V}{U}$$

equation (5) may be written in the form

$$\left[1-\frac{\gamma-1}{2}M^2(v^2-1)\right]\nabla^2\phi=\frac{1}{2}M^2\left(\frac{\partial\phi}{\partial r}\frac{\partial v^2}{\partial r}+\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial v^2}{\partial\theta}\right) \tag{6}$$

where

$$v^2=\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 \tag{7}$$

Following the method of Rayleigh and Janzen, assume that ϕ may be developed in a series of ascending powers of M^2 so that

$$\phi=\phi_0+M^2\phi_1+M^4\phi_2+\ .\ .\ .\ .\ . \tag{8}$$

After substitution of equation (8) in equation (7), elementary calculations show that

$$v^2=v_0^2+v_1^2M^2+v_2^2M^4+\ .\ .\ .\ .\ . \tag{9}$$

where

$$v_0^2=\left(\frac{\partial\phi_0}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\phi_0}{\partial\theta}\right)^2 \tag{10a}$$

$$v_1^2=2\left\{\frac{\partial\phi_0}{\partial r}\frac{\partial\phi_1}{\partial r}+\frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial\phi_1}{\partial\theta}\right\} \tag{10b}$$

$$v_2^2=2\left\{\frac{\partial\phi_0}{\partial r}\frac{\partial\phi_2}{\partial r}+\frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial\phi_2}{\partial\theta}\right\}+\left\{\left(\frac{\partial\phi_1}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\phi_1}{\partial\theta}\right)^2\right\} \tag{10c}$$

In a similar manner, equations (8) and (9) may be substituted in equation (6) and on equating coefficients of the same powers of M, the following relations for ϕ_0, ϕ_1, ϕ_2, . . . result:

$$\nabla^2\phi_0=0 \tag{11a}$$

$$\nabla^2\phi_1=\frac{1}{2}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial v_0^2}{\partial r}+\frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial v_0^2}{\partial\theta}\right) \tag{11b}$$

$$\nabla^2\phi_2=\frac{1}{2}(\gamma-1)(v_0^2-1)\nabla^2\phi_1+\frac{1}{2}\left(\frac{\partial\phi_0}{\partial r}\frac{\partial v_1^2}{\partial r}+\frac{1}{r^2}\frac{\partial\phi_0}{\partial\theta}\frac{\partial v_1^2}{\partial\theta}\right)+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial r}\frac{\partial v_0^2}{\partial r}+\frac{1}{r^2}\frac{\partial\phi_1}{\partial\theta}\frac{\partial v_0^2}{\partial\theta}\right) \tag{11c}$$

If the equations (11a), (11b), and (11c) can be solved successively for ϕ_0, ϕ_1, ϕ_2, . . . the values may be substituted in equation (8) to get the potential function for the flow of a compressible fluid. A step-by-step procedure is therefore established whereby any desired degree of approximation to ϕ may be obtained, provided the value of M is within the region of convergence of the resulting series. Equation (11a) is the differential equation satisfied by the potential function in the case of incompressibility. Once this potential function is known it is used to evaluate the right-hand member of the second equation, the solution of which furnishes the second term in the development of ϕ. The method of obtaining further terms follows the same general procedure.

Consider now the case of a right circular cylinder of infinite length in a compressible fluid, the axis of the cylinder being at right angles to the direction of steady flow. In determining the velocity distribution about the cylinder, the problem may be treated two dimensionally with a circle as the boundary curve and the equations established in the Rayleigh-Janzen method may be applied directly in the following manner. The radius of the circle is arbitrarily assumed equal to one, and a polar coordinate system is chosen with origin at the center of the circle and polar axis extending downstream. The flow about the circle is assumed to be that resulting from the combination of uniform stream velocity and circulation about the cylinder. Under these conditions the classical expression for ϕ_0 is well known. It may be written

$$\phi_0=\left(r+\frac{1}{r}\right)\cos\theta-\frac{\Gamma}{2\pi U}\theta$$

where Γ is the circulation around the circle, measured positive in a clockwise direction. For ease of computation it is convenient to set

$$\frac{\Gamma}{\pi U}=K$$

and, as a consequence,

$$\phi_o=\left(r+\frac{1}{r}\right)\cos\theta-\frac{K}{2}\theta \tag{12}$$

The boundary conditions, in general, are

$$\frac{\partial\phi}{\partial r}=0 \text{ for } r=1 \tag{13a}$$

and

$$\frac{\partial\phi}{\partial r}=\cos\theta \text{ for } r=\infty \tag{13b}$$

From equations (12) and (10a)

$$v_o^2=\left(1+\frac{1}{r^4}\right)-\frac{2}{r^2}\cos 2\theta+K\left(\frac{1}{r}+\frac{1}{r^3}\right)\sin\theta+\frac{K^2}{4r^2} \tag{14}$$

This result, together with equation (11b), gives

$$\nabla^2\phi_1=\left(\frac{-4}{r^5}+\frac{2}{r^7}\right)\cos\theta+\frac{2}{r^3}\cos 3\theta+$$
$$K\sin 2\theta\left(\frac{-1}{2r^2}-\frac{2}{2r^4}+\frac{1}{2r^6}\right)-\frac{K^2}{2r^3}\cos\theta \tag{15}$$

The more elementary methods of integration lead to certain difficulties when an attempt is made to solve for ϕ_1, in equation (15). These difficulties result from nonperiodic terms in the particular integral and resultant trouble in determining such constants of integration that the necessary periodicity, in terms of θ, is maintained in the final expression for the potential function. This difficulty may be obviated, however, by established methods. (See appendix.) It follows that the solution of

$$\nabla^2\Omega=\frac{\sin m\theta}{r^s} \tag{16}$$

satisfying the boundary conditions

$$\left(\frac{\partial\Omega}{\partial r}\right)_{r=1}=0,\left(\frac{\partial\Omega}{\partial r}\right)_{r=\infty}=0$$

is

$$\Omega=\frac{\sin m\theta}{m(m-s+2)(m+s-2)}\left\{\frac{(s-2)}{r^m}-\frac{m}{r^{s-2}}\right\}(\text{when } m+2\neq s) \tag{17a}$$

and

$$\Omega=\frac{-\sin m\theta}{mr^m}\left\{\frac{1}{2m}+\frac{1}{2}\log r\right\}(\text{when } m+2=s) \tag{17b}$$

The veracity of these solutions, together with analogous ones existing when $\sin m\theta$ is replaced by $\cos m\theta$, may be checked easily by substitution in equation (16).

Since equation (15) is a linear differential equation, its solution is determined by considering each term of the right-hand member and summing the individual integrals obtained by means of equations (17a) and (17b). The final result is

$$\phi_1=\cos\theta\left(\frac{13}{12r}-\frac{1}{2r^3}+\frac{1}{12r^5}\right)+$$
$$\cos 3\theta\left(\frac{-1}{4r}+\frac{1}{12r^3}\right)+K\sin 2\theta\left(\frac{1}{8}+\frac{1}{6r^2}+\frac{1}{24r^4}+\frac{\log r}{2r^2}\right)+$$
$$K^2\cos\theta\left(\frac{1}{4r}+\frac{\log r}{4r}\right) \tag{18}$$

In the evaluation of ϕ_2 the calculation follows the same pattern of development. From equation (10b), together with equations (12) and (18)

$$v_1^2=\left(\frac{+19}{6r^4}-\frac{7}{3r^6}+\frac{1}{2r^8}\right)+\cos 2\theta\left(\frac{-8}{3r^2}+\frac{1}{r^4}-\frac{1}{r^6}+\frac{1}{3r^8}\right)+$$
$$\cos 4\theta\left(\frac{1}{r^2}\right)+K\sin\theta\left(\frac{+1}{4r}+\frac{11}{6r^3}-\frac{5}{12r^5}+\frac{1}{3r^7}+\frac{2\log r}{r^5}\right)+$$
$$K\sin 3\theta\left(-\frac{1}{4r}-\frac{7}{6r^3}-\frac{1}{2r^5}+\frac{1}{12r^7}-\frac{2\log r}{r^3}\right)+$$
$$K^2\left(\frac{1}{4r^2}+\frac{1}{4r^4}+\frac{\log r}{2r^4}\right)+K^2\cos 2\theta\left(\frac{-1}{2r^2}-\frac{7}{12r^4}-\frac{1}{12r^6}-\right.$$
$$\left.\frac{\log r}{2r^2}-\frac{\log r}{r^4}\right)+K^3\sin\theta\left(+\frac{1}{4r^3}+\frac{\log r}{4r^3}\right) \tag{19}$$

This result, together with equations (12), (14), and (18), substituted in equation (11c), gives

$$\nabla^2\phi_2=(\gamma-1)\left\{\cos\theta\left(-\frac{1}{r^5}+\frac{2}{r^7}-\frac{3}{r^9}+\frac{1}{r^{11}}\right)+\cos 3\theta\left(\frac{3}{r^7}-\frac{1}{r^9}\right)+\right.$$
$$\cos 5\theta\left(\frac{-1}{r^5}\right)+K\sin 2\theta\left(\frac{-1}{2r^4}-\frac{7}{4r^6}-\frac{3}{2r^8}+\frac{3}{4r^{10}}\right)+$$
$$K\sin 4\theta\left(\frac{3}{4r^4}+\frac{3}{2r^6}-\frac{1}{4r^8}\right)+K^2\cos\theta\left(\frac{-1}{8r^3}-\frac{3}{8r^5}-\frac{9}{8r^7}+\frac{3}{8r^9}\right)+$$
$$K^2\cos 3\theta\left(\frac{1}{8r^3}+\frac{9}{8r^5}+\frac{3}{8r^7}-\frac{1}{8r^9}\right)+K^3\sin 2\theta\left(\frac{-3}{16r^4}-\frac{3}{8r^6}+\frac{1}{16r^8}\right)+$$
$$\left.K^4\cos\theta\left(\frac{-1}{16r^5}\right)\right\}+\left\{\cos\theta\left(\frac{-32}{3r^5}+\frac{39}{2r^7}-\frac{15}{r^9}+\frac{11}{3r^{11}}\right)+\right.$$
$$\cos 3\theta\left(\frac{19}{6r^3}+\frac{3}{r^7}-\frac{5}{3r^9}+\frac{1}{2r^{11}}\right)+\cos 5\theta\left(\frac{-3}{2r^3}-\frac{1}{r^5}\right)+$$
$$K\sin 2\theta\left(\frac{-1}{4r^2}-\frac{11}{3r^4}+\frac{25}{12r^6}-\frac{14}{3r^8}+\frac{3}{2r^{10}}-\frac{6\log r}{r^6}+\frac{4\log r}{r^8}\right)+$$

$$K\sin 4\theta\left(\frac{1}{4r^2}+\frac{11}{4r^4}+\frac{2}{r^6}-\frac{5}{12r^8}+\frac{1}{12r^{10}}+\frac{3\log r}{r^4}\right)+$$
$$K^2\cos\theta\left(\frac{-3}{8r^3}-\frac{2}{3r^5}-\frac{13}{24r^7}-\frac{\log r}{r^5}+\frac{\log r}{2r^7}\right)+$$
$$K^2\cos 3\theta\left(\frac{5}{8r^3}+\frac{2}{r^5}+\frac{3}{4r^7}-\frac{1}{4r^9}+\frac{\log r}{2r^3}+\frac{3\log r}{r^5}-\frac{\log r}{r^7}\right)+$$
$$K^3\sin 2\theta\left(\frac{-1}{2r^4}-\frac{1}{3r^6}-\frac{\log r}{2r^4}\right)+K^4\cos\theta\left(\frac{-1}{16r^5}\right)\Bigg\} \qquad (20)$$

To integrate, formulas (17a) and (17b) are again resorted to. The method of integration given in the appendix also provides integrals corresponding to the new type of terms appearing in the right-hand member of equation (20). Thus, the solution of

$$\nabla^2\Omega=\frac{\log r\sin m\theta}{r^s} \qquad (21)$$

satisfying boundary conditions

$$\left(\frac{\partial\Omega}{\partial r}\right)_{r=1}=0,\ \left(\frac{\partial\Omega}{\partial r}\right)_{r=\infty}=0,$$

is

$$\Omega=-\frac{\sin m\theta}{2mr^m}\left\{\frac{1}{(m+s-2)^2}+\frac{1}{(m-s+2)^2}\right\}-\frac{\sin m\theta}{2mr^{s-2}}\left\{\frac{2m\log r}{(m+s-2)(m-s+2)}+\frac{1}{(m+s-2)^2}-\frac{1}{(m-s+2)^2}\right\} \qquad (22a)$$

when $m\neq s-2$. When $m=s-2$,

$$\Omega=\frac{-\sin m\theta}{2m\,r^m}\left(\frac{1}{2}\log^2 r+\frac{1}{2m}\log r+\frac{1}{2m^2}\right) \qquad (22b)$$

Proceeding directly with the integration results in the following expression:

$$\phi_2=(\gamma-1)\Bigg\{\cos\theta\left(\frac{17}{60r}-\frac{1}{8r^3}+\frac{1}{12r^5}-\frac{1}{16r^7}+\frac{1}{80r^9}\right)+$$
$$\cos 3\theta\left(\frac{-61}{240r^3}+\frac{3}{16r^5}-\frac{1}{40r^7}\right)+\cos 5\theta\left(\frac{+1}{16r^3}-\frac{3}{80r^5}\right)+$$
$$K\sin 2\theta\left(\frac{427}{960r^2}-\frac{7}{48r^4}-\frac{3}{64r^6}+\frac{1}{80r^8}+\frac{\log r}{8r^2}\right)+$$
$$K\sin 4\theta\left(\frac{-1}{16r^2}+\frac{1}{320r^4}-\frac{1}{80r^6}-\frac{3\log r}{16r^4}\right)+$$
$$K^2\cos\theta\left(\frac{49}{128r}-\frac{3}{64r^3}-\frac{3}{64r^5}+\frac{1}{128r^7}+\frac{\log r}{16r}\right)+$$
$$K^2\cos 3\theta\left(\frac{-1}{64r}-\frac{57}{640r^3}+\frac{3}{128r^5}-\frac{1}{320r^7}-\frac{3\log r}{16r^3}\right)+$$
$$K^3\sin 2\theta\left(\frac{41}{512r^2}-\frac{1}{32r^4}+\frac{1}{512r^6}+\frac{3\log r}{64r^2}\right)+$$
$$K^4\cos\theta\left(\frac{+3}{128r}-\frac{1}{128r^3}\right)\Bigg\}+\Bigg\{\cos\theta\left(\frac{+137}{80r}-\frac{4}{3r^3}+\frac{13}{16r^5}-\frac{5}{16r^7}+\frac{11}{240r^9}\right)+\cos 3\theta\left(\frac{-19}{48r}-\frac{5}{48r^3}+\frac{3}{16r^5}-\frac{1}{24r^7}+\frac{1}{144r^9}\right)+$$
$$\cos 5\theta\left(\frac{1}{16r}+\frac{1}{16r^3}-\frac{1}{20r^5}\right)+K\sin 2\theta\left(\frac{1}{16}+\frac{2267}{2880r^2}-\frac{23}{144r^4}-\frac{19}{192r^6}+\frac{1}{40r^8}+\frac{11\log r}{12r^2}-\frac{\log r}{2r^4}+\frac{\log r}{8r^6}\right)+$$
$$K\sin 4\theta\left(\frac{-1}{64}-\frac{7}{48r^2}-\frac{7}{288r^4}-\frac{1}{48r^6}+\frac{1}{576r^8}-\frac{\log r}{4r^2}-\frac{\log r}{4r^4}\right)+$$
$$K^2\cos\theta\left(\frac{197}{288r}-\frac{17}{96r^3}-\frac{1}{72r^5}+\frac{3\log r}{16r}-\frac{\log r}{8r^3}+\frac{\log r}{48r^5}\right)+$$
$$K^2\cos 3\theta\left(\frac{-1}{16r}-\frac{911}{5760r^3}+\frac{1}{128r^5}-\frac{1}{160r^7}-\frac{\log r}{16r}-\frac{5\log r}{12r^3}-\frac{\log r}{16r^5}-\frac{\log^2 r}{4r^3}\right)+K^3\sin 2\theta\left(\frac{77}{576r^2}-\frac{1}{36r^4}+\frac{5\log r}{32r^2}+\frac{\log^2 r}{16r^2}\right)+$$
$$K^4\cos\theta\left(\frac{3}{128r}-\frac{1}{128r^3}\right)\Bigg\} \qquad (23)$$

APPLICATIONS OF THEORY

Using the expressions for ϕ_0, ϕ_1, and ϕ_2 the two-term approximation for velocity potential is

$$\Phi=U(\phi_0+\phi_1M^2+\phi_2M^4) \qquad (24)$$

and from this function the values of velocity at any point in the plane may be computed. Of particular interest is the evaluation of

$$-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}$$

for this gives velocity normal to the radius vector of the point in question and thus, when $r=1$, is equal to the velocity at the surface of the cylinder.

Neglecting all terms containing powers of $1/r$ greater than the first, Glauert (reference 1) has given his well known result

$$\left(-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)=U\left(\sin\theta+\frac{K}{2r}\frac{\sqrt{1-M^2}}{1-M^2\sin^2\theta}\right)$$

and under the same restrictions equation (24) gives

$$\left(-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)=U\left(\sin\theta+\frac{K}{2r}-\frac{KM^2\cos 2\theta}{4r}-\frac{KM^4\cos 2\theta}{8r}+\frac{KM^4\cos 4\theta}{16r}\right)$$

These results are identical to the order of M^4.

Velocity at the surface of the cylinder is

$$V(1,\theta)=U\Bigg\{2\sin\theta+\frac{K}{2}+M^2\left[\frac{2\sin\theta}{3}-\frac{\sin 3\theta}{2}-\frac{2K\cos 2\theta}{3}+\frac{K^2\sin\theta}{4}\right]+M^4\Bigg[(\gamma-1)\Bigg(\frac{23}{120}\sin\theta-\frac{11}{40}\sin 3\theta+\frac{1}{8}\sin 5\theta-\frac{127}{240}K\cos 2\theta+$$
$$\frac{23}{80}K\cos 4\theta+\frac{19}{64}K^2\sin\theta-\frac{81}{320}K^2\sin 3\theta-\frac{13}{128}K^3\cos 2\theta+\frac{1}{64}K^4\sin\theta\Bigg)+\Bigg(\frac{37}{40}\sin\theta-$$
$$\frac{25}{24}\sin 3\theta+\frac{3}{8}\sin 5\theta-\frac{887}{720}K\cos 2\theta+\frac{59}{72}K\cos 4\theta+\frac{71}{144}K^2\sin\theta-\frac{631}{960}K^2\sin 3\theta-$$
$$\frac{61}{288}K^3\cos 2\theta+\frac{1}{64}K^4\sin\theta\Bigg)\Bigg]\Bigg\} \qquad (25)$$

Results derived from this equation will be presented in terms of pressure coefficient which is defined as

$$\frac{p-p_0}{1/2\rho_0 U^2} \tag{26}$$

where the zero subscripts refer to free-stream conditions. By means of Bernoulli's equation it follows that

$$P_{M=0}=1-\left(\frac{V}{U}\right)^2, \tag{27}$$

where $P_{M=0}$ denotes the pressure coefficient for incompressible fluids. If P denotes pressure coefficient for a compressible fluid obeying the adiabatic law, then

$$P=\frac{2}{\gamma M^2}\left\{\left[1+\frac{\gamma-1}{2}M^2\left(1-\frac{V^2}{U^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1\right\} \tag{28}$$

where M is the Mach number of the free stream and γ is the ratio of specific heats (1.40 for air).

As an approximation for P the Glauert-Prandtl result (reference 2) is given by equation (2) and the Kármán-Tsien result (reference 2) is given by equation (1).

The velocity at the topmost part of the cylinder may be found by setting $\theta=90°$ in equation (25) and the resultant expression is a function of K and M. In figure 1, pressure coefficient at this point is plotted against M for $K=0$. As a test for rapidity of convergence the expressions for velocity, using only M^2 (one-term approximation) as well as M^2 and M^4 (two-term approximation), are used. It is to be noted that the curves diverge greatly near the critical Mach number, but that for smaller values of M, the curves derived from equation (25) are together and definitely lie between the results derived from the Glauert-Prandtl and Kármán-Tsien relations. Figures 2 and 3 show the same equations applied for $K=1/4$ and $1/2$, respectively. It thus appears from these calculations that the true value of P lies somewhere between the approximations applied. On the other hand experimental data, as determined from airfoils, have shown a better agreement with the Kármán-Tsien equation than have the theoretical results obtained here for the cylinder.

In figures 4 and 5 the same point on the cylinder is under consideration, but circulation is made negative by setting K equal to $-1/4$ and $-1/2$ in the two cases. As the pressure

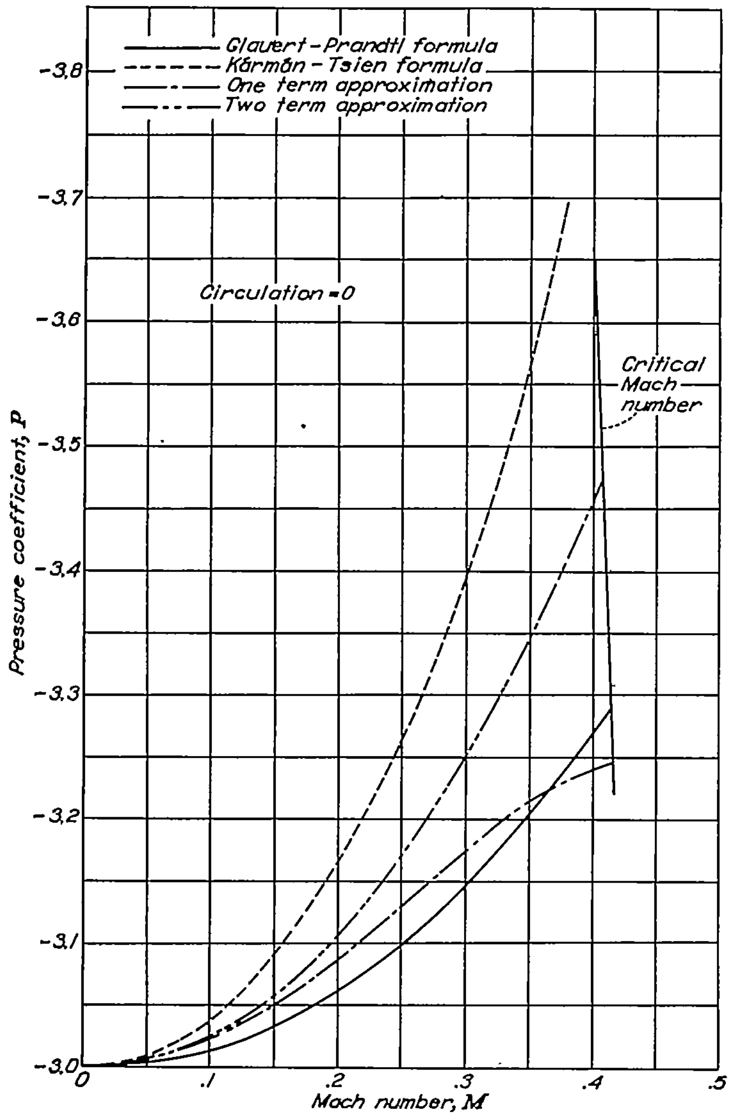

FIGURE 1.—Variation of minimum pressure coefficient with Mach number when circulation is 0.

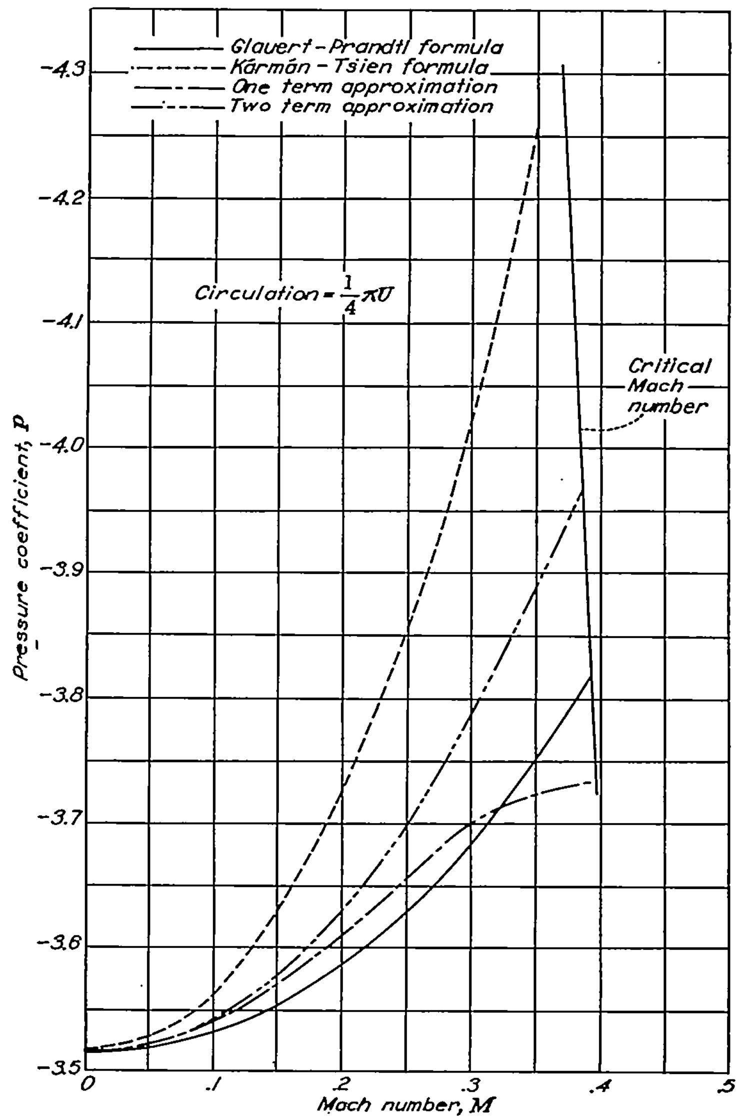

FIGURE 2.—Variation of minimum pressure coefficient with Mach number when circulation is $\frac{1}{4}\pi U$.

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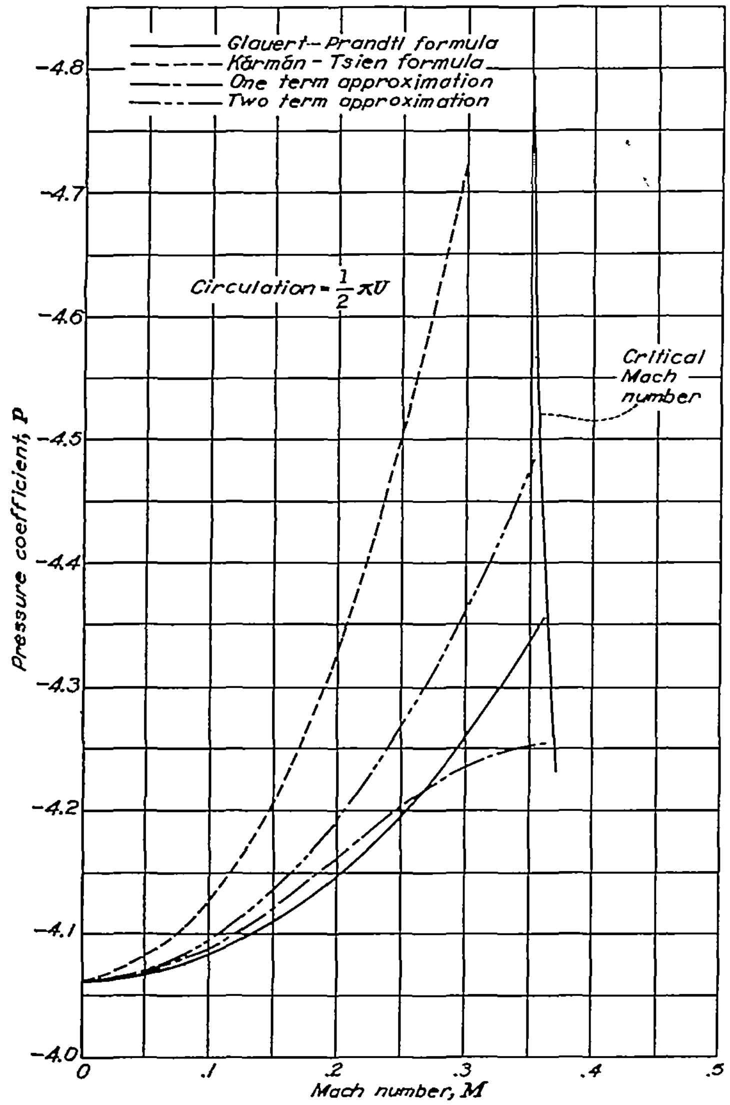

FIGURE 3.—Variation of minimum pressure coefficient with Mach number when circulation is ½ π U.

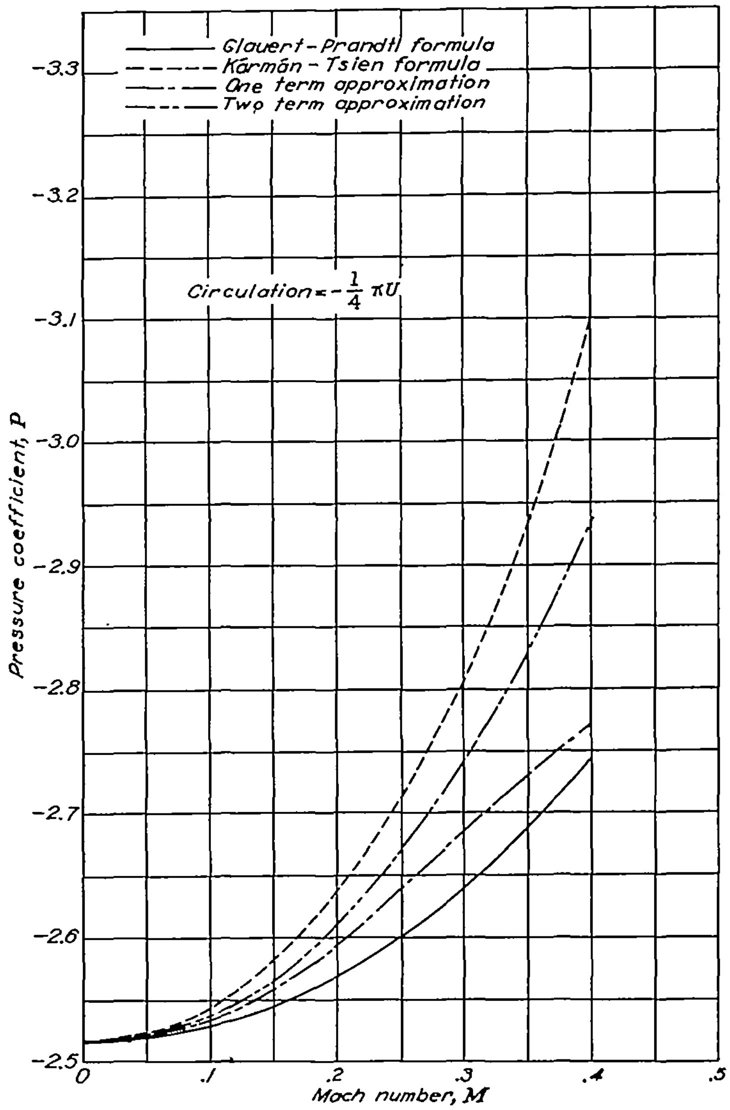

FIGURE 4.—Variation of pressure coefficient with Mach number at topmost point of cylinder when circulation is −¼ π U.

coefficient gets smaller in absolute value, the theoretical data agree more nearly with equation (1). For all the calculations, the one-term and two-term approximations diverge widely as the Mach number increases. This is to be expected for, as has been pointed out by Messrs. G. I. Taylor and C. F. Sharman in reference 14, the convergence of the series fails when M reaches its critical value. For near-critical velocities, several more terms would be required to furnish an accurate evaluation of the true potential-flow pressure coefficient.

Figure 6 shows the value of pressure coefficient at all points on the surface of the cylinder. These results were derived from equation (25) with K set equal to ¼ and at a Mach number of ⅖. Crosses on the graph are at positions obtained from equation (1) and the circles were determined by equation (2). The disagreement at the extreme pressure coefficients is again in evidence.

APPENDIX

For the integration of the differential equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=f(r,\theta) \tag{29}$$

with boundary conditions

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=1}=0 \text{ and } \left(\frac{\partial\phi}{\partial r}\right)_{r=\infty}=0 \tag{30}$$

it is assumed, as in Poggi's method, that the unit circle with center at the origin is surrounded externally by a continuous distribution of sources such that the source strength of an element $RdRd\omega$ is

$$f(R,\omega)RdRd\omega \tag{31}$$

Equation (29) may then be interpreted as the equation for incompressible flow in such a region.

The velocity potential of an incompressible fluid at point (r, θ) due to a unit source at (R,ω) may be calculated by the method of images. If this potential is denoted by $\overline{\phi}$, then

$$\overline{\phi}=\frac{1}{2\pi}\left\{\frac{1}{2}\log\left[r^2+R^2-2Rr\cos(\theta-\omega)\right]+\right.$$

$$\left.\frac{1}{2}\log\left[\frac{1}{r^2}+R^2-\frac{2R}{r}\cos(\theta-\omega)\right]+\log\frac{1}{R}\right\} \tag{32}$$

The required potential ϕ, satisfying equation (29), is therefore

$$\phi(r,\theta)=\frac{1}{4\pi}\int_s\int\Big\{\log\,[r^2+R^2-2Rr\cos\,(\theta-\omega)]+\log\left[\frac{1}{r^2}+R^2-\frac{2R}{r}\cos\,(\theta-\omega)\right]+2\log\frac{1}{R}\Big\}f(R,\omega)\,RdRd\omega \tag{33}$$

where the integration extends over the region of the plane lying external to the unit circle.

In the equations under consideration in this report the function $f\,(R,\,\omega)$ is restricted to one of the forms

$$\frac{\sin m\omega}{R^s} \qquad \frac{\cos m\omega}{R^s}$$

$$\log R\,\frac{\sin m\omega}{R^s} \qquad \log R\,\frac{\cos m\omega}{R^s}$$

$$m,s\geqq 1$$

As an example of the integration process, take the first case listed. Then, set $r'=1/r$, which results in

$$\phi(r,\theta)=\frac{1}{4\pi}\int_1^\infty\int_0^{2\pi}\log\Big(r^2+R^2-2Rr\cos(\theta-\omega)\Big)\frac{\sin m\omega}{R^s}RdRd\omega+\frac{1}{4\pi}\int_1^\infty\int_0^{2\pi}\log\Big(r'^2+R^2-2Rr'\cos(\theta-\omega)\Big)\frac{\sin m\omega}{R^s}RdRd\omega-\frac{1}{2\pi}\int_1^\infty\int_0^{2\pi}\log R\,\frac{\sin m\omega}{R^s}\,RdRd\omega$$
$$=I_1+I_2+I_3 \tag{34}$$

Integrating I_3 first with respect to ω shows immediately that its value is zero.

For purposes soon evident I_2 is written in the form

$$I_2=\frac{1}{4\pi}\int_1^\infty\int_0^{2\pi}\Big\{\log R^2+\log\left[1+\left(\frac{r'}{R}\right)^2\cos\,(\theta-\omega)\right]\Big\}\frac{\sin m\omega}{R^s}\,RdRd\omega \tag{35}$$

Since the log R^2 term vanishes, after integration with respect to ω, the expression for I_2 may be simplified further by the substitution

$$\log\left[1+\left(\frac{r'}{R}\right)^2-2\frac{r'}{R}\cos\,(\theta-\omega)\right]=-2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r'}{R}\right)^n\cos n(\theta-\omega) \tag{36}$$

since $r'<R$, and

$$I_2=-\frac{1}{2\pi}\int_1^\infty\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{rR}\right)^n\frac{1}{R^{s-1}}\sin m\omega\,(\cos n\theta\cos n\omega+\sin n\theta\sin n\omega)\,dRd\omega=-\frac{1}{2\pi}\int_1^\infty\frac{1}{m}\,\frac{1}{r^mR^{m+s-1}}\,\pi\sin m\theta dR$$
$$=-\frac{1}{2}\,\frac{1}{mr^m}\,\frac{\sin m\theta}{m+s-2} \tag{37}$$

To integrate I_1, the region exterior to the unit circle is broken into two parts. The first part is a circular ring external to the unit circle and extending to the fixed point r, the second part is the remaining portion of the plane and extends infinitely. Then

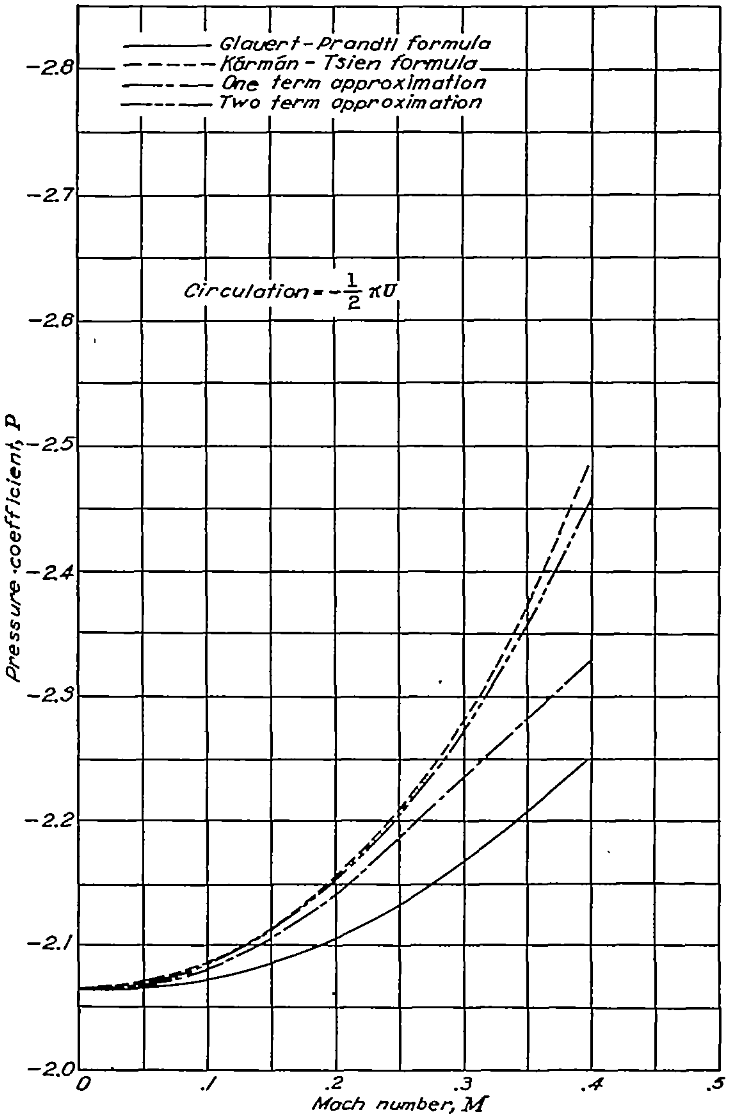

FIGURE 5.—Variation of pressure coefficient with Mach number at topmost point of cylinder when circulation is $-\frac{1}{2}\pi U$.

$$I_1=\frac{1}{4\pi}\int_1^r\int_0^{2\pi}\Big\{\log r^2+\log\left[1+\left(\frac{R}{r}\right)^2-2\frac{R}{r}\cos\,(\theta-\omega)\right]\Big\}\frac{\sin m\omega}{R^s}RdRd\omega+\frac{1}{4\pi}\int_r^\infty\int_0^{2\pi}\Big\{\log R^2+\log\left[1+\left(\frac{r}{R}\right)^2-2\left(\frac{r}{R}\right)\cos\,(\theta-\omega)\right]\Big\}\frac{\sin m\omega}{R^s}RdRd\omega$$
$$=J_1+J_2 \tag{38}$$

By use of the same series expansion as was previously used,

$$J_1=-\frac{1}{2\pi}\int_1^r\frac{\pi}{m}\frac{R^{m-s+1}}{r^m}\sin m\theta dR$$
$$=-\frac{\sin m\theta}{2mr^m}\,\frac{1}{m-s+2}\,(r^{m-s+2}-1)\text{ when }m-s+2\neq0$$
$$=-\frac{\sin m\theta}{2mr^m}\log r\text{ when }m-s+2=0 \tag{39}$$

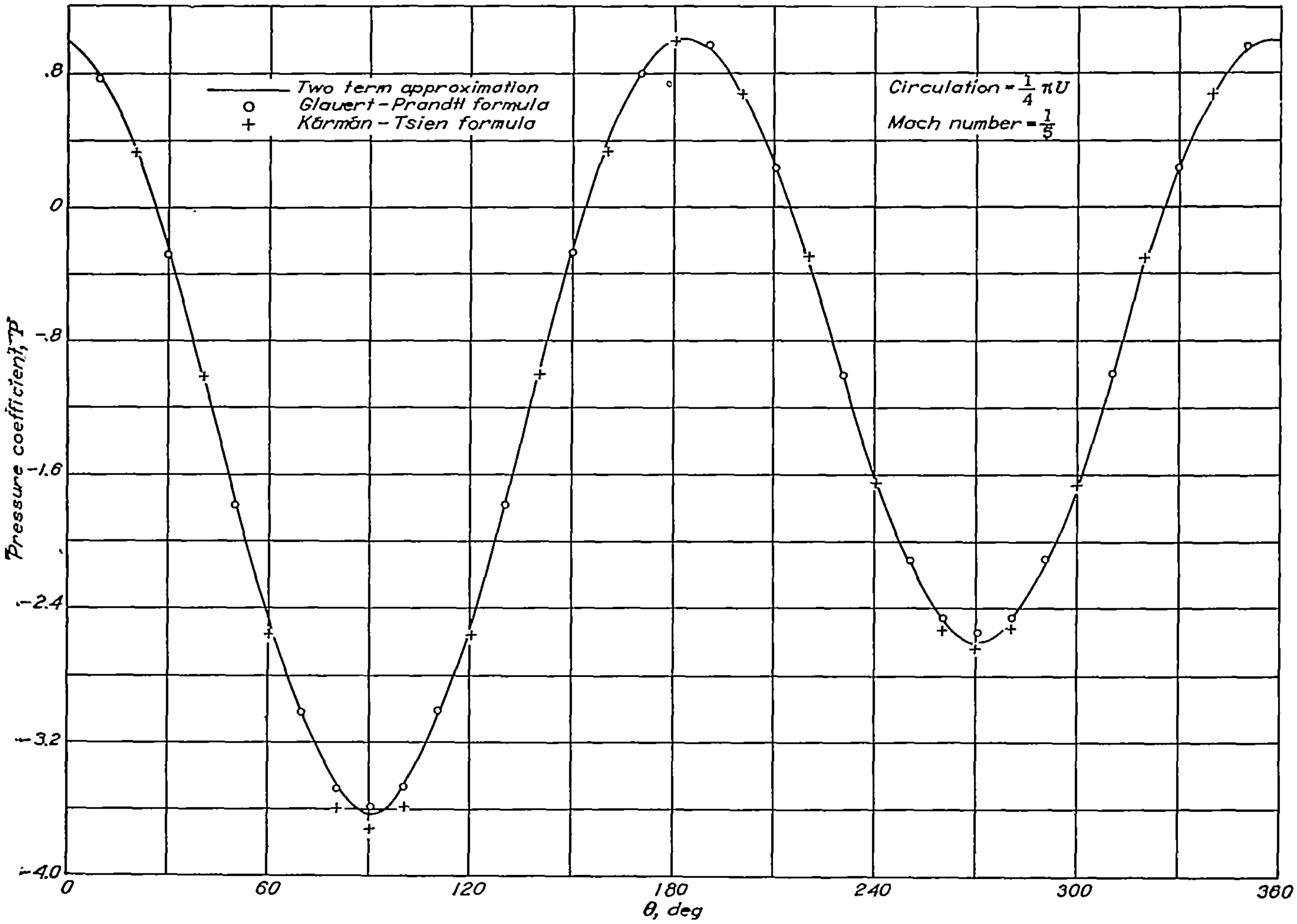

FIGURE 6.—Pressure coefficient on surface of cylinder for compressible flow with circulation.

In the same manner

$$J_2=-\frac{1}{2\pi}\int_r^\infty \frac{\pi}{m}\frac{r^m}{R^{m+s-1}}\sin m\theta dR$$

$$=-\frac{r^m}{2m}\frac{\sin m\theta}{(m+s-2)r^{m+s-2}} \tag{40}$$

From equations (37), (38), (39), and (40), the solution of equation (29), for the case in which

$$f(r,\theta)=\frac{\sin m\theta}{r^s},$$

is consequently

$$\phi=\frac{\sin m\theta}{m(m-s+2)(m+s-2)}\left\{\frac{(s-2)}{r^m}-\frac{m}{r^{s-2}}\right\} \text{ when } m+2\neq s$$

$$\phi=\frac{-\sin m\theta}{mr^m}\left\{\frac{1}{2m}+\frac{1}{2}\log r\right\} \qquad \text{when } m+2=s \tag{41}$$

For the other cases the integration process follows exactly the same procedure.

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